Revised Chapman-Enskog analysis for a class of forcing schemes in the lattice Boltzmann method

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In the lattice Boltzmann (LB) method, the forcing scheme, which is used to incorporate an external or internal force into the LB equation, plays an important role. It determines whether the force of the system is correctly implemented in an LB model and affects the numerical accuracy. In this paper we aim to clarify a critical issue about the Chapman-Enskog analysis for a class of forcing schemes in the LB method in which the velocity in the equilibrium density distribution function is given by $\mathbf{u} = \sum_{\alpha} \mathbf{e}_{\alpha} f_{\alpha} / \rho$, while the actual fluid velocity is defined as $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F} / (2\rho)$. It is shown that the usual Chapman-Enskog analysis for this class of forcing schemes should be revised so as to derive the actual macroscopic equations recovered from these forcing schemes. Three forcing schemes belonging to the above class are analyzed, among which Wagner's forcing scheme [A. J. Wagner, *Phys. Rev. E* **74**, 056703 (2006)] is shown to be capable of reproducing the correct macroscopic equations. The theoretical analyses are examined and demonstrated with two numerical tests, including the simulation of Womersley flow and the modeling of flat and circular interfaces by the pseudopotential multiphase LB model.

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I. INTRODUCTION

The lattice Boltzmann (LB) method, which is a special discrete solver for mimicking the kinetic Boltzmann equation, has been developed into an efficient mesoscopic numerical approach for simulating fluid flow and heat transfer [1,2]. Owing to its kinetic features, the LB method has attracted significant attention in the past two decades and has been applied in a large variety of fields [3–5]. In many systems, an external or internal force is involved. For these systems, a forcing scheme is needed to incorporate the force into the LB equation. In the literature, various forcing schemes have been developed, such as the forcing schemes proposed by Shan and Chen [6,7], He *et al.* [8], Buick and Greated [9], Ladd and Verberg [10], and Guo *et al.* [11]. To date, it has been generally accepted that the forcing scheme of Guo *et al.* is able to recover the correct unsteady macroscopic equations at the Navier-Stokes level.

Moreover, in the study of Guo *et al.* [11] a theoretical framework is given for deriving the macroscopic equations recovered from the forcing schemes that can be expanded in a power series in the lattice velocity $\mathbf{e}_{\alpha}$. Some of the previous forcing schemes have been analyzed in Ref. [11]. After the study of Guo *et al.*, several alternative forcing schemes were developed. For example, Wagner [12] proposed a forcing scheme by conducting a Taylor expansion analysis for the LB equation and Kupershtokh *et al.* [13] presented a forcing scheme based on the work of He *et al.* [8]. It can be found that these two forcing schemes share the following feature: The velocity in the equilibrium density distribution function is given by $\mathbf{u} = \sum_{\alpha} \mathbf{e}_{\alpha} f_{\alpha} / \rho$, while the actual fluid velocity is defined as $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F} / (2\rho)$, where $\mathbf{F}$ is the force and δ_t is

the time step. In fact, some previous forcing schemes such as a forcing scheme proposed by Ladd and Verberg [10], which was analyzed in Ref. [11], also belong to this class of forcing schemes.

According to the available Chapman-Enskog analysis in the literature (e.g., Refs. [9–11]), for this class of forcing schemes, it can be found that Wagner's forcing scheme cannot recover the correct Navier-Stokes equations. However, Wagner showed [12] that his forcing scheme was obtained based on the Taylor expansion analysis for recovering the correct macroscopic equations. To clarify such an issue, in the present study we demonstrate that the usual Chapman-Enskog analysis in the literature for the aforementioned class of forcing schemes should be modified in order to derive the actual macroscopic equations recovered from these forcing schemes.

The revised Chapman-Enskog analysis shows that Wagner's forcing scheme can correctly recover the macroscopic equations at the Navier-Stokes level, confirming the consistency between the Chapman-Enskog analysis and the Taylor expansion analysis. Besides Wagner's forcing scheme, another two forcing schemes are also analyzed. Two numerical tests are employed to validate the theoretical analysis. The rest of the present paper is organized as follows. The forcing schemes are introduced in Sec. II. The theoretical and numerical analyses are presented in Secs. III and IV, respectively. Finally, a brief summary is given in Sec. V.

II. FORCING SCHEMES

For the aforementioned class of forcing schemes, the LB equation is given as follows [10,12,13]:

$$f_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha} \delta_t, t + \delta_t) - f_{\alpha}(\mathbf{x}, t) = -\frac{1}{\tau} [f_{\alpha} - f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})] + F_{\alpha}, \quad (1)$$

where f_{α} is the density distribution function, f_{α}^{eq} is the equilibrium density distribution function, τ is the nondimensional relaxation time, and F_{α} is the forcing term. Using the D2Q9 lattice model, the equilibrium distribution function can be

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given by

$$f_\alpha^{\text{eq}}(\rho, \mathbf{u}) = \omega_\alpha \rho \left[1 + \frac{\mathbf{e}_\alpha \cdot \mathbf{u}}{c_s^2} + \frac{\mathbf{u}\mathbf{u} : (\mathbf{e}_\alpha \mathbf{e}_\alpha - c_s^2 \mathbf{I})}{2c_s^4} \right], \quad (2)$$

where the weights ω_α are $\omega_0 = 4/9$, $\omega_{1-4} = 1/9$, and $\omega_{5-8} = 1/36$. The velocity $\mathbf{u}$ and the actual fluid velocity $\hat{\mathbf{u}}$ are given by, respectively,

$$\rho \mathbf{u} = \sum_\alpha \mathbf{e}_\alpha f_\alpha, \quad \rho \hat{\mathbf{u}} = \rho \mathbf{u} + \frac{\delta_t}{2} \mathbf{F}. \quad (3)$$

The forcing schemes proposed by Wagner [12], Kupershtokh *et al.* [13], and Ladd and Verberg [10] fall into the above category, although the forcing terms F_α of these schemes are different. By conducting the Taylor expansion analysis of Eq. (1), Wagner [12] pointed out that the forcing term in Eq. (1) should satisfy the following constraints [see Eqs. (19)–(21) and (59) in Ref. [12]]:

$$\sum_\alpha F_{\alpha, \text{Wa}} = 0, \quad \sum_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Wa}} = \delta_t \mathbf{F}, \quad (4)$$

$$\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Wa}} = \delta_t \left[(\mathbf{u}\mathbf{F} + \mathbf{F}\mathbf{u}) + \delta_t \left(1 - \frac{1}{4\tau} \right) \frac{\mathbf{F}\mathbf{F}}{\rho} \right]. \quad (5)$$

Here $F_{\alpha, \text{Wa}}$ represents the forcing term of Wagner's forcing scheme. Note that a higher-order term has been neglected in Eq. (5).

