Similar size of slums caused by a Turing instability of migration behavior

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It is a remarkable fact that the size of slums is similar across the globe, regardless of city, country, or culture [Friesen *et al.*, Habitat Int. **73**, 79 (2018)]. The main thesis of this paper is that this universal scale is intrinsic to the slum-city system and is independent from external factors. By interpreting reaction and diffusion as long-and short-distance migration, our paper explains this universal length scale as resulting from a Turing instability of the interaction of two social groups: poor and rich.

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I. INTRODUCTION

Friesen *et al.* [1] point out that slums in Mumbai, India; Manila, the Philippines; Cape Town, South Africa; and Rio de Janeiro, Brazil all show the same scale, independent from city, country, or culture: on average, a slum occupies an area of 15 800 m², i.e., the length scale is $\tilde{\phi} \approx 125$ m; cf. Table I. The globally universal scale for intraurban structures $\tilde{\phi}$ is astonishing, provoking the question of whether there is a deeper reason for this similarity based on a general behavior of the migrants.

In social sciences, there are plenty of different models to describe slum formation using cellular automata or agentbased models [2,3]. Many of these models try to describe the development of slums with an ever increasing number of influencing factors. This trend in sociology to refine models with ever smaller nuances was criticized at the beginning of the 19th century by well-known representatives of this branch of science, including Menger [4] or Weber [5], and it was criticized in recent years by Healy [6]. We propose describing the development of slums and their characteristic size using as simple a model as possible to gain a principal insight into the formation of urban structure.

We hypothesize that the intraurban pattern can be explained by a Turing instability [7] occurring through the interaction of two social groups: rich and poor. This division is appropriate due to the great inequalities, especially in today's megacities such as Sao Paulo (cf. Fig. 1). The hypothesis followed in this paper is that slums are part of a self-emergence of order within a city. A convincing argument for this may be that the scale emerges out of a stability analysis. In other words, it is an intrinsic scale of the system. Slums are a Turing pattern.

This hypothesis is not as far-fetched as it seems at first glance, since Turing patterns are not limited to chemical

TABLE I. The size of slums in different cities [1].

City	$\widetilde{\phi_i}$ (m)
Mumbai, India	131
Manila, the Philippines	92
Cape Town, South Africa	132
Rio de Janeiro, Brazil	141

systems [8] or the patterning of organisms [9]. Theraulaz *et al.* [10] showed that spatial clustering in ant colonies is a Turing pattern, emerging through self-organization of these social insects. Consequently, Turing patterns do not just emerge in organisms but also through the interaction of organisms.

Moreover, for reaction-diffusion systems with two dominant components, the Turing mechanism can be shown to be the only possible patterning mechanism [11]. As proposed by Gierer and Meinhardt [12], the emergence of Turing patterns in a system with two dominant components, one being the activator and the other the inhibitor, can be understood as



FIG. 1. The western part of the slum (*favela*) Paraisopolis in Sao Paulo, and the penthouses located in close proximity. The image was captured using Google Maps (2018).

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the interaction of strong local self-enhancement coupled with long-range inhibition. From a nearly homogeneous initial state, the self-enhancing activator growth is initiated by small fluctuations, and, after a while, is limited by the inhibition mechanism.

By using the Turing mechanism to model the similar sizes of slums, the concept of continuum mechanics is transferred to urban modeling. Although this approach has some limitations and is not sufficient to describe all aspects of the phenomenon of slum emergence, the major characteristics can be modeled.

II. STABILITY ANALYSIS OF MIGRATION BEHAVIOR

We start by dividing the population into two agents: poor and rich. The number of poor inhabitants per area at the spatial point \tilde{x}_i (j = 1, 2) and time \tilde{t} is given by the density field $\widetilde{u}_1(\widetilde{x}_i, \widetilde{t})$. Equally, the density field of the rich inhabitants at the same point and same time is $\widetilde{u}_2(\widetilde{x}_1, \widetilde{t})$. Hence the number of inhabitants in a finite area A of the city is $N_i = \int_A \widetilde{u}_i dA$. The increase rate N_i of either poor (i = 1) or rich (i = 2)inhabitants has three contributions: (i) birth and death of poor and rich within the area A; (ii) migration due to flight from the land or movement from one city district to another; (iii) migration flux toward lower concentrations across the closed boundary C of the area A. The latter contribution is a diffusive flux referred to here as short-distance migration. For shortness, we call the sum of the two former contributions long-distance migration, recognizing that birth and death are different from migration. Hence the balance equations for poor and rich read

rate of = long-distance migration + short-distance density change (including birth and death) migration, $\dot{N}_1 = \frac{\partial}{\partial \tilde{t}} \int_A \tilde{u}_1 dA = \int_A \widehat{U} R f_1(u_1, u_2) dA - \oint_C \vec{J}_1 \cdot \vec{n} \, dC,$ $\dot{N}_2 = \frac{\partial}{\partial \tilde{t}} \int_A \widetilde{u}_2 dA = \int_A \widehat{U} R f_2(u_1, u_2) dA - \oint_C \vec{J}_2 \cdot \vec{n} \, dC.$

