Origin of the Six-Gluon Amplitude in Planar $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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We study the maximally-helicity-violating six-gluon scattering amplitude in planar $\mathcal{N}=4$ super-Yang-Mills theory at finite coupling when all three cross ratios are small. It exhibits a double logarithmic scaling in the cross ratios, controlled by a handful of "anomalous dimensions" that are functions of the coupling constant alone. Inspired by known seven-loop results at weak coupling and the integrability-based pentagon operator product expansion, we present conjectures for the all-order resummation of these anomalous dimensions. At strong coupling, our predictions agree perfectly with the string theory analysis. Intriguingly, the simplest of these anomalous dimensions coincides with one describing the lightlike limit of the octagon, namely, the four-point function of large-charge Bogomol'nyi–Prasad–Sommerfield (BPS) operators.

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Introduction.—The scattering of massless gluons in maximally supersymmetric gauge theory, $\mathcal{N}=4$ super-Yang-Mills theory (SYM), exhibits remarkable simplifications in the planar limit of a large number of colors. Scattering amplitudes for n gluons become dual to null polygonal Wilson loops [1–5] and consequently they depend essentially only on $3n-15$ dual conformal cross ratios [6,7], out of the $3n-10$ Mandelstam invariants. Powerful bootstrap techniques [8–12] allow the construction of the six-gluon maximally-helicity-violating (MHV) amplitude through seven loops, and the next-to-MHV amplitude through six loops [13]. Seven-point amplitudes have also been bootstrapped through four loops [14–16] at the level of the symbol [17].

For generic values of the cross ratios, the perturbative results can be expressed in terms of generalized polylogarithms to all orders, but resumming the results into a finite-coupling expression remains challenging. In the near-collinear limit, a finite-coupling description is available, based on integrability and the pentagon operator product expansion (OPE) [18–25].

In this Letter we will provide a (conjectural) finite-coupling description for another kinematical limit of the six-gluon MHV amplitude, where all three cross ratios become small. The "origin" is reached, roughly speaking, by taking three adjacent pairs of gluon momenta to be parallel (collinear) simultaneously. However, it is a Euclidean limit, which cannot be achieved for real Minkowski momenta. Our description of the amplitude at the origin is based on resumming the OPE for a gas of gluonic flux-tube excitations. It involves a "tilted" version of the Beisert-Eden-Staudacher (BES) kernel entering the finite-coupling formula for the cusp anomalous dimension [26]. Different tilt angles generate different anomalous dimensions controlling logarithmically enhanced terms in the amplitude. Intriguingly, one of the anomalous dimensions also appears in the lightlike limit of the octagon [27–32], a correlation function of four operators with large R charge. We also predict the nonlogarithmic term, as well as the coefficient ρ controlling a "cosmic" amplitude normalization [33]. Our key results are Eqs. (17)–(20) for the anomalous dimensions and constant terms, and Eq. (29) for $\mathcal{N}=\rho$.

More precisely, we consider the MHV amplitude normalized by the BDS-like ansatz [11,12,34,35], which remains finite as the dimensional regulator $\epsilon = 2-\frac{1}{2}D \to 0$,

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0}\frac{\mathcal{A}_6(s_{ij},\epsilon)}{\mathcal{A}_6^{\mathrm{BDS-like}}(s_{ij},\epsilon)} = \exp\left[\mathcal{R}_6 + \frac{1}{4}\Gamma_{\mathrm{cusp}}\mathcal{E}^{(1)}\right]. \tag{1}$$

The notation and normalization (for now) follow Ref. [12], where Γ_{cusp} is the cusp anomalous dimension, $\mathcal{R}_6$ is the remainder function, and $\mathcal{E}^{(1)} = \sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)$ is the one-loop amplitude with Li_2 the dilogarithm. The normalized amplitude is a function of three cross ratios,

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \tag{2}$$

TABLE I. Coefficients of expansions in g^2 of the main coefficients through $L = 5$ loops.

