What Spatial Geometries do $(2 + 1)$ -Dimensional Quantum Field Theory Vacua Prefer?

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We consider relativistic $(2 + 1)$ -dimensional quantum field theories (QFTs) on a product of time with a two-space and study the vacuum free energy as a functional of the temperature and spatial geometry. We focus on free scalar and Dirac fields on arbitrary perturbations of flat space, finding that the free energy difference from flat space is finite and always negative to leading order in the perturbation. Thus, free $(2 + 1)$ -dimensional QFTs appear to always energetically favor a crumpled space on all scales; this is true both as a purely quantum effect at zero temperature and as a purely thermal effect at high temperature. Importantly, we show that this quantum effect is non-negligible for the relativistic Dirac degrees of freedom on monolayer graphene even at room temperature, so we argue that this vacuum energy effect should be included for a proper analysis of the equilibrium configuration of graphene or similar materials.

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Introduction.—The presence of matter gives a surface embedded in an ambient space an energy. This matter may be external to the surface—like the pressure of air on a soap bubble—or may comprise the material nature of the surface itself—like a membrane with surface tension and bending energy. These energies determine the equilibrium (i.e., static) configuration of such a surface: for instance, the presence of surface tension tends to make membranes favor (smooth) minimal-area configurations, while finite-temperature thermodynamic effects may render membranes unstable to crumpling or rippling [1–3].

In this Letter, we initiate a study of the free energy contribution to the equilibrium configuration of a surface due to free relativistic quantized matter fields living on it. In particular, we include zero-temperature (Casimir) effects. Such relativistic quantum fields occur in various physical settings: for example, in graphene and related materials, the electronic structure gives rise to an effective description in terms of relativistic Dirac fermions propagating on the two-dimensional crystal [4–6]. In cosmology, domain wall defects may exist [7] and could carry upon them relativistic degrees of freedom (d.o.f.). More exotically, in braneworld models, our Universe is itself a surface on which the standard model fields live [8,9].

The setting is then $(2 + 1)$ -dimensional quantum field theory (QFT) on a product of time with a two-space. By studying both the free nonminimally coupled scalar and the free Dirac fermion, we will see that such a field *lowers* the

free energy of the surface on which it lives when the surface is deformed away from being intrinsically flat. [Note that free massless vector fields are equivalent to free minimally coupled massless scalar fields by duality in $(2 + 1)$ dimensions, so our analysis indirectly includes massless vector fields as well. Our analysis was motivated by holographic considerations: for $(2 + 1)$ -dimensional conformal field theories (CFTs), flat space is energetically disfavored at zero temperature. This was shown globally in holographic CFTs and perturbatively in general [10,11].] This energy difference is UV finite (and thus well defined) and present at any temperature including $T = 0$ (in which case, it can be interpreted as a Casimir effect), both for massless and massive fields, and any scalar nonminimal coupling. It is then natural to wonder whether a classical membrane action is able to counteract this quantum tendency to crumple. We will perform a naïve analysis of this question for monolayer graphene, which is indeed seen to ripple on short scales [12,13]. We show that at room temperature the quantum vacuum energy of the Dirac fermions gives a scale at which one would expect crumpling effects on the order of the lattice spacing. The effective membrane description that would validate our analysis breaks down at this scale, so our results make no definitive statement about the rippling of graphene. However, they do indicate that a careful consideration of these quantum effects is important for a proper treatment of equilibrium configurations of graphene and similar materials even at room temperature.

Free energy difference.—We consider a spacetime that is a product of time with a two-space Σ (for now taken to be general). Since we are interested in QFT at finite-temperature T , we work in Euclidean time, so the metric is

$$ds^2 = d\tau^2 + ds_\Sigma^2, \quad (1)$$

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with τ periodic with period $\beta = 1/T$. [Unless otherwise stated, we use natural units $\hbar = c = k_B = 1$, with c the “effective” speed of light of the relativistic fields (not necessarily equal to the actual speed of light).] We will consider a free scalar ϕ and Dirac spinor ψ , with equations of motion

$$(-\nabla^2 + \xi R + M^2)\phi = 0, \quad (\not{D} + M)\psi = 0, \quad (2)$$

respectively, where R is the Ricci scalar and $\not{D}$ is understood as being defined by the spin connection (our conventions can be found in the Supplemental Material [14]).

