Supersymmetric Solutions and Borel Singularities for $\mathcal{N} = 2$ Supersymmetric Chern-Simons Theories

Masazumi Honda*

Department of Particle Physics, Weizmann Institute of Science, Rehovot 7610001, Israel

(Received 8 November 2017; revised manuscript received 5 May 2018; published 9 July 2018)

In supersymmetric (SUSY) field theory, there exist configurations that formally satisfy SUSY conditions but are not on the original path integral contour. We refer to such configurations as complexified supersymmetric solutions (CSS). In this Letter we discuss that CSS provide important information on the large order behavior of the weak coupling perturbative series in SUSY field theories. We conjecture that CSS with a bosonic (fermionic) free parameter give poles (zeros) of the Borel transformation of perturbative series whose locations are uniquely determined by actions of the solutions. We demonstrate this for various SUSY observables in 3D $\mathcal{N} = 2$ SUSY Chern-Simons matter theories on a sphere. First we construct an infinite number of CSS in the general 3D $\mathcal{N} = 2$ SUSY theory with a Lagrangian where an adjoint scalar in the vector multiplet takes a complex value and matter fields are nontrivial. Then we compare their actions with Borel transformations of perturbative expansions by inverse Chern-Simons levels for the observables and see agreement with our conjecture. It turns out that the CSS explain all the Borel singularities for this case.

DOI: [10.1103/PhysRevLett.121.021601](https://doi.org/10.1103/PhysRevLett.121.021601)

Introduction.—Supersymmetric (SUSY) field theories have provided good laboratories to find and sharpen ideas in theoretical physics. In SUSY field theories, configurations preserving a part of the SUSY play prominent roles because of their stabilities, restricted quantum corrections, and so on. While most attention has been paid to SUSY solutions that take values on the original path integral contour, there also exist configurations that formally satisfy SUSY conditions but are not on the original contour. We refer to such configurations as complexified supersymmetric solutions (CSS). Compared to standard SUSY solutions, CSS have been much less appreciated. The purpose of this Letter is to point out the physical role of CSS.

Since SUSY conditions are often sufficient to satisfy the saddle point equation, CSS are also often complex saddle points that are saddle points but are not on the original path integral contour. Therefore, it is worth noting historical situations on complex saddle points in quantum field theory (QFT). While saddle points along the original contour always contribute to the path integral, complex saddles may or may not contribute since this is determined by the details on the steepest descents associated with them. Historically most physicists have not taken this possibility into account in saddle point analysis of path integral despite its importance

was pointed out long time ago [1]. Namely, people have usually considered only saddles on the original contour and not discussed whether or not complex saddles contribute. Recently there appeared relatively more examples in the context of resurgence where some complex saddles actually give important contributions to path integral, and are necessary to obtain unambiguous answer from resummation of weak coupling perturbative series [2]. Although the examples are much simpler than typical interacting QFT, which are either quantum mechanics, two dimensional, or topological, there is no *a priori* reason to doubt that similar things happen in a more general setup. However, it is usually difficult to check this explicitly since we need to construct complex saddles and find large order behaviors of perturbative series around all the contributing saddles. Therefore, SUSY field theories would provide good next steps to understand the roles of complex saddles in QFT.

The above facts suggest that CSS may importantly contribute to the path integral. Indeed, recent works on SUSY localization [3] imply that CSS sometimes give important contributions [4–6]. In this Letter we do not discuss whether CSS are contributing saddles or not. Here we propose that independently of this question, CSS provide important information on large order behavior of weak coupling perturbative series in SUSY field theories.

The perturbative series in QFT is typically nonconvergent [7]. One of the standard ways to resum nonconvergent series is with Borel resummation. Given a formal series $P(g) = \sum_{\ell=0}^{\infty} c_{\ell} g^{a+\ell}$, its Borel resummation along the direction θ is defined by

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$$S_\theta P(g) = \int_0^{\infty e^{i\theta}} dt e^{-\frac{t}{g}} \mathcal{B}P(t), \quad (1)$$

where $\mathcal{B}P(t)$ is an analytic continuation of the formal Borel transformation $\sum_{\ell=0}^{\infty} (c_\ell/\Gamma(a+\ell)) t^{a+\ell-1}$. In general, Borel transformations have singularities in the complex t plane and their locations contain important information. Technically they determine whether or not the perturbative series is Borel summable given θ , and large order behavior of perturbative series as they give the radius of convergence of the Borel transformation. Physically, according to Lipatov's argument [8], Borel singularities are interpreted as nontrivial saddle points and their actions determine the locations of the singularities.

