Angular Dynamics of a Small Particle in Turbulence

F. Candelier,¹ J. Einarsson,² and B. Mehlig²

¹Aix-Marseille University—IUSTI (UMR CNRS 7343), 13 453 Marseille Cedex, France

²Department of Physics, Gothenburg University, SE-41296 Gothenburg, Sweden

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We compute the angular dynamics of a neutrally buoyant nearly spherical particle immersed in an unsteady fluid. We assume that the particle is small, that its translational slip velocity is negligible, and that unsteady and convective inertia are small perturbations. We derive an approximation for the torque on the particle that determines the first inertial corrections to Jeffery's equation. These corrections arise as a consequence of local vortex stretching and can be substantial in turbulence, where local vortex stretching is strong and closely linked to the irreversibility of turbulence.

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The angular dynamics of small nonspherical particles in flows is often discussed in terms of Jeffery's theory [1–4], neglecting the effects of particle and fluid inertia. It is assumed that the instantaneous torque on the particle vanishes at every instant in time, so that the angular velocity of a small spherical particle equals half of the fluid vorticity at the particle position. But this is no longer true when the particle is so large that inertial effects become important. It is straightforward to take into account particle inertia [3,5–7]. This may be a good approximation for heavy particles, but for neutrally buoyant particles the acceleration of the surrounding fluid cannot be neglected. How to model the effect of fluid inertia on the angular motion of a suspended particle is an important question [4,7] yet difficult to answer. Past theoretical studies have therefore often concentrated on simple cases, for example, on the angular dynamics of axisymmetric particles in shear flows, analyzing the stability of Jeffery orbits under inertial perturbations [8–11].

In this Letter, we estimate the first inertial contributions to the angular dynamics of a nearly spherical particle in a time-dependent and spatially varying flow. Our three most important assumptions are that the particle is small, almost neutrally buoyant, and nearly spherical (details are given below). We find that a spherical particle rotates at an angular velocity different from the local flow vorticity. The difference is caused by local vortex stretching. For nearly spherical particles, we find additional contributions that depend in a more complex manner on the local fluid-velocity gradients. In turbulence, these effects can alter the angular particle dynamics substantially. Recently, the question was raised how the dynamics of tracer particles reflects the irreversibility of turbulence [12–15]. Our results show how the *inertial* angular dynamics of a small particles

is linked to the breaking of time-reversal invariance in turbulence.

Formulation of the problem.—Consider a small, nearly neutrally buoyant, spheroidal particle (with rotational symmetry axis $\mathbf{n}$) in a fluid. We assume that the particle is nearly spherical, and its aspect ratio λ is close to unity: $\lambda \equiv 1 + \epsilon$, where ϵ is a small parameter determining the eccentricity of the particle— $\epsilon > 0$ for prolate particles, while $\epsilon < 0$ for oblate particles. We assume that the particle is small: $\kappa \equiv 2a/\ell \ll 1$. Here $2a$ is the length of the symmetry axis of the particle, and ℓ is the length over which the flow can be linearized near the particle.

We assume that inertial effects matter but that they are weak. Convective inertia due to the fluid-velocity gradients is characterized by the shear Reynolds number $\text{Re}_s \equiv a^2 s / \nu$, where ν is the kinematic viscosity of the fluid and s measures the magnitudes of the fluid-velocity gradients, the strain rate. The effect of unsteady inertia depends on the time scale τ_c that describes how fast the boundary conditions change. The ratio of the magnitude of unsteady and convective inertia defines the Strouhal number $\text{Sl} \equiv (s\tau_c)^{-1}$. The effect of particle inertia on the angular dynamics is determined by the Stokes number, the ratio of the rate of change of angular momentum and the torque: $\text{St} \equiv (\rho_p a^5 s) / (\tau_c \mu s a^3) = (\rho_p / \rho_f) \text{Re}_s \text{Sl}$. Here ρ_f and ρ_p are the mass densities of the fluid and the particle, respectively, and $\mu = \rho_f \nu$ is the dynamic viscosity.

