Three-Loop Quark Jet Function

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We calculate the massless quark jet function to three-loop order. The quark jet function is a universal ingredient in SCET factorization for many collider and decay processes with quark initiated final state jets. Our three-loop result contributes to the resummation for observables probing the invariant mass of final state quark jets at primed next-to-next-to-next-to-leading-logarithmic accuracy. It represents the first complete three-loop result for a factorization ingredient describing collinear radiation. Furthermore it constitutes a major component of the N -jettiness subtraction method at next-to-next-to-next-to-leading order accuracy, which eventually may enable the calculation of fully differential cross sections with a colorful final state at this order.

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Introduction.—In QCD processes involving highly energetic partons factorization plays a crucial role. Most importantly it provides a means to disentangle perturbative physics from nonperturbative physics. Ideally, the non-perturbative effects can thus be absorbed into universal (process-independent) functions as, e.g., the parton distribution functions. More generally, whenever there is a strong hierarchy of scales one can hope to establish a factorization formula at leading order in the small scale ratio(s) that separates the physics happening at the different scales. Besides a considerable simplification, this usually allows us to resum large logarithms of the small scale ratio(s) to all orders in perturbation theory via renormalization group equations (RGEs) for the individual factorization ingredients. Such factorization formulas are conveniently derived in soft-collinear effective theory (SCET) [1–6].

In the following we will consider decay or scattering processes with final state jets and a large scale hierarchy between the jet invariant masses ($\sim\tau$) and the total center of mass energy ($\sim Q$). The cross section differential in a generic observable τ that constrains the jet invariant mass then schematically takes the factorized form

$$\frac{d\sigma}{d\tau} = H(Q) \times [B_a \otimes B_b \otimes J_{i_1} \otimes \cdots \otimes J_{i_N} \otimes S](\tau), \quad (1)$$

at leading order in τ/Q and all orders in α_s . The $\otimes$ symbol denotes a convolution of the type

$$A(\tau) \otimes B(\tau) \equiv \int d\tau' A^i(\tau - \tau') B(\tau'). \quad (2)$$

For concreteness, we assume here a process with two incoming (a, b) and N outgoing partons involved in the hard interaction, which is described by the hard function $H(Q)$, as, e.g., observed in proton-proton collisions at the LHC. For $\tau \ll Q$ the initial state radiation is then collimated along the two incoming beam directions and the final state radiation is collimated along N different jet directions. Wide-angle soft emissions are taken into account by the soft function $S(\tau)$. The beam functions $B_i(\tau)$ and the jet functions $J_i(\tau)$ describe the effects of collinear radiation in the beam and final state jets, respectively. The functions S , B_i , and J_i are universal in the sense that they are independent of the details of the hard process (e.g., the colorless final state). The collinear functions B_i and J_i are furthermore equal for any observable that in the collinear limit effectively reduces to a measurement of the jet invariant mass ($\tau \rightarrow \sqrt{s}$). A prime example for such an observable obeying factorization [7] as in Eq. (1) is the N -jettiness event shape [13] including the special cases beam thrust [14] (0-jettiness) and thrust [15] (~ 2 -jettiness in lepton collisions).

In this Letter we focus on the jet function $J_q(s)$ for the case that the corresponding hard parton initiating the jet is a massless (anti-)quark ($i = q$). The SCET (quark) jet function was introduced in Ref. [4]. It can be defined in terms of standard QCD fields as [16]

$$J_q(s) = \frac{1}{\pi N_c} \text{Im} \left[\frac{i}{\bar{n} \cdot p} \int d^d x e^{-ip \cdot x} \times \langle 0 | T \text{Tr} \left[\frac{\not{n}}{4} W^\dagger(0) \psi(0) \bar{\psi}(x) W(x) \right] | 0 \rangle \right], \quad (3)$$

where T is the time-ordering operator, n^μ is the lightlike jet direction ($\bar{n} \cdot n = 2$, $n^2 = \bar{n}^2 = 0$), p^μ is the jet momentum

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($s \equiv p^2$), the trace is over color ($N_c = 3$) and spinor indices, and

$$W(x) = P \exp \left[ig \int_{-\infty}^0 ds \bar{n} \cdot A(x + s\bar{n}) \right] \quad (4)$$

denotes a (n -collinear) Wilson line. This definition implies that we can use standard QCD Feynman rules in the calculation of the jet function as soft radiation has already been decoupled from the (leading order) collinear SCET Lagrangian by means of a field redefinition [4]. At one loop the quark jet function ($i = q$) was computed in Refs. [17,18] and the gluon jet function ($i = g$) in Ref. [19]. The two-loop results for $J_q(s)$ and $J_g(s)$ were obtained in Ref. [16] and Ref. [20], respectively.