The free energy $F[\Sigma]$ is a functional of the geometry Σ (and temperature T) and is given in terms of the partition function $Z[\Sigma]$ as $F = -T \ln Z$. We are specifically interested in the difference ΔF between the free energy on Σ and some reference background space $\bar{\Sigma}$ at the same temperature, which satisfies

$$e^{-\beta \Delta F} = \frac{Z[\Sigma]}{Z[\bar{\Sigma}]} = \frac{\int \mathcal{D}\Phi e^{-S_\Sigma[\Phi]}}{\int \mathcal{D}\Phi e^{-S_{\bar{\Sigma}}[\Phi]}} = \langle e^{-\Delta S} \rangle_{\bar{\Sigma}}, \quad (3)$$

where Φ stands for the matter field (scalar or fermion) being integrated over in the path integral, $\Delta S = S_\Sigma - S_{\bar{\Sigma}}$ is the difference between the action on $S^1 \times \Sigma$ and $S^1 \times \bar{\Sigma}$, and the expectation value is defined by the path integral on the background geometry $S^1 \times \bar{\Sigma}$. To evaluate ΔF , recall that for free fields the path integrals in (3) yield functional determinants, giving

$$Z = (\det \mathcal{L})^q \quad \text{with} \quad \mathcal{L} = -\partial_\tau^2 + \mathcal{O} + M^2, \quad (4)$$

where $q = -1/2$ (+1) for the scalar (fermion), $\mathcal{O}$ is an elliptic self-adjoint scalar operator on Σ given explicitly in (11) below, and the determinant is evaluated over Matsubara frequencies on the thermal circle (with appropriate periodicity or antiperiodicity in the scalar and fermion cases, respectively). For the scalar, (4) is obtained straightforwardly. The fermion case is more subtle, and we leave full details to the Supplemental Material [14]. In short, a direct path integral yields $Z = \det(i\not{D} - iM)$. However, by exploiting the direct product structure of the metric (1) along with the fact that the two-dimensional rotation group only has a single generator, we may eliminate the spinor structure and reduce the determinant to that of an elliptic operator of the form (4) with the determinant taken over the space of complex functions with antiperiodicity on the thermal circle.

The free energy can then be evaluated via heat kernel methods [15]: defining the heat kernel as $K_{\mathcal{L}}(t) \equiv \text{Tr}(e^{-t\mathcal{L}}) = \sum_i e^{-t\lambda_i}$ (with λ_i as the eigenvalues of $\mathcal{L}$), one obtains

$$\beta \Delta F = q \int_0^\infty \frac{dt}{t} \Delta K_{\mathcal{L}}(t), \quad (5)$$

where $\Delta K_{\mathcal{L}}(t) \equiv K_{\mathcal{L}}(t) - K_{\bar{\mathcal{L}}}(t)$. This expression is UV divergent unless $\Delta K_{\mathcal{L}}(t)$ vanishes at $t = 0$; this condition

can be ensured by an appropriate choice of background $\bar{\Sigma}$. Specifically, the heat kernel expansion gives [15]

$$K_{\mathcal{L}}(t) = \beta \left(\frac{c_1 V_\Sigma}{t^{3/2}} + \frac{c_2 \chi_\Sigma + c_3 V_\Sigma M^2}{t^{1/2}} + O(t^{1/2}) \right), \quad (6)$$

