Quark-Hadron Continuity beyond the Ginzburg-Landau Paradigm

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(Received 4 December 2018; revised manuscript received 7 February 2019; published 30 May 2019)

Quark-hadron continuity is a scenario in which hadronic matter is continuously connected to a color superconductor without phase transitions as the baryon chemical potential increases. This scenario is based on Landau's classification of phases, since they have the same symmetry breaking pattern. We address the question of whether this continuity is true as quantum phases of matter, which requires treatment beyond the Ginzburg-Landau description. To examine the topological nature of a color superconductor, we derive a dual effective theory for U(1) Nambu-Goldstone (NG) bosons and vortices of the color-flavor locked phase and discuss the fate of emergent higher-form symmetries. The theory has the form of a topological BF theory coupled to NG bosons, and fractional statistics of test quarks and vortices arise as a result of an emergent $\mathbb{Z}_3$ two-form symmetry. We find that this symmetry cannot be spontaneously broken, indicating that quark-hadron continuity is still a consistent scenario.

DOI: 10.1103/PhysRevLett.122.212001

Introduction.—One of the most fundamental questions in nuclear physics is to identify possible phases of quantum chromodynamics (QCD) [1–5]. Understanding the phase structures at finite baryon densities is relevant to the physics inside neutron stars and has been of interest to nuclear and astrophysicists [6]. Hadronic matter is expected to exhibit nucleon superfluidity at finite densities. At very high densities, color superconductivity [7,8] appears with a symmetric-pairing pattern among light three flavors, called the color-flavor locked (CFL) phase [9]. As classical many-body physics, phases of matter are classified by the pattern of spontaneous symmetry breaking. Based on this view, it is proposed that nucleon superfluidity and the CFL phases are connected with a smooth crossover, since they have the same symmetry breaking pattern: this is the quark-hadron continuity [10].

The question we would like to address here is whether this continuity holds beyond Ginzburg-Landau (GL) paradigm. Classification of quantum phases, i.e., zero-temperature phases of quantum many-body systems, requires beyond-GL description, because local order parameters cannot capture topological order. Importance of topology has been recognized in understanding gapped quantum phases [11,12]. A microscopic picture of topological order is given by long-range entanglement [13–15], and its low-energy description has spontaneously broken higher-form global symmetry [16]. An important consequence is that states with different topological orders cannot be continuously connected and there should be a quantum phase transition between them. In recent years, the role of topology for gapless quantum systems has also been gradually taken into account, and it potentially has an impact for understanding cuprate superconductors [17,18].

The quark-hadron continuity has recently been examined in the presence of superfluid vortices [19–21]. In the CFL phase, the minimal superfluid circulation of vortices is a fractional number $1/3$ [22–24]. In addition to U(1) circulation, they also carry color holonomies. It is pointed out that this is a physical observable using color Wilson loops and results in fractional statistics between test quarks and vortices [21]. This has a certain similarity with topologically ordered phases in condensed matter physics, which poses doubt on quark-hadron continuity as quantum phases of matter [21].

In this Letter, we carefully examine the role of topology in the CFL phase [25]. We first derive the low-energy effective field theory of the CFL phase starting from the gauged GL model. It describes Nambu-Goldstone (NG) bosons associated with the breaking of U(1) symmetry and superfluid vortices in a unified way. In particular, the effective theory correctly encodes the relation between the superfluid vortex and Wilson loop, which is responsible for the fractional statistics of colored test particles and vortices. We clarify that the $\mathbb{Z}_3$ fractional phase is a consequence of an emergent $\mathbb{Z}_3$ two-form symmetry in the effective theory, generated by color Wilson loops. The charged object under this symmetry is nothing but the CFL vortices. We show that this emergent two-form symmetry cannot be spontaneously broken, and thus the emergent two-form gauge field is confined. This means that the CFL phase has a trivial topological structure, and we conclude that the quark-hadron continuity scenario is alive also as quantum phases of matter.

