Derivation of Holographic Negativity in $\text{AdS}_3/\text{CFT}_2$

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We present a derivation of the holographic dual of logarithmic negativity in $\text{AdS}_3/\text{CFT}_2$ that was recently conjectured in *Phys. Rev. D* **99**, 106014 (2019). This is given by the area of an extremal cosmic brane that terminates on the boundary of the entanglement wedge. The derivation consists of relating the recently introduced Rényi reflected entropy to the logarithmic negativity in holographic conformal field theories. Furthermore, we clarify previously mysterious aspects of negativity at a large central charge seen in conformal blocks and comment on generalizations to generic dimensions, dynamical settings, and quantum corrections.

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Introduction.—The von Neumann entropy of the reduced density matrix is an excellent measure of the entanglement between bipartite subsystems in a pure state. In particular, it has played a major role in the understanding of how bulk geometry holographically emerges from microscopic degrees of freedom in the AdS/CFT correspondence. This is due to the fact that the von Neumann entropy of a boundary subregion A is equal to the area of the extremal bulk surface γ_A that is homologous to A [1–3],

$$S_{vN}(\rho_A) = \frac{\text{area}(\gamma_A)}{4G_N}, \quad (1)$$

where G_N is the bulk Newton constant. For mixed states ρ_{AB} , the von Neumann entropy fails to serve as a correlation measure because $S_{vN}(\rho_A) \neq S_{vN}(\rho_B)$ and the corresponding mutual information $[I_{AB} \equiv S_{vN}(\rho_A) + S_{vN}(\rho_B) - S_{vN}(\rho_{AB})]$ fails to quantify the entanglement between A and B . Rather, the mutual information captures classical correlations, such as thermodynamic entropy, on top of the quantum correlations.

In general, one may be interested in characterizing the entanglement structure of many-body systems in mixed states. In particular, it is an interesting question to ask if and how mixed state entanglement manifests itself geometrically in the bulk in AdS/CFT. For this purpose, we study the logarithmic negativity, a suitable measure of

entanglement for mixed states [4], based on the positive partial transpose criterion [5–10]. While most mixed state entanglement measures are defined in terms of intractable optimization procedures, the logarithmic negativity is operationally defined and generally computable. For a bipartite density matrix ρ_{AB} , the partial transpose is an operation that transposes just one of the subsystems

$$\langle i_A, j_B | \rho_{AB}^{T_B} | k_A, l_B \rangle = \langle i_A, l_B | \rho_{AB} | k_A, j_B \rangle, \quad (2)$$

where i_A, j_B, k_A , and l_B are bases for subsystems A and B . The logarithmic negativity is then defined as

$$\mathcal{E}(\rho_{AB}) \equiv \log |\rho_{AB}^{T_B}|_1, \quad (3)$$

where $|\mathcal{O}|_1 = \text{Tr} \sqrt{\mathcal{O} \mathcal{O}^\dagger}$ is the trace norm.

With motivations from quantum error-correcting codes and preliminary examples in 2D conformal field theory, two of the authors conjectured that logarithmic negativity in holographic conformal field theories is dual to a back-reacted entanglement wedge cross section in asymptotically anti-de Sitter (AdS) spacetimes [11], giving a concrete proposal for how mixed state entanglement is geometrized in the bulk. In this Letter, we provide a proof of the conjecture. For symmetric configurations, we provide direct comparisons between conformal field theory (CFT) computations of negativity and entanglement wedge cross sections.

Holographic reflected entropy.—Additional quantities have been shown to be related to the entanglement wedge cross section in holographic theories, including the entanglement of purification, odd entropy, and reflected entropy [12–15]. While each quantity is intriguing in its own right,

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we will use the reflected entropy for our purposes in deriving the holographic dual for logarithmic negativity in $\text{AdS}_3/\text{CFT}_2$.

We start by reviewing the construction of the reflected entropy and then state the conjecture for the holographic dual of logarithmic negativity [13]. Generically, mixed density matrices may be decomposed into a sum of pure states

$$\rho_{AB} = \sum_a p_a \rho_{AB}^{(a)}. \quad (4)$$

We may then perform a Schmidt decomposition on each pure state

$$\rho_{AB}^{(a)} = \sum_{i,j} \sqrt{\lambda_i^{(a)} \lambda_j^{(a)}} |i^{(a)}\rangle \langle j^{(a)}|_A \otimes |i^{(a)}\rangle \langle j^{(a)}|_B. \quad (5)$$

