Dual Conformal Structure Beyond the Planar Limit

Zvi Bern, Michael Enciso, Chia-Hsien Shen, and Mao Zeng

*Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy,
University of California at Los Angeles, Los Angeles, California 90095, USA*

(Received 4 July 2018; published 21 September 2018)

The planar scattering amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory display symmetries and structures which underlie their relatively simple analytic properties such as having only logarithmic singularities and no poles at infinity. Recent work shows in various nontrivial examples that the simple analytic properties of the planar sector survive into the nonplanar sector, but this has yet to be understood from underlying symmetries. Here, we explicitly show that for an infinite class of nonplanar integrals that covers all subleading-color contributions to the two-loop four- and five-point amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory, symmetries analogous to dual conformal invariance exist. A natural conjecture is that this continues to all amplitudes of the theory at any loop order.

DOI: [10.1103/PhysRevLett.121.121603](https://doi.org/10.1103/PhysRevLett.121.121603)

Introduction.—Recent years have seen significant advances in constructing scattering amplitudes, especially for planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. A key feature of planar $\mathcal{N} = 4$ SYM theory that makes this progress possible is its remarkable symmetries and structures. These include dual conformal symmetry [1], Yangian symmetry [2], integrability [3], a dual interpretation of scattering amplitudes in terms of Wilson loops [4], uniform transcendentality [5], structures that aid various bootstraps [6,7], and even an all-loop resummation of four- and five-point amplitudes [8]. Scattering amplitudes have been reformulated using on-shell diagrams and the positive Grassmannian [9], which culminated in the geometric concept of the amplituhedron [10]. Some of these advances have been helpful in quantum chromodynamics relevant for collider physics, including improved ways for dealing with polylogarithms that arise in multiloop computations [11] and for finding good choices [12–14] of integral bases that simplify their evaluation. In fact, the integrals we analyze here for the two-loop five-point amplitude [14,15] are useful choices for the basis of master integrals for 2-to-3 scattering in generic theories [16].

These symmetries and structures impose nontrivial constraints on the analytic properties of planar $\mathcal{N} = 4$ SYM amplitudes. In particular, the loop-level color-ordered amplitudes $\mathcal{M}_{123,\dots,n}$ can be written as

$$\mathcal{M}_{123,\dots,n} = \text{PT}_{123,\dots,n} \int \mathcal{I}, \quad (1)$$

where the integrand $\mathcal{I}$ has only logarithmic singularities, no poles at infinity [9], and unit leading singularities [17] as tied to the amplituhedron [10]. The prefactor $\text{PT}_{123,\dots,n}$ is the standard Parke-Taylor factor [18], as defined in, e.g., Ref. [14].

It is unclear how to define dual conformal symmetry in the nonplanar sector given the lack of dual variables to define the symmetry. However, as shown in a variety of examples [13,14,19], the key analytic properties of the planar sector implied by its symmetries carry over to the nonplanar sector, even if the symmetries are unclear. In each example, the full amplitude can be expressed as [20]

$$\mathcal{M} = \sum_{k,\sigma,j} a_{\sigma,k,j} c_k \text{PT}_{\sigma} \int \mathcal{I}^j, \quad (2)$$

where the $a_{\sigma,k,j}$ are rational numbers, the c_k are color factors, the PT_{σ} are the Parke-Taylor factors corresponding to an ordering σ of external particles, and the $\mathcal{I}^j$ are integrands with only logarithmic singularities, no poles at infinity, and unit leading singularities. Equation (2) is a natural extension of Eq. (1) to the nonplanar sector. It is nontrivial that such a representation exists where each integrand is expressed in terms of local diagrams. Some structures of the nonplanar sector were also explored at the level of on-shell diagrams [21,22].

In the present Letter we address the following question: Can we identify a hidden symmetry associated with the simple analytic properties for the nonplanar sector uncovered in Refs. [13,14,19]? Building on the initial studies in Ref. [23], we answer this question affirmatively and demonstrate that the integrands $\mathcal{I}^j$ in (2) encoding the simple analytic structure of the full two-loop four- and five-point amplitudes all have hidden symmetries related to dual conformal invariance. These are not hidden symmetries of

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

FIG. 1. Planar double box with dual coordinates and the crossed-box related to it by moving leg 3 to the central rung.

the full amplitude, but of individual components of the amplitudes, analogous to the situation with dual conformal symmetry in the planar case (1). We also identify an infinite class of nonplanar integrands with the hidden symmetry. In many cases these symmetries rely on nontrivial identities, making it all the more striking that a symmetry actually exists.