In this Letter we conjecture a relation between CSS and the analytic properties of the Borel transformation in SUSY field theories. Let us consider a weak coupling perturbative expansion $P(g)$ around a saddle point with certain topological numbers. In general there exist two types of CSS with the same topological numbers: one has a bosonic parameter, while the other has a fermionic one. We refer to them as bosonic and fermionic complexified supersymmetric solutions, respectively. We propose that bosonic CSS basically give poles of Borel transformation of $P(g)$ while fermionic CSS give zeros. More precisely, if there are n_B bosonic and n_F fermionic solutions with the topological number and the action $S = S_c/g$, then we conjecture that the Borel transformation has a pole for $n_B \geq n_F$ and zero for $n_B \leq n_F$ with degree $|n_B - n_F|$ at $t = S_c$. Namely, the Borel transformation includes the following factor:

$$\mathcal{B}P^{(t)}(t) \supset \prod_{\text{solutions}} \frac{1}{(t - S_c)^{n_B - n_F}}. \quad (2)$$

Note that CSS gives information not only on locations of singularities but also on their degrees. This leads us to further insights into nonperturbative structures of path integrals.

In the next sections we demonstrate the above story for various SUSY observables in 3D $\mathcal{N} = 2$ SUSY Chern-Simons (CS) matter theory on a sphere [9]. In the section of this Letter on construction of complexified supersymmetric solutions, we construct an infinite number of CSS in general 3D $\mathcal{N} = 2$ theory with a Lagrangian. They are described by zero modes of differential operators appearing in SUSY transformations of chiral multiplets [11]. We cannot have such zero modes for most cases when the adjoint scalar σ in the vector multiplet is real, i.e., on the original path integral contour, but relaxing the reality condition allows such zero modes. In the section of this Letter on complexified supersymmetric solutions and the analytic property of the Borel transformation, we compare their actions with the explicit Borel transformation found in [13] (see also [12, 14]) and we see the agreement with our conjecture.

Construction of complexified supersymmetric solutions.— Let us construct complexified supersymmetric solutions in

general 3D $\mathcal{N} = 2$ SUSY gauge theories with Lagrangians on a squashed S^3 . We choose the squashed S^3 to be the ellipsoid S_b^3 defined as the hypersurface in $\mathbb{R}^4$: $(x_1^2 + x_2^2) + b^2(x_3^2 + x_4^2) = \text{const}$. The round S^3 corresponds to the limit $b \rightarrow 1$. As we will see, the relation between CSS and Borel singularities is more transparent for generic b rather than the $b \rightarrow 1$ case because the actions of many solutions become coincident as $b \rightarrow 1$.

Vector multiplet: First we find CSS for the 3D $\mathcal{N} = 2$ vector multiplet, which consists of gauge field A_μ , adjoint scalar σ , gaugino $(\lambda, \bar{\lambda})$, and auxiliary field D . For this purpose, we need only the SUSY transformations [15]:

$$\begin{aligned} \delta A_\mu &= -\frac{1}{2} \bar{\lambda} \gamma_\mu \epsilon - \frac{1}{2} \bar{\epsilon} \gamma_\mu \lambda, \\ \delta \sigma &= \frac{i}{2} \bar{\lambda} \epsilon + \frac{i}{2} \bar{\epsilon} \lambda, \\ \delta \lambda &= \left(\frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} - D_\mu \sigma \right) \gamma^\mu \epsilon - i D \epsilon - \frac{i}{f(\vartheta)} \sigma \epsilon, \\ \delta \bar{\lambda} &= \left(\frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} + D_\mu \sigma \right) \gamma^\mu \bar{\epsilon} + i D \bar{\epsilon} + \frac{i}{f(\vartheta)} \sigma \bar{\epsilon}, \\ \delta D &= \frac{1}{2} \bar{\epsilon} \gamma^\mu D_\mu \lambda - \frac{1}{2} D_\mu \bar{\lambda} \gamma^\mu \epsilon - \frac{1}{2} [\bar{\epsilon} \lambda, \sigma] + \frac{1}{2} [\bar{\lambda} \epsilon, \sigma] \\ &\quad - \frac{i}{4f(\vartheta)} (\bar{\epsilon} \lambda + \bar{\lambda} \epsilon), \end{aligned} \quad (3)$$

where we take the coordinate $(x_1, x_2, x_3, x_4) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta \cos \chi, \sin \vartheta \sin \chi)$ and $f(\vartheta) = \sqrt{b^{-2} \sin^2 \vartheta + b^2 \cos^2 \vartheta}$. Our SUSY condition is simply vanishing of all the transformations (3). As a sufficient condition, we can solve the SUSY condition by [16]

$$F_{\mu\nu} = 0, \quad \sigma = \text{const}, \quad D = -\frac{\sigma}{f(\vartheta)}, \quad \lambda = \bar{\lambda} = 0. \quad (4)$$

This is just the Coulomb branch solution for real σ , but the SUSY condition can be formally satisfied even for complex σ . Next we find CSS for the chiral multiplet under the configuration (4). It turns out that σ is fixed to a particular complex value by the SUSY condition for the chiral multiplet.