We treat the effect of inertia perturbatively; this requires Re_s and $\text{Re}_s \text{Sl}$ to be small (but not too small; see below). We disregard the effect of translational slip. This is justified for small particle Reynolds number $\text{Re}_p \equiv a v_s / \nu \ll \text{Re}_s^{1/2}$. Here v_s is the slip velocity. To ensure that it is small enough, we assume that the particle is approximately neutrally buoyant.

Equations of motion.—The equations that govern the angular particle dynamics read

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \wedge \mathbf{n}, \quad \left(\mathbb{I} \cdot \frac{d\boldsymbol{\omega}}{dt} + \frac{d\mathbb{I}}{dt} \cdot \boldsymbol{\omega} \right) = \mathbf{T}. \quad (1)$$

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Here $\mathbb{I}$ is the moment-of-inertia tensor of the particle, with elements $I_{ij} = A^I n_i n_j + B^I (\delta_{ij} - n_i n_j)$, A^I and B^I are the moments of inertia around and transverse to the axis $\mathbf{n}$, and $\mathbf{T}$ is the hydrodynamic torque:

$$\mathbf{T} = \int_{S_p} \mathbf{r} \wedge \boldsymbol{\sigma} \cdot d\mathbf{s}. \quad (2)$$

The integral is over the particle surface S_p , $d\mathbf{s}$ is the outward normal surface element, $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the stress tensor of the fluid at position $\mathbf{x}$, and $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_p$, where $\mathbf{x}_p$ is the particle position. The torque (2) is determined by the solution of Navier-Stokes equations. We decompose the stress as $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty + \boldsymbol{\sigma}^{(1)}$, where $\boldsymbol{\sigma}^\infty$ is the stress tensor of the undisturbed Eulerian fluid velocity, denoted by $\mathbf{U}^\infty(\mathbf{x}, t)$. The second term, $\boldsymbol{\sigma}^{(1)}$, is the contribution to the stress tensor from the disturbance flow. The torque is decomposed in a similar way: $\mathbf{T} = \mathbf{T}^\infty + \mathbf{T}^{(1)}$. We compute these two contributions to the torque separately.

Torque due to stress of undisturbed flow.—The undisturbed Eulerian fluid velocity $\mathbf{U}^\infty(\mathbf{x}, t)$ satisfies Navier-Stokes equations in the laboratory frame:

$$\nabla \cdot \boldsymbol{\sigma}^\infty = \rho_f [(\partial_t \mathbf{U}^\infty)_x + (\mathbf{U}^\infty \cdot \nabla) \mathbf{U}^\infty] \equiv \rho_f \frac{D}{Dt} \mathbf{U}^\infty. \quad (3)$$

The last equality defines the Lagrangian derivative along $\mathbf{U}^\infty$, ∇ denotes the spatial derivative with respect to $\mathbf{x}$, and the partial time derivative is evaluated at fixed $\mathbf{x}$. The torque due to the undisturbed stress can be expressed as a volume integral using Eq. (3):

$$\mathbf{T}^\infty = \rho_f \int_{V_p} d\mathbf{v} \mathbf{r} \wedge \frac{D\mathbf{U}^\infty}{Dt}. \quad (4)$$

To evaluate this expression further, we expand $D\mathbf{U}^\infty(\mathbf{x}, t)/Dt$ around the particle position $\mathbf{x}_p$:

$$\begin{aligned} \left. \frac{D\mathbf{U}_k^\infty}{Dt} \right|_{\mathbf{x}} &= \left. \frac{D\mathbf{U}_k^\infty}{Dt} \right|_{\mathbf{x}_p} + r_m \frac{\partial}{\partial x_m} \left. \frac{D\mathbf{U}_k^\infty}{Dt} \right|_{\mathbf{x}_p} \\ &+ r_m r_n \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n} \left. \frac{D\mathbf{U}_k^\infty}{Dt} \right|_{\mathbf{x}_p} + O(r^3/\ell^3). \end{aligned} \quad (5)$$