Dual coordinates and conformal symmetry.—To set up our discussion of hidden symmetries in the nonplanar sector, we first briefly review dual conformal symmetry in the planar sector [1]. In general, the momenta (corresponding to edges or lines) in *any* planar diagram can be represented as the difference of adjacent dual coordinates (corresponding to regions). For example, the momenta in the planar double-box diagram on the left of Fig. 1 can be expressed as

$$\begin{aligned} p_1 &= x_2 - x_1, & p_2 &= x_3 - x_2, & p_3 &= x_4 - x_3, \\ p_4 &= x_1 - x_4, & l_5 &= x_5 - x_1, & l_6 &= x_1 - x_6, \end{aligned} \quad (3)$$

where the p_i are external momenta, l_5 and l_6 are the loop momenta, and x_i are the dual coordinates with all Lorentz indices omitted. We can perform infinitesimal conformal transformations on these dual coordinates,

$$\delta x_i^\mu = \frac{1}{2} x_i^2 b^\mu - (x_i \cdot b) x_i^\mu, \quad (4)$$

where b^μ is an infinitesimal boost vector. The transformation of the square of proper distance is

$$\delta(x_i - x_j)^2 = -b \cdot (x_i + x_j)(x_i - x_j)^2. \quad (5)$$

In general, if a quantity f transforms as $\delta f = w f$ with w a local function, we say f *rescales* under the transformation with weight w . Thus, under dual conformal transformations, $(x_i - x_j)^2$ carries a weight $-b \cdot (x_i + x_j)$. Note that all massless external legs remain on shell after the transformation. All the inverse propagators have the form $(x_i - x_j)^2$. This implies that locality is maintained for planar loop integrals under dual conformal transformations and allows us to construct simple functions that are invariant.

As a simple illustration, consider an integral associated with the planar double box,

$$I = \int d^D x_5 d^D x_6 \frac{s^2 t}{\prod_k \rho_k}, \quad (6)$$

where $s = (x_1 - x_3)^2 = (p_1 + p_2)^2$ and $t = (x_2 - x_4)^2 = (p_2 + p_3)^2$. The inverse Feynman propagators ρ_k in dual coordinates are

$$\begin{aligned} \rho_1 &= (x_5 - x_1)^2, & \rho_2 &= (x_5 - x_2)^2, & \rho_3 &= (x_5 - x_3)^2, \\ \rho_4 &= (x_5 - x_6)^2, & \rho_5 &= (x_6 - x_1)^2, & \rho_6 &= (x_6 - x_4)^2, \\ \rho_7 &= (x_6 - x_3)^2. \end{aligned} \quad (7)$$

In what follows, we will be interested in the integrand $\mathcal{I}$, defined by $I = \int \mathcal{I}$. With this numerator the integrand has a hidden symmetry exposed by using the dual variables [1]. Performing the dual conformal transformation on the integrand (6) yields

$$\delta \mathcal{I} = -(D - 4)[b \cdot (x_5 + x_6)] \mathcal{I}, \quad (8)$$

where we used

$$\delta(d^D x_i) = \left(\frac{\partial \delta x_i^\mu}{\partial x_i^\mu} \right) d^D x_i = -D(b \cdot x_i) d^D x_i. \quad (9)$$

For $D = 4$ space-time dimensions this integrand is invariant under dual conformal transformations, which is what motivated the choice of numerator $s^2 t$. Outside $D = 4$, this is reminiscent of ϵ -form differential equations [12], but without doubled propagators on the right hand side before reduction to a basis [23,24].

What is the relevance of this symmetry? It turns out that *all* integrands of planar $\mathcal{N} = 4$ SYM amplitudes possess this property, which then leads to nontrivial constraints on the amplitude after integration. This is the celebrated *dual conformal symmetry* [1] which has spurred many developments. In the following we identify an analogous symmetry in a class of nonplanar diagrams.