Chiral multiplet: The 3D $\mathcal{N} = 2$ chiral multiplet consists of scalars $(\phi, \bar{\phi})$, fermions $(\psi, \bar{\psi})$, and auxiliary fields $(F, \bar{F})$, whose SUSY transformations are

$$\begin{aligned} \delta \phi &= i \bar{\epsilon} \psi, & \delta \bar{\phi} &= i \epsilon \bar{\psi}, \\ \delta \psi &= -\gamma^\mu \epsilon D_\mu \phi - \epsilon \sigma \phi - \frac{i \Delta}{f(\vartheta)} \epsilon \phi + i \bar{\epsilon} F, \\ \delta \bar{\psi} &= -\gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} - \bar{\phi} \sigma \bar{\epsilon} - \frac{i \Delta}{f(\vartheta)} \bar{\phi} \bar{\epsilon} + i \bar{F} \epsilon, \\ \delta F &= \epsilon (-\gamma^\mu D_\mu \psi + \sigma \psi + \lambda \phi) + \frac{i(2\Delta - 1)}{2f(\vartheta)} \epsilon \psi, \\ \delta \bar{F} &= \bar{\epsilon} (-\gamma^\mu D_\mu \bar{\psi} + \bar{\psi} \sigma - \bar{\phi} \bar{\lambda}) + \frac{i(2\Delta - 1)}{2f(\vartheta)} \bar{\epsilon} \bar{\psi}, \end{aligned} \quad (5)$$

where we have assigned the $U(1)_R$ charges $(-\Delta, \Delta, 1 - \Delta, \Delta - 1, 2 - \Delta, \Delta - 2)$ to $(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})$ respectively. Under the vector multiplet configuration (4), we find two types of CSS where one of them has a bosonic parameter while the other has a fermionic parameter. Below we take σ to be the Cartan part by using the gauge symmetry.

Bosonic solutions.—First we look for solutions with trivial $(\psi, \bar{\psi}, F, \bar{F})$ but nontrivial $(\phi, \bar{\phi})$, namely,

$$\psi = \bar{\psi} = F = \bar{F} = 0, \quad (6)$$

which leads us to $\delta\phi = \delta\bar{\phi} = \delta F = \delta\bar{F} = 0$. A nontrivial condition for ϕ comes from $\delta\psi = 0$. Decomposing the fields into components associated with weight vector ρ of the gauge group representation, the condition for the component ϕ^ρ is

$$\gamma^\mu \epsilon D_\mu \phi^\rho + \epsilon(\rho \cdot \sigma) \phi^\rho + \frac{i\Delta}{f(\vartheta)} \epsilon \phi^\rho = 0. \quad (7)$$

We can easily find solutions for this condition because the eigenvalue problem of the differential operator above is already solved in [17] in the context of SUSY localization of the S_b^3 partition function. In order to compute its one-loop determinant for the chiral multiplet, [17] studied the eigenvalue problem

$$\gamma^\mu \epsilon D_\mu \Phi + \epsilon(\rho \cdot \sigma) \Phi + \frac{i\Delta}{f(\vartheta)} \epsilon \Phi = M \epsilon \Phi, \quad (8)$$

where M is the eigenvalue given by

$$M = M_{m,n} = \rho \cdot \sigma + i \left(mb + nb^{-1} + \frac{Q}{2} \Delta \right), \quad (9)$$

$$m, n \in \mathbb{Z}_{\geq 0}, \quad Q = b + b^{-1}.$$

The product of the eigenvalues roughly gives the “denominator” of the one-loop determinant. Our SUSY condition is satisfied for $M = 0$, but the real σ cannot satisfy this for generic (b, Δ) . If we relax this, denoting the eigenmode with the eigenvalue $M_{m,n}$ by $\Phi_{m,n}$, we have the SUSY solutions

$$\phi = \Phi_{m,n}, \quad \rho \cdot \sigma = \rho \cdot \sigma_{m,n}^{\text{bos}} = -i \left(mb + nb^{-1} + \frac{Q}{2} \Delta \right). \quad (10)$$

Note that these conditions do not uniquely determine value of ϕ since we have the freedom to change an overall constant.

Fermionic solutions.—Next we find solutions with trivial $(\phi, \bar{\phi}, F, \bar{F})$ but nontrivial $(\psi, \bar{\psi})$, namely,

$$\phi = \bar{\phi} = F = \bar{F} = 0, \quad (11)$$

which gives $\delta\psi = \delta\bar{\psi} = 0$. Then nontrivial SUSY conditions for ψ come from the vanishing of $\delta\phi$ and δF :

$$\bar{\epsilon} \psi^\rho = 0, \quad \epsilon(-\gamma^\mu D_\mu + \rho \cdot \sigma) \psi^\rho + \frac{i(2\Delta - 1)}{2f(\vartheta)} \epsilon \psi^\rho = 0. \quad (12)$$