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$\Gamma_{\rm oct}$	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
$\Gamma_{\rm cusp}$	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$	$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$
$\Gamma_{\rm hex}$	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$	$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$

constructed from the Mandelstam invariants $s_{i\ldots j} = (p_i + \cdots + p_j)^2$.

The logarithm of the amplitude $\mathcal{E}$, or, equivalently, the remainder function $\mathcal{R}_6$, exhibits logarithmic scaling when all cross ratios $\to 0$,

$$\ln \mathcal{E} = -\frac{\Gamma_{\rm oct}}{24}\ln^2(u_1u_2u_3) - \frac{\Gamma_{\rm hex}}{24}\sum_{i=1}^{3}\ln^2\left(\frac{u_i}{u_{i+1}}\right) + C_0 + \mathcal{O}(u_i), \tag{3}$$

with $u_4 \equiv u_1$ and where $\Gamma_{\rm oct}$, $\Gamma_{\rm hex}$, and C_0 are functions of the coupling constant $g^2 = \lambda/(4\pi)^2$ of the planar theory. The simpler $\ln^2 u$ behavior on the diagonal $u_1 = u_2 = u_3 = u$, where $\Gamma_{\rm hex}$ drops out, was conjectured [35] to hold at any coupling for a function $h = -\frac{3}{8}(\Gamma_{\rm oct} - \Gamma_{\rm cusp})$ appearing in $\mathcal{R}_6$, based on two-loop results in gauge theory and strong coupling behavior in string theory. The more general behavior (3) for unequal u_i was observed through seven loops [13], up to power corrections in the u_i. Its structure is reminiscent of Sudakov double logarithms.

The subleading power corrections to Eq. (3) do not exponentiate simply; in $\ln \mathcal{E}$ at L loops at finite u_1 there are terms with up to L powers of $\{\ln u_2, \ln u_3\}$. From this observation, based on results in Ref. [13], we expect the simplest finite-coupling resummation, apart from OPE limits, to be when all three cross ratios are small.

Weak coupling evidence.—The first evidence for Eq. (3) comes from weak coupling. The hexagon function bootstrap enables the analytic determination of $\mathcal{R}_6$ through seven loops [8,9,12,13,36], throughout the entire kinematical space. At the origin, the remainder function admits a simple representation, through at least seven loops [13],

$$\mathcal{R}_6 = c_1P_1 + c_2P_2 + c_0 + \mathcal{O}(u_i), \tag{4}$$

in terms of the two symmetric quadratic polynomials in $\ln u_i$,

$$P_1 = P_2 + \sum_{i=1}^{3}\ln^2 u_i, \qquad P_2 = \sum_{i=1}^{3}\ln u_i \ln u_{i+1}. \tag{5}$$

There is no term linear in $\ln u_i$. Close to the origin, $\mathcal{E}^{(1)} = -\frac{1}{2}\sum_i \ln^2 u_i - 3\zeta_2$, and using Eq. (1), one finds

$$\Gamma_{\rm oct} = \Gamma_{\rm cusp} - 16c_1 - 8c_2, \quad \Gamma_{\rm hex} = \Gamma_{\rm cusp} - 4c_1 + 4c_2, \tag{6}$$

and $C_0 = c_0 - \frac{3}{4}\zeta_2\Gamma_{\rm cusp}$. Perturbative results in Sec. 4.2 of Ref. [13] yield the numbers in Table I for the expansion in g^2, truncated here to 5 loops due to space limitations, where $\zeta_n = \zeta(n)$ is the Riemann zeta function. Note that $\Gamma_{\rm oct}$ has an expansion in powers of π^2 only (through 7 loops at least). Furthermore, it agrees with the exact [31] anomalous dimension controlling the lightlike limit of the octagon [27–30],

$$\Gamma_{\rm oct} = \frac{2}{\pi^2}\ln\cosh(2\pi g). \tag{7}$$

The other quantities are more complicated. Their perturbative expansions contain products of odd Riemann zeta values, much like the cusp anomalous dimension, which is recalled in the table.