Kupershtokh *et al.* [13] presented a forcing scheme, namely, the so-called exact-difference-method forcing scheme, that obeys Eqs. (1)–(3) with the following forcing term:

$$F_{\alpha, \text{EDM}} = f_\alpha^{\text{eq}}(\rho, \mathbf{u} + \Delta \mathbf{u}) - f_\alpha^{\text{eq}}(\rho, \mathbf{u}), \quad (6)$$

where $\Delta \mathbf{u} = \mathbf{F} \delta_t / \rho$. Substituting the expression of f_α^{eq} into Eq. (6), it can be found that $F_{\alpha, \text{EDM}}$ is identical to [14]

$$F_{\alpha, \text{EDM}} = \omega_\alpha \delta_t \left[\frac{\mathbf{F} \cdot \mathbf{e}_\alpha}{c_s^2} + \frac{(\mathbf{v}_{\text{EDM}} \mathbf{F} + \mathbf{F} \mathbf{v}_{\text{EDM}}) : (\mathbf{e}_\alpha \mathbf{e}_\alpha - c_s^2 \mathbf{I})}{2c_s^4} \right], \quad (7)$$

where $\mathbf{v}_{\text{EDM}} = \mathbf{u} + \Delta \mathbf{u} / 2 = \mathbf{u} + \delta_t \mathbf{F} / (2\rho)$. Evaluating the moments of $F_{\alpha, \text{EDM}}$, we can obtain

$$\sum_\alpha F_{\alpha, \text{EDM}} = 0, \quad \sum_\alpha \mathbf{e}_\alpha F_{\alpha, \text{EDM}} = \delta_t \mathbf{F}, \quad (8)$$

$$\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{\alpha, \text{EDM}} = \delta_t \left((\mathbf{u}\mathbf{F} + \mathbf{F}\mathbf{u}) + \delta_t \frac{\mathbf{F}\mathbf{F}}{\rho} \right). \quad (9)$$

Comparing Eq. (9) with Eq. (5), we can see that the second-order moments of the forcing terms of Wagner and Kupershtokh *et al.* are almost the same except that the coefficients in front of the term $\mathbf{F}\mathbf{F}/\rho$ are different. Similarly, it can be found that Ladd and Verberg's forcing term yields the following moments [see Eq. (28) and Sec. 5.3 in Ref. [10]]:

$$\sum_\alpha F_{\alpha, \text{Ladd}} = 0, \quad \sum_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Ladd}} = \delta_t \mathbf{F}, \quad (10)$$

$$\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Ladd}} = \delta_t (\mathbf{u}\mathbf{F} + \mathbf{F}\mathbf{u}). \quad (11)$$

Unlike Eqs. (5) and (9), Eq. (11) does not contain the term related to $\mathbf{F}\mathbf{F}/\rho$.

For comparison, the forcing scheme of Guo *et al.* is also given here, which consists of the following LB equation [11]:

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \delta_t, t + \delta_t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [f_\alpha - f_\alpha^{\text{eq}}(\rho, \hat{\mathbf{u}})] + F_{\alpha, \text{Guo}}, \quad (12)$$

with the forcing term

$$F_{\alpha, \text{Guo}} = \omega_\alpha \left(1 - \frac{1}{2\tau} \right) \delta_t \left[\frac{\mathbf{F} \cdot \mathbf{e}_\alpha}{c_s^2} + \frac{(\hat{\mathbf{u}}\mathbf{F} + \mathbf{F}\hat{\mathbf{u}}) : (\mathbf{e}_\alpha \mathbf{e}_\alpha - c_s^2 \mathbf{I})}{2c_s^4} \right], \quad (13)$$

which leads to the following moments:

$$\sum_\alpha F_{\alpha, \text{Guo}} = 0, \quad \sum_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Guo}} = \delta_t \left(1 - \frac{1}{2\tau} \right) \mathbf{F}, \quad (14)$$

$$\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{\alpha, \text{Guo}} = \delta_t \left(1 - \frac{1}{2\tau} \right) (\hat{\mathbf{u}}\mathbf{F} + \mathbf{F}\hat{\mathbf{u}}). \quad (15)$$

For the forcing scheme of Guo *et al.*, the actual fluid velocity and the velocity in the equilibrium density distribution function $f_\alpha^{\text{eq}}(\rho, \hat{\mathbf{u}})$ and the forcing term $F_{\alpha, \text{Guo}}$ are all given by $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F} / (2\rho)$, with $\mathbf{u} = \sum_\alpha \mathbf{e}_\alpha f_\alpha / \rho$.