where V_Σ and χ_Σ are the volume (suitably IR regulated if Σ is noncompact) and Euler number of Σ , respectively, and c_1 , c_2 , and c_3 are dimensionless constants independent of the geometry (though they depend on the choice of matter field). Thus, requiring that ΔF be UV finite only imposes that we choose a background geometry $\bar{\Sigma}$ with the same volume and topology as Σ . It is worth emphasizing that, although the undifferentiated functional determinant $\det \mathcal{L}$ is UV divergent, we do not need to invoke any regularization to evaluate the *differenced* free energy. (Note that since $\det \mathcal{L}$ is UV divergent, ΔF is not necessarily the same as a difference of separately renormalized free energies on Σ and $\bar{\Sigma}$, which could be scheme dependent and therefore unphysical.) It is also worth noting that, in higher dimensions, the expansion (6) contains nontopological curvature invariants of Σ ; thus, obtaining an UV-finite free energy difference would require a careful matching of these invariants on Σ and $\bar{\Sigma}$ (in contrast with the heuristic expectation that “energy differences are always UV finite”).

Now we specialize to our case of interest. Ultimately, we wish to take Σ to be a deformation of flat space $\bar{\Sigma}$. Since these are two-dimensional, we introduce conformally flat coordinates x^A , in terms of which the metrics on Σ and $\bar{\Sigma}$ take the form

$$ds_\Sigma^2 = e^{2f(x)} \delta_{AB} dx^A dx^B, \quad ds_{\bar{\Sigma}}^2 = \delta_{AB} dx^A dx^B. \quad (7)$$

In order to have good control over the spectrum of $\mathcal{L}$ (which is essential for computing the heat kernel), we compactify these to tori Σ_L and $\bar{\Sigma}_L$ via the identifications $x^A \sim x^A + L_A$ with $L_1 = L$ and $L_2 = rL$. We consider a family of deformations $f_L(x)$ so that as $L \rightarrow \infty$ (with r fixed) we recover (7) with the x^A uncompactified. Moreover, at any finite L , we may choose f_L such that $V_{\Sigma_L} = V_{\bar{\Sigma}_L}$. By the arguments above, this condition will ensure that, for every L , the free energy difference between the deformed and flat torus will be UV finite. (For finite L , one may choose between periodic and antiperiodic boundary conditions for the fermion on the torus cycles; since our torus is only an IR regulator, and this distinction vanishes in the limit $L \rightarrow \infty$, we take the periodic case for simplicity.)

Our object of interest is the free energy difference with this IR regulator removed,

$$\beta \Delta F_\infty[f] \equiv q \lim_{L \rightarrow \infty} \int_0^\infty \frac{dt}{t} \Delta K_{\mathcal{L}}[f_L; L](t), \quad (8)$$

with $\Delta K_{\mathcal{L}}[f_L; L](t) \equiv K_{\mathcal{L}}[f_L; L](t) - K_{\mathcal{L}}[0; L](t)$. For notational convenience, we will henceforth forego writing

the arguments of these functionals explicitly, and we will drop the subscripts L on f_L and ∞ on ΔF_∞ . Using (4), we finally obtain

$$\beta \Delta F[f] = q \lim_{L \rightarrow \infty} \int_0^\infty \frac{dt}{t} e^{-M^2 t} \Theta(T^2 t) \Delta K_{\mathcal{O}}(t), \quad (9)$$

where the sum over Matsubara frequencies yields

$$\Theta(\zeta) = \sum_{n=-\infty}^{\infty} e^{-(2\pi)^2(n-q+1/2)^2\zeta}, \quad (10)$$

and the operators $\mathcal{O}$ are given explicitly in terms of f as

$$\mathcal{O}_s = -e^{-2f} [\bar{\nabla}^2 + 2\xi(\bar{\nabla}^2 f)], \quad (11a)$$

$$\mathcal{O}_f = \mathcal{O}_s|_{\xi=1/4} - e^{-2f} \left(i\bar{\star}(df \wedge d) - \frac{(\bar{\nabla}_A f)^2}{4} \right), \quad (11b)$$

where $\bar{\nabla}_A$ and $\bar{\star}$ are the covariant derivative and Hodge dual on the flat background $f = 0$, and the subscripts s and f denote the scalar and fermion.