The two-loop quark jet function contributes to a number of important cross section predictions with resummation beyond next-to-next-to-leading-logarithmic accuracy, e.g., in DIS [21], for thrust [22–24], C parameter [25,26], and heavy jet mass [27]. To improve their precision to the full primed next-to-next-to-next-to-leading-logarithmic ($N^3\text{LL}'$) accuracy level [28] (in the peak region) the three-loop correction to the quark jet function is required. This is particularly desirable for the latter three observables as they are used for precise determinations of α_s from e^+e^- data. Another good motivation to calculate $J_q(s)$ at three loops is the perspective to extend the N -jettiness (infrared) subtraction (slicing) method [12,30], which has been applied successfully to several next-to-next-to-leading-order processes with final state jets [30–34], to next-to-next-to-next-to-leading order ($N^3\text{LO}$).

Calculation.—We work in general covariant gauge with gauge parameter ξ , where $\xi = 0$ corresponds to the Feynman gauge, and use dimensional regularization ($d = 4 - 2\epsilon$). We generate the three-loop Feynman diagrams for $J_q(s)$ with `qgraf` [35]. The output is then further processed using a custom code that does the color, Lorentz, and Dirac algebra. Our code also performs partial fractioning of products of the eikonal Wilson line propagators following the strategy outlined in Ref. [36], and finally maps the resulting terms onto (scalar) integral topologies with twelve linearly independent propagators or numerators.

For the integrals in each topology we then perform an integration-by-parts (IBP) reduction to master integrals (MIs) with the public computer program `FIRE` [37] in combination with `LiteRed` [38,39]. Next, we identify MIs that are related across the different topologies by shifts of the loop momenta. In this way we find 34 MIs in five topologies. This set of MIs, however, turns out to be redundant. Four extra (one-to-one) IBP relations among the MIs are revealed by the following tricks:

The first is to search for identities between integrals based on their Feynman parameter representation using the algorithm of Ref. [36], which is implemented in the

`FindRules` command of `FIRE`. In practice we apply `FindRules` to a large list of (test) integrals. The output are a number of equalities among these integrals, which must hold after IBP reduction. In our case this required two more independent relations among the 34 MI candidates.

Even more relations are found via dimensional recurrence [40–42]. Compact formulas that relate a generic d -dimensional integral to a linear combination of either $(d + 2)$ -dimensional or $(d - 2)$ -dimensional integrals with the same kind (but different powers) of propagators have been derived using Baikov’s representation [43] of Feynman integrals in Refs. [41,42]. They are implemented in the `RaisingDDR` and `LoweringDDR` commands of `LiteRed`, respectively. By applying `RaisingDDR` we can thus directly express the 34 d -dimensional MI candidates as linear combinations of integrals in $d + 2$ dimensions. We then IBP reduce the output, and lower the dimension of the resulting integrals back to d using the `LoweringDDR` command. Comparing the result to the original integrals after another IBP reduction yields four equalities among the 34 MIs, including the two relations found with the `FindRules` trick.

We have checked all four extra relations analytically to the required order in ϵ . We are thus left with 30 MIs, which have maximally eight quadratic and two linear (Wilson line) propagators. Expressing the full three-loop amplitude of the quark jet function in terms of these MIs the dependence on the gauge parameter ξ manifestly vanishes as expected. This provides a first cross check of our setup.