The components are relative to a fixed Cartesian basis in the laboratory frame. Substituting this expansion into Eq. (4), we find

$$T_i^\infty = \rho_f \int_{V_p} d^3r \varepsilon_{ijk} r_j \left(\frac{\partial}{\partial x_m} \left. \frac{D\mathbf{U}_k^\infty}{Dt} \right|_{\mathbf{x}_p} \right) r_m + O(r^4/\ell^4). \quad (6)$$

The order is $O(r^4/\ell^4)$, because terms odd in $\mathbf{r}$ vanish upon integration. The partial derivative in the integrand of Eq. (6)

evaluates to $\nabla(D\mathbf{U}^\infty/Dt)|_{\mathbf{x}_p} = (D\mathbb{A}^\infty/Dt)|_{\mathbf{x}_p} + \mathbb{A}_p^\infty \cdot \mathbb{A}_p^\infty$, where the elements of $\mathbb{A}_p^\infty$ are the gradients of the undisturbed fluid velocity $\mathbf{U}^\infty$ at the particle center: $(\mathbb{A}_p^\infty)_{ij} \equiv \partial U_i^\infty / \partial x_j|_{\mathbf{x}=\mathbf{x}_p}$.

Finally, to perform the integral in Eq. (6) we use the definition of the moment-of-inertia tensor. We obtain

$$\begin{aligned} \mathbf{T}^\infty &= \frac{\rho_f}{\rho_p} \left\{ \mathbb{I} \cdot \left(\left. \frac{D\mathbb{A}^\infty}{Dt} \right|_{\mathbf{x}_p} - \mathbb{S}_p^\infty \cdot \mathbb{Q}_p^\infty \right) \right. \\ &\quad \left. + (A^I - B^I) \left[\left(\mathbb{S}_p^\infty \cdot \mathbb{S}_p^\infty + \mathbb{Q}_p^\infty \cdot \mathbb{Q}_p^\infty + \left. \frac{D\mathbb{S}^\infty}{Dt} \right|_{\mathbf{x}_p} \right) \cdot \mathbf{n} \right] \wedge \mathbf{n} \right\}. \end{aligned} \quad (7)$$

Here we have decomposed the matrix $\mathbb{A}_p^\infty$ into its symmetric and antisymmetric parts, the strain-rate matrix $\mathbb{S}_p^\infty$, and $\mathbb{Q}_p^\infty$. The matrix $\mathbb{Q}_p^\infty$ is linked to $\mathbb{Q}_p^\infty \equiv \frac{1}{2} \nabla \wedge \mathbf{U}^\infty$ through $\mathbb{Q}^\infty \cdot \mathbf{r} = \mathbf{U}^\infty \wedge \mathbf{r}$. Equation (7) is valid for a spheroid with an arbitrary aspect ratio (provided that $\kappa \ll 1$).

Using the vorticity equation $(D\mathbb{Q}^\infty/Dt) - \mathbb{S}^\infty \cdot \mathbb{Q}^\infty = \nu \nabla^2 \mathbb{Q}^\infty$ evaluated at the particle position, we can express the difference in the first row of Eq. (7) in terms of the Laplacian of $\mathbb{Q}^\infty$ which vanishes for a strictly linear flow. In general, however, the nonlinearity of $\mathbf{U}^\infty(\mathbf{x}, t)$ results in a nonzero value of $\nabla^2 \mathbb{Q}^\infty$ at the particle position.

Disturbance torque.—We calculate the torque due to the disturbed fluid in the perturbation theory, assuming that inertial effects are small and neglecting translational slip. This dictates how the problem must be dedimensionalized: $t = \tau_c t'$, $\mathbf{r} = a \mathbf{r}'$, $\mathbf{U} = sa \mathbf{U}'$, and $\boldsymbol{\sigma} = \mu s \boldsymbol{\sigma}'$. In the remainder of this Letter, we use these dimensionless variables. To simplify the notation, we drop the primes; all equations below are written in dimensionless variables. The disturbance caused by the particle has the flow velocity $\mathbf{w}(\mathbf{r}, t) \equiv \mathbf{U}(\mathbf{r} + \mathbf{x}_p, t) - \mathbf{U}^\infty(\mathbf{r} + \mathbf{x}_p, t)$. It is defined to be a function of $\mathbf{r}(t) \equiv \mathbf{x} - \mathbf{x}_p(t)$ (Fig. 1). In dimensionless variables, the disturbance problem reads