Again we can easily find solution of this condition because [17] also solved the eigenvalue problem

$$\epsilon(-\gamma^\mu D_\mu \Psi + \rho \cdot \sigma \Psi) + \frac{i(2\Delta - 1)}{2f(\vartheta)} \epsilon \Psi = M \epsilon \Psi,$$

$$M = M_{m,n} = \rho \cdot \sigma + i \left(mb + nb^{-1} - \frac{Q(\Delta - 2)}{2} \right), \quad (13)$$

to compute the “numerator” of the one-loop determinant. The SUSY condition is satisfied for $M = 0$, but again we need to take the complex σ . Thus, denoting the eigenmode with the eigenvalue $M_{m,n}$ by $\Psi_{m,n}$, we have the SUSY solutions

$$\bar{\epsilon} \psi^\rho = 0, \quad \psi^\rho = \Psi_{m,n},$$

$$\rho \cdot \sigma = \rho \cdot \sigma_{m,n}^{\text{fer}} = i \left(mb + nb^{-1} - \frac{Q(\Delta - 2)}{2} \right). \quad (14)$$

So far we have focused on the particular component ρ of a single chiral multiplet. In general we have various chiral multiplets with various representations, and each of those has multiple components associated with weights. We can always construct a CSS where only one component of a chiral multiplet is (10) or (14) and all the others are trivial. This solution forces σ to be in a $[\text{rank}(G) - 1]$ -dimensional hypersurface in σ space. Hence σ is not completely determined for $\text{rank}(G) > 1$, and we can also take nontrivial configuration (10) or (14) for other components. In general there are solutions with multiple nontrivial components unless σ is “overdetermined,” namely, SUSY conditions set more than $\text{rank}(G)$ linearly independent hypersurfaces in the σ space.

Complexified supersymmetric solutions and the analytic property of the Borel transformation.—Let us consider 3D $\mathcal{N} = 2$ SUSY CS matter theory coupled to chiral multiplets of representations $\{\mathbf{R}_a\}$ with R charges $\{\Delta_a\}$. We take the gauge group to be a product of the semisimple gauge groups: $G = G_1 \times \cdots \times G_n$ with CS levels $k_1, \dots, k_n$, respectively. We are interested in perturbative expansion by $g_p \propto 1/|k_p|$ and a relation between its Borel transformation and the CSS.

Squashed sphere: In [13], the author found explicit finite dimensional integral representations of Borel transformations for S_b^3 partition functions. As we briefly review below, the Borel transformations are somehow hidden in the localization formula given by [17,18]

$$Z_{S_b^3}(g) = \int_{-\infty}^{\infty} d^{\text{rank}(G)} \sigma Z_{\text{cl}}(\sigma) Z_{1\text{loop}}(\sigma), \quad (15)$$

where Z_{cl} is the classical contribution and $Z_{1\text{loop}}$ is the one-loop determinant in the localization procedure

$$Z_{\text{cl}}(\sigma) = \exp \left[\sum_{p=1}^n \frac{i \text{sgn}(k_p)}{g_p} \text{tr}(\sigma^{(p)})^2 \right],$$

$$Z_{1\text{loop}}(\sigma) = \frac{\prod_{\alpha \in \text{root}_+} 4 \sinh(\pi b \alpha \cdot \sigma) \sinh(\pi b^{-1} \alpha \cdot \sigma)}{\prod_a \prod_{\rho_a \in \mathbf{R}_a} s_b(\rho_a \cdot \sigma - \frac{iQ(1-\Delta_a)}{2})},$$

$$s_b(z) = \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \frac{mb + nb^{-1} + Q/2 - iz}{mb + nb^{-1} + Q/2 + iz}. \quad (16)$$

The basic idea for finding the Borel transformation is quite simple [19]. First we take the polar coordinate for each gauge group G_p : $\sigma_i^{(p)} = \sqrt{\tau_p} \hat{x}_i^{(p)}$ with $\tau_p = i \text{sgn}(k_p) t_p$ and $\hat{x}_i^{(p)} \in S^{\text{rank}(G_p)-1}$ [20]. Then the partition function takes the form

$$\begin{aligned} Z_{S_b^3} &= \left(\prod_{p=1}^n i \text{sgn}(k_p) \int_0^{-i \text{sgn}(k_p) \infty} dt_p e^{-\frac{t_p}{g_p}} \right) f(i \text{sgn}(k) t), \\ f(\tau) &= \frac{\prod_{p=1}^n \tau_p^{\frac{\dim(G_p)-1}{2}}}{2^n} \int_{\text{spheres}} d\hat{x} h(\tau, \hat{x}), \\ h(\tau, \hat{x}) &= Z_{\text{1loop}}(\sigma) \prod_{p=1}^n \tau_p^{-\frac{\dim(G_p)-\text{rank}(G_p)}{2}} \Big|_{\sigma_i^{(p)} = \sqrt{\tau_p} \hat{x}_i^{(p)}}. \end{aligned} \quad (17)$$

This is similar to the form of Borel resummation (1) for multiple couplings, and it is natural to ask whether the function $f(\tau)$ is related to the Borel transformation. We can actually prove that $f(\tau)$ is related to the Borel transformation of the perturbative series by [13]

$$\left[\prod_{p=1}^n i \text{sgn}(k_p) \right] f(\{\tau_p\}) = \mathcal{B}Z_{S_b^3}(\{-i \text{sgn}(k_p) \tau_p\}). \quad (18)$$