III. THEORETICAL ANALYSIS

A. Multiscale expansions

The Chapman-Enskog analysis has been widely utilized in the LB community to derive the macroscopic equations recovered from LB models. To perform such an analysis, the following multiscale expansions are usually introduced [9]:

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + \dots, \quad (16)$$

$$\partial_t = \varepsilon \partial_{t1} + \varepsilon^2 \partial_{t2}, \quad \nabla = \varepsilon \nabla_1, \quad \mathbf{F} = \varepsilon \mathbf{F}_1, \quad (17)$$

where ε is the expansion parameter. According to the expansion of the force $\mathbf{F}$, the forcing term is also expanded as $F_\alpha = \varepsilon F_{1\alpha}$. Using these expansions, the LB equation with a forcing term can be written in the consecutive order of ε ,

$$O(\varepsilon^0): f_\alpha^{(0)} = f_\alpha^{\text{eq}}, \quad (18)$$

$$O(\varepsilon^1): (\partial_{t1} + \mathbf{e}_\alpha \cdot \nabla_1) f_\alpha^{(0)} = -\frac{1}{\tau \delta_t} f_\alpha^{(1)} + \frac{1}{\delta_t} F_{1\alpha}, \quad (19)$$

$$O(\varepsilon^2): \partial_{t2} f_\alpha^{(0)} + (\partial_{t1} + \mathbf{e}_\alpha \cdot \nabla_1) f_\alpha^{(1)} + \frac{\delta_t}{2} (\partial_{t1} + \mathbf{e}_\alpha \cdot \nabla_1)^2 f_\alpha^{(0)} = -\frac{1}{\tau \delta_t} f_\alpha^{(2)} \quad (20)$$

for $O(\varepsilon^0)$, $O(\varepsilon^1)$, and $O(\varepsilon^2)$, respectively. With the aid of Eq. (19), Eq. (20) can be rewritten as

$$\begin{aligned} \partial_{t2} f_\alpha^{(0)} + (\partial_{t1} + \mathbf{e}_\alpha \cdot \nabla_1) \left(1 - \frac{1}{2\tau} \right) f_\alpha^{(1)} + \frac{1}{2} (\partial_{t1} + \mathbf{e}_\alpha \cdot \nabla_1) F_{1\alpha} \\ = -\frac{1}{\tau \delta_t} f_\alpha^{(2)}. \end{aligned} \quad (21)$$

In the literature [10,11], the above equations were applied to both Eqs. (1) and (12) by setting f_α^{eq} in Eq. (18) to $f_\alpha^{\text{eq}}(\rho, \mathbf{u})$ and $f_\alpha^{\text{eq}}(\rho, \hat{\mathbf{u}})$, respectively.

B. Original analysis

Here the original analysis denotes the usual Chapman-Enskog analysis of Eq. (1), in which f_α^{eq} in Eq. (18) is taken as $f_\alpha^{\text{eq}}(\rho, \mathbf{u})$, which leads to $\sum_\alpha \mathbf{e}_\alpha f_\alpha^{(n)} = 0$ ($n \neq 0$) since $\rho \mathbf{u} = \sum_\alpha \mathbf{e}_\alpha f_\alpha^{\text{eq}} = \sum_\alpha \mathbf{e}_\alpha f_\alpha$. Meanwhile, it can be found that $\sum_\alpha f_\alpha^{(n)} = 0$ ($n \neq 0$). With these results, the following equations can be obtained by taking the summations of Eqs. (19) and (21), respectively,

$$\partial_{t1}\rho + \nabla_1 \cdot (\rho \mathbf{u}) = 0, \quad (22)$$

$$\partial_{t2}\rho + \frac{1}{2}\nabla_1 \cdot (\delta_t \mathbf{F}_1) = 0. \quad (23)$$

Combining Eq. (22) with Eq. (23) using the relationships in Eq. (17), we have

$$\partial_t \rho + \nabla \cdot (\rho \hat{\mathbf{u}}) = 0, \quad (24)$$

where $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F}/(2\rho)$ is the actual fluid velocity. The above equation is the continuity equation. Similarly, the first-order moments of Eqs. (19) and (21) yield, respectively,

$$\partial_{t1}(\rho \mathbf{u}) + \nabla_1 \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla_1 p + \mathbf{F}_1, \quad (25)$$

$$\begin{aligned} \partial_{t2}(\rho \mathbf{u}) + \nabla_1 \cdot \left[\left(1 - \frac{1}{2\tau} \right) \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(1)} \right] \\ + \frac{\delta_t}{2} \partial_{t1} \mathbf{F}_1 + \frac{1}{2} \nabla_1 \cdot \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{1\alpha} = 0, \end{aligned} \quad (26)$$

where $p = \rho c_s^2$. In addition, from Eq. (19) the following equation can be obtained:

$$\begin{aligned} \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(1)} = -\tau \left[\delta_t \partial_{t1} \left(\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(0)} \right) + \delta_t \nabla_1 \cdot \left(\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(0)} \right) - \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{1\alpha} \right]. \end{aligned} \quad (27)$$

Substituting Eq. (27) into Eq. (26) gives

$$\partial_{t2}(\rho \mathbf{u}) + \nabla_1 \cdot \mathbf{\Pi}^{(1)} + \frac{\delta_t}{2} \partial_{t1} \mathbf{F}_1 + \nabla_1 \cdot \tau \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_{1\alpha} = 0, \quad (28)$$

where $\mathbf{\Pi}^{(1)}$ is given by

$$\begin{aligned} \mathbf{\Pi}^{(1)} = -\left(\tau - \frac{1}{2} \right) \delta_t \left[\partial_{t1} \left(\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(0)} \right) \right. \\ \left. + \nabla_1 \cdot \left(\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha f_\alpha^{(0)} \right) \right]. \end{aligned} \quad (29)$$

According to the treatment $f_\alpha^{(0)} = f_\alpha^{\text{eq}}(\rho, \mathbf{u})$ as well as Eqs. (22) and (25), the following expression can be obtained:

$$\mathbf{\Pi}^{(1)} = -\left(\tau - \frac{1}{2} \right) \delta_t \left\{ \rho c_s^2 [\nabla_1 \mathbf{u} + (\nabla_1 \mathbf{u})^T] + \mathbf{u} \mathbf{F}_1 + \mathbf{F}_1 \mathbf{u} \right\}. \quad (30)$$

Substituting Eq. (30) into Eq. (28) and then combining the result with Eq. (25), we can obtain

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot \{ \rho v [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \} + \mathbf{F} - \frac{\delta_t}{2} \varepsilon \partial_{t1} \mathbf{F} + \nabla \cdot \left[\left(\tau - \frac{1}{2} \right) \delta_t (\mathbf{u} \mathbf{F} + \mathbf{F} \mathbf{u}) - \tau \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_\alpha \right], \quad (31)$$

where $v = (\tau - 0.5)c_s^2 \delta_t$ is the kinematic viscosity. The final form of Eq. (31) depends on the second-order moment of the forcing term, namely, $\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_\alpha$. For example, if $\sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_\alpha = 0$, we can obtain Eq. (21b) in Ref. [11].