Perturbation theory.—We introduce a perturbation parameter ϵ to expand our deformation Σ of the flat $\bar{\Sigma}$ as

$$f = \epsilon f^{(1)} + \epsilon^2 f^{(2)} + O(\epsilon^3). \quad (12)$$

Preservation of the volume requires

$$\int d^2x f^{(1)} = 0, \quad \int d^2x [f^{(2)} + (f^{(1)})^2] = 0. \quad (13)$$

We denote by λ_i and $h_i(x)$ the eigenvalues and eigenfunctions of $\mathcal{O}$, so in order to compute $\Delta K_{\mathcal{O}}(t)$ perturbatively, we must compute the perturbative corrections to λ_i . We expand $\mathcal{O}$ as

$$\mathcal{O} = -\bar{\nabla}^2 + \epsilon \mathcal{O}^{(1)} + \epsilon^2 \mathcal{O}^{(2)} + O(\epsilon^3); \quad (14)$$

explicit expressions for $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ can be obtained by expanding the form of $\mathcal{O}$ given in (11). Continuing to use bars to denote unperturbed objects, we likewise expand its eigenvalues and eigenfunctions as

$$\lambda_i = \bar{\lambda}_i + \epsilon \lambda_i^{(1)} + \epsilon^2 \lambda_i^{(2)} + O(\epsilon^3), \quad (15a)$$

$$h_i = \bar{h}_i + \epsilon \sum_j b_{ij}^{(1)} \bar{h}_j + \epsilon^2 \sum_j b_{ij}^{(2)} \bar{h}_j + O(\epsilon^3), \quad (15b)$$

so the $\bar{h}_i$ are eigenfunctions of the flat space Laplacian with corresponding eigenvalues $\bar{\lambda}_i$. We choose these eigenfunctions to be normalized as

$$\langle \bar{h}_i | \bar{h}_j \rangle \equiv \int d^2x \bar{h}_i^* \bar{h}_j = \delta_{ij}. \quad (16)$$

Then defining

$$P_{ij}^{(1)} = \langle \bar{h}_i | \mathcal{O}^{(1)} \bar{h}_j \rangle, \quad (17a)$$

$$P_{ij}^{(2)} = \langle \bar{h}_i | \mathcal{O}^{(2)} \bar{h}_j \rangle + \sum_{k: \bar{\lambda}_k \neq \bar{\lambda}_i} \frac{P_{ik}^{(1)} P_{kj}^{(1)}}{\bar{\lambda}_i - \bar{\lambda}_k}, \quad (17b)$$

standard perturbation theory yields the eigenvalue shifts

$$\lambda_i^{(1)} = P_{ii}^{(1)}, \quad \lambda_i^{(2)} = P_{ii}^{(2)} \quad (\text{no sum}). \quad (18)$$

Note that we have glossed over a subtlety: recall from quantum mechanics perturbation theory that the presence of degenerate subspaces imposes additional constraints on the unperturbed eigenfunctions $\bar{h}_i$ for the expansion (15b) to be consistent. The first order eigenvalue problem requires we arrange our basis $\bar{h}_i$ such that $P_{ij}^{(1)}$ is diagonal within such subspaces (i.e., if $\bar{\lambda}_i = \bar{\lambda}_j$, but $i \neq j$, then $P_{ij}^{(1)} = 0$). If any degeneracies remain at first order, we must further ensure at second order that $P_{ij}^{(2)}$ be diagonal in the remaining degenerate subspaces. We discuss this issue explicitly in the Supplemental Material [14].