Pentagon OPE.—Insight at higher loops is provided by the pentagon OPE [19]. It generates a systematic expansion of the amplitude around the collinear limit, $u_2 \to 0$, $u_1 + u_3 \to 1$, see Fig. 1, based on (flux tube) excitations of the dual two-dimensional string theory of 't Hooft surfaces that emerges in the large N_c, planar limit. The collinear limit is $\tau \to \infty$ at fixed σ and φ with the parametrization

$$u_2 = \frac{1}{e^{2\tau}+1}, \qquad u_1 = e^{2\tau+2\sigma}u_2u_3,$$
$$u_3 = \frac{1}{1 + e^{2\sigma} + 2e^{\sigma-\tau}\cosh\varphi + e^{-2\tau}}. \tag{8}$$

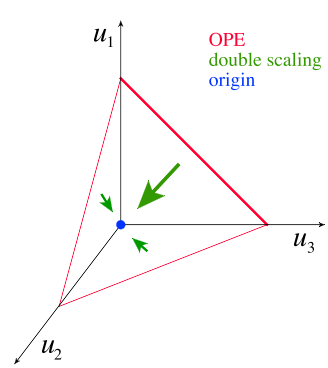

FIG. 1. Six-gluon kinematics. The collinear OPE is an expansion around one edge of the triangle, e.g., around $u_2 = 0$ and $u_1 + u_3 = 1$. The latter condition must be relaxed to get to the origin, as discussed below Eq. (8).

Because τ is conjugate to the energy, or twist, of flux-tube excitations, the collinear limit is controlled by the lowest-twist excitations that can propagate from one side of the hexagon to the other. These include gluonic, fermionic, and scalar excitations. Higher-twist contributions are suppressed by additional powers of $e^{-\tau}$.

We can move from the collinear limit toward the origin by first considering the limit where φ, τ are taken to be large, keeping their difference finite [22,37]. In this double-scaling limit, $u_2 \to 0$, but u_1 and u_3 are generic. The hyperbolic angle φ is conjugate to the helicity of the particles exchanged in the OPE channel. As $\varphi \to \infty$, the OPE is dominated by gluonic excitations, which have the highest helicity for a given twist. They form a family labeled by an integer $a = 1, 2, \ldots$, and each carries a rapidity u for its energy $E_a(u)$ and momentum $p_a(u)$ (conjugate to σ).

The OPE is naturally expressed in terms of the framed Wilson-loop expectation value $\mathcal{W}_6$ [18], which is related to $\mathcal{E}$ by

$$\mathcal{W}_6 = \mathcal{E} \exp\left[\frac{1}{2}\Gamma_{\text{cusp}}(\sigma^2 + \tau^2 + \zeta_2)\right]. \tag{9}$$

In the double-scaling limit, where only gluonic excitations contribute, $\mathcal{W}_6$ takes the form

$$\mathcal{W}_6 = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\mathbf{a}} e^{\varphi \sum_{k=1}^{N} a_k} \int \frac{d\mathbf{u}}{(2\pi)^N} \frac{e^{-\tau E + i\sigma P} \prod_k \mu_k}{\prod_{k<l} P_{kl} P_{lk}}, \tag{10}$$

where $\mathbf{a} = (a_1, \ldots, a_N)$ are positive integers and $d\mathbf{u} = du_1 \ldots du_N$ with $u_k \in \mathbb{R}$. The total energy and momentum of the N-gluon flux-tube state are $E = \sum_k E_{a_k}(u_k)$ and $P = \sum_k p_{a_k}(u_k)$. The integrand is built out of the pentagon transitions $P_{kl} = P_{a_k|a_l}(u_k|u_l)$ and measures $\mu_k = \mu_{a_k}(u_k)$, which have been conjectured to all orders in the coupling constant [22]. This concludes our review of the double-scaling limit.