In the above-mentioned class of forcing schemes, the actual fluid velocity is defined as $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F}/2\rho$. Hence a transformation can be made from Eq. (31) to

$$\begin{aligned} \partial_t(\rho \hat{\mathbf{u}}) + \nabla \cdot (\rho \hat{\mathbf{u}} \hat{\mathbf{u}}) = -\nabla p + \nabla \cdot \{ \rho v [\nabla \hat{\mathbf{u}} + (\nabla \hat{\mathbf{u}})^T] \} + \mathbf{F} - \nabla \cdot \{ \rho v [\nabla \mathbf{a} + (\nabla \mathbf{a})^T] \} + \frac{\delta_t}{2} \varepsilon^2 \partial_{t2} \mathbf{F} \\ + \nabla \cdot \left[\tau \delta_t (\mathbf{u} \mathbf{F} + \mathbf{F} \mathbf{u}) + \delta_t^2 \frac{\mathbf{F} \mathbf{F}}{4\rho} - \tau \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha F_\alpha \right], \end{aligned} \quad (32)$$

where $\mathbf{a} = \hat{\mathbf{u}} - \mathbf{u} = \delta_t \mathbf{F}/2\rho$. The following relationships have been utilized in deriving Eq. (32):

$$\begin{aligned} \partial_t \mathbf{F} &= \varepsilon \partial_{t1} \mathbf{F} + \varepsilon^2 \partial_{t2} \mathbf{F}, \\ \rho \hat{\mathbf{u}} \hat{\mathbf{u}} &= \rho \mathbf{u} \mathbf{u} + \frac{\delta_t}{2} (\mathbf{u} \mathbf{F} + \mathbf{F} \mathbf{u}) + \delta_t^2 \frac{\mathbf{F} \mathbf{F}}{4\rho}. \end{aligned} \quad (33)$$

The left-hand side of Eq. (32) together with the first three terms on the right-hand side of Eq. (32) constitutes the correct macroscopic momentum equation and the remaining terms

are regarded as error terms. For Wagner's forcing scheme, the following error terms would be obtained according to Eqs. (5) and (32):

$$\begin{aligned} \mathbf{Er}_{\text{Wa}} &= \frac{\delta_t}{2} \varepsilon^2 \partial_{t2} \mathbf{F} - \nabla \cdot \{ \rho v [\nabla \mathbf{a} + (\nabla \mathbf{a})^T] \} \\ &+ \nabla \cdot \left[\delta_t^2 \left(\frac{1}{2} - \tau \right) \frac{\mathbf{F} \mathbf{F}}{\rho} \right]. \end{aligned} \quad (34)$$

Similarly, for the forcing schemes of Kupershtokh *et al.* and Ladd and Verberg, the following error terms are obtained,

respectively:

$$\begin{aligned} \mathbf{Er}_{\text{EDM}} &= \frac{\delta_t}{2} \varepsilon^2 \partial_{t2} \mathbf{F} - \nabla \cdot \{ \rho v [\nabla \mathbf{a} + (\nabla \mathbf{a})^T] \} \\ &\quad + \delta_t^2 \nabla \cdot \left[\left(\frac{1}{4} - \tau \right) \frac{\mathbf{FF}}{\rho} \right], \end{aligned} \quad (35)$$

$$\begin{aligned} \mathbf{Er}_{\text{Ladd}} &= \frac{\delta_t}{2} \varepsilon^2 \partial_{t2} \mathbf{F} - \nabla \cdot \{ \rho v [\nabla \mathbf{a} + (\nabla \mathbf{a})^T] \} \\ &\quad + \delta_t^2 \nabla \cdot \left(\frac{\mathbf{FF}}{4\rho} \right). \end{aligned} \quad (36)$$

According to the above analysis, there are many error terms in the macroscopic momentum equations recovered from the forcing schemes proposed by Wagner [12], Kupershtokh *et al.* [13], and Ladd and Verberg [10]. In other words, the above analysis tells us that all of these forcing schemes cannot recover the correct macroscopic momentum equation at the Navier-Stokes level. However, as mentioned in the Introduction, Wagner's forcing scheme was constructed based on the Taylor expansion analysis for recovering the correct macroscopic equations. According to Wagner's work [12], the following constraint should be satisfied so as to recover the momentum equation at the Navier-Stokes level [see Eq. (59) in the reference and neglect a high-order term]:

$$\tau \Psi = \delta_t \left(\tau - \frac{1}{4} \right) \frac{\mathbf{FF}}{\rho}, \quad (37)$$

where $\Psi = \sum_{\alpha} (\mathbf{e}_{\alpha} - \mathbf{u})(\mathbf{e}_{\alpha} - \mathbf{u}) F_{\alpha} \equiv \sum_{\alpha} \mathbf{e}_{\alpha} \mathbf{e}_{\alpha} F_{\alpha} - \delta_t (\mathbf{uF} + \mathbf{Fu})$. It can be verified that Eq. (37) is satisfied when $\sum_{\alpha} \mathbf{e}_{\alpha} \mathbf{e}_{\alpha} F_{\alpha}$ is defined by Eq. (5). Therefore, the question is raised of whether the Taylor expansion analysis is inconsistent with the Chapman-Enskog analysis even at the Navier-Stokes level.

C. Revised analysis

To clarify the above inconsistency, we would like to introduce an effective forcing term for Wagner's forcing scheme. Since the actual fluid velocity is defined as $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F}/(2\rho)$, it is natural to rewrite $f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})$ in Eq. (1) as follows:

$$f_{\alpha}^{\text{eq}}(\rho, \mathbf{u}) \equiv f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}}) - [f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}}) - f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})]. \quad (38)$$

Correspondingly, Eq. (1) can be rewritten as

$$f_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha} \delta_t, t + \delta_t) - f_{\alpha}(\mathbf{x}, t) = -\frac{f_{\alpha} - f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}})}{\tau} + \hat{F}_{\alpha}, \quad (39)$$

where the effective forcing term $\hat{F}_{\alpha}$ is given by

$$\hat{F}_{\alpha} = F_{\alpha} - \frac{f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}}) - f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})}{\tau}. \quad (40)$$

It can be seen that Eq. (39) takes the same form as Eq. (12), although the forcing term may be different.

