Toward a Definition of Complexity for Quantum Field Theory States

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(Received 6 September 2017; revised manuscript received 24 November 2017; published 22 March 2018)

We investigate notions of complexity of states in continuous many-body quantum systems. We focus on Gaussian states which include ground states of free quantum field theories and their approximations encountered in the context of the continuous version of the multiscale entanglement renormalization ansatz. Our proposal for quantifying state complexity is based on the Fubini-Study metric. It leads to counting the number of applications of each gate (infinitesimal generator) in the transformation, subject to a state-dependent metric. We minimize the defined complexity with respect to momentum-preserving quadratic generators which form $\mathfrak{su}(1,1)$ algebras. On the manifold of Gaussian states generated by these operations, the Fubini-Study metric factorizes into hyperbolic planes with minimal complexity circuits reducing to known geodesics. Despite working with quantum field theories far outside the regime where Einstein gravity duals exist, we find striking similarities between our results and those of holographic complexity proposals.

DOI: 10.1103/PhysRevLett.120.121602

Introduction.—Applications of quantum information concepts to high-energy physics and gravity have recently led to many far-reaching developments. In particular, it has become apparent that special properties of entanglement in holographic [1] quantum field theory (QFT) states are crucial for the emergence of smooth higher-dimensional (bulk) geometries in the gauge-gravity duality [2]. Much of the progress in this direction was achieved by building on the holographic entanglement entropy proposal by Ryu and Takayanagi [3], which geometrizes the von Neumann entropy of a reduced density matrix of a QFT in a subregion in terms of the area of codimension-2 bulk minimal surfaces anchored at the boundary of this subregion (see, e.g., Ref. [4] for a recent overview). However, Ryu-Takayanagi surfaces are often unable to access the whole holographic geometry [5–7]. This observation has led to significant interest in novel, from the point of view of quantum gravity, codimension-1 (volume) and codimension-0 (action) bulk quantities, whose behavior suggests conjecturing a link with the information theory notion of quantum state complexity [8–14]. In fact, a certain identification between complexity and action was originally suggested by Toffoli [15,16] outside the context of holography.

Quantum state complexity originates from the field of quantum computations, which are usually modeled in a finite Hilbert space as the application of a sequence of gates chosen from a discrete set. In this context, the complexity of a unitary U is roughly associated with the minimal number of gates necessary to realize (or approximate) U. Notable progress has been made in connecting this notion to distances in Riemannian geometries derived from a set of generators [17]. The complexity of a target state $|T\rangle$ is usually subordinated to unitary complexity by specifying a "simple" reference state $|R\rangle$ and minimizing the complexity of U subject to $|T\rangle = U|R\rangle$ [18,19]. Our approach differs in defining state complexity directly.

In the context of holography, the organization of discrete tensor networks (seen as a quantum circuit U) has been suggested to give a qualitative picture of how quantum states give rise to emergent geometries [20]. This heuristic analysis was applied to the multiscale entanglement renormalization ansatz (MERA) [21], employed to find ground states of critical physical theories presenting a tensor network structure reminiscent of an anti–de Sitter (AdS) time slice. This motivated proposing "complexity equals volume" (CV) [9] and "complexity equals action" (CA) [11,12] as new entries in the holographic dictionary. However, in holography, one naturally considers continuum setups, QFTs, and there are shortcomings of traditional approaches to complexity when attempting to address field theory states. The aim of this Letter is to bridge a pressing gap by exploring complexity-motivated distance measures in QFTs.

The main challenges in providing a workable definition of complexity in the continuum are related to choosing (a) a reference state $|R\rangle$, (b) a set of allowed gates (correspondingly infinitesimal generators), (c) a measure for how such gates contribute to the resulting distance function and a procedure for how to minimize it, and (d) a way to regulate ultraviolet (UV) divergences. Our proposed choice for (c) is to measure the path length by integrating the Fubini-Study (FS) line element along a path from $|R\rangle$ to $|T\rangle$ associated with an allowed realization of U. Minimizing the path will amount to studying geodesics on the manifold of quantum states induced by allowed gates acting on the reference state. In this way, our approach derives complexity from the projective structure of the Hilbert space in a universal way. In the FS prescription, directions which modify the state by an overall phase have no effect on the complexity. Simultaneously with our work, Ref. [22] appeared, which considers a different approach based on unitary complexity [17] (see Sec. F in Ref. [23] for a comparison).