$$\begin{aligned} \nabla_{\mathbf{r}} \cdot \boldsymbol{\sigma}^{(1)} &= \text{Re}_s [\text{Sl}(\partial_t \mathbf{w})_{\mathbf{r}} + (\mathbb{A}_p^\infty \cdot \mathbf{r}) \cdot \nabla_{\mathbf{r}} \mathbf{w} + \mathbb{A}_p^\infty \cdot \mathbf{w} \\ &\quad + (\mathbf{w} \cdot \nabla_{\mathbf{r}}) \mathbf{w}], \\ \mathbf{w}(\mathbf{r}, t) &= -(\mathbb{A}_p^\infty \cdot \mathbf{r} - \boldsymbol{\omega} \wedge \mathbf{r}) \quad \text{for } \mathbf{r} \in S_p, \\ \mathbf{w}(\mathbf{r}, t) &= 0 \quad \text{as } |\mathbf{r}| \rightarrow \infty. \end{aligned} \quad (8)$$

The partial time derivative is evaluated at fixed $\mathbf{r}$, and we have linearized $\mathbf{U}^\infty(\mathbf{x}, t)$ around $\mathbf{x}_p$, $\mathbf{U}^\infty(\mathbf{x}, t) = \mathbf{U}^\infty(\mathbf{x}_p, t) + \mathbb{A}_p^\infty \cdot \mathbf{r}$. When is it justified to use this linear form in the disturbance problem (8)? The disturbance caused by the particle decays exponentially at distances larger than the Saffman length $\ell_S \equiv 1/\sqrt{\text{Re}_s}$, so that we must require $\ell_S < \ell$, where ℓ is the length scale over which the flow can be linearized. This condition is more restrictive than $\kappa \ll 1$. In other words, the shear Reynolds number Re_s

FIG. 1. Schematic of vectors used to describe the particle motion. $\mathbf{x}$ is a position in the laboratory frame, $\mathbf{O}$ is its origin, and $\mathbf{x}_p(t)$ is the particle position at time t . The perturbation theory leading to Eq. (13) assumes that the particle follows the Lagrangian fluid trajectory.

should not be too small, because convective inertia causes the disturbance to decay at ℓ_s .

We use the reciprocal theorem [16] to find the hydrodynamic torque on the particle, given an “auxiliary” Stokes solution in the same geometry [9,10,17,18]. In dimensionless variables, the reciprocal theorem reads

$$\mathbf{T} = \mathbf{T}^\infty + \mathbf{T}^{(0)} - \text{Re}_s \int_V dv \mathbb{M}^T \cdot \mathbf{f}(\mathbf{w}). \quad (9)$$

The first term on the right-hand side is the torque due to the undisturbed fluid stresses, Eq. (7), expressed in dimensionless variables. The second term is Jeffery’s torque [1]: $\mathbf{T}^{(0)} = -\mathbb{K} \cdot (\boldsymbol{\omega} - \boldsymbol{\Omega}_p^\infty) + \mathbb{H} : \mathbb{S}_p^\infty$, where $\mathbb{K}$ and $\mathbb{H}$ are Brenner’s resistance tensors [19]. The third term in Eq. (9) is the torque due to the disturbance flow beyond the Stokes approximation. The tensor $\mathbb{M}(\mathbf{r})$ is determined by the known auxiliary Stokes solution ($\tilde{\mathbf{u}} = \mathbb{M} \cdot \tilde{\boldsymbol{\omega}}$), the Stokes flow around the particle rotating with angular velocity $\tilde{\boldsymbol{\omega}}$ in a quiescent fluid; see Supplemental Material [20]. The integration domain in (9) is the fluid volume outside the particle, and $\text{Re}_s \mathbf{f}$ is defined as the right-hand side of the first Eq. (8).