Symmetry of color-flavor locking.—We consider a three-flavor QCD with degenerate quark masses [30]. The system has the global symmetry

$$\frac{\mathrm{SU}(3)_f \times \mathrm{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3}, \tag{1}$$

where $\mathrm{SU}(3)_f$ is the vectorlike flavor symmetry, U(1) is the quark-number symmetry, and two $\mathbb{Z}_3$ factors in the denominators are introduced to remove the redundancies among $\mathrm{SU}(3)_f$, U(1), and SU(3) color gauge invariance [31–33]. In the presence of large chemical potential, quarks form the Fermi surface with large Fermi momentum. Since QCD has asymptotic freedom, the presence of this typical large energy scale suggests that the system is weakly coupled and semiclassical computation becomes reliable [9]. Within the one-gluon exchange, quark-quark interaction is attractive in the antisymmetric channel, indicating the Cooper instability of the Fermi surface. Motivated by this observation, it is quite useful to introduce the diquark operator Φ using the quark field q by

$$\Phi_{c_1 f_1} = \varepsilon_{c_1 c_2 c_3} \varepsilon_{f_1 f_2 f_3} (q^t_{c_2 f_2} i\gamma_0 \gamma_2 \gamma_5 q_{c_3 f_3}). \tag{2}$$

Here, c_i and f_i represent the color and flavor labels, respectively. The diquark field Φ is in the antifundamental representation for $\mathrm{SU}(3)_c$ color and $\mathrm{SU}(3)_f$ flavor symmetry, and it has charge 2 under U(1).

Using this diquark field, the simplest effective Lagrangian is given by the gauged Ginzburg-Landau model

$$S = \frac{1}{2g_{\mathrm{YM}}^2}|G|^2 + \frac{1}{2}|(d + i a_{\mathrm{SU}(3)})\Phi|^2 + V_{\mathrm{eff}}(\Phi^\dagger \Phi, \det(\Phi)), \tag{3}$$

where $a_{\mathrm{SU}(3)}$ is the $\mathrm{SU}(3)_c$ color gauge field, G is its field strength, the effective potential V_{eff} depends only on the color-singlet order parameters, $\Phi^\dagger\Phi$ and $\det(\Phi)$, and V_{eff} has the symmetry $[\mathrm{SU}(3)_f \times \mathrm{U}(1)]/[\mathbb{Z}_3 \times \mathbb{Z}_3]$. For simplicity of discussion, we neglect the effect of the absence of Lorentz symmetry due to the chemical potential, but the extension will be straightforward. Let us now assume that V_{eff} has the minima at

$$\Phi^\dagger \Phi = \Delta_0^2 \mathbf{1}. \tag{4}$$

Taking the determinant of both sides, we get $|\det\Phi| = \Delta_0^3$. In the gauge-invariant language [34,35], classical vacua break the global symmetry spontaneously as

$$\frac{\mathrm{SU}(3)_f \times U(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3} \to \frac{\mathrm{SU}(3)_f \times \mathbb{Z}_6}{\mathbb{Z}_3 \times \mathbb{Z}_3} = \frac{\mathrm{SU}(3)}{\mathbb{Z}_3} \times \mathbb{Z}_2. \tag{5}$$

Picking up a classical vacuum with $\det(\Phi) = \Delta_0^3$, we can fix the gauge of SU(3) color group so that

$$\Phi = \Delta_0 \mathbf{1}. \tag{6}$$

Since Φ is in the bi-(anti-)fundamental representation of $\mathrm{SU}(3)_c \times \mathrm{SU}(3)_f$, the symmetry breaking pattern in this fixed gauge looks like

$$\frac{\mathrm{SU}(3)_c \times \mathrm{SU}(3)_f \times \mathrm{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3} \to \frac{\mathrm{SU}(3)_{c+f} \times \mathbb{Z}_6}{\mathbb{Z}_3 \times \mathbb{Z}_3}, \tag{7}$$

where $\mathrm{SU}(3)_{c+f}$ is the diagonal subgroup of $\mathrm{SU}(3)_c \times \mathrm{SU}(3)_f$. This is why it is called color-flavor locking [9].