Nonplanar extension.—While there are no known global variables for generic nonplanar diagrams, it is natural to require that, as for the planar case, a nonplanar analog of dual conformal transformations also maintains the local structure for inverse propagators, $\delta \rho_k \propto \rho_k$ [23]. We start by considering a nonplanar diagram that can be made planar by moving the location of one external leg carrying momentum p_k^μ . This is an infinite class of nonplanar integrals, and includes all the nonplanar integrals at two loops with five or fewer external legs. In particular, all of the nonplanar integrals in Fig. 2 are of this type. For example, diagram (a) can be made planar by moving external leg 3. Under this, the momenta of the propagators are modified compared to the planar case at most by adding or subtracting a single external momentum p_k^μ . Thus, the inverse propagators ρ_l therein can be written as either $(x_i - x_j)^2$, or $(x_i - x_j \pm p_k)^2$, when using the dual

FIG. 2. Diagrams (a)–(i) compose the two-loop five-point amplitude in Ref. [14].

coordinates of the planar cousin. The key observation here is that if the infinitesimal boost vector b^μ is proportional to a massless external leg p_k^μ , then $(x_i - x_j \pm p_k)^2$ transforms in the same way as $(x_i - x_j)^2$ for any x_i^μ and x_j^μ . Specifically,

$$\frac{\delta(x_i - x_j \pm p_k)^2}{(x_i - x_j \pm p_k)^2} = \frac{\delta(x_i - x_j)^2}{(x_i - x_j)^2} = -b \cdot (x_i + x_j), \quad (10)$$

implying that all the propagators in this class of nonplanar diagrams satisfy $\delta\rho_k \propto \rho_k$ for this conformal boost.

As a simple first example, consider the crossed double-box diagram on the right of Fig. 1, with numerator $N_1 = su(l_5 + p_4)^2$, which is one of the nonplanar pure integrands found in Ref. [19] as a building block of the full amplitude:

$$I^{(\text{NP})} = \int \mathcal{I}^{(\text{NP})} = \int d^D l_5 d^D l_6 \frac{N_1}{\prod_k \rho_k}, \quad (11)$$

where the ρ_k are the inverse propagators. This diagram can be obtained from the planar double box in Fig. 1 by moving the external leg 3 to the central rung. Using the dual coordinates of the planar double box, we can write the nonplanar integrand as

$$\mathcal{I}^{(\text{NP})} = d^D x_5 d^D x_6 \frac{(x_1 - x_3)^2 (x_2 - x_1 + p_3)^2 (x_5 - x_4)^2}{\prod_k \rho_k}, \quad (12)$$

where the propagators are given by

$$\begin{aligned} \rho_1 &= (x_5 - x_1)^2, & \rho_2 &= (x_5 - x_2)^2, & \rho_3 &= (x_5 - x_3)^2, \\ \rho_4 &= (x_5 - x_6)^2, & \rho_5 &= (x_6 - x_1)^2, & \rho_6 &= (x_6 - x_4)^2, \\ \rho_7 &= (x_5 - x_6 + p_3)^2, \end{aligned} \quad (13)$$

with the x_i defined in Eq. (3). Applying a dual conformal transformation to the integrand with the boost vector $b^\mu \propto p_3^\mu$ and using Eq. (10) we find that

$$\delta \mathcal{I}^{(\text{NP})} = -(D-4)[b \cdot (x_5 + x_6)] \mathcal{I}^{(\text{NP})}, \quad (14)$$

exposing a hidden symmetry in $D = 4$.

A similar analysis holds for the numerator $N_2 = st(l_5 + p_3)^2$, corresponding to the other pure integrand found in Ref. [19]. One can also obtain the crossed box from the planar double box by moving the leg 4 to the central rung (and making a change in the momentum routing of the planar double box), giving a new conformal boost with $b^\mu \propto p_4^\mu$. As can be straightforwardly checked, both numerators N_1 and N_2 give integrands that are invariant in $D = 4$ under this transformation as well.

While we propose these transformations as a natural extension of the planar dual conformal symmetry, it is striking that the numerators N_1 and N_2 are precisely the correct numerators of the building blocks for the two-loop four-point amplitude in $\mathcal{N} = 4$ SYM theory that unveil their analytic properties [19]. Here, we see that we can constrain these numerators from symmetry considerations instead of from imposing desired analytic properties on the integrands. Similar symmetry considerations can be used to match the numerators of a subset of three-loop four-point diagrams in Ref. [13] that can be obtained from planar ones by moving a single external line.