To get to the origin from the double-scaling limit, we must then take $\varphi - \tau \to \infty$. While this limit lies outside of the radius of convergence of the OPE series (10), we may nevertheless reach it by analytically continuing in the helicity a, and replacing the sum by a contour integral with the help of the Sommerfeld-Watson transform,

$$\sum_{a \geq 1} (-1)^a f(a) \to \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{i f(a) da}{2 \sin(\pi a)}, \tag{11}$$

with $\epsilon \in (0, 1)$ [38]. Next we deform the contour of the a integral to the left, picking up residues from poles with $\text{Re}(a) \leq 0$. Because of the factor $e^{\varphi \sum_{k=1}^{N} a_k}$ in Eq. (10), poles with $\text{Re}(a) < 0$ are suppressed by powers of the u_i near the origin at weak coupling. That is, computing the $a = 0$ residue alone suffices to obtain the logarithmic and constant terms at the origin.

Take for illustration the one-loop $N = 1$ result [18,39],

$$\mu_a(u) = (-1)^a \frac{g^2 \Gamma(\frac{a}{2} + iu)\Gamma(\frac{a}{2} - iu)}{(\frac{a^2}{4} + u^2)\Gamma(a)} + O(g^4), \tag{12}$$

with $E_a = a + O(g^2)$ and $p_a = 2u + O(g^2)$. This integrand vanishes at $a = 0$. Nonetheless, the u integral diverges as $1/a^2$ due to pinch singularities at $u = \pm ia/2$. Accordingly, the dominant contribution is obtained by considering the residue around either one of these singularities, say the one at $u = ia/2$. Doing the u integral around $ia/2$ and then the a integral around 0, we get

$$\begin{aligned} i \oint \frac{da du}{(2\pi)^2} e^{a\varphi - a\tau + 2iu\sigma} \frac{\Gamma(1-a)\Gamma(\frac{a}{2} + iu)\Gamma(\frac{a}{2} - iu)}{\frac{a^2}{4} + u^2} \\ = \sigma^2 - (\varphi - \tau)^2 - \zeta_2 = -\ln u_1 \ln u_3 - \zeta_2, \end{aligned} \tag{13}$$

in agreement with the one-loop result $\mathcal{E}^{(1)} + 2(\sigma^2 + \tau^2 + \zeta_2)$ close to the origin, $u_i \to 0$, where we have

$$u_1 \sim e^{\tau - \varphi + \sigma}, \qquad u_2 \sim e^{-2\tau}, \qquad u_3 \sim e^{\tau - \varphi - \sigma}. \tag{14}$$

The above analysis remains unchanged as we increase the loop order or particle number: The amplitude at the origin may be obtained to all loops as the contour integral of the OPE integrand first around $u_k = ia_k/2$, and then around $a_k = 0$, for $k = 1, \ldots, N$. Since N-particle states are suppressed as g^{2N^2}, by restricting to $N \leq 2$ and applying the techniques of [37,40,41] we indeed reproduce all existing data, and obtain new predictions at 8 loops.

At finite coupling, the pole at $u = ia/2$ is replaced by a square-root branch cut between $\pm 2g + ia/2$, and the recipe is to integrate u closely around this cut. Equivalently, we may bring the contour through the cut to the so-called Goldstone sheet [22,42], where the flux-tube ingredients greatly simplify.

In particular, all Γ functions in the integrand (10) disappear when passing to the Goldstone sheet, which in turn allows us to reexpress $\mathcal{W}_6$ as a simpler, infinite-dimensional integral. Strikingly, this integral is secretly Gaussian in the vicinity of the origin (but not away from it). As a result, it is entirely characterized by a small number of moments, for which we were able to obtain conjectures that matched the structure through four loops. The details of this technical analysis are relegated to Sec. A of the Supplemental Material [43]. After some elementary algebra, we can recast our conjectures for the moments as very concise expressions for the anomalous dimensions and constants appearing at the origin, to be described next. They feature the celebrated BES kernel [26] which enters the all-loop formula for the cusp anomalous dimension and is ubiquitous in the flux-tube dynamics.