Now we compare the analytic property of the Borel transformation with the CSS in the last section. When the gauge group is a product of rank-1 gauge groups, we can easily see this since $f(\tau)$ is no longer the integral representation. For example, let us consider a $U(1)$ CS theory with CS level $k > 0$ coupled to charge $\{q_a\}$ chiral multiplets with R charges $\{\Delta_a\}$. Borel transformation of perturbative expansion of $Z_{S_b^3}$ by $g = 1/\pi k$ in this theory is

$$\mathcal{B}Z_{S_b^3}(t) = \frac{1}{2\sqrt{-it} \prod_{a=1}^{N_f} s_b(q_a \sqrt{-it} - \frac{iQ(1-\Delta_a)}{2})}. \quad (19)$$

This has simple poles and zeros at

$$\begin{aligned} t_{\text{pole}}^{m,n} &= -\frac{i}{q_a^2} \left(mb + nb^{-1} + \frac{Q}{2} \Delta_a \right)^2, \\ t_{\text{zero}}^{m,n} &= -\frac{i}{q_a^2} \left(mb + nb^{-1} - \frac{Q(\Delta_a - 2)}{2} \right)^2, \end{aligned} \quad (20)$$

which are parametrized by two integers $m, n \in \mathbb{Z}_{\geq 0}$. Let us compare these with the actions of the CSS. Although the theory has various terms in general, the only relevant term for us is the SUSY CS term

$$\begin{aligned} S_{\text{CS}} &= \frac{ik}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right. \\ &\quad \left. - \bar{\lambda} \lambda + 2D\sigma \right], \end{aligned} \quad (21)$$

because the CSS have vanishing Yang-Mills and matter actions [21]. In this theory, the bosonic and fermionic CSS associated with the a th chiral multiplet have

$$\begin{aligned} \sigma_{m,n}^{\text{bos}} &= -\frac{i}{q_a} \left(mb + nb^{-1} + \frac{Q}{2} \Delta_a \right), \\ \sigma_{m,n}^{\text{fer}} &= +\frac{i}{q_a} \left(mb + nb^{-1} - \frac{Q(\Delta_a - 2)}{2} \right), \end{aligned} \quad (22)$$

respectively. Therefore, their actions are given by

$$\begin{aligned} S_{\text{bos}} &= \frac{i\pi k}{q_a^2} \left(mb + nb^{-1} + \frac{Q}{2} \Delta_a \right)^2 = \frac{t_{\text{pole}}^{m,n}}{g}, \\ S_{\text{fer}} &= \frac{i\pi k}{q_a^2} \left(mb + nb^{-1} - \frac{Q(\Delta_a - 2)}{2} \right)^2 = \frac{t_{\text{zero}}^{m,n}}{g}. \end{aligned} \quad (23)$$

Thus, a single bosonic (fermionic) CSS with the action $S = S_c/g$ gives a simple pole (zero) of the Borel transformation at $t = S_c$.

For the general rank, it is more involved since there are integrals for $f(\tau)$ and it is not easy to find the precise locations of the poles and zeros of the Borel transformation. However, it is always true that the origins of all the Borel singularities are the CSS because all the singularities are technically coming from the poles of the one-loop determinant whose locations are the same as the values of σ on the CSS. Although we have focused on the partition function, the same result holds also for SUSY Wilson loops [22] because its effect on the Borel transformation is insertion of a function without poles and zeros into $h(\tau, \hat{x})$ [13].

Round sphere limit: For $b = 1$, the building block $s_b(z)$ of the one-loop determinant becomes a product over a single integer [23]:

$$s_1(z) = \prod_{n=1}^{\infty} \left(\frac{n - iz}{n + iz} \right)^n. \quad (24)$$

Because of this, the poles and zeros of the Borel transformation become degenerate. For example, in the $U(1)$ CS theory considered for the squashed case, the Borel transformations have the poles and zeros with degree n at

$$t_{\text{pole}}^n = -\frac{i(n + \Delta_a - 1)^2}{q_a^2}, \quad t_{\text{zero}}^n = -\frac{i(n - \Delta_a + 1)^2}{q_a^2}, \quad (25)$$

with $n \in \mathbb{Z}_+$. Hence the poles and zeros parametrized by the two integers for general b become coincident and are described by a single integer for $b = 1$. This degeneration is reflected as the change of the degrees of the poles and zeros.

Correspondingly, the eigenvalues (9) and (13) become degenerate for $b \rightarrow 1$ and the CSS with the same $m + n$ have the same actions. Therefore, we have n bosonic and fermionic solutions with the actions

$$S_{\text{bos}} = \frac{t_{\text{pole}}^n}{g}, \quad S_{\text{fer}} = \frac{t_{\text{zero}}^n}{g}, \quad (26)$$

respectively. Thus the n -bosonic (fermionic) solutions with the coincident action $S = S_c/g$ give a degree- n pole (zero) of the Borel transformation at $t = S_c$. The same result holds

also for various observables studied in [13]. Especially, for the superconformal case, one can obtain the same conclusion for the bremsstrahlung function [24], two point functions [25] of the stress tensor, and the $U(1)$ flavor symmetry current on $\mathbb{R}^3$. Therefore, we expect that for the superconformal case (not limited to the 3D $\mathcal{N} = 2$ theory) CSS on the sphere provide information on the analytic property of the Borel transformation even for the flat space case.