The next step is to compute the MIs to high enough order in ϵ . For this we adopt a strategy similar to the one described in Refs. [44–46]. First, we note that the dependence of the MIs on the two external scalar products $s = p^2$ and $\bar{n} \cdot p$ is completely fixed by scaling properties. From reparametrization invariance [47,48] and dimensional counting in Eq. (3) we even know that the full three-loop contribution equals $[(-s - i0)^{-1-3\epsilon} \times \text{const}]$ before taking the imaginary part. In order to determine the constant we can therefore safely set $s = -1$ and $\bar{n} \cdot p = 1$ in the evaluation of the MIs for convenience.

We then switch to a new basis of MIs that are quasifinite [45] in some integer dimension, in our case $d = 4$ or $d = 6$. By “quasifinite” we mean integrals that are either convergent or their divergence can be factored out in a simple way. We allow, e.g., divergences ($\propto 1/\epsilon^n$) contained in the prefactor of the corresponding Feynman parameter representation of the integral or generated as an overall prefactor by integrating over the Feynman parameters associated with the linear propagators. The latter integrations are always straightforward to carry out. We are thus left with up to eight convergent Feynman parameter integrals to be done.

A quasifinite basis can be constructed as follows. The basic idea is to increase the dimension of a given IR divergent integral to $d = n - 2\epsilon$ ($n \in \mathbb{Z}$, $n > 4$) in order to

render it IR (quasi-)finite. For our MIs $n = 6$ turned out to be sufficient. One can then carefully increase the power of some propagators (by one) to decrease the degree of UV divergence without generating new IR singularities. [49] Once a quasifinite integral is found in this way it can be related to the original MI plus integrals with less propagators in $d = 4 - 2\epsilon$ dimensions by dimensional recurrence and another IBP reduction. In some (exceptional) cases the quasifinite integral does not reduce to the original MI and one has to try another quasifinite candidate. An algorithm that for a given integral automatically determines a desired number of proper quasifinite integrals in shifted spacetime dimensions is implemented in the public program REDUZE [50].

To perform the remaining convergent Feynman parameter integrals of the quasifinite MIs we first expand the integrands to sufficiently high order in ϵ . After that we integrate them using the HYPERINT package [51]. This code automatically evaluates linearly reducible convergent (Feynman) integrals in terms of multiple polylogarithms. With the outlined procedure we were able to compute all 30 MIs analytically to the required order in ϵ . We have checked all MI results numerically using the sector decomposition programs FIESTA [52] and PYSECDEC [53]. For many of the MIs we also have obtained analytic results with the Mellin Barnes technique [54,55] employing the MB package [56,57] as well as the PSLQ algorithm [58]. We found perfect agreement in all cases.

To complete the calculation of the bare three-loop contribution to $J_q(s)$ we have to take the imaginary part according to Eq. (3) and consistently expand in ϵ . To this end we use

$$\text{Im}[(s - i0)^{-1-a\epsilon}] = -\sin(\pi a\epsilon)\theta(s)s^{-1-a\epsilon} \quad (5)$$

and

$$\mu^{2a\epsilon}\theta(s)s^{-1-a\epsilon} = -\frac{\delta(s)}{a\epsilon} + \sum_{n=0}^{\infty} \frac{(-a\epsilon)^n}{n!} \frac{1}{\mu^2} \mathcal{L}_n\left(\frac{s}{\mu^2}\right) \quad (6)$$

with the usual plus distributions defined as

$$\mathcal{L}_n(x) = \left[\frac{\theta(x) \ln^n x}{x} \right]_+ = \lim_{\epsilon \rightarrow 0} \frac{d}{dx} \left[\theta(x - \epsilon) \frac{\ln^{n+1} x}{n+1} \right]. \quad (7)$$

Convolutions among the $\mathcal{L}_n(s)$ take the form

$$(\mathcal{L}_m \otimes \mathcal{L}_n)(s) = V_{-1}^{mn} \delta(s) + \sum_{k=0}^{m+n+1} V_k^{mn} \mathcal{L}_k(s). \quad (8)$$

A generic expression for V_k^{mn} is given in Ref. [59].