A canonical purification of the original mixed state may then be constructed in the doubled Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{A^*} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B^*}$

$$|\sqrt{\rho_{AB}}\rangle \equiv \sum_{a,i,j} \sqrt{p^{(a)} \lambda_i^{(a)} \lambda_j^{(a)}} |i^{(a)}\rangle_A |j^{(a)}\rangle_{A^*} |i^{(a)}\rangle_B |j^{(a)}\rangle_{B^*}. \quad (6)$$

The reflected entropy is then defined by [15]

$$S_R(A:B) \equiv S_{vN}(\rho_{AA^*}), \quad (7)$$

where S_{vN} is the von Neumann entropy and ρ_{AA^*} is the reduced density matrix on $\mathcal{H}_A \otimes \mathcal{H}_{A^*}$. In holographic conformal field theories, this was shown to be dual to twice the area of the entanglement wedge cross section [15]

$$S_R = 2E_W. \quad (8)$$

This was proven for time reflection symmetric states using the Lewkowycz-Maldacena gravitational replica trick [16]. It is natural to think that such a correspondence holds in generic time-dependent settings. In fact, many nontrivial checks of the time-dependent conjecture have been performed in Ref. [17].

We consider the Rényi reflected entropies

$$S_R^{(n)}(A:B) \equiv S^{(n)}(\rho_{AA^*}). \quad (9)$$

As shown by Dong [18], the Rényi entropies are related to the modular entropies, which are dual to cosmic branes in the bulk gravity theory

$$\tilde{S}_R^{(n)} \equiv n^2 \partial_n \left(\frac{n-1}{n} S^{(n)} \right) = \frac{\text{area}(\text{cosmic brane}_n)}{4G_N}. \quad (10)$$

The cosmic branes are codimension-two objects with tension

$$T_n = \frac{n-1}{4nG_N}. \quad (11)$$

The modular reflected entropies are then dual to twice the area of cosmic branes that terminate on the entanglement wedge [19]

$$\tilde{S}_R^{(n)} \equiv n^2 \partial_n \left(\frac{n-1}{n} S_R^{(n)} \right) = 2 \frac{\text{area}(\text{cosmic brane}_n)}{4G_N} \Big|_{E_W}. \quad (12)$$

While this was not explicitly stated in Ref. [15], it is a simple corollary of their gravitational construction of the state $|\sqrt{\rho_{AB}}\rangle$.

Holographic negativity conjecture.—In Ref. [11], it was conjectured that the logarithmic negativity in holographic conformal field theories is dual to a backreacting entanglement wedge cross section. The conjecture may be concisely stated in terms of the Rényi reflected entropy as

$$\mathcal{E} = \frac{S_R^{(1/2)}}{2} \quad (13)$$

for holographic conformal field theories. In the special cases where both the subregion configurations and the states are spherically symmetric, this backreaction may be accounted for by [20]

$$\mathcal{E} = \mathcal{X}_d E_W, \quad (14)$$

where $\mathcal{X}_d$ is a constant that depends on the dimension of the CFT

$$\mathcal{X}_d = \left(\frac{1}{2} x_d^{d-2} (1 + x_d^2) - 1 \right), \quad (15)$$

$$x_d = \frac{2}{d} \left(1 + \sqrt{1 - \frac{d}{2} + \frac{d^2}{4}} \right). \quad (16)$$

We will consider 2D CFTs where $\mathcal{X}_2 = 3/2$. We will explicitly compute the holographic negativity for disjoint intervals in the vacuum state and a single interval at finite temperature from conformal blocks, finding precise agreement with (14).

Deriving holographic negativity.—We now provide a simple derivation of (13) in 2D CFTs. We consider two arbitrary subsystems

$$A = \bigcup_{i=1}^{n_A} [u_i, v_i], \quad B = \bigcup_{i=1}^{n_B} [w_i, y_i]. \quad (17)$$

The logarithmic negativity in the vacuum state (and generic conformal transformations from the vacuum) may be computed by a correlation function of twist fields

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \log \left\langle \prod_i^{n_A} (\sigma_{n_e}(u_i) \bar{\sigma}_{n_e}(v_i)) \prod_i^{n_B} (\bar{\sigma}_{n_e}(w_i) \sigma_{n_e}(y_i)) \right\rangle. \quad (18)$$