(1) The *long-distance migration* is determined by the product of a dimensionless so called reaction term $f_i(u_j)$ and the reaction rate $\widehat{U}R$. The dimension of \widehat{U} is inhabitants per area. The dimension of R is that of a rate, i.e., 1 over time. $u_j := \widetilde{u}_j/\widehat{U}$ is a dimensionless population density being measured in multiples of the reference density \widehat{U} . \widehat{U} may be the maximal possible density. By using this transformation, it is possible to compare cities with completely different population densities, like Manila and Cape Town.

The *short-distance migration* is driven by a density gradient. Neglecting cross diffusion, the simplest model is Fick's first law $\vec{J_1} = -D_1 \nabla \tilde{u_1}$ and $\vec{J_2} = -D_2 \nabla \tilde{u_2}$ with constant diffusion coefficients D_1 and D_2 for poor and rich, respectively. By using Gauss' theorem, the two interacting reaction diffusion equations are derived:

$$\frac{\partial \widetilde{u}_1}{\partial \widetilde{t}} = \widehat{U} R f_1(u_1, u_2) + D_1 \Delta \widetilde{u}_1,$$
$$\frac{\partial \widetilde{u}_2}{\partial \widetilde{t}} = \widehat{U} R f_2(u_1, u_2) + D_2 \Delta \widetilde{u}_2.$$
(2)

The transformation $t := R\tilde{t}, x_j := \tilde{x}_j \sqrt{R/D_1}, u_i := \tilde{u}_i/\hat{U}$ yields the dimensionless reaction diffusion system (written in Einstein's index notation)

$$\frac{\partial u_i}{\partial t} = f_i(u_j) + d_{ij} \frac{\partial^2 u_j}{\partial x_k \partial x_k}, \ (d_{ij}) = \begin{pmatrix} 1 & 0\\ 0 & d \end{pmatrix}$$
(3)

with $d := D_2/D_1$.

Following well-known literature [13], we conduct a stability analysis of the system. We linearize Eq. (3) with $u_j = U_j + \delta u_j$, where U_j is the homogeneous solution of $f_i(U_j) = 0$. Introduction of the ansatz $\delta u_j = \mathcal{R}[\delta \hat{u}_j \exp(\sigma t + ik_k x_k)]$ results in an eigenvalue problem and hence a dispersion relation between the eigenvalue σ and the Euclidian length $k = \sqrt{k_i k_i}$ of the wave vector k_i . Linear stability of a uniform distribution of poor and rich without short-distance migration is gained for negative real parts of the eigenvalues. For $d_{ij} = 0$, the eigenvalue problem reads $(\sigma \delta_{ij} - a_{ij})\delta \hat{u}_j = 0$, with the Jacobian $a_{ij} := \partial f_i / \partial u_j$ and the Kronecker delta δ_{ij} . The characteristic equation is $\sigma^2 - a_{ii}\sigma + \det(a_{ij}) = 0$ and the eigenvalues are $2\sigma_{1,2} = a_{ii} \pm \sqrt{a_{ii}^2 - 4} \det(a_{ij})$. The real part of the eigenvalues is only negative provided the trace of the Jacobian a_{ij} is negative,

$$a_{ii} = a_{11} + a_{22} < 0, \tag{4}$$

and the determinant of the Jacobian is positive,

$$\det (a_{ij}) = a_{11}a_{22} - a_{12}a_{21} > 0.$$
(5)

Hence, an initial distribution of poor and rich is only stable if the Jacobian assumes the form

$$(a_{ij}) = \begin{pmatrix} a_{11} > 0 & a_{12} > 0 \\ a_{21} < 0 & a_{22} < 0 \end{pmatrix}.$$
 (6)

There are three alternative classes of systems, but they are not relevant for migration. As is well known [13], the conditions (4) and (5) lead to four qualitatively different Jacobians with pairwise equal properties:

$$\begin{pmatrix} + & + \\ - & - \end{pmatrix}, \begin{pmatrix} - & - \\ + & + \end{pmatrix}, \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \begin{pmatrix} - & + \\ - & + \end{pmatrix}.$$
 (7)

For the first two cases, the resulting concentrations are spatially out of phase (segregation of rich and poor as explained below), whereas the last two matrices lead to a concentration that is spatially in phase, which is not plausible for the model and thus have been neglected.

