Comment on “Relativistic Electron Vortices”

The appearance of vortices in [1] raises the question of the interpretation of the electron wave function in the Foldy-Wouthuysen representation. We show explicitly that when the wave functions are transformed back into the standard Dirac representation, the vortex lines are smeared out into a continuous distribution of vorticity; the circulation vanishes when we approach the beam center.

In a recent Letter [1] Stephen Barnett wrote in the Abstract “electron vortices do indeed exist in the relativistic theory and that the charge of such a vortex is simply related to a conserved orbital part of the total angular momentum” and in the text “Recent developments have demonstrated, beyond reasonable doubt, the existence of propagating electrons with an on-axis vortex.” These claims contradict our statement [2] that “the vorticity in the wave packet described by the Dirac equation has no singularities; it is distributed continuously in space.”

The resolution of this discrepancy rests on the definition of vortices in relativistic quantum mechanics of electrons. In all experiments with beams endowed with angular momentum the preparation and the detection of electrons is based solely on electromagnetic interactions. Therefore, the electron charge density ρ and the electron current $\mathbf{j}$ built from the Dirac wave functions,

$$\rho = e\psi^\dagger\psi, \quad \mathbf{j} = e\psi^\dagger\boldsymbol{\alpha}\psi, \quad (1)$$

are the relevant characteristics for the coupling to the electromagnetic field. This coupling is described by the term $j^\mu A_\mu$ where the current is defined in (1). Of course, the current is built from the electron wave functions so that it has only a probabilistic meaning. The existence of well-defined charge and current densities makes it possible to introduce the standard definitions of the velocity $\mathbf{v} = \mathbf{j}/\rho$ ($c = 1$) and its circulation Γ ,

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l}, \quad (2)$$

of the same form as in hydrodynamics [3] or in non-relativistic quantum mechanics. The circulation is a perfect tool to search for vortex lines. In the presence of a vortex with the topological charge ℓ the circulation evaluated along the contour encircling the vortex line must have the value $\ell\hbar/m$. Of course, we could express our conclusions in terms of vorticity $\nabla \times \mathbf{v}$, instead of circulation, but by Stokes’s theorem these descriptions are mathematically equivalent. In search for “an on-axis vortex” of Ref. [1] with the charge ℓ we take two exact solutions of the Dirac

FIG. 1. Circulation (measured in units of $\hbar/m$) as a function of the circle radius (measured in Compton wave lengths). Both curves are obtained for $J_z = 3/2\hbar$ but they differ in having in the nonrelativistic approximation the opposite helicities $H = \pm 1/2$.

equation belonging to the same value of the total angular momentum in the direction of propagation: $J_z = 3/2\hbar$. The bispinors describing these solutions are [4]

$$\psi_+ = \begin{bmatrix} (m + E_-)f_l \\ -ip_\perp f_{l+1} \\ (m - E_-)f_l \\ ip_\perp f_{l+1} \end{bmatrix}, \quad \psi_- = \begin{bmatrix} ip_\perp f_l \\ (m + E_+)f_{l+1} \\ -ip_\perp f_l \\ (m - E_+)f_{l+1} \end{bmatrix}, \quad (3)$$

where $E_\pm = \sqrt{m^2 + p_\perp^2 + p_z^2} \pm p_z$ and f_l is the Bessel solution of the Klein-Gordon equation [2]. In the non-relativistic approximation only the terms $m + E_\pm$ survive and the solutions (3) differ by having *opposite helicities*; i.e. the spin is either parallel or antiparallel to the orbital angular momentum. The circulation for these two solutions is plotted in Fig. 1 as a function of the contour radius. When the radius tends to zero, the circulation also tends to zero for both helicities. The same behavior of the circulation is obtained for the approximate solution of the Dirac equation (Eq. (16) of Ref. [1]). Again, the “on-axis vortex,” predicted in [1], is not present; the circulation at the beam center vanishes.

Thus, our conclusion in [2] is fully confirmed. There are no relativistic wave functions with singular vortex lines; vorticity is never concentrated on isolated vortex lines but it is always spread in space.

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- [1] S. M. Barnett, *Phys. Rev. Lett.* **118**, 114802 (2017).
- [2] I. Bialynicki-Birula and Z. Bialynicka-Birula, *Phys. Rev. Lett.* **118**, 114801 (2017).

- [3] P. G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, England, 1992), p. 8.
- [4] We transcribed here the solutions given by Eqs. (11) and (12) of Ref. [2] to the Dirac representation of γ matrices. There was a misprint in Eq. (12): minus should be replaced by plus.