For Wagner's forcing scheme, the following moments of the effective forcing term $\hat{F}_{\alpha, \text{wa}}$ can be obtained from Eqs. (4), (5),

and (40) with the relationship $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F}/(2\rho)$:

$$\sum_{\alpha} \hat{F}_{\alpha, \text{wa}} = 0, \quad \sum_{\alpha} \mathbf{e}_{\alpha} \hat{F}_{\alpha, \text{wa}} = \delta_t \left(1 - \frac{1}{2\tau} \right) \mathbf{F}, \quad (41)$$

$$\sum_{\alpha} \mathbf{e}_{\alpha} \mathbf{e}_{\alpha} \hat{F}_{\alpha, \text{wa}} = \delta_t \left(1 - \frac{1}{2\tau} \right) \left[(\mathbf{uF} + \mathbf{Fu}) + \delta_t \frac{\mathbf{FF}}{\rho} \right]. \quad (42)$$

Since $(\mathbf{uF} + \mathbf{Fu}) + \delta_t \mathbf{FF}/\rho \equiv (\hat{\mathbf{u}}\mathbf{F} + \mathbf{F}\hat{\mathbf{u}})$, it can be found that Eqs. (41) and (42) are identical to the results of the forcing scheme of Guo *et al.*, namely, Eqs. (14) and (15), respectively. Actually, according to Eqs. (4) and (5), the original forcing term of Wagner's forcing scheme can be defined as

$$\begin{aligned} F_{\alpha, \text{wa}} &= \omega_{\alpha} \delta_t \left\{ \frac{\mathbf{F} \cdot \mathbf{e}_{\alpha}}{c_s^2} + \left[(\mathbf{uF} + \mathbf{Fu}) + \delta_t \left(1 - \frac{1}{4\tau} \right) \frac{\mathbf{FF}}{\rho} \right] \right. \\ &\quad \left. : \frac{(\mathbf{e}_{\alpha} \mathbf{e}_{\alpha} - c_s^2 \mathbf{I})}{2c_s^4} \right\}. \end{aligned} \quad (43)$$

Meanwhile, with the aid of Eq. (2), we can obtain

$$\begin{aligned} &\frac{f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}}) - f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})}{\tau} \\ &= \frac{\rho \omega_{\alpha}}{\tau} \left[\frac{\mathbf{a} \cdot \mathbf{e}_{\alpha}}{c_s^2} + \frac{(\mathbf{ua} + \mathbf{au} + \mathbf{aa}) : (\mathbf{e}_{\alpha} \mathbf{e}_{\alpha} - c_s^2 \mathbf{I})}{2c_s^4} \right] \\ &= \omega_{\alpha} \delta_t \left[\frac{1}{2\tau} \frac{\mathbf{F} \cdot \mathbf{e}_{\alpha}}{c_s^2} + \left(\frac{1}{2\tau} (\mathbf{uF} + \mathbf{Fu}) + \frac{\delta_t}{4\tau} \frac{\mathbf{FF}}{\rho} \right) \right. \\ &\quad \left. : \frac{(\mathbf{e}_{\alpha} \mathbf{e}_{\alpha} - c_s^2 \mathbf{I})}{2c_s^4} \right], \end{aligned} \quad (44)$$

where $\mathbf{a} = \hat{\mathbf{u}} - \mathbf{u} = \delta_t \mathbf{F}/2\rho$. Then the following effective forcing term can be established for Wagner's forcing scheme by substituting Eqs. (43) and (44) into Eq. (40):

$$\begin{aligned} \hat{F}_{\alpha, \text{wa}} &= \omega_{\alpha} \left(1 - \frac{1}{2\tau} \right) \delta_t \left[\frac{\mathbf{F} \cdot \mathbf{e}_{\alpha}}{c_s^2} + \left(\mathbf{uF} + \mathbf{Fu} + \delta_t \frac{\mathbf{FF}}{\rho} \right) \right. \\ &\quad \left. : \frac{(\mathbf{e}_{\alpha} \mathbf{e}_{\alpha} - c_s^2 \mathbf{I})}{2c_s^4} \right]. \end{aligned} \quad (45)$$

Using the relationship below Eq. (42), it can be readily verified that the above forcing term is just the forcing term of Guo *et al.* defined by Eq. (13). Based on Eqs. (39) and (45), we can conclude that Wagner's forcing scheme is theoretically identical to the forcing scheme of Guo *et al.* at the Navier-Stokes level. In other words, these two forcing schemes are theoretically equivalent, although they are constructed according to the Taylor expansion analysis and the Chapman-Enskog analysis, respectively.

Now it is clear that the so-called inconsistency mentioned in the preceding section is attributed to the problem that $f_{\alpha}^{(0)}$ in Eq. (18) is treated as $f_{\alpha}^{(0)} = f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})$ in the usual Chapman-Enskog analysis for the aforementioned class of forcing schemes. According to the above analysis, a direct treatment for deriving the actual macroscopic equations from these forcing schemes is to utilize the transformation given by Eqs. (38) and (39). For such a treatment, Eqs. (18) and (19) together with Eq. (21) are still applicable. However, f_{α}^{eq} in Eq. (18) should be taken as $f_{\alpha}^{\text{eq}}(\rho, \hat{\mathbf{u}})$. Meanwhile, $F_{1\alpha}$ in Eqs. (19) and (21) should be replaced with $\hat{F}_{1\alpha}$.