The signs of the elements of the Jacobian are easily interpretable as sociological general rules. As long as the following four rules are satisfied on the one hand, and there is no short-distance migration by means of diffusion on the other hand, the initial distribution of poor and rich is stable:

(i) $a_{11} > 0$, *poor attract poor*. An increase of poor inhabitants, i.e., an increase of the concentration $u_1(x_j, t)$ at a point x_j in time t, results in an increase of poor people moving into the city and settling at that point x_j . One influential factor for the growth of cities is the migration of people from the countryside seeking opportunities. This is reasoned by informal networks: The migrants know people in cities that already live there. Thus, areas with a higher concentration of poor people tend to grow [14]. Equally, an increase of poor inhabitants results in an increased number of births [15]. In either case, the consequence is a long-distance migration of poor into the city.

(ii) $a_{12} > 0$, *rich attract poor*. An increase of rich inhabitants $u_2(x_j, t)$ results in an increase of poor people moving into the city and settling at point x_j . This is due to the higher financial resources generated by the wealthy citizens and the resulting increase in opportunities for poor people.

(iii) $a_{21} < 0$, *poor repel rich*. An increase of poor inhabitants at a point x_j results in long-distance migration of wealthy inhabitants leaving the location x_j . This is because for wealthy citizens, the attractiveness of an area is diminished with a high number of poor people. A consequence of this behavior are "gated communities," often seen in the global south [16].

(iv) $a_{22} < 0$, *rich repel rich*. An increase of rich inhabitants $u_2(x_j, t)$ at point x_j at time t may result in a long-distance migration of wealthy inhabitants. This is related to lower birth rates with better education [15] and a lower population density due to greater financial resources. The underlying assumption is analogous to a "monolayer" of molecules. This assumption of course would not be valid for extreme cases like Manhattan in New York.

In the following, the role of *short-distance migration*, i.e., diffusion, is studied. As is known [13], diffusion can lead to Turing instability [7]. From statistical mechanics [17] it is known that the diffusion coefficient, D_1 or D_2 , is the product of specific energy k_BT (Boltzmann constant k_B , temperature T) and mobility μ of an entity: for a system with constant temperature, the ratio d equals the ratio of the mobility $d = \mu_2/\mu_1$. It is feasible that mobility increases with wealth. Hence, d shall be greater than 1. In the following, the diffusion-driven instability is recaptured [13] but interpreted for a social system. Using the abbreviation $b_{ij} := a_{ij} - d_{ij}k^2$, the eigenvalue problem reads $(\sigma \delta_{ij} - b_{ij})\delta \hat{u}_j = 0$. The two eigenvalues are

$$2\sigma_{1,2} = b_{ii} \pm \sqrt{b_{ii}^2 - 4 \det(b_{ij})}.$$
 (8)

A perturbation of U_j is only stable if the real part of the largest eigenvalue associated with the largest wave number is smaller than zero: max $\mathcal{R}(\sigma(k)) < 0$. This is the case for

$$b_{ii} = a_{ii} - k^2 (1+d) < 0 \tag{9}$$

and

$$\det(b_{ij}) = (a_{11} - k^2)(a_{22} - dk^2) - a_{12}a_{21} > 0.$$
(10)

Since $a_{ii} < 0$ due to the stable initial state, the trace $b_{ii} = a_{ii} - k^2(1+d)$ is smaller than zero as well. Hence, instability may only be caused by a violation of the condition det $(b_{ij}) = (a_{11} - k^2)(a_{22} - dk^2) - a_{12}a_{21} > 0$. There is a change in sign at det $(b_{ij}) = 0$ for $(a_{11} - k^2)(a_{22} - dk^2) - a_{12}a_{21} = 0$. This is a quadratic equation in k^2 . The minimal value of the determinant, det $(b_{ij}) = \det(a_{ij}) - (a_{11}d + a_{22})^2/4d$, is reached for

$$k_{\rm dom}^2 = \frac{1}{2} \Big(a_{11} + \frac{a_{22}}{d} \Big). \tag{11}$$

Hence, the determinant det (b_{ij}) is negative for $a_{11}d + a_{22} > 2\sqrt{d \det(a_{ij})}$. With $2\sqrt{d \det(a_{ij})} > 0$, the necessary condition for Turing instability reads $a_{11}d + a_{22} > 0$. Conse-

quently, the initial stable distribution may become unstable for

$$\frac{a_{11}}{\mu_1} > -\frac{a_{22}}{\mu_2} \text{ or } a_{11} > -\frac{a_{22}}{d}.$$
 (12)

In other words, a stable social system may become Turingunstable when "the generalized attraction of poor" dominates the "generalized repulsion of rich." When *d* exceeds this threshold, a Turing instability resulting in a Turing pattern with the dominant wave number k_{dom} will emerge. The analysis of the dispersion relation (8) leads to the expression (11) for k_{dom} .