Hypermultiplets on round sphere: When matters are only hypermultiplets, the one-loop determinant becomes much simpler:

$$Z_{1\text{loop}}(\sigma) = \frac{\prod_{\alpha \in \text{root}_+} 4 \sinh^2(\pi \alpha \cdot \sigma)}{\prod_a \prod_{\rho_a \in \mathbf{R}_a} 2 \cosh(\pi \rho_a \cdot \sigma)}, \quad (27)$$

because of the identity

$$\frac{1}{s_1(z - i/2)s_1(-z - i/2)} = \frac{1}{2 \cosh(\pi z)}. \quad (28)$$

This simplification comes from the cancellations of bosonic and fermionic contributions in a pair of conjugate representations with the R charges $1/2$.

For example, if we consider a $U(1)$ CS theory with CS level $k > 0$ coupled to charge $\{q_a\}$ hypermultiplets, then the Borel transformation does not have zeros but has simple poles at

$$t_{\text{pole}}^n = -\frac{i(2n-1)^2}{4q_a^2 g}, \quad n \in \mathbb{Z}_+. \quad (29)$$

Correspondingly, we have n_B bosonic and n_F fermionic solutions with the following actions:

$$S_{\text{bos}} = \frac{i(2n_B-1)^2}{4q_a^2 g}, \quad S_{\text{fer}} = \frac{i(2n_F+1)^2}{4q_a^2 g}, \quad (30)$$

with $n_B, n_F \in \mathbb{Z}_+$. For solutions with the action t_{pole}^n/g , we have n bosonic and $(n-1)$ fermionic solutions and hence always $n_B - n_F = 1$ for any n . Thus the CSS predict that the degree- n poles at the $t = t_{\text{pole}}^n$ are canceled by the degree- $(n-1)$ zeros and become simple poles. This agrees with our conjecture.

Conclusion and discussions.—In this Letter we have pointed out the physical importance of complexified supersymmetric solutions that satisfy SUSY conditions but are not on the original path integral contour. We have conjectured that if there are n_B bosonic and n_F fermionic CSS with the action $S = S_c/g$, then the Borel transformation has a pole for $n_B \geq n_F$ and zero for $n_B \leq n_F$ with degree $|n_B - n_F|$ at $t = S_c$. We have shown this statement for various SUSY observables in the 3D $\mathcal{N} = 2$ SUSY CS matter theories on the sphere. We have constructed the CSS in the general Lagrangian 3D $\mathcal{N} = 2$ theory and compared their

actions with the analytic property of the Borel transformations to agree with our conjecture.

We have seen that the fermionic CSS give information on the zeros of the Borel transformation. To our knowledge, the physical implication of Borel zeros has not been explored in the literature. Naively, Borel zeros do not seem to be related to the large order behavior of the perturbative series unless they are the ends of branch cuts. There might be a general argument for a relation between the Borel zeros and the fermionic saddle points as for the relation between the Borel singularities and the saddle points [8,26].

We have not discussed whether the CSS are contributing saddle points or not. In order to answer this question, it is appropriate to perform Lefschetz thimble analysis for 3D $\mathcal{N} = 2$ theories. More generally, since most of the analyses of SUSY localization in the literature have not performed serious complex saddle point analysis to our knowledge, there is a logical possibility that we are missing contributions from complex saddles in most setups. Perhaps the setups in [4–6] would be good starting points to address this issue.

It is known that in the (Coulomb branch) localization formula for $Z_{S^3_b}$, picking up poles of the one-loop determinant in a half complex plane gives rise to the Higgs branch representation of the partition function including a product of vortex and antivortex partition functions for some theories [4,5,27,28]. Since we have seen the correspondence between the poles and bosonic CSS, presumably the CSS are related to the Higgs branch formula or vortices. It is interesting to make this intuition more precise.

Recently there appeared multicut solutions in matrix models obtained by localization of the S^3 partition function of 3D $\mathcal{N} = 2$ CS theories [29]. In the solutions, poles and zeros in the one-loop determinant play important roles to gather eigenvalues of the matrix models. It would be illuminating to study the connection between the solutions and our CSS.

A part of this work was presented in RIKEN, and lectures at Fudan University. The author is grateful to B. Assel, C. Closset, S. Cremonesi, T. Fujimori, K. Hosomichi, S. Kamata, T. Misumi, T. Morita, M. Nitta, and N. Sakai for useful discussions. The author would like to thank the Centro de Ciencias de Benasque, CERN, Fudan University, KITP, National Taiwan University, RIKEN, and YITP for their hospitality.