Tilted BES kernel.—The BES kernel can be described [20,26,44,45], after an expansion in terms of Bessel functions, as a semi-infinite matrix,

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}, \tag{15}$$

where $J_i(z)$ is the ith Bessel function of the first kind. We can spell out our finite-coupling conjectures for the origin in terms of this matrix. To this end, let us partition $\mathbb{K}$ into four blocks according to whether i and j in Eq. (15) are odd or even. After reshuffling lines and columns, we write

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{\circ\circ} & \mathbb{K}_{\circ\bullet} \\ \mathbb{K}_{\bullet\circ} & \mathbb{K}_{\bullet\bullet} \end{bmatrix}, \tag{16}$$

with $\mathbb{K}_{\circ\circ}$ the odd-odd block, built out of overlaps of odd Bessel functions (J_{2i-1}), $\mathbb{K}_{\circ\bullet}$ the odd-even one, and so on.

The tilted kernel is defined by

$$\mathbb{K}(\alpha) = 2\cos\alpha \begin{bmatrix} \cos\alpha\, \mathbb{K}_{\circ\circ} & \sin\alpha\, \mathbb{K}_{\circ\bullet} \\ \sin\alpha\, \mathbb{K}_{\bullet\circ} & \cos\alpha\, \mathbb{K}_{\bullet\bullet} \end{bmatrix}. \tag{17}$$

It reduces to the BES kernel (16) when $\alpha = \pi/4$, that is $\mathbb{K} = \mathbb{K}(\pi/4)$. Our conjectures are that the coefficients in Eq. (3) are given by

$$\Gamma_\alpha = 4g^2 \left[\frac{1}{1 + \mathbb{K}(\alpha)}\right]_{11} \tag{18}$$

with $\alpha = 0$, $\pi/4$ and $\pi/3$ for $\Gamma_{\rm oct}$, $\Gamma_{\rm cusp}$, and $\Gamma_{\rm hex}$, respectively, where the "11" subscript denotes the top left component of the semi-infinite matrix.

The constant C_0 is more complicated as it arises from determinants of quadratic forms appearing in the secretly Gaussian integral. Using formulas for the determinants of block matrices, we get

$$C_0 = -\frac{\zeta_2}{2}\Gamma_{\rm cusp} + D(\pi/4) - D(\pi/3) - \frac{1}{2}D(0), \tag{19}$$

where

$$D(\alpha) \equiv \ln\det[1 + \mathbb{K}(\alpha)] = \mathrm{tr}\ln[1 + \mathbb{K}(\alpha)]. \tag{20}$$

These formulae can be verified easily at weak coupling, since the matrix elements $\mathbb{K}_{ij} = O(g^{i+j})$. (See, e.g., Appendix A.2 in Ref. [22] for explicit expressions.) The inversion in Eq. (18) is done by expanding the geometric series in $\mathbb{K}(\alpha)$. Through four loops we get

$$\frac{\Gamma_\alpha}{4g^2} = 1 - 4c^2\zeta_2 g^2 + 8c^2(3 + 5c^2)\zeta_4 g^4 - 8c^2[(25 + 42c^2 + 35c^4)\zeta_6 + 4s^2\zeta_3^2]g^6 + \cdots, \tag{21}$$

$$D(\alpha) = 4c^2\zeta_2 g^2 - 4c^2(3 + 5c^2)\zeta_4 g^4 + \frac{8}{3}c^2[(30 + 63c^2 + 35c^4)\zeta_6 + 12s^2\zeta_3^2]g^6 + \cdots, \tag{22}$$

where $c = \cos\alpha$, $s = \sin\alpha$, and we verify agreement with the numbers in Table I using Eq. (19). Higher loops are easily generated. We provide results through 25 loops in an ancillary file. From the growth rate of their perturbative coefficients, all these quantities appear to have the same radius of convergence, $g_c^2 = 1/16$, as $\Gamma_{\rm cusp}$ [26].