While the FS prescription is quite general, our choices for (a), (b), and (d) render the necessary calculations tractable. Some of these choices are inspired by the continuous MERA (cMERA) approach to free QFTs [24–26], which we briefly review in Sec. A of Ref. [23]. Similarly to the states in cMERA, our choices for the reference state $|R\rangle$ and target state $|T\rangle$ will be *pure Gaussian states,* and allowed generators will be subsets of quadratic operators. Our choices include cMERA in the set of allowed circuits, letting us test its optimality. We perform our analysis in momentum space and ignore frequencies above the UV cutoff Λ, which equips us with a notion of approximation. Unlike in cMERA, Λ need not coincide with the reference state characteristic scale M, defined below, since the freedom of choosing the reference state is a part of the definition of complexity and is *a priori* independent from a notion of cutoff or regulator (this observation is due to R. C. Myers).

As a first step, we consider the two-mode squeezing operator for each pair of opposite momenta $\pm\vec{k}$. We then extend our analysis to include the full set of momentum preserving quadratic generators which form $\mathfrak{su}(1,1)$ algebras. In this case, the study of minimal complexity reduces to the study of geodesics on a product of hyperbolic planes.

While a full literature review is outside the scope of this Letter, there is a substantial body of important recent developments which include, e.g., Refs. [27–35].

Complexity from the Fubini-Study metric.—We are interested in considering unitary operators U arising from iterating generators $G(s)$ taken from some elementary set of Hermitian operators $\mathcal{G}$. The allowed transformations U can then be represented as path ordered exponentials:

$$U(\sigma) = \mathcal{P}e^{-i\int_{s_i}^{\sigma} G(s)ds}. \tag{1}$$

Here, s parametrizes progress along a path, starting at s_i and ending at s_f, and $\sigma \in [s_i, s_f]$ is some intermediate value of s. The path-ordering $\mathcal{P}$ is required for noncommuting generators $G(s)$. We seek a path achieving $|T\rangle \approx U(s_f)|R\rangle$, where $(\approx)$ indicates that states coincide for momenta below a cutoff Λ. According to the FS line element (see, e.g., [36])

$$ds_{\mathrm{FS}}(\sigma) = d\sigma\sqrt{|\partial_\sigma|\Psi(\sigma)\rangle|^2 - |\langle\Psi(\sigma)|\partial_\sigma|\Psi(\sigma)\rangle|^2}, \tag{2}$$

the length of a path going via states $|\Psi(\sigma)\rangle$ is

$$\ell[|\Psi(\sigma)\rangle] = \int_{s_i}^{s_f} ds_{\mathrm{FS}}(\sigma). \tag{3}$$

For a path $|\Psi(\sigma)\rangle = U(\sigma)|R\rangle$, with $U(\sigma)$ given by Eq. (1), the line element of Eq. (3) can be reexpressed as

$$ds_{\mathrm{FS}}(\sigma) = d\sigma\sqrt{\langle G^2(\sigma)\rangle_{\Psi(\sigma)} - \langle G(\sigma)\rangle^2_{\Psi(\sigma)}} \tag{4}$$

and is independent of path reparametrizations.

If the path $|\Psi(\sigma)\rangle$ is unrestricted, the unique unitarily invariant distance measure $d_{R,T} = \arccos|\langle R|T\rangle| \leq \pi/2$ is obtained. However, by restricting the allowed generators $G(s)$, highly nontrivial notions of distance deserving the name complexity may be obtained. Our proposal is to define the complexity $\mathcal{C}$ as the minimal length according to Eq. (3) of a path from $|\Psi(s_i)\rangle \approx |R\rangle$ to $|\Psi(s_f)\rangle \approx |T\rangle$ driven by generators $G(s)$ in $\mathcal{G}$:

$$\mathcal{C}(|R\rangle, |T\rangle, \mathcal{G}, \Lambda) = \min_{G(s)} \ell[|\Psi(\sigma)\rangle]. \tag{5}$$

The proposed complexity $\mathcal{C}$ inherits the properties of a distance function from the FS metric.