Two-loop five-point case.—As the central nontrivial example consider the two-loop five-point $\mathcal{N} = 4$ SYM amplitude first obtained in Ref. [15]. This amplitude was rewritten in a desired form where each diagram composing the amplitude contains only logarithmic singularities and no pole at infinity [14], as follows from dual conformal symmetry in the planar case. The diagrams composing this amplitude are given in Fig. 2. These diagrams are either planar, or in the nonplanar class of diagrams discussed above, so our discussion immediately applies.

Consider diagrams (a), (d), (h), and (i), which can be made planar by moving the external leg 3, corresponding to choosing $b^\mu \propto p_3^\mu$. Using the dual coordinates

$$\begin{aligned} p_1 &= x_3 - x_2, & p_2 &= x_4 - x_3, & p_3 &= x_2 - x_1, \\ p_4 &= x_5 - x_4, & p_5 &= x_1 - x_5, \\ l_6 &= x_6 - x_1, & l_7 &= x_1 - x_7, \end{aligned} \quad (15)$$

in the diagram on the right of Fig. 3, the propagators in the original nonplanar diagrams are a subset of

FIG. 3. Dual variables useful for the two-loop planar pentabox and the nonplanar integrals in Fig. 2.

$$\begin{aligned}\rho_1 &= (x_6 - x_1)^2, & \rho_2 &= (x_6 - x_3 + p_3)^2, \\ \rho_3 &= (x_6 - x_4 + p_3)^2, & \rho_4 &= (x_7 - x_4)^2, \\ \rho_5 &= (x_7 - x_5)^2, & \rho_6 &= (x_7 - x_1)^2, \\ \rho_7 &= (x_6 - x_7)^2, & \rho_8 &= (x_6 - x_7 + p_3)^2.\end{aligned}\quad (16)$$

A crucial difference between integrands at four points and five points is the appearance of spinor helicity variables, which makes the transformation properties less clear. We therefore restrict to $D = 4$ from now on, and the convention for spinors is chosen such that $s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ji] = \langle i|j|i\rangle = (\langle i|_a |j\rangle^{\dot{a}})([i]^a |j]_a) = \langle i|p_j|i\rangle$. A complete set of numerators for the diagrams in Fig. 2 is given in Table 3 of Ref. [14].

To warm up, consider the numerator in diagram (i)

$$N^{(i)} = \langle 2|4|3\rangle\langle 3|5|2\rangle - \langle 3|4|2\rangle\langle 2|5|3\rangle. \quad (17)$$

This numerator is constructed to follow the S_3 symmetry among legs 2,3,5 of the diagram (up to a sign). By choosing to move leg 3 to make the diagram planar and using the coordinates in Eq. (15), we recast the numerator as

$$N^{(i)} = \langle 3|x_{54}x_{43}x_{32}|3\rangle + \langle 3|x_{23}x_{34}x_{45}|3\rangle, \quad (18)$$

under momentum conservation and spinor identities. To see that this numerator only rescales with a local weight under the transformation with $b^\mu \propto p_3^\mu$, we need a nontrivial identity

$$\frac{\delta \langle b|x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n} |b\rangle}{\langle b|x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n} |b\rangle} = -b \cdot (x_{i_1} + \dots + x_{i_n}), \quad (19)$$

where $x_{ij} \equiv x_i - x_j$ and $\langle b|x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n} |b\rangle = (\langle b|_{\dot{a}})(x_{i_1 i_2}^{\dot{a}a})(x_{i_2 i_3, ab}^{\dot{a}b}) \dots (x_{i_{n-1} i_n}^{\dot{a}d})(|b]_d)$. One can prove this using conformal inversions irrespective of whether the x_{ij} 's are null separated or not. We have directly confirmed Eq. (19) numerically through $n = 8$. Therefore, the numerator in Eq. (18) is manifestly rescaled under the transformation with weight $-b \cdot (x_2 + x_3 + x_4 + x_5)$. Moreover, accounting for the transformation of the propagators and measure using Eqs. (9), (10), and (16), this is precisely the weight needed to make the integrand invariant.

We can also make diagram (i) planar by moving the leg carrying momentum p_2 or p_5 , giving a total of three choices of b^μ for the conformal boosts. We have checked that these three transformations are independent symmetry generators, corresponding to three hidden symmetries of this nonplanar integrand.