Derivation of a unified theory of U(1) NG bosons and CFL vortices.—In the CFL phase, there are massless NG bosons associated with the spontaneous breaking of U(1) baryon number symmetry, and we can construct the phenomenological Lagrangian by nonlinear realization. Because of the quark masses, CFL pions are massive and they can be neglected at low energies. Starting from the gauged GL theory with $\mathrm{SU}(3)_c$ color gauge group, we derive the effective low-energy theory that satisfies this requirement.

In order to correctly describe the possible low-energy excitations including a higher-dimensional object, it is important to take into account the topology of the ground-state manifold. Here, we take the gauge so that the diquark field Φ is a diagonal matrix,

$$\Phi = \Delta_0 \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \tag{8}$$

where ϕ_i is 2π periodic scalar fields. This realizes (4) and hence indicates the symmetry breaking pattern (5). This choice of gauge is an analog of the maximal Abelian gauge in Yang-Mills theory with adjoint scalars [36]. In this gauge fixing, the local gauge redundancy becomes the Cartan subgroup of $\mathrm{SU}(3)_c$,

$$\frac{\mathrm{U}(1)_{\tau_3} \times \mathrm{U}(1)_{\tau_8}}{\mathbb{Z}_2} \subset \mathrm{SU}(3)_c. \tag{9}$$

Here, $\mathrm{U}(1)_{\tau_3}$ and $\mathrm{U}(1)_{\tau_8}$ are U(1) groups generated by $\tau_3 = \mathrm{diag}[1, -1, 0]$ and by $\tau_8 = \mathrm{diag}[1, 1, -2]$, respectively. Since the rotations by π in $\mathrm{U}(1)_{\tau_3}$ and $\mathrm{U}(1)_{\tau_8}$ give the same transformation matrix, $\mathrm{diag}[e^{i\pi}, e^{i\pi}, 1]$, the group structure is divided by $\mathbb{Z}_2$. Let us denote the corresponding U(1) gauge fields by a_3 and a_8, and then the low-energy effective action (3) becomes

$$S=\frac{1}{2g_0^2}(|d\phi_1+a_3+a_8|^2+|d\phi_2-a_3+a_8|^2+|d\phi_3-2a_8|^2). \tag{10}$$

Here, we omit the kinetic term of gauge fields since they become heavy by Higgs mechanism, and $g_0 = \Delta_0^{-1}$.

Each scalar ϕ_i is not gauge invariant, and the only gauge-invariant combination is

$$\varphi = \phi_1 + \phi_2 + \phi_3, \tag{11}$$

and this corresponds to the NG boson associated with the spontaneous breaking of U(1) symmetry. Another important remark is that each Wilson loop of a_3 and a_8 is not observable since the gauge group is $[\mathrm{U}(1)_{\tau_3} \times \mathrm{U}(1)_{\tau_8}]/\mathbb{Z}_2$ instead of $\mathrm{U}(1)_{\tau_3} \times \mathrm{U}(1)_{\tau_8}$ [37]. Observable Wilson lines are generated by

$$W_3(C)^2, \qquad W_8(C)^2, \qquad W_3(C)W_8(C), \tag{12}$$

where $W_3(C)=\exp(i\int_C a_3)$ and $W_8(C)=\exp(i\int_C a_8)$. As a related fact, the normalization of gauge fields a_3 and a_8 has to be modified from a canonical choice of U(1) gauge fields as

$$\int da_3 \in \pi\mathbb{Z}, \qquad \int da_8 \in \pi\mathbb{Z}, \tag{13}$$

with the constraint

$$\int da_3 = \int da_8 \mod 2\pi. \tag{14}$$

We are interested in the role of vortex configurations in the CFL phase, and they are realized as the defect of the scalar field in the gauged GL description. For description of topological defects, it is convenient to take an Abelian duality [38]. As preparation, let us derive [40] the Abelian dual of the following model in four dimensions:

$$S = \frac{1}{2g_0^2}(d\phi + ka) \wedge \star(d\phi + ka), \tag{15}$$

where ϕ is the 2π periodic scalar field, a is the U(1) gauge field, and $k \in \mathbb{Z}$ is the U(1) charge. We can rewrite this theory by introducing the $\mathbb{R}$-valued three-form field h as

$$S = \frac{g_0^2}{8\pi^2} h \wedge \star h - \frac{i}{2\pi} h \wedge (d\phi + ka). \tag{16}$$

Solving the equation of motion of h, we get $h = (2\pi i/g_0^2) \star (d\phi + ka)$ and obtain the original action by substitution. Instead of integrating out h, we solve the equation of motion of ϕ first, and then we obtain

$$h = db, \tag{17}$$

with a U(1) two-form gauge field b. The action becomes

$$S = \frac{g_0^2}{8\pi^2}|db|^2 + i\frac{k}{2\pi} b \wedge da. \tag{18}$$

This is the dual action of the Abelian Higgs model with charge k.

Applying this procedure to the effective action (10) for the CFL phase, we obtain

$$S_{\mathrm{eff}} = \frac{g_0^2}{8\pi^2}\sum_{i=1}^{3}|db_i|^2 + \frac{i}{2\pi}\sum_{i=1}^{3}\sum_{A=3,8} K_{iA} b_i \wedge da_A, \tag{19}$$

where the matrix K is given by

$$K = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{pmatrix}. \tag{20}$$

This is the low-energy effective gauge theory describing the NG boson, vortices, and color Wilson lines [41]. It has a structure of a topological BF theory coupled with massless NG bosons. General properties of this theory will be discussed elsewhere [42].

Fractional statistics and an emergent two-form symmetry.—The effective theory derived here encodes the relation between the color holonomies and superfluid circulations. In the dual description, we can define the vortex operator as the Wilson surface operator

$$V_i(M_2) = \exp\left(i\int_{M_2} b_i\right), \tag{21}$$

where M_2 is a vortex world sheet. Using (19), one can show that the braiding statistics between the vortex V_i and test quarks W_A is given by [43]

$$\frac{\langle V_i(M_2)W_A(C)\rangle}{\langle V_i(M_2)\rangle} = \exp\left[2\pi i K^+_{Ai}\mathrm{link}(C, M_2)\right], \tag{22}$$

where $\mathrm{link}(C, M_2) \in \mathbb{Z}$ is the linking number of C and M_2, and K^+_{Ai} is the Moore-Penrose inverse of K,

$$K^+ = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}. \tag{23}$$

Now, let us recall that the physical Wilson loops consist only of W_3^2, W_8^2, and W_3W_8. We find that $W_3^2 = 1$, $W_8^2 = (W_3W_8)^{-1}$, and

$$\frac{\langle V_i(M_2)W_8(C)^2\rangle}{\langle V_i(M_2)\rangle} = \exp\left(\frac{2\pi i}{3}\mathrm{link}(C, M_2)\right). \quad (24)$$

This reproduces the observation made in a recent paper [21]. Equation (24) indicates the emergence of a $\mathbb{Z}_3$ two-form symmetry [16] in the CFL phase, where the generators are Wilson loops W_8^2, and charged objects are CFL vortices V_i. The explicit transformation of this two-form symmetry is given by

$$b_1 \mapsto b_1 + \frac{1}{3}\lambda, \quad b_2 \mapsto b_2 + \frac{1}{3}\lambda, \quad b_3 \mapsto b_3 - \frac{2}{3}\lambda, \quad (25)$$

where λ is a flat two-form U(1) connection with $\int_{M_2}\lambda \in 2\pi\mathbb{Z}$ for $\partial M_2 = \emptyset$. Under this transformation, the action changes as

$$\Delta S = \frac{i}{2\pi}\int \lambda \wedge (2da_8) \in 2\pi i\mathbb{Z}, \quad (26)$$

using the fact that $\int\lambda \in 2\pi\mathbb{Z}$ and $\int da_8 \in \pi\mathbb{Z}$. Since $\exp(-\Delta S) = 1$, we have confirmed that this two-form transformation is the symmetry. Note that there is no one-form symmetry for a, unlike the case of the BF theory with level k. This is because $\dim(\mathrm{coker}K) \neq 0$, which is equivalent to the existence of massless NG modes.