In summary, for the aforementioned class of forcing schemes, the velocity $\mathbf{u}$ can be used in the equilibrium distribution function f_α^{eq} in Eq. (1), but in the Chapman-Enskog analysis we should ensure that $f_\alpha^{(0)}$ in the multiscale expansion (16) is equal to $f_\alpha^{\text{eq}}(\rho, \hat{\mathbf{u}})$, which leads to $\sum_\alpha \mathbf{e}_\alpha f_\alpha^{(1)} = -\delta_t \mathbf{F}_1/2$. In other words, the multiscale expansion of f_α should be based on the actual fluid velocity $\hat{\mathbf{u}}$. The incorrectness of the original analysis lies in the treatment $f_\alpha^{(0)} = f_\alpha^{\text{eq}}(\rho, \mathbf{u})$ with $\sum_\alpha \mathbf{e}_\alpha f_\alpha^{(1)} = 0$. By applying the revised treatment, it can be found that the momentum equation recovered from the aforementioned class of forcing schemes takes the following form:

$$\begin{aligned} \partial_t(\rho \hat{\mathbf{u}}) + \nabla \cdot (\rho \hat{\mathbf{u}} \hat{\mathbf{u}}) \\ = -\nabla p + \nabla \cdot \hat{\mathbf{\Pi}} + \mathbf{F} \\ + \nabla \cdot \left[\left(\tau - \frac{1}{2} \right) \delta_t (\hat{\mathbf{u}} \mathbf{F} + \mathbf{F} \hat{\mathbf{u}}) - \tau \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha \hat{F}_\alpha \right], \end{aligned} \quad (46)$$

where $\hat{\mathbf{\Pi}} = \rho \nu [\nabla \hat{\mathbf{u}} + (\nabla \hat{\mathbf{u}})^T]$. Obviously, for Wagner's forcing scheme [see Eq. (42)], the macroscopic momentum equation at the Navier-Stokes level can be correctly recovered.

According to Eq. (40) and Eqs. (9) and (11), the following moments can be obtained for the forcing schemes of Kupershtokh *et al.* and Ladd and Verberg, respectively,

$$\begin{aligned} \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha \hat{F}_{\alpha, \text{EDM}} \\ = \delta_t \left(1 - \frac{1}{2\tau} \right) \left[(\mathbf{u} \mathbf{F} + \mathbf{F} \mathbf{u}) + \delta_t \frac{\mathbf{F} \mathbf{F}}{\rho} \right] + \frac{\delta_t^2}{4\tau} \frac{\mathbf{F} \mathbf{F}}{\rho}, \quad (47) \\ \sum_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha \hat{F}_{\alpha, \text{Ladd}} \\ = \delta_t \left(1 - \frac{1}{2\tau} \right) \left[(\mathbf{u} \mathbf{F} + \mathbf{F} \mathbf{u}) + \delta_t \frac{\mathbf{F} \mathbf{F}}{\rho} \right] - \delta_t^2 \left(1 - \frac{1}{4\tau} \right) \frac{\mathbf{F} \mathbf{F}}{\rho}. \end{aligned} \quad (48)$$

With the above formulations, the actual momentum equations recovered from the forcing schemes of Kupershtokh *et al.* and Ladd and Verberg can be written as, respectively,

$$\partial_t(\rho \hat{\mathbf{u}}) + \nabla \cdot (\rho \hat{\mathbf{u}} \hat{\mathbf{u}}) = -\nabla p + \nabla \cdot \hat{\mathbf{\Pi}} + \mathbf{F} - \delta_t^2 \nabla \cdot \left(\frac{\mathbf{F} \mathbf{F}}{4\rho} \right), \quad (49)$$

$$\begin{aligned} \partial_t(\rho \hat{\mathbf{u}}) + \nabla \cdot (\rho \hat{\mathbf{u}} \hat{\mathbf{u}}) \\ = -\nabla p + \nabla \cdot \hat{\mathbf{\Pi}} + \mathbf{F} + \delta_t^2 \nabla \cdot \left[\left(\tau - \frac{1}{4} \right) \frac{\mathbf{F} \mathbf{F}}{\rho} \right]. \end{aligned} \quad (50)$$

The last terms on the right-hand sides of Eqs. (49) and (50) are spatial error terms, which are second order in space since $\delta_t = \delta_x$ in the standard LB method ($c = \delta_x/\delta_t = 1$).

Specifically, Eqs. (35) and (36) involve a time-dependent error term $\partial_{t2} \mathbf{F}$ and an error term like the viscous term in the momentum equation, whereas there are no such error terms in Eqs. (49) and (50). Moreover, it can be found that the error terms in Eqs. (49) and (50) are proportional to $\nabla \cdot (\mathbf{F} \mathbf{F}/\rho)$. Therefore, these terms cannot be neglected when $\nabla \cdot (\mathbf{F} \mathbf{F}/\rho)$

is relatively large and are negligible when the force $\mathbf{F}$ is spatially uniform and the variation of density in space is small. Meanwhile, Eqs. (49) and (50) show that the error term (at the Navier-Stokes level) resulting from the forcing scheme of Kupershtokh *et al.* is not affected by the relaxation time τ , while the error term caused by the forcing scheme of Ladd and Verberg depends on τ . However, according to Eqs. (35) and (36), a contrary conclusion would be drawn. The features of the error terms in Eqs. (49) and (50) will be examined in numerical simulations.

IV. NUMERICAL ANALYSIS

A. Womersley flow

Two numerical tests are employed to validate the above theoretical analysis. In simulations, we still use Eqs. (1) and (3) as well as the original forcing terms of the three forcing schemes. First, the Womersley flow bounded by two parallel surfaces is considered, which is an unsteady flow driven by a periodic pressure gradient or a periodic force $F_x(t) = -A_p \cos(\omega t)$, where A_p is the amplitude and $\omega = 2\pi/T$ is the frequency. Obviously, F_x is spatially uniform but dependent on time. The analytical solution for the velocity of Womersley flow is given by [15]

$$U(y, t) = \text{Re} \left(i \frac{A_p}{\rho_0 \omega} \left\{ 1 - \frac{\cosh[(1+i)\sqrt{(\omega/2\nu)y}]}{\cosh[(1+i)\sqrt{(\omega/2\nu)L}]} \right\} e^{i\omega t} \right), \quad (51)$$

where $y \in [-L, L]$ with $2L$ being the channel width, ν is the kinematic viscosity, and Re denotes the real part of the complex number.