A more involved example is diagram (a) in Fig. 2. The numerator yielding the desired analytic properties given in Ref. [14] is

$$N_1^{(a)} = \langle 13 \rangle \langle 24 \rangle ([24][13](l_7 - l_7^*)(l_6 - l_6^*)^2 - (1 \leftrightarrow 2)), \quad (20)$$

where $l_7^* = \frac{[54]}{[24]}|5\rangle[2|$ and $l_6^* = p_1 + \frac{[23]}{[13]}|2\rangle[1|$. How this numerator transforms is far from clear in the above form. In fact, the first or second term alone does not rescale with a local weight. However, by using on-shell conditions and Schouten identities it can be rewritten as

$$\begin{aligned}N_1^{(a)} &= -\langle 3|x_{23}x_{34}x_{45}|3\rangle\rho_4\rho_1 \\ &\quad + \langle 3|x_{23}x_{34}x_{45}x_{57}x_{76}x_{61}x_{14}|3\rangle,\end{aligned}\quad (21)$$

using the dual coordinates in Eq. (15). With the help of Eq. (19), each of the two terms in Eq. (21) above transforms with the weight necessary to make the integrand invariant in $D = 4$. After canceling the propagators, the first term gives rise to the daughter diagram (i) in Fig. 2, and the numerator $\langle 3|x_{23}x_{34}x_{45}|3\rangle$ also matches to one of the components in Eq. (18).

Similarly, we can rewrite the original numerators of diagrams (d) and (h) using the dual coordinates in the diagram on the left of Fig. 3 as

$$\begin{aligned}N_1^{(d)} &= s_{34}(s_{34} + s_{35})\left(l_7 - \frac{\langle 54 \rangle}{\langle 34 \rangle}|3\rangle[5]\right)^2 \\ &= s_{34}(s_{34} + s_{35})\rho_6 + \langle 3|x_{71}x_{15}x_{54}|3\rangle,\end{aligned}\quad (22)$$

and

$$\begin{aligned}N_1^{(h)} &= \langle 15 \rangle [35] \langle 23 \rangle [12] \left(l_6 - \frac{\langle 12 \rangle}{\langle 32 \rangle}|3\rangle[1]\right)^2 \\ &= (s_{23}s_{35} - \langle 3|x_{34}x_{45}x_{51}|3\rangle)\rho_1 - s_{12}\langle 3|x_{62}x_{23}x_{35}|3\rangle, \\ N_3^{(h)} &= -s_{12}\langle 3|p_1 p_5 l_6|3\rangle = -s_{12}\langle 3|x_{35}x_{51}x_{16}|3\rangle.\end{aligned}\quad (23)$$

In addition there are numerators simply related via diagram symmetries. Using Eqs. (9), (10), and (19), we see that these numerators have weights that make the integrand invariant under the dual conformal boost with $b^\mu \propto p_3^\mu$.

Diagrams (c) and (f) can be made planar by moving the external leg carrying momentum p_4 , corresponding to $b^\mu \propto p_4^\mu$. The dual coordinates are defined according to the left of Fig. 3, analogous to Eq. (15). The propagators in the original nonplanar diagrams are a subset of

$$\begin{aligned}\rho_1 &= (x_6 - x_1)^2, & \rho_2 &= (x_6 - x_2)^2, & \rho_3 &= (x_6 - x_3)^2, \\ \rho_4 &= (x_6 - x_4)^2, & \rho_5 &= (x_7 - x_5)^2, & \rho_6 &= (x_7 - x_1)^2, \\ \rho_7 &= (x_6 - x_7)^2, & \rho_8 &= (x_6 - x_7 + p_4)^2.\end{aligned}\quad (24)$$

The numerator of diagram (f) is $N_1^{(f)} = s_{14}s_{45}(l_6 + p_5)^2$ which manifestly rescales with local weight under the transformation. To see the conformal property of diagram (c), we need

$$\begin{aligned}N_1^{(c)} &= \langle 15 \rangle [54] \langle 43 \rangle [13] (l_6 - l_6^*)^2 (l_6 + p_4)^2 \\ &= (-s_{51}s_{45}\rho_3 + \langle 4|x_{46}x_{63}x_{32}x_{21}x_{15}|4\rangle)(l_6 + p_4)^2,\end{aligned}\quad (25)$$

with the same l_6^* as defined below Eq. (20). After canceling the propagator, the first term matches $N_2^{(f)}$ of Ref. [14] which is related to $N_1^{(f)}$ under $4 \leftrightarrow 5$.