Implication for quark-hadron continuity.—If the CFL is a superfluid phase with topological order, there should be an emergent higher-form symmetry, and it has to be spontaneously broken. We have seen that there exists an emergent $\mathbb{Z}_3$ two-form symmetry, whose charged objects are CFL vortices V_i. However, these vortices show the logarithmic confinement, and $\langle V_i\rangle$ vanishes as vortex world sheets become larger. This implies that the $\mathbb{Z}_3$ two-form symmetry is unbroken. Consequently, there is no deconfined topological excitation and the emergent two-form symmetry does not change the topological structure of ground states. Therefore, it does not rule out the possibility that the CFL phase is continuously connected to the nucleon superfluidity.

This can be further supported by a general theorem of quantum field theory, without relying on the mean field approximation. Since the U(1) symmetry is spontaneously broken, interaction of the low-energy Lagrangian should be written by the derivative of the NG boson, $(1/2\pi)d\varphi = (g_0^2/4\pi^2 i) \star d(b_1 + b_2 + b_3)$. If the vortex fluctuation is heavy enough, then the topological defect of φ is negligible in the path integral, and $(1/2\pi)d\varphi$ is a conserved U(1) current, generating the U(1) two-form symmetry. There is a subgroup $\mathbb{Z}_3 \subset \mathrm{U}(1)$, which could be a different symmetry from Eq. (25). Incidentally, those two symmetries act in the same way on the physical observable $\exp(i\int b_i)$ as $2\pi i/3$ phase rotations. A generalized version [16] of the Coleman-Mermin-Wagner theorem [44,45] states that the U(1) p-form symmetry cannot be broken in less than or equal to $p+2$ dimension, and thus U(1) two-form symmetry cannot be broken in our four-dimensional spacetime. Consequently, its subgroup $\mathbb{Z}_3 \subset \mathrm{U}(1)$ is unbroken. Since this symmetry has the same order parameter as the emergent $\mathbb{Z}_3$ two-form symmetry, it cannot be broken either in the CFL phase. This suggests the quark-hadron continuity beyond Ginzburg-Landau paradigm.

Breaking $\mathrm{SU}(3)_f$ *flavor symmetry.*—Let us consider the effect of explicit $\mathrm{SU}(3)_f$ breaking. To see this, we assume that V_{eff} has the minimum at $\Phi^\dagger\Phi = \mathrm{diag}(\Delta_1^2, \Delta_2^2, \Delta_3^2)$. After gauge fixing, the diquark field is

$$\Phi = \begin{pmatrix} \Delta_1 e^{i\phi_1} & 0 & 0 \\ 0 & \Delta_2 e^{i\phi_2} & 0 \\ 0 & 0 & \Delta_3 e^{i\phi_3} \end{pmatrix}, \quad (27)$$

instead of (8) (see, e.g., [46]). The absence of $\mathrm{SU}(3)_f$ symmetry is translated as $\Delta_i \neq \Delta_j$ for different i, j. Correspondingly, the dual effective action is changed as

$$S_{\mathrm{eff}} = \frac{1}{8\pi^2}\sum_{i=1}^{3} g_i^2 |db_i|^2 + \frac{i}{2\pi}\sum_{i=1}^{3}\sum_{A=3,8} K_{iA} b_i \wedge da_A, \quad (28)$$

with $g_i = 1/\Delta_i$.