Finally, we expand the heat kernel as

$$\Delta K_{\mathcal{O}}(t) = \epsilon K^{(1)}(t) + \epsilon^2 K^{(2)}(t) + O(\epsilon^3), \quad (19)$$

with

$$K^{(1)}(t) = -t \sum_i e^{-\bar{\lambda}_i t} P_{ii}^{(1)}, \quad (20a)$$

$$K^{(2)}(t) = t \sum_i e^{-\bar{\lambda}_i t} \left[\frac{t}{2} (P_{ii}^{(1)})^2 - P_{ii}^{(2)} \right]. \quad (20b)$$

Results.—In order to perform our computations, we Fourier decompose the perturbation

$$f^{(1)}(x) = \frac{(2\pi)^2}{rL^2} \sum_{\vec{N}} \tilde{f}_{\vec{N}}^{(1)} e^{2\pi i(n_1 x^1 + n_2 x^2/r)/L}, \quad (21a)$$

$$\rightarrow \int d^2k \tilde{f}^{(1)}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \quad \text{as } L \rightarrow \infty, \quad (21b)$$

where the sum runs over all pairs of integers $\vec{N} = \{n_1, n_2\}$, and the second line defines $k_A = \lim_{L \rightarrow \infty} 2\pi n_A / L_A$.

An explicit calculation on the torus for fixed L reveals that (for both the scalar and fermion), while the eigenvalues are indeed shifted at first order in ϵ , their contribution to the heat kernel vanishes: $K^{(1)} = 0$. The leading order perturbation to the heat kernel is then the second order term $K^{(2)}$. A lengthy but straightforward computation yields the finite- L expressions presented in the Supplemental Material [14]; in the limit $L \rightarrow \infty$, they become

$$K^{(2)}(t) = t \int d^2k k^4 |\tilde{f}^{(1)}(\vec{k})|^2 I(k^2 t), \quad (22)$$

with $k = |\vec{k}|$,

$$I_s(\zeta) = -\frac{\pi}{4\zeta^2} \left[6 + \zeta(1 - 8\xi) - (6 + 2\zeta(1 - 4\xi)) \right. \\ \left. + \frac{\zeta^2}{2}(1 - 4\xi)^2 \mathcal{F}\left(\frac{\sqrt{\zeta}}{2}\right) \right], \quad (23a)$$

$$I_f(\zeta) = \frac{\pi}{4\zeta^2} \left[(6 + \zeta) \mathcal{F}\left(\frac{\sqrt{\zeta}}{2}\right) - 6 \right], \quad (23b)$$

and $\mathcal{F}(\zeta) = \zeta^{-1} e^{-\zeta^2} \int_0^\zeta d\zeta' e^{(\zeta')^2}$. Thus, using (9) we find

$$\Delta F = -\epsilon^2 \int d^2k a(k) |\tilde{f}^{(1)}(\vec{k})|^2, \quad (24)$$

with

$$a(k) \equiv -qTk^4 \int_0^\infty dt e^{-M^2 t} \Theta(T^2 t) I(k^2 t). \quad (25)$$

A few comments are in order. First, we see the leading variation in ΔF is quadratic in ϵ . Next, as $L \rightarrow \infty$, the volume constraint $V_{\Sigma_L} = V_{\Sigma_L}$ becomes the condition that the variation of the volume $\int d^2x \sqrt{g}$ vanishes; for $f^{(1)}$ this simply imposes no constant Fourier component. We have that $I(\zeta)$ is finite and $\Theta(\zeta)$ is $O(\zeta^{-1/2})$ at small ζ , and thus ΔF is UV finite. Likewise, since $I(\zeta)$ and $\Theta(\zeta)$ are $O(1)$ at large ζ , ΔF is also IR finite for $M > 0$; in fact, the large- ζ decay of $I(\zeta)$ also implies IR finiteness in the massless case $M = 0$ for both the fermion and minimally coupled scalar ($\xi = 0$). [For the massless scalar with nonminimal coupling $\xi \neq 0$, ΔF_s is IR divergent, since the flat space zero eigenvalue acquires a negative contribution due to the scalar curvature coupling. This is reflected in the $\ln(\ell_M/\ell)$ corrections mentioned in Table I.] Finally, a key physical point is that, as can be seen by explicitly plotting, $qI(\zeta) < 0$ for all $\zeta > 0$ (and all ξ for the scalar), implying that for any (nonvanishing) f the free energy difference is strictly negative to leading order in ϵ : $\Delta F < 0$. (It is possible to prove that $I_f < 0$ without resorting to plotting it; we have not been able to find as elegant of a proof for I_s .) The form of the expression (25), along with the asymptotic behaviors of $\Theta(\zeta)$ and $I(\zeta)$, makes it possible to derive scaling relations. Specifically, defining $\ell_M = \hbar/(cM)$ to be the (reduced) Compton wavelength, $\ell_T = \hbar c/(k_B T)$ to be a thermal wavelength, and ℓ to be the characteristic length scale of f , ΔF scales as in Table I. Thus, at small temperatures—in which ΔF becomes the energy difference ΔE —the effect is a purely quantum one: $\Delta E \sim -\epsilon^2 \hbar c/\ell$ for $\ell \ll \ell_M$. On the other hand, at high temperatures, the energy decrease is a thermal effect: e.g., $\Delta F_f \sim -\epsilon^2 k_B T$ for $\ell \ll \ell_M$. We are exploring numerical methods to verify these perturbative results and also understand the non-perturbative regime [16].