Gaussian states in free QFTs.—We consider a theory of free relativistic bosons in $(d+1)$-spacetime dimensions defined by the quadratic Hamiltonian

$$H_m = \int d^d x : \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_{\vec{x}}\phi)^2 + \frac{1}{2}m^2\phi^2\right\} : \tag{6}$$

with commutation relations $[\phi(\vec{x}), \pi(\vec{x}')] = i\delta^d(\vec{x} - \vec{x}')$. This theory describes noninteracting particles created and annihilated by operators $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ obeying $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^d(\vec{k} - \vec{k}')$. These are related to the field and momentum operators via ($\omega_k \equiv \sqrt{k^2 + m^2}$)

$$\phi(\vec{k}) = \frac{1}{\sqrt{2\omega_k}}(a_{\vec{k}} + a^\dagger_{-\vec{k}}) \quad \text{and} \quad \pi(\vec{k}) = \frac{\sqrt{\omega_k}}{\sqrt{2}i}(a_{\vec{k}} - a^\dagger_{-\vec{k}}) \tag{7}$$

and diagonalize the Hamiltonian: $H_m = \int d^d k \omega_k a^\dagger_{\vec{k}} a_{\vec{k}}$. For $m = 0$ we obtain a free *conformal field theory* (CFT).

A general translation-invariant *pure Gaussian state* $|S\rangle$ with momentum space correlation functions

$$\langle S|\phi(\vec{k})\phi(\vec{k}')|S\rangle = \frac{1}{2\alpha_k}\delta^{(d)}(\vec{k}+\vec{k}') \tag{8}$$

is specified by its nullifiers (annihilation operators):

$$\left(\sqrt{\frac{\alpha_k}{2}}\phi(\vec{k}) + i\frac{1}{\sqrt{2\alpha_k}}\pi(\vec{k})\right)|S\rangle = 0. \tag{9}$$

The ground state $|m\rangle$ of the free theory (6) is a pure Gaussian state corresponding to $\alpha_k = \omega_k$. The ground state $|m\rangle$ is a product of vacuum states in momentum space without particles according to the number operators $n_{\vec{k}} \equiv a^\dagger_{\vec{k}} a_{\vec{k}}$. In momentum space, the only nontrivial correlations in $|S\rangle$ are between $\vec{k}$ and $(-\vec{k})$ modes. In real space, the $\vec{k}$-dependent factor on the rhs of Eq. (8) leads to spatial correlations (and entanglement).

A natural choice for a reference state $|R(M)\rangle$ is the Gaussian state corresponding to

$$|R(M)\rangle: \quad \alpha_k = M. \tag{10}$$

Since here α_k is independent of k, this state is a product state with no spatial correlations; i.e., in real space, the two-point function of field operators takes the form $\langle R(M)|\phi(\vec{x})\phi(\vec{x}')|R(M)\rangle = [1/(2M)]\delta^d(\vec{x}-\vec{x}')$. Nevertheless, in the basis associated with energy eigenstates of H_m, momentum sectors $\vec{k}$ and $-\vec{k}$ are pairwise entangled according to (8). We will show that the reference state scale M is related to certain ambiguities encountered in the context of holographic complexity. The annihilation and creation operators $b_{\vec{k}}$ and $b^\dagger_{\vec{k}}$ associated with the state $|R(M)\rangle$ can be related to those of the vacuum state $|m\rangle$ by the following Bogoliubov transformation:

$$b_{\vec{k}} = \beta^+_k a_{\vec{k}} + \beta^-_k a^\dagger_{-\vec{k}}; \quad b_{\vec{k}}|R(M)\rangle = 0;$$
$$\beta^+_k = \cosh 2r_k; \quad \beta^-_k = \sinh 2r_k; \quad r_k \equiv \log\sqrt[4]{\frac{M}{\omega_k}}. \tag{11}$$

As our target state, we consider the approximate ground state $|m^{(\Lambda)}\rangle$ characterized by the UV momentum cutoff Λ which corresponds to

$$|m^{(\Lambda)}\rangle: \quad \alpha_k = \begin{cases} \omega_k, & k < \Lambda \quad \text{(QFT vacuum)}, \\ M, & k \geq \Lambda \quad \text{(product state)}, \end{cases} \tag{12}$$

with correlation functions interpolating between those of the vacuum state $|m\rangle$ and the reference state $|R(M)\rangle$ as the momentum increases according to Eq. (8). This state is, in fact, identical to the real ground state $|m\rangle$ up to the cutoff momentum. When $M = \omega_\Lambda$, this state is identical to the one obtained by cMERA circuits [24,25] (see, e.g., Ref. [37]).

