Erratum: Experimental Test of Entropic Noise-Disturbance Uncertainty Relations for Spin-1/2 Measurements [Phys. Rev. Lett. 115, 030401 (2015)]

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The Letter reports experiments that test two information-theoretic noise-disturbance inequalities:

$$N(\mathcal{M}, \sigma_z) + D(\mathcal{M}, \sigma_y) \geq 1, \quad g[N(\mathcal{M}, \sigma_z)]^2 + g[D(\mathcal{M}, \sigma_y)]^2 \leq 1, \quad (1)$$

for complementary qubit observables σ_y and σ_z . Here $g[x]$ denotes the inverse of the monotonic-decreasing function $h(x)$ on the interval $x \in [0, 1]$ given by $h(x) := -(1+x/2)\log_2(1+x/2) - (1-x/2)\log_2(1-x/2)$. The first inequality in Eq. (1) is valid for all measurements $\mathcal{M}$ [1]. The second inequality in Eq. (1), i.e., Eq. (6) of the Letter, was similarly claimed to be valid for all measurements. However, the variational proof given for this claim in the Supplemental Material of the Letter is incorrect: the displayed equation following Eq. (5) of the Supplemental Material is satisfied for $\theta_m = 0, \pi/2$ and any value of p_m , thus invalidating Eq. (6) (where the latter corresponds to dividing by zero for this case).

Below we give a new proof for the second noise-disturbance relation in Eq. (1), that demonstrates its validity for any measurement apparatus that performs a 2-outcome measure-and-prepare qubit measurement. This includes, in particular, the class of projective qubit measurements, and hence the noise-disturbance relation is valid for the experiments reported in the Letter. More generally, however, there are 3-outcome qubit measurements for which the relation does not hold [2].

Recall that the measurement apparatus is modeled as an instrument $\mathcal{M}$, i.e., a collection of completely positive maps $\{\mathcal{M}^m\}$ such that $\sum_m \mathcal{M}^m$ is trace-preserving and m labels the outcomes of the measurement [1]. We denote the input quantum system by S , the output quantum system by S' , and the random variable corresponding to the classical outcome by M . All systems are finite dimensional, in particular, $\dim \mathcal{H}_S = d$. We first establish a useful lemma.

Lemma 1. Given are a nondegenerate observable A on $\mathcal{H}_S$ and a measurement apparatus $\mathcal{M}$ with d outcomes. If $\mathcal{M}$ is of the measure-and-prepare form, then the minimum possible disturbance of A is equal to the noise, i.e.,

$$D_{\text{opt}}(\mathcal{M}, A) = N(\mathcal{M}, A). \quad (2)$$

Note that, if the positive operator valued measure (POVM) corresponding to $\mathcal{M}$ is rank-one, then $\mathcal{M}$ is necessarily of the measure-and-prepare form.

Proof.—By definition, $N(\mathcal{M}, A) = H(A|M)$ and $D_{\text{opt}}(\mathcal{M}, A) = \min_C H(A|A')$, where the minimum is taken over all measurements $\mathcal{C}$ acting on the joint quantum-classical output $S'M$ and having, as the labeling index, the set of eigenvalues of A (thus, $\mathcal{C}$ produces an estimate of A , typically incorporating an error correction operation as discussed in the Letter). As shown in Ref. [1] (where D_{opt} is denoted by D), $D_{\text{opt}}(\mathcal{M}, A) \geq H(A|S'M)$. Now, since $\mathcal{M}$ is assumed to be of the measure-and-prepare form, $S \rightarrow M \rightarrow S'$ forms a Markov chain. As a consequence, $D_{\text{opt}}(\mathcal{M}, A) \geq H(A|S'M) = H(A|M) = N(\mathcal{M}, A)$. Finally, since M is also assumed to have d possible outcomes, i.e., as many outcomes as the eigenvalues of A , the POVM corresponding to M already is a possible estimate of A (a part from an irrelevant one-to-one mapping of the labels m into the eigenvalues a of A). Therefore, $N(\mathcal{M}, A) = H(A|M) \geq \min_C H(A|A') = D_{\text{opt}}(\mathcal{M}, A)$.

In particular, Eq. (2) holds for any 2-outcome measure-and-prepare qubit measurement. Now, in the special case where $\mathcal{M}$ corresponds to measuring a projective qubit observable $M = \vec{m} \cdot \vec{\sigma}$ with $|\vec{m}| = 1$, the corresponding noise of any observable $A = \vec{a} \cdot \vec{\sigma}$ is easily calculated to be $N(\mathcal{M}, A) = h(\vec{m} \cdot \vec{a})$ [cf. Eq. (11) of the Letter]. Further, if $\mathcal{M}'$ is a general 2-outcome measure-and-prepare qubit measurement, its POVM necessarily has the form $\{\lambda|m\rangle\langle m| + (1-\lambda)q_m\mathbb{1}\}$, where $|m\rangle$ is the eigenstate of $\vec{m} \cdot \vec{\sigma}$ with eigenvalue $m = \pm 1$, for some unit vector $\vec{m}$ and $0 \leq \lambda, q_m \leq 1$ with $q_+ + q_- = 1$. This is equivalent to (i) measuring $M = \vec{m} \cdot \vec{\sigma}$, and then (ii) equating the outcome with m with probability λ and otherwise sampling m from the distribution $\{q_{\pm}\}$ with probability $1 - \lambda$. This means that the measurement $\mathcal{M}'$ essentially amounts to mixing $\mathcal{M}$ with extra random noise, thus resulting in a larger noise than that incurred by $\mathcal{M}$ alone [1]. In formula, $N(\mathcal{M}', A) \geq N(\mathcal{M}, A) = H(A|M) = h(\vec{m} \cdot \vec{a})$. We hence arrive at

$D(\mathcal{M}', \sigma_y) \geq D_{\text{opt}}(\mathcal{M}', \sigma_y) = N(\mathcal{M}', \sigma_y) \geq N(\mathcal{M}, \sigma_y) = D_{\text{opt}}(\mathcal{M}, \sigma_y)$, which, together with the fact that g is monotonic decreasing, yields the desired result:

$$g[N(\mathcal{M}', \sigma_z)]^2 + g[D(\mathcal{M}', \sigma_y)]^2 \leq g[N(\mathcal{M}, \sigma_z)]^2 + g[D_{\text{opt}}(\mathcal{M}, \sigma_y)]^2 = (m_z)^2 + (m_y)^2 \leq \vec{m} \cdot \vec{m} = 1.$$

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- [1] F. Buscemi, M. J. W. Hall, M. Ozawa, and M. M. Wilde, *Phys. Rev. Lett.* **112**, 050401 (2014).
- [2] A. A. Abbott and C. Branciard, [arXiv:1607.00261](#).