To find the statistics, let us consider the equation of motion under the presence of $V_3(M_2)$ vortex, which again has $1/3$ circulation. Equations of motion of a_3 and a_8 are

$$db_1 = db_2 = db_3. \quad (29)$$

Equations of motion of b_1, b_2, and b_3 say

$$\begin{aligned} \frac{g_1^2}{4\pi^2} d\star db_1 &= \frac{i}{2\pi} d(a_3 + a_8), \\ \frac{g_2^2}{4\pi^2} d\star db_2 &= \frac{i}{2\pi} d(-a_3 + a_8), \\ \frac{g_3^2}{4\pi^2} d\star db_3 &= \frac{i}{2\pi} d(-2a_8) - i\delta^\perp(M_2), \end{aligned} \quad (30)$$

where $\delta^\perp(M_2)$ is the two-form-valued delta function whose support is M_2. As a result, e.g., we find

$$\frac{\langle V_3(M_2)W_8(C)^2\rangle}{\langle V_3(M_2)\rangle} = \exp\left(\frac{2\pi i g_3^2}{g_1^2 + g_2^2 + g_3^2}\mathrm{link}(C, M_2)\right), \quad (31)$$

which is not quantized to the $\mathbb{Z}_3$ phase unless we require $g_1 = g_2 = g_3$ coming out of SU(3) flavor symmetry. In the absence of $\mathrm{SU}(3)_f$ symmetry, two-form symmetry generated by Wilson loops becomes an infinite group, in general. Since this may be regarded approximately as U(1) two-form symmetry, the vortices should be confined by the generalized Coleman-Mermin-Wagner theorem.

Summary and conclusions.—We have derived the effective gauge theory of the CFL phase describing the NG bosons and vortices. The fractional statistics between vortices and colored test particles is shown to be a result of an emergent $\mathbb{Z}_3$ two-form symmetry. Color Wilson loops are the generator of symmetry, and the charged objects are superfluid vortices. This emergent two-form symmetry is unbroken since the vortex-vortex interaction shows logarithmic confinement. This is also supported by the generalized Coleman-Mermin-Wagner theorem since we can find that $\mathbb{Z}_3$ two-form symmetry is a subgroup of the emergent U(1) two-form symmetry generated by $(1/2\pi)d\varphi$. Therefore, the symmetry breaking pattern of the CFL phase is the same as that of nucleon superfluidity, not only for ordinary symmetries, but also for higher-form symmetries. The effect of explicit $\mathrm{SU}(3)_f$ breaking is also studied, and we check that no higher-form symmetry is spontaneously broken. Our analysis indicates that the quark-hadron continuity scenario is consistent also as quantum phases of matter.

Our analysis suggests that there is some continuous local deformation of the QCD Hamiltonian at finite densities that connects hadronic superfluid and the CFL phase without quantum phase transition. It is important to point out, however, that we do not know if the chemical potential direction corresponds to this continuous deformation, so there may exist phase transition when we change the baryon chemical potential. The answer for this question requires knowledge of the dynamics of finite-density QCD, and one must go beyond the kinematical approach based on symmetry, anomaly matching, etc.

Finally, let us make several comments. The current Letter is based on a Lagrangian in the mean field approximation; however the whole analysis is translated into the language of generalized global symmetry. This indicates that the result of our analysis does not change under the effect of perturbative fluctuations. Vortices can appear as excited states (by rotation, for example). There are Majorana-fermionic excitations inside them [27,47–49]. Roles and consequences of possible physics from those states inside neutron stars are to be understood.

Y. T. appreciates useful discussions with Aleksey Cherman and Srimoyee Sen. The work of Y. T. was partly supported by Special Postdoctoral Researcher Program of RIKEN and also by a JSPS Overseas Fellowship. The work of Y. H. was supported in part by the Korean Ministry of Education, Science and Technology, Gyeongsangbuk-do and Pohang City for Independent Junior Research Groups at the Asia Pacific Center for Theoretical Physics.

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