To simplify the calculation of the third term in Eq. (9), we assume that the particle is nearly spherical [18], $\lambda = 1 + \epsilon$, and expand in the small parameter ϵ . To the order of ϵ^2 , for example, the moments of inertia around and transverse to the particle symmetry axis [18] read in dimensionless variables $A^I = (8\pi/15)(1 - 4\epsilon + 6\epsilon^2)$ and $B^I = (8\pi/15)(1 - 3\epsilon + (7/2)\epsilon^2)$. For small values of ϵ , the contribution of the volume integral in Eq. (9) can be evaluated. Details are given in Supplemental Material [20].

For a spherical particle ($\epsilon = 0$), we find to the order of $O(\text{Re}_s)$

$$\mathbf{T} = -8\pi(\boldsymbol{\omega} - \boldsymbol{\Omega}_p^\infty) + 8\pi \text{Re}_s \left(\frac{\text{Sl}}{15} \frac{D\boldsymbol{\Omega}^\infty}{Dt} \Big|_{x_p} - \frac{2}{5} \mathbb{S}_p^\infty \cdot \boldsymbol{\Omega}_p^\infty \right). \quad (10)$$

We now use Eqs. (10) and (1) to determine the particle angular velocity $\boldsymbol{\omega}$ to the order of $O(\text{Re}_s)$. For a spherical

particle, $\mathbb{I}$ has the elements $I_{ij} = 8\pi\delta_{ij}/15$, in dimensionless variables. In the perturbation theory, $d\boldsymbol{\omega}/dt = d\boldsymbol{\Omega}_p^\infty/dt = D\boldsymbol{\Omega}^\infty/Dt|_{x_p}$. We must also require that $\rho_p \approx \rho_f$, so that Re_p remains small. To the order of $O(\text{Re}_s)$ we find

$$\boldsymbol{\omega} = \boldsymbol{\Omega}_p^\infty + \frac{1}{15}(\text{Re}_s \text{Sl} - \text{St}) \frac{D\boldsymbol{\Omega}^\infty}{Dt} \Big|_{x_p} - \frac{2}{5} \text{Re}_s \mathbb{S}_p^\infty \cdot \boldsymbol{\Omega}_p^\infty. \quad (11)$$

Equation (11) is the main result of this Letter. We see that a very small particle in a viscous flow rotates with half the fluid vorticity, $\boldsymbol{\Omega}_p^\infty$, as expected. The first inertial correction term, proportional to $D\boldsymbol{\Omega}^\infty/Dt|_{x_p}$, resembles the form of the slip velocity of a small particle subject to particle inertia and to the force due to the undisturbed pressure gradients. A small-St expansion gives

$$\mathbf{v}_s = \text{St}(\rho_f/\rho_p - 1) D\mathbf{U}^\infty/Dt|_{x_p} \quad (12)$$

for $\text{Sl} = 1$. Maxey used this approximation to conclude that heavy particles are centrifuged out of vortical regions in turbulence [22,23]. The expression for $\mathbf{v}_s$ is of the same form as the second term on the right-hand side of Eq. (11), since $\text{Re}_s = (\rho_f/\rho_p)\text{St}$ for $\text{Sl} = 1$. This term predicts that a particle that is slightly heavier than the fluid rotates a little bit more slowly, because it cannot keep up with the fluid acceleration. A lighter particle, by contrast, rotates faster than $\boldsymbol{\Omega}_p^\infty$. But these arguments disregard the $\mathbb{S}_p^\infty \cdot \boldsymbol{\Omega}_p^\infty$ -term in Eq. (11). This inertial term connects the angular particle dynamics to vortex stretching in the undisturbed flow. This term vanishes in steady linear flows such as a simple shear [24,25] and for planar flows because vorticity is orthogonal to the flow plane.

