Comment on “Strong Quantum Darwinism and Strong Independence Are Equivalent to Spectrum Broadcast Structure”

In their Letter [1], the authors introduce the notions of (a) “strong quantum Darwinism” and (b) “strong independence.” It is proposed that, taken together, conditions (a) and (b) are equivalent to the so-called spectrum broadcast structure (SBS) for the system-environment quantum state ρ_{SE} (throughout this Comment, we follow the notations and definitions used in the Letter). By definition, if ρ_{SE} has a SBS, the state of $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_F$ relative to $\mathcal{S}$ is fully factorized and has *no correlations whatsoever*; and the authors aimed at capturing this fully factorized structure by an information-theory criterion of strong independence. In this Comment, we show that condition (b), while ensuring *pairwise* independence of the fragments $\mathcal{E}_i$ when conditioning on the system (a necessary condition for having the SBS), does not rule out the existence of higher-order correlations and, as such, is insufficient to imply a SBS. We first construct a (completely classical) counterexample of a state with no SBS yet satisfying conditions (a) and (b). We then propose a stronger condition (b’), ensuring the equivalence of conditions (a) and (b’) with SBS. At a conceptual level, the main conclusion of the Letter, namely the equivalence between, on the one hand, the SBS, and on the other, the conjunction of strong quantum Darwinism (as defined by the authors) and strong independence [as now defined by (b’)], remains therefore unaltered.

Counterexample.—We consider a qubit system $\mathcal{S}: \{|0\rangle, |1\rangle\}$, and the environment $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3$, with three qutrit fragments $\mathcal{E}_i: \{|0_i\rangle, |1_i\rangle, |2_i\rangle\}$ ($i \in \{1, 2, 3\}$). We define the projectors $\Pi_a^{(S)} = |a\rangle\langle a|$ and $\Pi_a^{(i)} = |a_i\rangle\langle a_i|$. Finally, we define the projectors $\Pi_{abc} = \Pi_a^{(1)} \otimes \Pi_b^{(2)} \otimes \Pi_c^{(3)}$. Our counterexample is the state ($0 \leq p \leq 1$)

$$\rho_{SE} = (1-p)\Pi_0^{(S)} \otimes \rho_{\mathcal{E}}^{(0)} + p\Pi_1^{(S)} \otimes \rho_{\mathcal{E}}^{(1)}, \quad (1)$$

with the relative states

$$\rho_{\mathcal{E}}^{(0)} = \Pi_{000}, \quad (2)$$

$$\rho_{\mathcal{E}}^{(1)} = \frac{1}{4}[\Pi_{111} + \Pi_{122} + \Pi_{212} + \Pi_{221}], \quad (3)$$

$$\rho_{\mathcal{E}_i \mathcal{E}_j}^{(1)} = \frac{1}{2}[\Pi_1^{(i)} + \Pi_2^{(i)}] \otimes \frac{1}{2}[\Pi_1^{(j)} + \Pi_2^{(j)}], \quad (4)$$

$$\rho_{\mathcal{E}_i}^{(1)} = \frac{1}{2}[\Pi_1^{(i)} + \Pi_2^{(i)}]. \quad (5)$$

As the relative states $\rho_{\mathcal{E}_i}^{(0)} = \Pi_0^{(i)}$ and $\rho_{\mathcal{E}_i}^{(1)} = [\Pi_1^{(i)} + \Pi_2^{(i)}]/2$ have disjoint supports, it is clear that each fragment has full access to the system’s state. Therefore, $I(\mathcal{S}:\mathcal{E}_i) = I_{\text{acc}}(\mathcal{S}:\mathcal{E}_i) = \chi(\mathcal{S}^\Pi:\mathcal{E}_i) = H(\mathcal{S})$ [condition (a)]. The state also fulfills condition (b) (pairwise independence conditioned on $\mathcal{S}$). Indeed, the relative states $\rho_{\mathcal{E}_i \mathcal{E}_j}^{(0)} = \Pi_0^{(i)} \otimes \Pi_0^{(j)}$ and $\rho_{\mathcal{E}_i \mathcal{E}_j}^{(1)}$ [Eq. (4)] are product states, which implies that $I(\mathcal{E}_i:\mathcal{E}_j|\mathcal{S}) = 0$ (as can be checked by direct computation on $\rho_{SE_i \mathcal{E}_j}$). Yet, the state $\rho_{SE_1 \mathcal{E}_2 \mathcal{E}_3}$ does *not* admit a SBS.