Result.—Bare and renormalized jet functions are related by ($i = q, g$)

$$J_i^{\text{bare}}(s) = Z_J^i(s, \mu) \otimes J_i(s, \mu). \quad (9)$$

Throughout this work we employ the $\overline{\text{MS}}$ renormalization scheme. The RGE of the jet function reads

$$\mu \frac{d}{d\mu} J_i(s, \mu) = \gamma_J^i(s, \mu) \otimes J_i(s, \mu), \quad (10)$$

with the anomalous dimension

$$\gamma_J^i(s, \mu) = -(Z_J^i)^{-1}(s, \mu) \otimes \mu \frac{d}{d\mu} Z_J^i(s, \mu) \quad (11)$$

$$= -2\Gamma_{\text{cusp}}^i(\alpha_s) \frac{1}{\mu^2} \mathcal{L}_0\left(\frac{s}{\mu^2}\right) + \gamma_J^i(\alpha_s) \delta(s). \quad (12)$$

The collinear jet anomalous dimension $\gamma_J^i(\alpha_s)$ is equal to the one of the (virtuality-dependent) beam function [60]. In the following we use the expansions

$$\begin{aligned} \Gamma_{\text{cusp}}^i(\alpha_s) &= \sum_{n=0}^{\infty} \Gamma_n^i \left(\frac{\alpha_s}{4\pi} \right)^{n+1}, \quad \gamma_J^i(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n^i \left(\frac{\alpha_s}{4\pi} \right)^{n+1}, \\ J_i(s, \mu) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n J_i^{(n)}(s, \mu). \end{aligned} \quad (13)$$

The coefficients of the (lightlike) cusp anomalous dimension (Γ_n^q) [61–63] and the collinear jet anomalous dimension (γ_n^q) [21] for $n = 0, 1, 2$ are, e.g., listed in Ref. [60]. The jet function coefficients have the form

$$J_i^{(m)}(s, \mu) = J_{i,-1}^{(m)} \delta(s) + \sum_{n=0}^{2m-1} J_{i,n}^{(m)} \frac{1}{\mu^2} \mathcal{L}_n\left(\frac{s}{\mu^2}\right). \quad (14)$$

The coefficients of the μ -dependent plus distributions in Eq. (14) can be expressed in terms of lower-loop coefficients and anomalous dimensions by iteratively solving the RGE in Eq. (10). Up to three loops we find

$$\begin{aligned} J_{i,1}^{(1)} &= \Gamma_0^i, \\ J_{i,0}^{(1)} &= -\frac{\gamma_0^i}{2}, \\ J_{i,3}^{(2)} &= \frac{(\Gamma_0^i)^2}{2}, \\ J_{i,2}^{(2)} &= -\frac{\Gamma_0^i}{2} \left(\frac{3\gamma_0^i}{2} + \beta_0 \right), \\ J_{i,1}^{(2)} &= \Gamma_1^i - (\Gamma_0^i)^2 \frac{\pi^2}{6} + \frac{\gamma_0^i}{2} \left(\frac{\gamma_0^i}{2} + \beta_0 \right) + \Gamma_0^i J_{i,-1}^{(1)}, \\ J_{i,0}^{(2)} &= (\Gamma_0^i)^2 \zeta_3 + \Gamma_0^i \gamma_0^i \frac{\pi^2}{12} - \frac{\gamma_1^i}{2} - \left(\frac{\gamma_0^i}{2} + \beta_0 \right) J_{i,-1}^{(1)}, \\ J_{i,5}^{(3)} &= \frac{(\Gamma_0^i)^3}{8}, \\ J_{i,4}^{(3)} &= -\frac{5}{12} (\Gamma_0^i)^2 \left(\frac{3}{4} \gamma_0^i + \beta_0 \right), \end{aligned}$$