Angular velocity and vortex stretching in turbulence.—Fully developed turbulent flows exhibit large local vortex stretching rates; this is how the turbulent kinetic energy is dissipated at small scales. Irreversibility of turbulence [12–15] implies that the stretching rate $\boldsymbol{\Omega}^\infty \cdot (\mathbb{S}^\infty \cdot \boldsymbol{\Omega}^\infty)$ does not average to zero [26,27]. This matrix element is positive on average, because $\boldsymbol{\Omega}^\infty$ tends to align with the middle eigenvector of $\mathbb{S}^\infty$ [12], and its eigenvalue is positive on average. Equation (11) shows that the inertial correction to the angular velocity of a neutrally buoyant sphere in turbulence is determined by the stretching rate,

$$|\boldsymbol{\omega}|^2/|\boldsymbol{\Omega}_p^\infty|^2 \approx 1 - \frac{4}{5} \text{Re}_s \boldsymbol{\Omega}_p^\infty \cdot (\mathbb{S}_p^\infty \cdot \boldsymbol{\Omega}_p^\infty)/|\boldsymbol{\Omega}_p^\infty|^2, \quad (13)$$

and thus linked to the breaking of time-reversal invariance in turbulence.

But are the conditions of validity summarized above met for a neutrally buoyant particle in turbulence? The shear rate, defined by $s = \langle \text{Tr} \mathbb{S} \cdot \mathbb{S} \rangle^{1/2}$, is of the same order as τ_K^{-1} , the inverse of the Kolmogorov time $\tau_K \equiv \langle \text{Tr} \mathbb{A}^T \cdot \mathbb{A} \rangle^{-1/2}$. This means that the shear Reynolds number is of the order of $\text{Re}_s \sim a^2/\eta_K^2$, where $\eta_K = \sqrt{\nu\tau_K}$ is the

FIG. 2. (a) Inertial correction $Y \equiv \tau_K \Omega_p^\infty \cdot \mathbb{S}_p^\infty \cdot \Omega_p^\infty / |\Omega_p^\infty|^2$ [Eq. (13)] along a Lagrangian path in turbulence (DNS at $\text{Re}_\lambda = 433$ using the JHU turbulence database [28,29]). (b) Vorticity along the same path. (c) Distribution of Y (DNS, lines). Also shown: experimental data (circles) at $\text{Re}_\lambda = 50$ read off from Fig. 2(d) in Ref. [26], rescaled by $\tau_K \equiv \langle \text{Tr} \mathbb{A}^\top \cdot \mathbb{A} \rangle^{-1/2}$; see the text. The experimental value (0.27 s^{-1}) was taken from Ref. [30].

Kolmogorov length. We conclude that the particle must be smaller than the Kolmogorov length for the perturbation theory in Re_s to be valid. For a nearly neutrally buoyant particle we can disregard gravity, and we assume that no other external forces act on the particle. Therefore, the time scale τ_c is that of the fluid, and we take $\tau_c = s^{-1} \sim \tau_K$, so that the Strouhal number is unity. The perturbation theory also requires that the Saffman length $\ell_S \equiv a/\sqrt{\text{Re}_s} \sim \eta_K$ is smaller than the length ℓ over which the flow can be linearized. For turbulent flow at Reynolds number $\text{Re}_\lambda \sim 500$, $\ell \sim 10\eta_K$, on average. So this condition is only marginally satisfied. Translational slip is negligible when the Oseen length is much larger than the Saffman length, that is $\text{Re}_p^2 \ll \text{Re}_s$. For small neutrally buoyant particles in turbulence, this condition is well satisfied.

We evaluate the inertial correction in Eq. (13) for a neutrally buoyant particle numerically, following a Lagrangian trajectory in fully developed turbulence. We use the Johns Hopkins turbulence database [28,29] that contains a direct numerical simulation of forced, isotropic turbulence at $\text{Re}_\lambda = 433$. The result is shown in Fig. 2(a). Figure 2(b) shows the vorticity along the same path. We observe the stretching of a vortex tube at $t \approx 20\tau_K$. We see that the inertial correction to the particle angular velocity can be substantial during vortex stretching, a factor of the order of unity times Re_s . Figure 2(c) shows that the distribution of the inertial correction has non-Gaussian tails that may give rise to large values of the inertial correction. A comparison with the experimental data at $\text{Re}_\lambda = 50$ ($\circ$), from Fig. 2(d) in Ref. [26], shows that the tails are quite robust even at moderate values of Re_λ , a consequence of the universality of dissipative-range turbulent fluctuations [31].