III. DISCUSSION

The model shows that patterns with a certain wavelength are formed. The solution of the equation leads to a regular concentration distribution with wavelength $\lambda = 2\pi/k$ depending on the Jacobian a_{ij} and the diffusion ratio d. Setting a threshold value u^* for the concentration u_i leads to a binary pattern similar to observed binary patterns of slums in a city. Within the binary pattern, two distinct but linked properties can be found: (i) the wavelength λ , which translates to the distance between slums; and (ii) the size of concentration peaks ϕ of the poor, which itself is proportional to λ and dependent on the threshold value u^* . ϕ can be interpreted as the size of slums. The characteristic intrinsic scale $\phi \propto \lambda$ results from simple rules of behavior of two social groups, by interpreting the Turing model in a sociological way. So far the analysis was performed dimensionless, i.e., ϕ is measured in multiples of the length scale $\sqrt{D_1/R}$: $\phi = \phi \sqrt{D_1/R}$. The characteristic length can be understood as the distance "molecules," e.g., humans of the first morphogen, i.e., poor humans, travel during a characteristic reaction time and thus connect short- and long-distance migration. We hypothesize that the length scale $\sqrt{D_1/R}$ is similar between cities since it is only dependent on the human scale: as the data show, the birth rate, which influences the reaction rate R, is similar between countries. In the countries considered in the Introduction, the birth rates are in the realm of 2.4 births per woman (India 2.3, the Philippines 2.9, South Africa 2.5, and Brazil 1.7 births per woman) [18]. It is a much harder task to support the diffusion coefficient D_1 with observations. In the sketched analogy to statistical mechanics, D_1 shall be proportional to the fluctuation energy of humans, measured by a "temperature." Assuming a similar fluctuation energy for humans in different countries, the model in its present form predicts similar dynamics and a similar characteristic length $\sqrt{D_1/R}$ for a characteristic slum size. Still, in future investigations this point should be verified.

The interpretation shows that short- and long-distance migration suffices for the self-emergence of slums from a homogeneous distribution of rich and poor. Self-organized pattern formation only occurs when short-distance migration is considered. According to this model, slums only emerge when there is a concentration-dependent effort by people to move away from the same social group in their surroundings. An interesting outcome is that the interaction behavior within the "poor" group differs between long and short distance. On the one hand, poor people attract poor people on long distance, but on the other hand they repel each other depending on the concentration in short distance.

Following these results, the role of the relation $\lambda \propto \phi$ remains unclear since it predicts a proportionality between the size and distance of slums, which is not found in the data of mapped slums. Analysis of slum data suggests no strong correlation between size and distance. One could argue that the reason for the differences between the model and reality is the relative size of the patterns and their building blocks. In the case of slums, the size of the pattern-building organisms (inhabitants) is in the realm of 10^0 m. The size of the patterns (slums) is about 10^2 m. Thus, slums are only 100 times larger than their builders, which calls into question the assumption of a continuum. In the aforementioned work of Theraulaz et al., the behavior of ants is modeled with a validated continuum model [10]. However, ants as the building blocks are ten times smaller than the features of the pattern, which is an argument for our continuum hypothesis. A more plausible reason for the differences may be the effects of the nonhomogeneous environment, which the model does not account for and which cannot be neglected. In [10] it was possible to exclude the effects of the environment on the ants' behavior in a laboratory experiment. In the case of slums, this is obviously not possible and thus may lead to the differences between the model and reality.

IV. CONCLUSION

We finish this paper with some general considerations. First, the derived rules for *short- and long-distance migration* that are necessary conditions for a Turing instability are surprisingly simple and feasible. Second, since $\phi \propto \lambda$ PHYSICAL REVIEW E 99, 022302 (2019)

emerges out of a stability analysis, the typical scale is intrinsic and not caused by external factors. The question arises as to whether the assumptions made here might be too strict to describe such a complex sociological phenomenon as slum formation in cities by means of two coupled, nonlinear reaction-diffusion equations.

We interpret a reaction-diffusion model to describe quantitatively measurable variables of the system. There has to be a general driving force for the similar size of slums to be independent of continent or culture. To determine the reason behind this phenomenon, one should use the most concise model rather than a detailed one. In this sense, our model does not attempt to depict the differences between slum developments in different cities. Instead, it attempts to outline the empirically observed similarities by describing the central forces underlying the respective behavior.

Our research shows that the interpretation of simple mathematical models in the context of social sciences can lead to a better understanding of possible basic driving forces behind a system as complex as a city, and such models lead to new impulses for the empirical research of slums. We believe the trend toward more nuance in social science models is a fallacy, and simple models such as the Turing mechanism suffice to explain astonishing facts like the similar sizes of slums.

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