*masazumi.honda@weizmann.ac.il

- [1] R. Balian, G. Parisi, and A. Voros, Quartic oscillator, *SACLAY-DPH-T-78-95*. A. Voros, The return of the quartic oscillator. The complex WKB method, *Ann. de l'I.H. Poincaré* **A39**, 211 (1983).
- [2] A. Cherman, D. Dorigoni, and M. Unsal, Decoding perturbation theory using resurgence: Stokes phenomena, new saddle points and Lefschetz thimbles, *J. High Energy Phys.* **10** (2015) 056; A. Behtash, T. Sulejmanpasic, T. Schafer,

- and M. Unsal, Hidden Topological Angles and Lefschetz Thimbles, *Phys. Rev. Lett.* **115**, 041601 (2015); A. Behtash, E. Poppitz, T. Sulejmanpasic, and M. Unsal, The curious incident of multi-instantons and the necessity of Lefschetz thimbles, *J. High Energy Phys.* **11** (2015) 175; A. Behtash, G. V. Dunne, T. Schaefer, T. Sulejmanpasic, and M. Unsal, Complexified Path Integrals, Exact Saddles and Supersymmetry, *Phys. Rev. Lett.* **116**, 011601 (2016); A. Behtash, G. V. Dunne, T. Schaefer, T. Sulejmanpasic, and M. Unsal, Toward Picard-Lefschetz theory of path integrals, complex saddles and resurgence, *Ann. Math. Sci. Appl.* **2**, 95 (2017); P. V. Buividovich, G. V. Dunne, and S. N. Valgushev, Complex Path Integrals and Saddles in Two-Dimensional Gauge Theory, *Phys. Rev. Lett.* **116**, 132001 (2016); T. Misumi, M. Nitta, and N. Sakai, Non-BPS exact solutions and their relation to bions in $\mathbb{CP}^{N-1}$ models, *J. High Energy Phys.* **05** (2016) 057; S. Gukov, M. Marino, and P. Putrov, Resurgence in complex Chern-Simons theory, [arXiv:1605.07615](#); T. Fujimori, S. Kamata, T. Misumi, M. Nitta, and N. Sakai, Nonperturbative contributions from complexified solutions in $\mathbb{CP}^{N-1}$ models, *Phys. Rev. D* **94**, 105002 (2016); Exact resurgent trans-series and multibion contributions to all orders, *Phys. Rev. D* **95**, 105001 (2017); T. Fujimori, S. Kamata, T. Misumi, M. Nitta, and N. Sakai, Resurgence structure to all orders of multi-bions in deformed SUSY quantum mechanics, *Prog. Theor. Exp. Phys.* **2017**, 083B02 (2017); G. V. Dunne and M. Unsal, Deconstructing zero: Resurgence, supersymmetry and complex saddles, *J. High Energy Phys.* **12** (2016) 002; C. Kozcaz, T. Sulejmanpasic, Y. Tanizaki, and M. Unsal, Cheshire cat resurgence, self-resurgence and quasi-exact solvable systems, [arXiv:1609.06198](#); A. Ahmed and G. V. Dunne, Transmutation of a trans-series: The Gross-Witten-Wadia phase transition, [arXiv:1710.01812](#).
- [3] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, *Commun. Math. Phys.* **313**, 71 (2012).
- [4] M. Fujitsuka, M. Honda, and Y. Yoshida, Higgs branch localization of 3d $\mathcal{N} = 2$ theories, *Prog. Theor. Exp. Phys.* **2014**, 123B02 (2014).
- [5] F. Benini and W. Peelaers, Higgs branch localization in three dimensions, *J. High Energy Phys.* **05** (2014) 030; H. Y. Chen, H. Y. Chen, and J. K. Ho, Connecting mirror symmetry in 3d and 2d via localization, *Int. J. Mod. Phys. A* **29**, 1530004 (2014).
- [6] F. Benini and A. Zaffaroni, A topologically twisted index for three-dimensional supersymmetric theories, *J. High Energy Phys.* **07** (2015) 127; F. Benini and A. Zaffaroni, Supersymmetric partition functions on Riemann surfaces, *Proc. Symp. Pure Math.* **96**, 13 (2017); M. Honda and Y. Yoshida, Supersymmetric index on $T^2 \times S^2$ and elliptic genus, [arXiv:1504.04355](#); C. Closset, S. Cremonesi, and D. S. Park, The equivariant A-twist and gauged linear sigma models on the two-sphere, *J. High Energy Phys.* **06** (2015) 076; C. Closset and H. Kim, Comments on twisted indices in 3d supersymmetric gauge theories, *J. High Energy Phys.* **08** (2016) 059.
- [7] F. J. Dyson, Divergence of perturbation theory in quantum electrodynamics, *Phys. Rev.* **85**, 631 (1952).
- [8] L. N. Lipatov, Divergence of the perturbation theory series and the quasiclassical theory, *Zh. Eksp. Teor. Fiz.* **72**, 411 (1977) [*Sov. Phys. JETP* **45**, 216 (1977)].
- [9] We consider theories with well-defined sphere partition functions, although ill-defined cases are also interesting [10].
- [10] T. Morita and V. Niarchos, F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories, *Nucl. Phys. B* **858**, 84 (2012); B. R. Safdi, I. R. Klebanov, and J. Lee, A crack in the conformal window, *J. High Energy Phys.* **04** (2013) 165; I. Yaakov, Redeeming bad theories, *J. High Energy Phys.* **11** (2013) 189; J. Lee and M. Yamazaki, Gauging and decoupling in 3d $\mathcal{N} = 2$ dualities, *J. High Energy Phys.* **06** (2016) 077; B. Assel and S. Cremonesi, The infrared physics of bad theories, *SciPost Phys.* **3**, 024 (2017); A. Arabi Ardehali, High-temperature asymptotics of supersymmetric partition functions, *J. High Energy Phys.* **07** (2016) 025; L. Di Pietro and M. Honda, Cardy formula for 4d SUSY theories and localization, *J. High Energy Phys.* **04** (2017) 055.
- [11] It was pointed out in [12] that for some rank-1 theories, Borel singularities come from zero modes of kinetic terms. Our CSS are also a part of them.
- [12] I. Aniceto, J. G. Russo, and R. Schiappa, Resurgent analysis of localizable observables in supersymmetric gauge theories, *J. High Energy Phys.* **03** (2015) 172.
- [13] M. Honda, How to resum perturbative series in 3d $\mathcal{N} = 2$ Chern-Simons matter theories, *Phys. Rev. D* **94**, 025039 (2016).
- [14] J. G. Russo, A note on perturbation series in supersymmetric gauge theories, *J. High Energy Phys.* **06** (2012) 038.
- [15] We follow the notation of [4].
- [16] This is not the most general solution. We could relate $F_{\mu\nu}$ to D . This class of solution includes vortices [4,5].
- [17] N. Hama, K. Hosomichi, and S. Lee, SUSY gauge theories on squashed three-spheres, *J. High Energy Phys.* **05** (2011) 014.
- [18] Y. Imamura and D. Yokoyama, $\mathcal{N} = 2$ supersymmetric theories on squashed three-sphere, *Phys. Rev. D* **85**, 025015 (2012); T. Nishioka and I. Yaakov, Supersymmetric Renyi entropy, *J. High Energy Phys.* **10** (2013) 155; A. Tanaka, Localization on round sphere revisited, *J. High Energy Phys.* **11** (2013) 103; C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, The geometry of supersymmetric partition functions, *J. High Energy Phys.* **01** (2014) 124.
- [19] M. Honda, Borel Summability of Perturbative Series in 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ Supersymmetric Theories, *Phys. Rev. Lett.* **116**, 211601 (2016).
- [20] If G_p is $SU(N)$, we consider $U(N)$ with insertion of $\delta(\sum_{j=1}^N \sigma_j^{(p)})$.
- [21] For their explicit forms, see (2.9) and (2.18) in [4].
- [22] A. Tanaka, Comments on knotted 1/2 BPS Wilson loops, *J. High Energy Phys.* **07** (2012) 097.
- [23] A. Kapustin, B. Willett, and I. Yaakov, Exact results for Wilson loops in superconformal Chern-Simons theories with matter, *J. High Energy Phys.* **03** (2010) 089; D. L. Jafferis, The exact superconformal R-symmetry extremizes Z , *J. High Energy Phys.* **05** (2012) 159; N. Hama, K.