See Ref. [21] for a thorough exposition of computing negativity in conformal field theory. The twist fields have conformal dimensions

$$h_{n_e} = \bar{h}_{n_e} = \frac{c}{24} \left(n_e - \frac{1}{n_e} \right), \quad (19)$$

$$h_{n_e}^{(2)} = \bar{h}_{n_e}^{(2)} = \frac{c}{12} \left(\frac{n_e}{2} - \frac{2}{n_e} \right), \quad (20)$$

where $h_{n_e}^{(2)}$ are the double twist fields that arise when fusing two twist fields of the same chirality. We compare (18) to the correlation function of generalized twist fields that computes half of the Rényi reflected entropy

$$S_R^{(1/2)}/2 = \lim_{m \rightarrow 1} \lim_{n \rightarrow 1/2} \log \left\langle \prod_i^{n_A} (\sigma_{g_A}(u_i) \sigma_{g_A}^{-1}(v_i)) \times \prod_i^{n_B} (\sigma_{g_B}(w_i) \sigma_{g_B}^{-1}(y_i)) \right\rangle_{\text{CFT}^{\otimes mn}}. \quad (21)$$

These generalized twist fields have the action of moving fields between sheets in two directions labeled by m and n . The $g_B g_A^{-1}$ twist field does not appear in the correlation function, rather in the conformal block, as it is the lowest weight primary operator in the operator product expansion (OPE) of g_B and g_A^{-1} . For more precise definitions, see Ref. [15]. The generalized twist fields have conformal dimensions

$$h_{g_B} = h_{g_A^{-1}} = \frac{cn(m^2 - 1)}{24m}, \quad (22)$$

$$h_{g_B g_A^{-1}} = \frac{2c(n^2 - 1)}{24n}. \quad (23)$$

The conformal dimensions, positions, and dominant intermediate channels of the operators in (18) and (21) precisely match. Thus, in the limit of a large central charge, we confirm (13) for the class of states that may be obtained by conformal transformations from the vacuum. For completely generic states that include primary operator insertions, we are also able to confirm (13), though we leave the details to the Supplemental Material [22].

Symmetric examples.—In symmetric configurations, we may use the simplification of (14). First, we focus on the negativity of disjoint intervals in the vacuum. The negativity is a conformally invariant quantity [23] and computed by

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \log \langle \sigma_{n_e}(\infty) \bar{\sigma}_{n_e}(1) \bar{\sigma}_{n_e}(x, \bar{x}) \sigma_{n_e}(0) \rangle, \quad (24)$$

where we set the two intervals to $A = [0, x]$ and $B = [1, \infty]$ for simplicity. We restrict ourselves to the holographic

FIG. 1. The blue line is the negativity that comes from the Virasoro block computed to order q^{500} using Zamolodchikov's recursion relation where q is the elliptic nome. The yellow line shows the minimal entanglement wedge cross section (25). Here we set $c = 10$ and $\epsilon = 10^{-2}$, and in this plot, we divide these quantities by c to rescale.

CFTs where the correlator can be approximated by a single conformal block. In the $x \rightarrow 1$ limit, the exchange operator is the identity and negativity is identically zero [21,24]. We are thus concerned with the $x \rightarrow 0$ limit where the exchanged operator in this block, $\sigma_{n_e}^2$, has a conformal dimension of the order of c ; therefore, the explicit form is unknown. Nevertheless, we can use the Zamolodchikov recursion relation [25,26] to evaluate the conformal block numerically to arbitrarily high precision.

On this background, the negativity should be compared to the entanglement wedge cross section, which is given by [12]

$$E_W = \begin{cases} \frac{c}{6} \log \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}, & 0 < x < 1/2, \\ 0, & 1/2 < x < 1. \end{cases} \quad (25)$$

In Fig. 1, we show the entanglement wedge cross section and the negativity calculated by the Zamolodchikov recursion relation. One can immediately find that the negativity perfectly matches the minimal entanglement wedge cross section. This result resolves the mysteries that arose when computing (24) using less direct approaches in Refs. [11,24,27].

We progress to the negativity of a single interval of length l at finite temperature β^{-1} , which can be calculated by [28,29]

$$\begin{aligned} \mathcal{E} &= \lim_{L \rightarrow \infty} \lim_{n_e \rightarrow 1} \log \langle \sigma_{n_e}(-L) \bar{\sigma}_{n_e}^2(-l) \sigma_{n_e}^2(0) \bar{\sigma}_{n_e}(L) \rangle_\beta \\ &\quad \times (2\epsilon)^{4h_{n_e}^{(2)}} (C_{\sigma_{n_e} \bar{\sigma}_{n_e}^2 \sigma_{n_e}^2})^2 \\ &= \frac{c}{2} \log \left(\frac{\beta}{2\pi\epsilon} e^{\pi l/\beta} \right) \\ &\quad + \lim_{L \rightarrow \infty} \lim_{n_e \rightarrow 1} \log \langle \sigma_{n_e}(\infty) \bar{\sigma}_{n_e}^2(1) \sigma_{n_e}^2(x, \bar{x}) \bar{\sigma}_{n_e}(0) \rangle, \end{aligned} \quad (26)$$