$$\begin{aligned}
 J_{i,3}^{(3)} &= \frac{\Gamma_0^i}{6} \left(5\beta_0\gamma_0^i + 2\beta_0^2 + \frac{3}{2}(\gamma_0^i)^2 - \pi^2(\Gamma_0^i)^2 + 6\Gamma_1^i \right. \\
 &\quad \left. + 3\Gamma_0^i J_{i,-1}^{(1)} \right), \\
 J_{i,2}^{(3)} &= \frac{5}{2}(\Gamma_0^i)^3 \zeta_3 + \frac{\pi^2}{4}(\Gamma_0^i)^2(\gamma_0^i + \beta_0) - \frac{\Gamma_0^i}{4}(3\gamma_1 + 2\beta_1) \\
 &\quad - \frac{\Gamma_0^i}{4}(3\gamma_0^i + 4\beta_0) - \frac{\gamma_0^i}{16}(6\beta_0\gamma_0^i + 8\beta_0^2 + (\gamma_0^i)^2) \\
 &\quad - \frac{\Gamma_0^i}{4}(3\gamma_0^i + 8\beta_0)J_{i,-1}^{(1)}, \\
 J_{i,1}^{(3)} &= \Gamma_2^i - \frac{\pi^4}{180}(\Gamma_0^i)^3 - \zeta_3(\Gamma_0^i)^2(3\beta_0 + 2\gamma_0^i) - \frac{\pi^2}{3}\Gamma_0^i\Gamma_1^i \\
 &\quad - \frac{\pi^2}{12}\Gamma_0^i\gamma_0^i(3\beta_0 + \gamma_0^i) + \frac{\gamma_1^i}{2}(2\beta_0 + \gamma_0^i) + \frac{\beta_1\gamma_0^i}{2} \\
 &\quad + \left(\frac{3}{2}\beta_0\gamma_0^i + 2\beta_0^2 + \frac{1}{4}(\gamma_0^i)^2 - \frac{\pi^2}{6}(\Gamma_0^i)^2 + \Gamma_1^i \right) J_{i,-1}^{(1)} \\
 &\quad + \Gamma_0^i J_{i,-1}^{(2)}, \\
 J_{i,0}^{(3)} &= \left(3\zeta_5 - \frac{\pi^2}{3}\zeta_3 \right) (\Gamma_0^i)^3 + \frac{\pi^4}{90}(\Gamma_0^i)^2 \left(\beta_0 + \frac{\gamma_0^i}{4} \right) \\
 &\quad + \Gamma_1^i \left(2\zeta_3\Gamma_0^i + \frac{\pi^2}{12}\gamma_0^i \right) \\
 &\quad + \Gamma_0^i \left(\frac{\pi^2}{12}\gamma_1 + \frac{\zeta_3}{2}\beta_0\gamma_0^i + \frac{\zeta_3}{4}(\gamma_0^i)^2 \right) - \frac{\gamma_2^i}{2} \\
 &\quad + \left(\zeta_3(\Gamma_0^i)^2 + \frac{\pi^2}{12}\Gamma_0^i(2\beta_0 + \gamma_0^i) - \beta_1 - \frac{\gamma_1}{2} \right) J_{i,-1}^{(1)} \\
 &\quad - \frac{1}{2}(4\beta_0 + \gamma_0^i)J_{i,-1}^{(2)}, \tag{15}
 \end{aligned}$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ and $\beta_1 = \frac{34}{3}C_A^2 - (\frac{20}{3}C_A + 4C_F)T_F n_f$, with n_f the number of active flavors. Our result for the three-loop quark jet function perfectly reproduces Eq. (15) for $i = q$. This provides another strong cross check and at the same time represents the first direct calculation of γ_2^q , which up to now has been inferred from RG consistency [21] using the three-loop results of Refs. [62,67]. For $m = 0, 1, 2$ the constants $J_{q,-1}^{(m)}$ are, e.g., collected in Ref. [12] in accordance with our conventions. The new result of our work is