The point $\alpha = 0$ corresponds to the octagon [27–30]. Here the off-diagonal blocks of the BES kernel drop out,

$$\mathbb{K}(\alpha = 0) = \begin{bmatrix} 2\mathbb{K}_{\circ\circ} & 0 \\ 0 & 2\mathbb{K}_{\bullet\bullet} \end{bmatrix}, \tag{23}$$

and with them all zeta values with odd arguments, leaving only powers of π^2. Nicely, in Eq. (18) these can be resummed exactly [31] into Eq. (7) for $\Gamma_{\rm oct}$, and similarly for the associated determinant,

$$D(0) = \frac{1}{4}\ln\left[\frac{\sinh(4\pi g)}{4\pi g}\right], \tag{24}$$

which also appears in the lightlike octagon [31].

In section B of the Supplemental Material [43], we analyze the strong-coupling behavior, and provide four terms in the expansion of Γ_α and two terms for $D(\alpha)$. Here we quote the leading-order expressions,

$$\Gamma_\alpha \approx \frac{8\alpha g}{\pi\sin(2\alpha)}, \qquad D(\alpha) \approx 4\pi g\left[\frac{1}{4} - \frac{\alpha^2}{\pi^2}\right]. \tag{25}$$

In Fig. 2, our weak- and strong-coupling expansions are compared with finite-coupling numerics. The agreement is excellent.

We can also validate our formulas at strong coupling through comparison with string theory, as described in more detail in the Supplemental Material [43], Sec. C. On the diagonal $u = u_1 = u_2 = u_3$, the string-theoretic analysis yields [35,46]

$$[\ln\mathcal{E}(u,u,u)]/\Gamma_{\rm cusp} = -\frac{3}{4\pi}\ln^2 u - \frac{\pi^2}{12} - \frac{\pi}{6} + \frac{\pi}{72}, \tag{26}$$

at small u, up to power corrections. With the help of Eq. (25), we can perfectly reproduce the above result, including the sphere contribution [46] of $+\pi/72$. Off the diagonal the behavior is richer at strong coupling. Nonetheless, following Ref. [47] we can also confirm the leading strong coupling behavior of $\Gamma_{\rm hex} = \Gamma_{\pi/3}$ in Eq. (25).

Cosmic normalization.—At last, let us remark about the normalization of the amplitude. The subtraction of divergences in the amplitude leaves a freedom in defining the finite part. Depending on the situation, it might prove convenient to subtract more than just the BDS-like amplitude. For example, in the collinear limit it is natural to work with the non-cyclic-invariant object $\mathcal{W}_6$. Another instance is provided by the so-called cosmic normalization for $\mathcal{E}$ introduced in the hexagon function bootstrap,

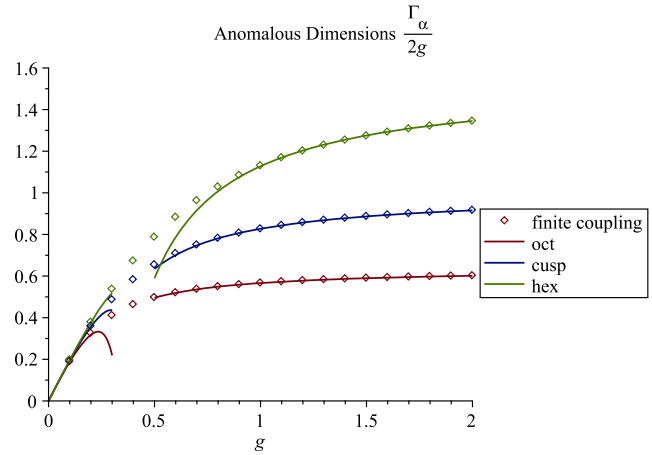

FIG. 2. Plot of $\Gamma_\alpha/2g$ as a function of g and comparison with weak and strong coupling expansions, Eqs. (21) and (30) from the Supplemental Material [43], respectively.