Indeed, the relative state $\rho_{\mathcal{E}}^{(1)}$ [Eq. (3)] does *not* factorize (while pairs of fragments are uncorrelated, the parity of the total number of fragments in state $|1_i\rangle$ is odd, yielding genuine three-partite correlations). This concludes our proof that the conditions (a) and (b) are insufficient to imply a SBS.

Proposed correct formulation of the theorem.—The above counterexample suggests to introduce a stronger condition (b’) ensuring the complete factorization of the fragments when conditioning on the system, which is a defining property of the SBS. Noticing that a multipartite state is factorized, $\rho_{\mathcal{E}_1 \dots \mathcal{E}_F} = \bigotimes_{i=1}^F \rho_{\mathcal{E}_i}$, if and only if the multipartite mutual information vanishes [2]: $I(\mathcal{E}_1, \dots, \mathcal{E}_F) := \sum_{i=1}^F H(\rho_{\mathcal{E}_i}) - H(\rho_{\mathcal{E}_1 \dots \mathcal{E}_F}) = 0$, we are led to propose the following:

Theorem.—The state ρ_{SE} has the SBS if and only if (a1) $I(\mathcal{S}:\mathcal{E}) = \chi(\mathcal{S}^\Pi, \mathcal{E})$ (classical-quantum state). (a2) $I_{\text{acc}}(\mathcal{S}:\mathcal{E}_i) = H(\mathcal{S})$ for all i (the information about $\mathcal{S}$ can be fully recovered by measuring any fragment). (b’) $I(\mathcal{E}_1, \dots, \mathcal{E}_F|\mathcal{S}) = 0$ (totally factorized relative states).

Proof.—If the state has the SBS, it is clear that conditions (a1), (a2), and (b’) are fulfilled. Conversely, condition (a1) (vanishing discord) implies that the system-environment state is of the form $\rho_{SE} = \sum_s p_s \Pi_s^{(S)} \otimes \rho_{\mathcal{E}}^{(s)}$ with $\{\Pi_s^{(S)}\}$ forming mutually orthogonal projectors for the system. Condition (b’) then implies that the relative states factorize: $\rho_{\mathcal{E}}^{(s)} = \bigotimes_{i=1}^F \rho_{\mathcal{E}_i}^{(s)}$. Finally, condition (a2) implies that, for each i , the states $\rho_{\mathcal{E}_i}^{(s)}$ are pairwise orthogonal. ■

We thank Thao P. Le and Alexandra Olaya-Castro for constructive feedback on our Comment. This work has been supported by the European Space Agency (Ariadna study 1912-01). I.F. acknowledges support from the Government of Spain (FIS2020-TRANQI and Severo Ochoa CEX2019-000910-S), Fundació Cellex and Fundació Mir-Puig through an ICFO-MPQ Postdoctoral Fellowship, Generalitat de Catalunya (CERCA and AGAUR SGR 1381).

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](#) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

Alexandre Feller,¹ Benjamin Roussel¹, Irénée Frérot^{2,3}
and Pascal Degiovanni⁴

¹European Space Agency–Advanced Concepts Team
ESTEC, Keplerlaan 1, 2201 AZ Noordwijk
Netherlands

²ICFO-Institut de Ciencies Fotoniques
The Barcelona Institute of Science and Technology
Avenue Carl Friedrich Gauss 3, 08860 Barcelona, Spain

³Max-Planck-Institut für Quantenoptik
D-85748 Garching, Germany

⁴Univ Lyon, Ens de Lyon, Université Claude Bernard Lyon 1
CNRS, Laboratoire de Physique, F-69342 Lyon, France

Received 22 January 2021; accepted 29 March 2021;
published 7 May 2021

DOI: [10.1103/PhysRevLett.126.188901](https://doi.org/10.1103/PhysRevLett.126.188901)

- [1] T. P. Le and A. Olaya-Castro, Strong Quantum Darwinism and Strong Independence are Equivalent to Spectrum Broadcast Structure, *Phys. Rev. Lett.* **122**, 010403 (2019).
- [2] D. Yang, K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, and W. Song, Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof, *IEEE Trans. Inf. Theory* **55**, 3375 (2009).