The target states (12) can be reached from the reference states (10) by a circuit constructed with *two-mode squeezing operators* which entangle the $\vec{k}$ and $-\vec{k}$ modes:

$$K(\vec{k}) = \phi(\vec{k})\pi(-\vec{k}) + \pi(\vec{k})\phi(-\vec{k})$$
$$= i(a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}} - a_{\vec{k}}a_{-\vec{k}}) = i(b^\dagger_{\vec{k}}b^\dagger_{-\vec{k}} - b_{\vec{k}}b_{-\vec{k}}). \tag{13}$$

This operator is the main building block in cMERA circuits and allows preparing the target state as follows:

$$|m^{(\Lambda)}\rangle = e^{-i\int_{k\leq\Lambda} d^dk\, r_k K(\vec{k})}|R(M)\rangle, \tag{14}$$

which is the starting point for our complexity analysis.

Ground state complexity with a single generator per pair of momenta $\pm\vec{k}$.—We start by evaluating our proposed complexity under the assumption that we allow for a single generator per pair of momenta $\pm\vec{k}$ which we take to be $K(\vec{k})$ of Eq. (13); i.e., $\mathcal{G} = \mathsf{Span}\{K(\vec{k})\}$, where Span is taken over the field of real numbers. These generators continue to achieve minimal complexity within the larger $\mathfrak{su}(1,1)$ class considered below. We consider circuits of the form (1) with

$$G(\sigma) = \int_{k\leq\Lambda} d^dk\, K(\vec{k})\, y_{\vec{k}}(\sigma). \tag{15}$$

Since all the $K(\vec{k})$ commute, the unitary $U(\sigma)$ of (1) is simply specified by the integrated values

$$Y_{\vec{k}}(\sigma) \coloneqq \int_{s_i}^{\sigma} y_{\vec{k}}(s)ds; \qquad Y_{\vec{k}}(s_f) = r_k, \tag{16}$$

where $Y_{\vec{k}}(s_f)$ was fixed to match Eq. (14). The commutation of generators allows the variance in the FS line element (4) to be evaluated at any state $|\Psi(\sigma)\rangle$ along the path. Furthermore, the variance is additive with respect to the different $K(\vec{k})$ contributions, because only equal or opposite momenta can be correlated. The complexity minimization of Eq. (5) then reduces to

$$\mathcal{C} = \min_{Y_{\vec{k}}(\sigma)} \int_{s_i}^{s_f} d\sigma \sqrt{2\mathrm{Vol}\int_{k\leq\Lambda} d^dk[\partial_\sigma Y_{\vec{k}}(\sigma)]^2}, \tag{17}$$

where $\mathrm{Vol} \equiv \delta^d(0)$ is the volume of the d-dimensional time slice. One recognizes a flat Euclidean geometry with coordinates $Y_{\vec{k}}(\sigma)$ continuously labeled by $\vec{k}$. To achieve minimal complexity, the generators for the different momenta must act simultaneously with the ratio dictated by Eq. (16) (straight path). A particularly simple affine parametrization for the path is

$$Y_{\vec{k}}(\sigma) = \frac{\sigma - s_i}{s_f - s_i} Y_{\vec{k}}(s_f); \quad y_k(\sigma) = \frac{1}{s_f - s_i} Y_{\vec{k}}(s_f). \tag{18}$$

As the corresponding cMERA circuit presents a σ-dependent ratio, the complexity associated with it will generically be larger (as shown in Sec. A of Ref. [23]). Evaluating (17) with (18), the minimal complexity reads

$$\mathcal{C}^{(2)} = \sqrt{2\mathrm{Vol} \int_{k\leq\Lambda} d^d k r_k^2}, \tag{19}$$

where the superscript (2) anticipates an interpretation of Eq. (19) as an L^2 norm.