As a final note, the small-temperature limit $\ell_T \gg \max[\ell, \ell_M]$ is in fact analytically tractable: Poisson

TABLE I. The scaling of ΔF for the minimally coupled free scalar field and Dirac fermion for different relative magnitudes of ℓ , ℓ_M , and ℓ_T . Note that, for the nonminimally coupled scalar (i.e., $\xi \neq 0$), factors of $\ln(\ell_M/\ell)$ appear in the last two rows.

	$-\Delta F/(\epsilon^2 \hbar c/\ell)$
$\ell_T \gg \ell \gg \ell_M$	ℓ_M/ℓ
$\ell_T \gg \ell_M \gg \ell$	1
$\ell \gg \ell_T \gg \ell_M$	ℓ_M/ℓ
$\ell \gg \ell_M \gg \ell_T$	$\ell_M^2/(\ell \ell_T)$
$\ell_M \gg \ell_T \gg \ell$	1
$\ell_M \gg \ell \gg \ell_T$	ℓ/ℓ_T

resummation gives $\Theta(T^2 t) = \beta/\sqrt{4\pi t}$ up to terms that are exponentially suppressed in β^2/t , which allows us to compute $a(k)$ explicitly as

$$a_s^{(T=0)}(k) = \frac{\pi k^3}{128} \left[\frac{2(3 - 32\xi)M}{k} - \frac{24M^3}{k^3} \right. \\ \left. + \left(3 - 32\xi + 128\xi^2 - 8(1 - 16\xi) \frac{M^2}{k^2} \right. \right. \\ \left. \left. + \frac{48M^4}{k^4} \right) \operatorname{arccot}\left(\frac{2M}{k}\right) \right], \quad (26a)$$

$$a_f^{(T=0)}(k) = \frac{\pi k^3}{64} \left[\frac{2M}{k} + \frac{24M^3}{k^3} + \left(1 - \frac{8M^2}{k^2} - \frac{48M^4}{k^4} \right) \right. \\ \left. \times \operatorname{arccot}\left(\frac{2M}{k}\right) \right]. \quad (26b)$$

(In the massless limit, these agree precisely with the energy in [11] for the massless scalar CFT ($\xi = 1/8$) and free fermion CFT [with their appropriate central charges $c_T = (3/2)/(4\pi)^2$ and $3/(4\pi)^2$, respectively].)

Membrane crumpling.—We have seen that free relativistic $(2+1)$ -dimensional d.o.f. on deformations of flat space that have UV-finite free energy difference from flat space always energetically prefer the deformation, for any temperature. Let us now consider how this effect competes with a membrane's bending energy (which at zero temperature favors a flat geometry) if it carries such d.o.f. Our goal is to exhibit membrane deformations that potentially make the total energy negative relative to a flat membrane, rather than consider *all* deformations. Hence, we consider the deformations of the previous section. Since these have vanishing volume variation, a nice consequence is that the tension contribution to membrane energy is unaffected.