We have checked all of the two-loop five-point nonplanar integrands from Ref. [14] that manifest the desired analytic properties of the full two-loop five-point amplitude and found that all of them have a hidden symmetry in $D = 4$ closely related to dual conformal symmetry. In cases where more than one conformal boost is available, as for diagrams (c), (f), (h), and (i) in Fig. 2, we have checked that all such choices of b^μ give symmetries of the integrand. While Eq. (10) guarantees that all the propagators transform with definite weight, the fact that all the corresponding numerators behave accordingly to make the integrand invariant appears miraculous.

Using Eqs. (10) and (19) we can generalize these results to integrals relevant for higher-point amplitudes. As a concrete example, consider diagram (a) in Fig. 2 but with legs 1,2,4,5 being massive or replaced with arbitrary collections of massless particles, while keeping leg 3 massless. Crucially, the identity in Eq. (19) holds even for $x_{i,i+1}^2 \neq 0$. This implies the numerator with the dual variables in Eq. (15)

$$\langle 3|x_{23}x_{34}x_{45}x_{57}x_{76}x_{61}x_{14}|3\rangle, \quad (26)$$

transforms with the proper weight to make the integrand invariant, providing a generalization of the second term in Eq. (21). Another possible numerator is

$$s_{12}s_{24}(\langle 3|x_{47}x_{76}x_{61}|3\rangle + \langle 3|x_{16}x_{67}x_{74}|3\rangle). \quad (27)$$

The latter example (27) is especially interesting since it vanishes in the collinear limit $x_{76}^\mu \propto p_3^\mu$ and gives an infrared-finite integral, for which the hidden symmetry is *exact* and free of anomalies from divergences. By working in six dimensions, additional finite integrals with the hidden symmetry can be found; such integrals are related to four-dimensional ones via dimension shifting relations [25].

Conclusions.—Following the four-point hints in Ref. [23], here we demonstrated that all sectors of the two-loop five-point $\mathcal{N} = 4$ SYM amplitude, including the nonplanar sector, possess new nontrivial hidden symmetries related to dual conformal symmetry. To show this we demonstrated that each integrand sector identified in Ref. [14] possessing simple analytic properties manifests a hidden symmetry. For some sectors the symmetry is rather unobvious. The construction used for the two-loop five-point amplitude extends to any number of loops and legs, giving an infinite class of integrands with new hidden symmetries. It would be interesting to check if these cases actually appear with nonzero coefficient in $\mathcal{N} = 4$ SYM amplitudes. Even for the cases studied here we can expect a larger set of symmetries than the ones we found; we expect this to be helpful for the important problem of identifying the hidden symmetries of more general cases beyond the ones studied here. It would be interesting to apply the symmetries to help identify nonplanar integrals of uniform transcendentality, which become nontrivial at high loop orders by directly checking leading singularities [26]. It would also be interesting to understand how the new symmetries described here relate to recent progress in extending integrability to nonplanar theories described in Ref. [27]. Given the useful role hidden symmetries have played in the planar sector of $\mathcal{N} = 4$ SYM theory, we should expect new progress from fully unraveling the corresponding symmetries of the nonplanar sector of the theory.

We thank Lance Dixon, Enrico Herrmann, Harald Ita, Julio Parra-Martinez, and Jaroslav Trnka for discussions. We especially thank Emery Sokatchev for pointing out the ease of proving Eq. (19) via conformal inversions. This work was supported in part by the Department of Energy under Award No. DE-SC0009937.

-
- [1] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, *J. High Energy Phys.* **01** (2007) 064; Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, *Phys. Rev. D* **75**, 085010 (2007); J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, *Nucl. Phys.* **B795**, 385 (2008); J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *Nucl. Phys.* **B828**, 317 (2010).
 - [2] I. Bena, J. Polchinski, and R. Roiban, *Phys. Rev. D* **69**, 046002 (2004); J. M. Drummond, J. M. Henn, and J. Plefka, *J. High Energy Phys.* **05** (2009) 046; N. Beisert, J. Broedel, and M. Rosso, *J. Phys. A* **47**, 365402 (2014).
 - [3] N. Beisert and M. Staudacher, *Nucl. Phys.* **B670**, 439 (2003).
 - [4] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *Nucl. Phys.* **B815**, 142 (2009); A. V. Belitsky, G. P. Korchemsky, and E. Sokatchev, *Nucl. Phys.* **B855**, 333 (2012); L. J. Mason and D. Skinner, *J. High Energy Phys.* **12** (2010) 018.