Nonspherical particles.—We have computed the angular velocity also for nearly spherical particles ($|\epsilon| \ll 1$). Neglecting inertial effects, we obtain an ϵ expansion of Jeffery's equation [1]: $\omega = \Omega_p^\infty + (\epsilon + (\epsilon^2/2))(\mathbf{n} \wedge \mathbb{S}_p^\infty \cdot \mathbf{n}) + O(\epsilon^3)$.

The form of the first inertial corrections to Jeffery's angular velocity is constrained by symmetries. To linear order in St and Re_s the corrections are quadratic in $\mathbb{S}_p^\infty$ and Ω_p^∞ and linear in time derivatives of $\mathbb{S}_p^\infty$ and Ω_p^∞ . The inertial corrections to ω must be invariant under $\mathbf{n} \rightarrow -\mathbf{n}$, and $\mathbb{S}_p^\infty$ is symmetric and traceless. Only the following terms can occur in the $O(\epsilon \text{Re}_s)$ correction $\delta\omega \equiv \epsilon \text{Re}_s \omega^{(\epsilon \text{Re}_s)} + \epsilon \text{St} \omega^{(\epsilon \text{St})}$ to Eq. (11):

$$\begin{aligned} \delta\omega = & \beta_1 \frac{D\Omega^\infty}{Dt} \Big|_{x_p} + \beta_2 \left(\frac{D\Omega^\infty}{Dt} \Big|_{x_p} \cdot \mathbf{n} \right) \mathbf{n} + \beta_3 \left(\frac{d\mathbb{S}_p^\infty}{dt} \cdot \mathbf{n} \right) \wedge \mathbf{n} \\ & + \beta_4 \mathbb{S}_p^\infty \cdot \Omega_p^\infty + \beta_5 [\mathbf{n} \cdot (\mathbb{S}_p^\infty \cdot \Omega_p^\infty)] \mathbf{n} \\ & + \beta_6 [\mathbf{n} \cdot (\mathbb{S}_p^\infty \cdot \mathbf{n})] \Omega_p^\infty \\ & + \beta_7 (\mathbf{n} \cdot \Omega_p^\infty) (\Omega_p^\infty \wedge \mathbf{n}) + \beta_8 (\mathbb{S}_p^\infty \cdot \mathbb{S}_p^\infty \cdot \mathbf{n}) \wedge \mathbf{n} \\ & + \beta_9 [\mathbb{S}_p^\infty \cdot (\Omega_p^\infty \wedge \mathbf{n})] \wedge \mathbf{n} + \beta_{10} (\mathbf{n} \cdot \Omega_p^\infty) \mathbb{S}_p^\infty \cdot \mathbf{n}. \end{aligned} \quad (14)$$

For $\text{Sl} = 1$ we find using the method described above (details in Supplemental Material [20])

$$\begin{aligned} \beta_1 = & -\frac{3\epsilon}{15} (\text{Re}_s - \text{St}), & \beta_2 = & -\frac{\epsilon}{15} (\text{Re}_s - \text{St}), \\ \beta_3 = & -\frac{\epsilon}{3} \left(\text{Re}_s - \frac{1}{5} \text{St} \right), & \beta_4 = & \frac{733\epsilon}{525} \text{Re}_s, \\ \beta_5 = & -\frac{\epsilon}{15} \left(\frac{4}{35} \text{Re}_s + \text{St} \right), & \beta_6 = & -\frac{\epsilon}{15} \left(\frac{30}{35} \text{Re}_s - \text{St} \right), \\ \beta_7 = & -\frac{\epsilon}{15} (\text{Re}_s - \text{St}), & \beta_8 = & -\frac{8\epsilon}{21} \text{Re}_s, \\ \beta_9 = & -\frac{\epsilon}{15} (4\text{Re}_s - \text{St}), & \beta_{10} = & \frac{3\epsilon}{35} \text{Re}_s. \end{aligned} \quad (15)$$