The simulations are carried out in a computational domain with $N_x \times N_y = 50 \times 100$. The periodic boundary condition is applied in the x direction and the no-slip boundary condition is imposed at the two solid surfaces (see Sec. III A in Ref. [16]). The period T is set to 1200, the kinematic viscosity is given by $\nu = 0.1$, and the amplitude A_p is taken as 0.0001. The initial density is chosen as $\rho = \rho_0 = 1$. The simulations begin with $u_x = 0$ and $u_y = 0$. Hence $\hat{u}_x$ is initially given by $\hat{u}_x = 0.5 F_x(0)$. The numerical results are obtained after running 20 periods.

The numerical results predicted by Wagner's forcing scheme at different times are shown in Fig. 1, from which it can be seen that the numerical results agree well with the analytical solution given by Eq. (51). Quantitatively, the average relative error of the eight cases is about 0.57%. The relative error is defined as $E_r = \sum_y |U - U_a| / \sum_y |U_a|$, in which the subscript a denotes the analytical solution. A comparison of the numerical results has been made between different forcing schemes. In all the cases we find that the numerical results of the three forcing schemes are completely the same as those produced by the forcing scheme of Guo *et al.* except for a very slight difference at $y^* = \pm 0.5$ (the solid walls). By taking the case $t = 3T/4$ as an example, Table I displays the numerical results obtained by different forcing schemes at $y^* = y/2L = 0, 0.15, 0.35, 0.49$, and 0.5 . For comparison, the analytical results are also listed in the table. The difference at $y^* = 0.5$ is due to the roundoff errors in numerical computations.

FIG. 1. Numerical (symbols) and analytical (solid line) results of Womersley flow at different times $t = nT/16$ with (a) $n = 4, 6, 8, 10$ and (b) $n = 0, 2, 12, 14$. The numerical results are obtained by Wagner's forcing scheme.

It is clear that if the macroscopic equations recovered from the three forcing schemes satisfy Eq. (32), the time dependence of F_x will yield numerical errors according to Eqs. (34)–(36). Correspondingly, their results will be different from the results of the forcing scheme of Guo *et al.* In contrast, if the macroscopic equations obey Eq. (46), the results of the three forcing schemes should be the same as those of the

forcing scheme of Guo *et al.* since F_x is spatially uniform and there are no time-dependent error terms in Eqs. (49) and (50). Obviously, our numerical results confirm the latter.

B. Static flat and circular interfaces

In this section we consider the simulations of static flat and circular interfaces by the pseudopotential multiphase LB model proposed by Shan and Chen [6,7], which is very sensitive to the forcing scheme and the relaxation time [5]. The pseudopotential interaction force is given by [17]

$$\mathbf{F} = -G\psi(x) \sum_{\alpha} w_{\alpha} \psi(\mathbf{x} + \mathbf{e}_{\alpha} \delta_t) \mathbf{e}_{\alpha}, \quad (52)$$

where ψ is the pseudopotential, G is the interaction strength, and w_{α} are the weights. For the nearest-neighbor interactions on the D2Q9 lattice, the weights $w_{\alpha} = 1/3$ for $|\mathbf{e}_{\alpha}|^2 = 1$ and $w_{\alpha} = 1/12$ for $|\mathbf{e}_{\alpha}|^2 = 2$. The pseudopotential is taken as $\psi(\rho) = \psi_0 \exp(-\rho_0/\rho)$. With such a choice, the thermodynamic consistency can be satisfied when the macroscopic equations are correctly recovered.

The coexistence densities (liquid density ρ_L and vapor density ρ_V) are related to G , ψ_0 , and ρ_0 . In the present study we adopt $\psi_0 = 1$, $\rho_0 = 1$, and $G = -10/3$, which leads to $\rho_L \approx 2.783$ and $\rho_V \approx 0.3675$ [18]. The analytical coexistence densities obey the Maxwell equal-area rule (also called the Maxwell construction). The simulations are carried out in a rectangular domain and the periodic boundary condition is applied in both the x and y directions. For flat interfaces, the lattice system is chosen as $N_x \times N_y = 100 \times 100$ and the initial density is set to $\rho = \rho_L$ when $0.25N_x \leq x \leq 0.75N_x$ and $\rho = \rho_V$ when $x > 0.75N_x$ or $x < 0.25N_x$. The numerical coexistence densities are displayed in Table II, with the relaxation time being examined at $\tau = 0.7, 1.1$, and 1.5 .

According to Table II, we can see that the numerical coexistence densities obtained by Wagner's forcing scheme are identical to those given by the forcing scheme of Guo *et al.* Their results are in excellent agreement with the analytical solution of the Maxwell construction ($\rho_L \approx 2.783$ and $\rho_V \approx 0.3675$) and are independent of the relaxation time τ , demonstrating the theoretical analysis in Sec. III. For the forcing scheme of Kupershtokh *et al.*, the numerical results obviously deviate from the analytical solution, although the results are τ -independent. For Ladd and Verberg's forcing scheme, the numerical coexistence densities significantly deviate from the analytical solution and the results vary with τ . Specifically, the deviation increases with the increase of τ . The performances of these two forcing schemes confirm the error terms in Eqs. (49) and (50), i.e., the error term (at

TABLE I. Numerical results obtained by different forcing schemes at $t = 3T/4$.

Scheme	$y^* = 0$	$y^* = 0.15$	$y^* = 0.35$	$y^* = 0.49$	$y^* = 0.5$
Wagner [12]	0.0191349	0.0190741	0.0203897	0.00304959	4.11605×10^{-19}
Kupershtokh <i>et al.</i> [13]	0.0191349	0.0190741	0.0203897	0.00304959	4.11605×10^{-19}
Ladd and Verberg [10]	0.0191349	0.0190741	0.0203897	0.00304959	4.12082×10^{-19}
Guo [11]	0.0191349	0.0190741	0.0203897	0.00304959	4.10864×10^{-19}
Eq. (51)	0.0191013	0.0190449	0.0203724	0.00306534	0

TABLE II. Simulation of flat surfaces. Here ρ_L and ρ_V were obtained by different forcing schemes.