Consider three-dimensional flat space with Cartesian coordinates $\{X^A, Z\}$ and parametrize a surface in it by $X^A = x^A + \epsilon v^A(x^B)$, $Z = \sqrt{\epsilon} h(x^A)$. Then for small ϵ and suitable v^A , the intrinsic metric on the membrane in the coordinates x^A is as in Eq. (7) with $-\bar{\nabla}^2 f = \epsilon \det(\partial_A \partial_B h)$. The constant part of f is not determined by this relation; thus,

we may choose it so that $f^{(1)}$ has no constant Fourier component. The bending energy due to extrinsic curvature is

$$H = \epsilon \kappa \int d^2x (\bar{\nabla}^2 h)^2, \quad (27)$$

where κ is the bending rigidity. If the membrane is deformed from flat over a region of characteristic size $\ell \ll \ell_M$, then the (positive) bending energy E_B and (negative) vacuum energy E_Q (at zero temperature) for N free relativistic quantum fields are parametrically given as

$$E_B \sim \epsilon \kappa, \quad E_Q \sim -\epsilon^2 N \frac{\hbar c}{\ell}. \quad (28)$$

The ground state equilibrium configuration of the membrane should minimize $E = E_B + E_Q$. One might expect that, because E_B is lower order in ϵ than E_Q , a perturbative analysis guarantees that $E > 0$ for any deformation of flat space. However, note that E_B and E_Q have different scale dependence, with E_Q dominating at sufficiently small scales. Defining $\ell_{\text{crumple}} \equiv N\hbar c/\kappa$ and noting that ϵ and $\ell_{\text{crumple}}/\ell$ are independent, if $\ell/\ell_{\text{crumple}} \lesssim \epsilon \ll 1$, then E_Q can be comparable to and even dominate E_B while still being in the perturbative regime. Whether or not E actually decreases for (sufficiently large) deformations of flat space—therefore implying that the membrane’s equilibrium configuration is crumpled at a sufficiently small scale relative to ℓ_{crumple} —then depends on nonlinear and higher-derivative contributions to its bending action and whether or not these are relevant at scales up to ℓ_{crumple} at amplitudes $O(\epsilon^2)$. Hence, ℓ_{crumple} defines a scale below which a membrane has the potential to crumple.

It is instructive to consider the case of a graphene monolayer, for which the bending rigidity is $\kappa \sim 1$ eV, the unit cell has size $\ell_{\text{cell}} \sim 1$ Å, and the relativistic fields are two Dirac fermions with effective speed $c \sim c_{\text{light}}/100$, with c_{light} the actual speed of light [5,13,17,18]. Our effective membrane description is valid for $\ell \gg \ell_{\text{cell}}$, while from Table I the scaling properties (28) are valid at room temperature for $\ell \ll \ell_{T=300\text{K}} \sim 10^3 \ell_{\text{cell}}$. Computing the potential crumpling scale, we find $\ell_{\text{crumple}} \sim 10 \ell_{\text{cell}}$, which is sufficiently close to ℓ_{cell} to make our effective membrane description suspect. Thus, while this naïve analysis is insufficient to imply the existence of a crumpled equilibrium configuration for graphene, it *does* indicate that long range quantum properties of the conduction electrons (which give rise to the effective Dirac fermions) are important for understanding the energetics of equilibrium monolayer graphene even at room temperature; such effects are presumably highly challenging to correctly incorporate into Monte Carlo or *ab initio* quantum simulations. Indeed,

it is intriguing to note that, for freely suspended graphene at room temperature, one does see low amplitude ripples on short scales ~ 50 Å, close to our ℓ_{crumple} [12].

We emphasize that, in future two-dimensional crystal materials, whose electronic structures similarly give rise to Dirac fermions (or perhaps scalars or vectors), the situation may be different. In particular, if one wishes to have such a monolayer material that is flat on scales above the unit cell scale ℓ_{cell} , one presumably requires $N\hbar c/\kappa \lesssim \ell_{\text{cell}}$, which may be regarded as a bound on the speed or number of relativistic species, given the bending mechanics of the crystal.

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