- Hosomichi, and S. Lee, Notes on SUSY gauge theories on three-sphere, *J. High Energy Phys.* **03** (2011) 127.
- [24] A. Lewkowycz and J. Maldacena, Exact results for the entanglement entropy and the energy radiated by a quark, *J. High Energy Phys.* **05** (2014) 025.
- [25] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, Contact terms, unitarity, and F-maximization in three-dimensional superconformal theories, *J. High Energy Phys.* **10** (2012) 053; Comments on Chern-Simons contact terms in three dimensions, *J. High Energy Phys.* **09** (2012) 091; C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, Supersymmetric field theories on three-manifolds, *J. High Energy Phys.* **05** (2013) 017.
- [26] P. Argyres and M. Unsal, A semiclassical realization of infrared renormalons, *Phys. Rev. Lett.* **109**, 121601 (2012);
- The semi-classical expansion and resurgence in gauge theories: New perturbative, instanton, bion, and renormalon effects, *J. High Energy Phys.* **08** (2012) 063.
- [27] L. F. Alday, D. Martelli, P. Richmond, and J. Sparks, Localization on three-manifolds, *J. High Energy Phys.* **10** (2013) 095.
- [28] S. Pasquetti, Factorisation of $\mathcal{N} = 2$ theories on the squashed 3-sphere, *J. High Energy Phys.* **04** (2012) 120; C. Beem, T. Dimofte, and S. Pasquetti, Holomorphic blocks in three dimensions, *J. High Energy Phys.* **12** (2014) 177; Y. Yoshida and K. Sugiyama, Localization of 3d $\mathcal{N} = 2$ supersymmetric theories on $S^1 \times D^2$, [arXiv:1409.6713](#).
- [29] T. Morita and K. Sugiyama, Multi-cut solutions in Chern-Simons matrix models, *Nucl. Phys.* **B929**, 1 (2018).