$$\mathcal{E}_{\text{cosmic}} = \mathcal{E}/\rho, \tag{27}$$

with $\rho = \rho(g^2)$ a function of the coupling constant. This function was determined iteratively in Refs. [13,33] by demanding that the spaces of functions in which the perturbative amplitudes live obey a coaction principle associated to a cosmic Galois group [48–50]. The implementation of this requirement fixes ρ order by order in perturbation theory,

$$\ln\rho = 8\zeta_3^2 g^6 - 160\zeta_3\zeta_5 g^8 + 16(-2\zeta_4\zeta_3^2 + 57\zeta_5^2 + 105\zeta_3\zeta_7)g^{10} + \cdots, \tag{28}$$

and two more loops can be found in Ref. [13]. Strictly speaking, $\ln\rho$ is fixed up to addition of pure even zeta values, which are trivial under the coaction, and in Eq. (28) all pure even zeta values $\zeta(2L)$ have been set to zero.

In the process of evaluating the infinite-dimensional integral in the Supplemental Material [43], Sec. A, a particular normalization factor emerges,

$$\mathcal{N} = \det[1 + \mathbb{K}]e^{-(1/2)\zeta_2\Gamma_{\text{cusp}}}, \tag{29}$$

Remarkably, the perturbative expansion of $\mathcal{N}$ bears a striking resemblance to ρ. To be precise, one has through at least seven loops

$$\ln\rho - \ln\mathcal{N} = \text{pure even zeta values}. \tag{30}$$

It is tempting to believe that Eq. (30) holds true to all orders in perturbation theory. It strongly suggests that the most natural normalization for the amplitude is simply to set $\rho = \mathcal{N}$. This ρ value shifts C_0 in Eq. (19) to $C_0 = -D(\pi/3) - \frac{1}{2}D(0)$, removing all $\alpha = \pi/4$ contributions from $\ln\mathcal{E}_{\text{cosmic}}$.

Conclusion and outlook.—We reported exact expressions for the anomalous dimensions and constant controlling the six-gluon MHV amplitude at the origin of the kinematical space. Our proposals rely on study of the weak coupling series on the field theory side and an extrapolation based on the pentagon OPE formulas. We evaluated our exact expressions to high orders in perturbation theory, numerically at finite coupling, as well as a few orders at strong coupling. The leading strong-coupling behavior was verified to agree with the string theory minimal surface analysis, plus a constant from the sphere determinant.

The main implication of our analysis is that the hexagon amplitude can be determined exactly at the origin, using the same ingredients needed for the cusp anomalous dimension, but tilted by an angle α. For now, the physical significance of α is unclear. Perhaps similar simplifications and extrapolations will be found for higher polygonal Wilson loops, utilizing other values of α. For example, one can define an "origin" of the heptagon by sending six of seven cross ratios to zero; the seventh is not independent and must go to unity. This limit is currently under investigation. Weak-coupling expansions generally feature coefficients of zeta values that are rational numbers. This consideration and Eq. (18) implies that $\sin^2\alpha \in \mathbb{Q}$. Our work also raises the hope of understanding the behavior at the origin for non-MHV amplitudes, and as one moves away from the origin for both MHV and non-MHV amplitudes, although in both cases it will not be as simple as the quadratic logarithmic behavior explored here. One could also study lightlike Wilson hexagons in other theories, to see whether the integrability of planar $\mathcal{N} = 4$ SYM is critical to this behavior.

We also observed an intriguing connection with the anomalous dimension which controls the lightlike limit of the correlator of four half-BPS operators dubbed the octagon [27,28,31]. It is reminiscent of the general correspondence between lightlike correlators and null polygonal Wilson loops [51]. It is not quite the same, however, since the Wilson loop studied here carries no R charge, while the octagon is full of it. It might be hinting at a connection between integrable descriptions based on the polygonalization of correlators [52–54] and amplitudes [19].

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