Suppose, on the other hand, that $\mathcal{G}$ contains only individual generators $K(\vec{k})$ and not their linear span. This is analogous to disallowing different elementary gates in a circuit to act simultaneously. Our path parameters in this case consist of σ and $\vec{k}$. The arguments leading to Eq. (17) continue to hold except that now the k integral must be pulled out of the square root and an extra $\sqrt{\mathrm{Vol}/2}$ factor appears. This leads to an L^1 norm (Manhattan distance) complexity

$$\mathcal{C}^{(1)} = \mathrm{Vol} \int_{k\leq\Lambda} d^d k |r_k|. \tag{20}$$

More generally, and without reference to the FS metric, one can postulate L^n norms as a measure of complexity:

$$\mathcal{C}^{(n)} = 2\sqrt[n]{\frac{\mathrm{Vol}}{2} \int_{k\leq\Lambda} d^d k |r_k|^n}. \tag{21}$$

The leading divergence in the complexity measures $\mathcal{C}^{(n)}$ is proportional to

$$\mathcal{C}^{(n)} \sim \mathrm{Vol}^{1/n} \Lambda^{d/n} \log(M/\Lambda), \tag{22}$$

when M and Λ are chosen independently, and to

$$\mathcal{C}^{(n)} \sim \mathrm{Vol}^{1/n} \Lambda^{d/n} \tag{23}$$

when $M = \Lambda$. See Sec. B of Ref. [23] for some additional details on evaluating the ground state complexities using the $\mathcal{C}^{(n)}$ measures. The $\mathcal{C}^{(1)}$ norm results carry a resemblance to those found in the context of holographic complexity as we explain below.

Ground state complexity using $\mathfrak{su}(1,1)$ generators.— Here, we extend our minimization to a larger set of generators $\mathcal{G}$ that transforms $|R(M)\rangle$ into $|m^{(\Lambda)}\rangle$. Namely, we consider momentum-preserving quadratic operators, which for each $\vec{k}$ are spanned by

$$\mathcal{G} = \mathsf{Span}\left\{K_0, K_1 \equiv \frac{K_+ + K_-}{2}, K_2 \equiv \frac{K_+ - K_-}{2i}\right\},$$
$$K_+ = \frac{b^\dagger_{\vec{k}} b^\dagger_{-\vec{k}}}{2}, \qquad K_- = \frac{b_{\vec{k}} b_{-\vec{k}}}{2}, \qquad K_0 = \frac{b^\dagger_{\vec{k}} b_{\vec{k}} + b_{-\vec{k}} b^\dagger_{-\vec{k}}}{4}. \tag{24}$$

These Hermitian operators form a larger (yet manageable), algebraically closed extension of the generators $K = -4K_2$ of Eq. (13) used in cMERA circuits. The algebra formed is an infinite product of $\mathfrak{su}(1,1)$ subalgebras of quadratic generators commuting with $n_{\vec{k}} - n_{-\vec{k}}$. The path in Eqs. (1), (15), and (18) is contained in this larger set. We prove that it continues to be minimal and determine its complexity, although we emphasize that this does not follow automatically from the results of the previous section. For instance, in Sec. D of Ref. [23], we study another constant generator $B(\vec{k}, M)$ which belongs to the extended $\mathfrak{su}(1,1)$ subalgebras but does not lead to a minimal length path. This generator has a bounded norm and drives constant period oscillations between the reference state $|R(M)\rangle$ and target state $|m^{(\Lambda)}\rangle$.

We will see that the manifold of states generated by each $\mathfrak{su}(1,1)$ is a hyperbolic plane, one for each pair of opposite momenta. Minimal complexity paths correspond to geodesics in the resulting tensor product manifold. At the level of the state $|\Psi(\sigma)\rangle$, the most general $\mathfrak{su}(1,1)$ path can always be recast in the form (see Sec. C of Ref. [23])

$$|\Psi(\sigma)\rangle = \mathcal{N}(\sigma) e^{\int d^d k \gamma_+(\vec{k},\sigma) K_+(\vec{k})} |R(M)\rangle, \tag{25}$$

where $\mathcal{N}(\sigma)$ is a complex normalization and σ is the path parameter from Eq. (1). This implies that the state $|\Psi(\sigma)\rangle$ can be conveniently parametrized by a single complex parameter $\gamma_+(\sigma)$. The existence of spurious parameters is a manifestation of the nonuniqueness of the unitary circuit $U(\sigma)$ achieving a minimal complexity path. Because of the noncommutative nature of the generators (24), there is a nontrivial relationship between their coefficients in the path ordered exponential in Eq. (1) and $\gamma_+(\vec{k}, \sigma)$. Our reference state $|R(M)\rangle$ corresponds to $\gamma_+(\vec{k}, s_i) = 0$, while the target state $|m^{(\Lambda)}\rangle$ corresponds to $\gamma_+(\vec{k}, s_f) = \tanh(2r_k)$.