Equation (14) shows that the inertial corrections to the angular velocity of nonspherical particles depend intricately on the relative alignment of the particle symmetry axis, of the vorticity, and of the eigensystem of the strain-rate matrix [2,3,32]. For a neutrally buoyant particle in incompressible isotropic homogeneous turbulence, we can average the correction. Since $\delta\omega$ contains a factor ϵ , we obtain the $O(\epsilon \text{Re}_s)$ result by averaging $\mathbf{n}$ uniformly over the unit sphere: $\langle n_i n_j \rangle = \delta_{ij}/3$. This gives $\langle \Omega_p^\infty \cdot \delta\omega \rangle = (4/3)\epsilon \text{Re}_s \langle \Omega_p^\infty \cdot (\mathbb{S}_p^\infty \cdot \Omega_p^\infty) \rangle$. On average, the form of the correction is similar to that in Eq. (11). We see that this correction is positive for slightly prolate neutrally buoyant particles but negative for oblate particles.

Conclusions.—We have computed the first inertial corrections to the angular velocity of a small, approximately neutrally buoyant particle in a space- and time-dependent flow. Our main prediction, Eq. (13), expresses the inertial correction to the angular velocity of a small sphere in terms of a matrix element that determines vortex stretching. This shows that the inertial angular dynamics of a small

neutrally buoyant sphere in turbulence picks up that time-reversal invariance is broken [12–15].

Our results demonstrate that convective and unsteady fluid inertia must be treated on an equal footing in turbulence. It is commonly argued that convective effects can be disregarded when the translational slip velocity $\mathbf{v}_s$ is negligible. But we have shown here that substantial inertial corrections may arise from convective terms due to turbulent strains. Such terms are likely to be important in the translational problem, too, so that the “Maxey-Riley” equations [33,34] cannot be used to describe the translational dynamics of small spheres in turbulence.

Particle-tracking experiments [35–37] and particle-resolving direct numerical simulations (DNS) required to test the predictions in this Letter have recently become possible [38,39], but they are still very challenging. In order to stimulate the substantial effort required for these measurements, we now give a concrete suggestion for how Eq. (13) could be tested in either an experiment or particle-resolving DNS. First, the particle should be neutrally buoyant, and its size a must be smaller than the Kolmogorov length η_K . Small a ensures that $\text{Re}_p^2 \ll \text{Re}_s$, since $\text{Re}_p \propto a^3$ [Eq. (12)] while $\text{Re}_s \propto a^2$. Small particle size may be a difficult technical requirement. It is easier to meet for lower rate of energy dissipation, so that η_K is larger. Figure 2(c) shows that the effect persists for lower Re_λ . Second, to determine the vorticity of the undisturbed flow, one must measure and interpolate the flow near the particle [35]. Then we suggest to consider the distribution of $|\boldsymbol{\omega}|^2/|\boldsymbol{\omega}_p^\infty|^2 - 1$. The width of this distribution must approach zero for a perfect tracer particle. Figure 2(c) shows that the first effect of inertia is to substantially widen this distribution and to slightly shift its mean value. Finally, our results for nearly spherical particles indicate that disks may be more sensitive to inertial corrections than rods.

Can our results be generalized to cases where $\mathbf{v}_s$ is not negligible? An important case is settling [40–43]. The settling of ice crystals, for instance, is important for rain initiation in cold turbulent clouds [44]. For a spatially constant flow, the effect of nonzero $\mathbf{v}_s$ was analyzed by Lovalenti and Brady [16]. Can one use their methods to compute lift forces [45] on small particles in turbulence? This is a difficult problem, because it requires singular perturbation theory [46]. Finally, larger particles pose other problems: They sense inertial-range turbulent fluctuations [47], and wakes may affect their dynamics [48].

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