where ϵ is an UV regulator, and the cross ratio is given by

$$x \xrightarrow{L \rightarrow \infty} e^{-(2\pi l/\beta)}. \quad (27)$$

We may fix the OPE coefficient in the replica limit by considering the pure state limits of (18) and (21)

$$\lim_{n_e \rightarrow 1} C_{\sigma_{n_e} \bar{\sigma}_{n_e} \sigma_{n_e}} = 2^{c/2}. \quad (28)$$

In the holographic CFT, this four-point function can be approximated by a single conformal block. The dominant conformal block has two candidates. (i) s channel: In the semiclassical limit, the s channel block is simplified because the only contribution to the intermediate state is the primary exchange [30]. Thus, the approximated correlator is given by

$$\left| \begin{array}{c} \bar{\sigma}_{n_e}^2 \quad \sigma_{n_e}^2 \\ | \quad | \\ \sigma_{n_e} \quad h_{n_e} \quad \bar{\sigma}_{n_e} \end{array} \right|^2 \left(C_{\sigma_{n_e} \bar{\sigma}_{n_e} \sigma_{n_e}} \right)^2 \xrightarrow{n_e \rightarrow 1} |x^{\frac{c}{8}}|^2 2^{-\frac{c}{2}}. \quad (29)$$

Substituting this into (26), we obtain

$$\mathcal{E} = \frac{c}{2} \log \frac{\beta}{\pi \epsilon}. \quad (30)$$

(ii) t channel: The t channel block is just the $HHLL$ block [30,31], whose explicit form is given by

$$\begin{aligned} \mathcal{F}_{LL}^{HH}(0|1-z) &= \frac{z^{2h_L}}{(1-z)^{2h_H}} \mathcal{F}_{HH}^{LL}(0|1-z) \\ &= \frac{z^{h_L(\delta+1)}}{(1-z)^{2h_H}} \left(\frac{1-z^\delta}{\delta} \right)^{-2h_L}, \end{aligned} \quad (31)$$

where $\delta = \sqrt{1 - (24/c)h_H}$. Using this, we obtain

$$\left| \begin{array}{c} \bar{\sigma}_{n_e}^2 \\ | \quad | \\ \sigma_{n_e} \quad 0 \quad \sigma_{n_e}^2 \\ | \quad | \\ \sigma_{n_e} \quad \bar{\sigma}_{n_e} \end{array} \right|^2 \xrightarrow{n_e \rightarrow 1} (1-x)^{\frac{c}{2}}. \quad (32)$$

This leads to the negativity

$$\mathcal{E} = \frac{c}{2} \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi l}{\beta} \right). \quad (33)$$

Thus, we can conclude that the negativity at finite temperature is

$$\mathcal{E} = \min \left[\frac{c}{2} \log \frac{\beta}{\pi \epsilon}, \frac{c}{2} \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi l}{\beta} \right) \right]. \quad (34)$$

This result perfectly matches the minimal entanglement wedge cross section in Bañados Teitelboim Zanelli black hole geometry [12] upon using (14).

Discussion.—In this Letter, we have derived the holographic dual of logarithmic negativity in $\text{AdS}_3/\text{CFT}_2$, confirming the conjecture from Ref. [11]. There are several directions for future understanding. In particular, it is important to prove the holographic conjecture in higher dimensions. 3D gravity and 2D conformal field theory are quite special, so we are far from a complete derivation. Such a derivation may require a clever implementation of the gravitational replica trick for the partially transposed density matrix or a comparison of correlation functions of higher-dimensional twist operators for negativity and Rényi reflected entropy. Furthermore, the reflected entropy is not proven to (though it is believed to) be dual to the entanglement wedge cross section in generic dynamical settings. Because we have used the reflected entropy as a crutch in our derivation, the dynamical proposal for logarithmic negativity in $\text{AdS}_3/\text{CFT}_2$ is also still conjectural. Finally, it would be interesting to investigate quantum corrections to the holographic formula in the sense of Ref. [32]. General expectations and hints from error-correcting codes lead us to guess that the leading correction comes from the logarithmic negativity between the bulk fields on either side of the entanglement wedge cross section.

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