Evaluating the FS line element (2) along the path (25) leads to the following remarkably simple form:

$$ds_{\mathrm{FS}}(\sigma) = d\sigma \sqrt{\frac{\mathrm{Vol}}{2} \int_\Lambda d^d \vec{k} \frac{\gamma'_+(\vec{k},\sigma)\gamma'^*_+(\vec{k},\sigma)}{[1 - |\gamma_+(\vec{k},\sigma)|^2]^2}} \tag{26}$$

(see Sec. C of Ref. [23] for the derivation). This line element corresponds to a direct product of Poincaré disks parametrized by the complex coordinates $\gamma_+(\vec{k}) = \gamma_+(-\vec{k})$ [$|\gamma_+(\vec{k})| < 1$], one for each pair of momenta $\pm\vec{k}$ (an

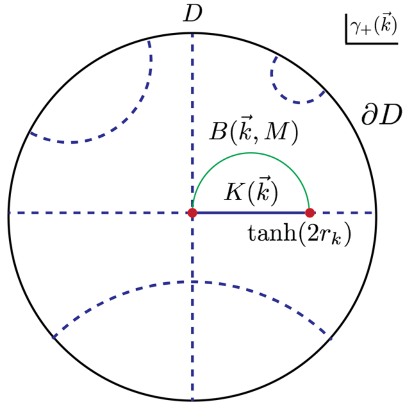

FIG. 1. The Poincaré disk, parametrized by real (horizontal) and imaginary (vertical) components of γ_+. Examples of geodesics appear as dashed lines. The two dots indicate the reference state (the center) and the target state (on the real axis). The geodesic connecting the two is the straight solid line along the diameter, corresponding to the generator $K(\vec{k})$. The solid semicircle is the nongeodesic path generated by $B(\vec{k}, M)$ (see Sec. D of Ref. [23]). The $\mathfrak{su}(1,1)$ algebra generates isometries on the hyperbolic plane.

example of such a disk is illustrated in Fig. 1). The Poincaré disk is the manifold naturally associated with the coset $SU(1,1)/U(1)$ (see, e.g., [38–40]), and its structure of geodesics is well known. Given an affinely parametrized geodesic on a Riemannian product manifold such as (26), its natural projections are affinely parametrized geodesics within each factor manifold. The relative speeds of these projections are coupled and will, as in (18), be fixed by the target state.

The geodesic connecting $|R(M)\rangle$ and $|m^{(\Lambda)}\rangle$ follows the radial direction on the Poincaré disk which corresponds to the affinely parametrized path $\gamma_+(\vec{k}, \sigma) = \tanh(2r_k\sigma)$, $\sigma \in [0, 1]$, generated by $K(\vec{k})$ defined in Eq. (13). Therefore, the path in Eqs. (15) and (18) leads to minimal complexity even within the larger class of $\mathfrak{su}(1,1)$ generators.