$$\begin{aligned}
 J_{q,-1}^{(3)} &= C_F^3 \left(274\zeta_3 + \frac{22\pi^2\zeta_3}{3} - \frac{400\zeta_3^2}{3} - 88\zeta_5 + \frac{1173}{8} \right. \\
 &\quad \left. - \frac{3505\pi^2}{72} + \frac{622\pi^4}{45} - \frac{9871\pi^6}{8505} \right) \\
 &\quad + C_A C_F^2 \left(-\frac{28241\zeta_3}{27} + \frac{2200\pi^2\zeta_3}{27} + \frac{424\zeta_3^2}{3} + \frac{560\zeta_5}{9} \right. \\
 &\quad \left. + \frac{206197}{324} - \frac{17585\pi^2}{72} + \frac{18703\pi^4}{1215} + \frac{1547\pi^6}{4860} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ C_A^2 C_F \left(-\frac{187951\zeta_3}{243} + \frac{394\pi^2\zeta_3}{9} + \frac{1528\zeta_3^2}{9} - \frac{380\zeta_5}{9} \right. \\
 &\quad \left. + \frac{50602039}{52488} - \frac{464665\pi^2}{4374} + \frac{1009\pi^4}{1620} + \frac{221\pi^6}{5103} \right) \\
 &\quad + C_A C_F n_f T_F \left(\frac{14828\zeta_3}{81} - \frac{64\pi^2\zeta_3}{9} + \frac{32\zeta_5}{3} - \frac{2942843}{6561} \right. \\
 &\quad \left. + \frac{136648\pi^2}{2187} - \frac{418\pi^4}{405} \right) \\
 &\quad + C_F^2 n_f T_F \left(\frac{22432\zeta_3}{81} - \frac{272\pi^2\zeta_3}{27} + \frac{160\zeta_5}{3} - \frac{261587}{486} \right. \\
 &\quad \left. + \frac{4853\pi^2}{54} - \frac{5876\pi^4}{1215} \right) \\
 &\quad + C_F n_f^2 T_F^2 \left(\frac{1504\zeta_3}{243} + \frac{249806}{6561} - \frac{1864\pi^2}{243} + \frac{8\pi^4}{45} \right). \tag{16}
 \end{aligned}$$

It is often convenient to work with the Laplace transform

$$\tilde{J}_q(\nu, \mu) = \int_0^\infty ds e^{-\nu s} J_q(s), \tag{17}$$

because the convolutions of Eq. (2) type turn in to simple products in Laplace space. The Laplace space equivalents to our Eqs. (14) and (15) can be read off from Ref. [22]. The new three-loop constant related to Eq. (16) in their notation is

$$\begin{aligned}
 c_3^J &= 25.06777873 C_F^3 + 32.81169125 C_A C_F^2 \\
 &\quad - 0.7795843561 C_A^2 C_F - 31.65196210 C_A C_F n_f T_F \\
 &\quad - 61.78995095 C_F^2 n_f T_F + 28.49157341 C_F n_f^2 T_F^2, \tag{18}
 \end{aligned}$$

where for brevity we have evaluated the exact analytical result to ten valid digits for each color factor. The constant c_3^J equals the position space coefficient j_3 affecting the α_s determinations in Refs. [23,24,26], where until now $j_3 = 0 \pm 3000$ has been assumed. Evaluating Eq. (18) for $N_c = 3$, $T_F = 1/2$, and $n_f = 5$ we have $j_3 = -128.6512525$.

In Ref. [68] the N³LO nonlogarithmic constant of the (normalized) thrust cumulant cross section in the singular limit was obtained from a fit to fixed-order data produced by the Monte Carlo program EERAD3 [69], albeit with large numerical errors. With our new three-loop jet function constant in Eq. (16) and the known three-loop hard function [23] at hand we can use this result to extract a rough estimate for the unknown thrust ($q\bar{q}$ channel) soft function constant at three loops. In Laplace (position) space and adopting the notation of Ref. [22] (c_3^S) and Ref. [23] (s_3) we find ($N_c = 3$, $T_F = 1/2$, $n_f = 5$)

$$c_3^S = 2s_3 + 691 = -19988 \pm 1440(\text{stat}) \pm 4000(\text{syst}). \tag{19}$$

Summary.—In this Letter we have presented our calculation of the quark jet function $J_q(s)$ at three loops. The main result is the three-loop contribution to the $\delta(s)$ coefficient and given in Eq. (16). All other terms at this order can be derived from RG consistency conditions in terms of previous results, see Eq. (15). The new contribution is a necessary ingredient to many N^3LL' resummed processes with final state jets. It has, e.g., a direct impact on existing α_s determinations from e^+e^- event shapes. Our calculation also represents the first step toward possible applications of the N -jettiness IR slicing (or subtraction) method at N^3LO .

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