Scheme	$\tau = 0.7$		$\tau = 1.1$		$\tau = 1.5$	
	ρ_L	ρ_V	ρ_L	ρ_V	ρ_L	ρ_V
Guo <i>et al.</i> [11]	2.785	0.369	2.785	0.369	2.785	0.369
Wagner [12]	2.785	0.369	2.785	0.369	2.785	0.369
Kupershtokh <i>et al.</i> [13]	2.840	0.402	2.840	0.402	2.840	0.402
Ladd and Verberg [10]	2.657	0.302	2.440	0.238	2.190	0.184

the Navier-Stokes level) resulting from the forcing scheme of Kupershtokh *et al.* is independent of the relaxation time, while the error term produced by Ladd and Verberg's forcing scheme varies with τ . In the pseudopotential LB model, the coexistence densities are governed by the mechanical stability condition, which is changed when an additional (or error) term is introduced into the macroscopic equations [14].

For circular interfaces, the lattice system is taken as $N_x \times N_y = 200 \times 200$ and a droplet of radius $r = 50$ is initially placed in the center of the computational domain. In comparison with flat interfaces, the following two issues should be noted about circular interfaces. First, according to the Laplace law, there is a pressure difference between the liquid and vapor phases in the presence of curved interfaces. Therefore, the Maxwell equal-area rule, which is established when the liquid pressure is equal to the vapor pressure, is theoretically invalid for curved interfaces. Correspondingly, the liquid and vapor densities will deviate from the results obtained by the simulation of flat interfaces, in which $\Delta p = 0$. Details about this issue can be found in Refs. [19,20]. Second, the spurious currents in the simulation of flat interfaces are very small. However, for circular interfaces the spurious currents are enlarged by many orders of magnitude, arising from the anisotropy of the high-order terms of the interaction force [21,22]. Meanwhile, with the significant increase of the spurious currents, the velocity-dependent error terms beyond the Navier-Stokes level will also be activated.

Some of the high-order error terms involve the density derivatives and the relaxation time. Accordingly, in the cases of curved interfaces the relaxation time will affect the numerical results even there are no τ -dependent error terms at the Navier-Stokes level. To suppress such an influence of high-order error terms, a low-density ratio has been adopted in the present study. The numerical coexistence densities obtained from the simulation of circular interfaces are shown in Table III.

TABLE III. Simulation of circular interfaces. Here ρ_L and ρ_V were obtained by different forcing schemes.

Scheme	$\tau = 0.7$		$\tau = 1.1$		$\tau = 1.5$	
	ρ_L	ρ_V	ρ_L	ρ_V	ρ_L	ρ_V
Guo <i>et al.</i> [11]	2.793	0.366	2.800	0.371	2.806	0.374
Wagner [12]	2.793	0.366	2.800	0.371	2.806	0.374
Kupershtokh <i>et al.</i> [13]	2.856	0.403	2.862	0.406	2.864	0.408
Ladd and Verberg [10]	2.627	0.297	2.480	0.250	2.320	0.209

FIG. 2. Simulation of circular interfaces: steady-state density contours ($\tau = 0.7$) obtained by the forcing schemes proposed by (a) Wagner [12], (b) Kupershtokh *et al.* [13], and (c) Ladd [10].

Obviously, the forcing schemes of Wagner and Guo *et al.* still yield the same results. Nevertheless, the results are different from those in Table II due to the influence of the pressure difference (the Laplace law). Moreover, a slight variation with τ can be observed in the results. Similarly, the results of the forcing scheme of Kupershtokh *et al.* also vary slightly with the relaxation time. The reason has just been illustrated. With the increase of the density ratio, the variations of ρ_L and ρ_V with τ (or the viscosity) will be more severe. An alternative approach is using the multi-relaxation-time collision operator [23–25] to reduce the variations caused by the relaxation times since most of the relaxation times can be fixed when varying the viscosity.

Finally, the steady-state density contours obtained by different forcing schemes are compared in Fig. 2. The dashed circle denotes the theoretical location of the interface (the radius $r = 50$). Obviously, Wagner's forcing scheme performs best among the three forcing schemes. Meanwhile, it can be seen that the droplet radius predicted by the forcing scheme of Kupershtokh *et al.* is a little smaller than $r = 50$ and the droplet produced by Ladd and Verberg's forcing scheme is noncircular, which shows the influence of the error term on the size and shape of the droplet.

V. SUMMARY

We have clarified a critical issue about the Chapman-Enskog analysis for a class of forcing schemes in the LB method, in which the velocity in the equilibrium density distribution function is given by $\mathbf{u} = \sum_{\alpha} \mathbf{e}_{\alpha} f_{\alpha}/\rho$, but the actual fluid velocity is defined as $\hat{\mathbf{u}} = \mathbf{u} + \delta_t \mathbf{F}/(2\rho)$. In the available Chapman-Enskog analysis of this class of forcing scheme, $f_{\alpha}^{(0)}$ in Eq. (18) was taken as $f_{\alpha}^{\text{eq}}(\rho, \mathbf{u})$. In the present work, we demonstrated that the usual Chapman-Enskog analysis for this class of forcing schemes should be revised in order to derive the actual macroscopic equations recovered from these forcing schemes.

It was shown theoretically and numerically that the forcing schemes of Wagner and Guo *et al.* are equivalent at the Navier-Stokes level, which confirms the consistency between the Chapman-Enskog analysis and the Taylor expansion analysis. Using the revised Chapman-Enskog analysis, the forcing schemes proposed by Ladd and Verberg [10] and Kupershtokh *et al.* [13] have also been analyzed. It is believed that the present analysis is useful for clarifying some inconsistencies

in the literature and we hope that this study would be helpful for general readers to better understand the forcing schemes in the LB community. Finally, it is worth mentioning that the Chapman-Enskog analysis for the forcing scheme proposed by Shan *et al.* [7,26] should be conducted in a similar way, in which the velocity in the equilibrium distribution function is give by $\mathbf{u}^{\text{eq}} = \mathbf{u} + \tau \delta_t \mathbf{F} / \rho$ and the actual fluid velocity is the same as above [26].

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