Comparison with holographic complexity proposals.—There are two proposals for the gravity dual of complexity in terms of maximal codimension-1 volumes (CV [9]) or on-shell actions of the Wheeler-DeWitt patch bounded by null hypersurfaces (CA [11,12]) in the dual bulk spacetime. The structure of the vacuum UV divergences of holographic complexity can be characterized by a UV regularization scheme [13,14,41] with the cutoff distance from the AdS boundary in Fefferman-Graham coordinates identified as $\delta \sim 1/\Lambda$. Equation (22) for $\mathcal{C}^{(1)}$ indicates a leading divergence of $\mathrm{Vol}\Lambda^d |\log(M/\Lambda)|$ (with M and Λ independent), which resembles the result of the CA proposal. In the holographic CA calculation, the leading logarithmic divergence is due to the codimension-2 joint action contributions associated with the intersection between the null and timelike hypersurfaces that bound the regulated Wheeler-DeWitt patch near the AdS boundary [42]. These contributions depend on the parametrization of null normals (Ref. [42] suggested working in an affine parametrization) and their overall rescaling. The latter gives rise to an extra freedom represented in Ref. [13] by a free parameter $\tilde{\alpha}$ inside the logarithm. In our calculation, the same type of ambiguity is related to the choice of the reference state scale M, and we can identify $M \sim \tilde{\alpha}/L_{\mathrm{AdS}}$, where L_{AdS} is the AdS scale. When $M = \omega_\Lambda$, the leading divergence becomes proportional to $\mathrm{Vol}\Lambda^d$, which is in agreement with the CV results [13] (or with the CA results when including a counterterm which renders the action reparametrization invariant; see [41]). It is interesting that, despite considering QFTs without semiclassical gravity duals (having a small central charge and no interactions), the $\mathcal{C}^{(1)}$ norm exhibits close similarity to the holographic calculations of leading UV divergences.

Summary and outlook.—We proposed a definition of state complexity in QFTs, independent from a notion of unitary complexity. This measure is derived from the FS metric by restricting to directions, in the space of states, generated by exponentiating allowed generators $\mathcal{G}$, on which our measure crucially depends. We identified unitary paths that map simple Gaussian reference states $|R(M)\rangle$ with no spatial correlations to approximate ground states of free QFTs, generated within $\mathfrak{su}(1,1)$ subalgebras of momentum-preserving quadratic generators and singled out the paths corresponding to minimal complexity according to our measure. Remarkably, for some instances, the evaluated complexity presents a qualitative agreement with holographic results.

We could verify using our methods that cMERA circuits are optimal in the $\mathcal{C}^{(1)}$ norm when interpreting the renormalization scale u of cMERA as the circuit parameter σ. In contrast, the $\mathcal{C}^{(2)}$ norm allows for lower FS complexity than that achieved by cMERA circuits by reorganizing the circuit in such a way that all the different momentum gates are active at every step along the circuit (see Sec. A of Ref. [23] for details). The $C^{(1)}$ norm results show a close resemblance to the holographic results, which suggests it is a better predictor of circuit complexity.

We worked in momentum space and restricted the generators to be quadratic. In position space, our generators are bilocal, which suggests an analogy to the two-qubit operations of traditional quantum circuits. However, our gates are spread in position space, and it would be interesting to explore the implications of working with local gates. Future directions include evaluating the complexity for fermionic systems and studying the time evolution of thermofield double states. Finally, it would be interesting to understand what universal data can be extracted from complexity, whether complexity in QFTs can serve as an order parameter, and if it plays a role in the context of RG flows.

We express first our special gratitude to R. C. Myers for numerous illuminating discussions and suggestions that helped shape the ideas and results of this Letter and for sharing with us his preliminary results with R. Jefferson on defining complexity in QFTs using Nielsen's approach. We are also particularly thankful to R. Jefferson for many illuminating discussions and for providing comments on the Supplemental Material comparing the two works, and to J. Eisert for providing numerous comments on this work. In addition, we thank G. Verdon-Akzam, J. de Boer, P. Caputa, M. Fleury, A. Franco, K. Hashimoto, Q. Hu, R. Janik, J. Jottar, T. Osborne, G. Policastro, K. Rejzner, D. Sarkar, V. Scholz, M. Spalinski, T. Takayanagi, K. Temme, J. Teschner, G. Vidal, F. Verstraete, and P. Witaszczyk for valuable comments and discussions. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. S. C. acknowledges additional support from an Israeli Women in Science Fellowship from the Israeli Council of Higher Education. The research of M. P. H. is supported by the Alexander von Humboldt Foundation and the Federal Ministry for Education and Research through the Sofja Kovalevskaja Award. M. P. H. is also grateful to Perimeter Institute, ETH, and the University of Amsterdam for stimulating hospitality during the completion of this project and to Kyoto University where this work was presented for the first time during the workshop *Quantum Gravity, String Theory and Holography*. F. P. also acknowledges the support of the Alexander von Humboldt Foundation.

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