- [5] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, *Phys. Lett. B* **595**, 521 (2004); **632**, 754(E) (2006); A. V. Kotikov and L. N. Lipatov, *Nucl. Phys.* **B769**, 217 (2007).
- [6] B. Basso, A. Sever, and P. Vieira, *Phys. Rev. Lett.* **111**, 091602 (2013); *J. High Energy Phys.* 01 (2014) 008.
- [7] L. J. Dixon, J. M. Drummond, C. Duhr, and J. Pennington, *J. High Energy Phys.* 06 (2014) 116; L. J. Dixon and M. von Hippel, *J. High Energy Phys.* 10 (2014) 065; L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou, and M. Spradlin, *J. High Energy Phys.* 02 (2017) 137; S. Caron-Huot, L. J. Dixon, A. McLeod, and M. von Hippel, *Phys. Rev. Lett.* **117**, 241601 (2016).
- [8] Z. Bern, L. J. Dixon, and V. A. Smirnov, *Phys. Rev. D* **72**, 085001 (2005); L. F. Alday and J. M. Maldacena, *J. High Energy Phys.* 06 (2007) 064.
- [9] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, and J. Trnka, *arXiv:1212.5605*.
- [10] N. Arkani-Hamed and J. Trnka, *J. High Energy Phys.* 10 (2014) 030; 12 (2014) 182.
- [11] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Phys. Rev. Lett.* **105**, 151605 (2010).
- [12] J. M. Henn, *Phys. Rev. Lett.* **110**, 251601 (2013); D. Chicherin, J. M. Henn, and E. Sokatchev, *Phys. Rev. Lett.* **121**, 021602 (2018).
- [13] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, and J. Trnka, *J. High Energy Phys.* 06 (2015) 202.
- [14] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, and J. Trnka, *J. High Energy Phys.* 06 (2016) 098.
- [15] J. J. M. Carrasco and H. Johansson, *Phys. Rev. D* **85**, 025006 (2012).
- [16] T. Gehrmann, J. M. Henn, and N. A. Lo Presti, *Phys. Rev. Lett.* **116**, 062001 (2016); **116**, 189903(E) (2016); D. Chicherin, J. Henn, and V. Mitev, *J. High Energy Phys.* 05 (2018) 164.
- [17] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, *J. High Energy Phys.* 01 (2011) 041.
- [18] S. J. Parke and T. R. Taylor, *Phys. Rev. Lett.* **56**, 2459 (1986); M. L. Mangano, S. J. Parke, and Z. Xu, *Nucl. Phys.* **B298**, 653 (1988).
- [19] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, and J. Trnka, *Phys. Rev. Lett.* **113**, 261603 (2014).
- [20] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. Postnikov, and J. Trnka, *J. High Energy Phys.* 06 (2015) 179.
- [21] S. Franco, D. Galloni, B. Penante, and C. Wen, *J. High Energy Phys.* 06 (2015) 199; J. L. Bourjaily, S. Franco, D. Galloni, and C. Wen, *J. High Energy Phys.* 10 (2016) 003.
- [22] R. Frassek and D. Meidinger, *J. High Energy Phys.* 05 (2016) 110.
- [23] Z. Bern, M. Enciso, H. Ita, and M. Zeng, *Phys. Rev. D* **96**, 096017 (2017).
- [24] M. Zeng, *J. High Energy Phys.* 06 (2017) 121; J. Bosma, K. J. Larsen, and Y. Zhang, *Phys. Rev. D* **97**, 105014 (2018).
- [25] Z. Bern, L. J. Dixon, and D. A. Kosower, *Phys. Lett. B* **302**, 299 (1993); **318**, 649(E) (1993); *Nucl. Phys.* **B412**, 751 (1994); O. V. Tarasov, *Phys. Rev. D* **54**, 6479 (1996).
- [26] R. H. Boels, T. Huber, and G. Yang, *J. High Energy Phys.* 01 (2018) 153.
- [27] B. Eden, Y. Jiang, D. le Plat, and A. Sfondrini, *J. High Energy Phys.* 02 (2018) 170; T. Bargheer, J. Caetano, T. Fleury, S. Komatsu, and P. Vieira, *arXiv:1711.05326*; R. Ben-Israel, A. G. Tumanov, and A. Sever, *arXiv:1802.09395*.