Erratum: Self-averaging and ergodicity of subdiffusion in quenched random media [Phys. Rev. E 93, 010101(R) (2016)]

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In our paper we studied self-averaging and ergodicity for anomalous diffusion in quenched random media. We concluded that diffusion is both self-averaging and ergodic in $d \geqslant 2$ and non-self-averaging and nonergodic in $d < 2$. Although our main results regarding self-averaging remain unchanged, we revise here the statement on ergodicity, correct the calculation that led to it, and develop the correct results. The mean square displacement exhibits in fact weak ergodicity breaking, which is consistent with Refs. [1–4], whereas in our paper, we assumed that the results on the self-averaging of $m(t)$ are equally valid for the ergodicity of $m(t)$ based on the following line of arguments.

(i) The noise variance of $m_\Delta(t)$ in a single disorder realization is 0. (ii) This implies that $m_\Delta(t)$ can be represented by its noise average $\langle m_\Delta(t)\rangle$. (iii) Thus, the variance of $m_\Delta(t)$ with respect to its disorder average can be measured by the disorder variance of $\langle m_\Delta(t)\rangle$.

Since the relative variance of $\langle m_\Delta(t)\rangle$ with respect to the disorder average is equal to the relative disorder variance of $m(t)$, see Eq. (6) in our paper, we concluded that the results for the self-averaging of $m(t)$ are equally valid for the ergodicity of $m_\Delta(t)$ with respect to its disorder average. This line of arguments, however, breaks down at point (i): The noise variance of $m_\Delta(t)$ is actually nonzero. In order to see this, we start from Eq. (11) in the Supplemental Material that gives $m_\Delta(t)$ in a single disorder realization for $\Delta/t \ll 1$. We can write this expression in a weak sense as

$$m_\Delta(t) = \frac{2\kappa\, d\Delta}{t}\int_0^t dt'\theta[\mathbf{x}(t')]^{-1} \equiv \frac{2\kappa\, d\Delta}{t}s(t), \tag{1}$$

where we set $w_\Delta(t)^2 = \langle w_\Delta(t)^2\rangle = d\Delta$ and disregard contributions of order higher than linear in Δ. Note that $\boldsymbol{w}_\Delta(t)$ is defined by Eq. (10) and the noise mean of $m_\Delta(t)$ is given by Eq. (12) in the Supplemental Material. The noise mean square of $m_\Delta(t)$, given by Eq. (13) in the Supplemental Material, is not correct. The correct expression reads as

$$\langle m_\Delta(t)^2\rangle = \frac{4\kappa^2 d^2\Delta^2}{t^2}\left\langle\int_0^t dt'\theta[\mathbf{x}(t')]^{-1}\int_0^t dt''\theta[\mathbf{x}(t'')]^{-1}\right\rangle = \frac{4\kappa^2 d^2\Delta^2}{t^2}\langle s(t)^2\rangle, \tag{2}$$

where we disregard contributions of order higher than Δ^2. The noise variance of $m_\Delta(t)$ defined by $\sigma_\Delta^2(t) = \langle m_\Delta(t)^2\rangle - \langle m_\Delta(t)\rangle^2 \neq 0$ is nonzero for strong disorder unlike stated in our paper. In the following, we derive the correct results for the variance $\sigma_\Delta^2(t)$.

First, we note that $\sigma_\Delta^2(t)$ itself fluctuates between disorder realizations. However, we have shown in our paper that the ensemble average is asymptotically a good estimator for the noise average, at least, for the mean square displacement $m(t)$ in $d \geqslant 2$ dimensions because $m(t)$ is self-averaging. Thus, it follows from Eq. (12) in the Supplemental Material that also $\langle m_\Delta(t)\rangle$ is self-averaging. Based on this, we use the ensemble average $\overline{\sigma_\Delta^2(t)}$ as an estimator for $\sigma_\Delta^2(t)$ and the ensemble average $\overline{\langle m_\Delta(t)\rangle}$ as an estimator for $\langle m_\Delta(t)\rangle$ in $d \geqslant 2$. Using (1) and (2), we obtain

$$\overline{\sigma_\Delta^2(t)} = \frac{\Delta^2 d^2 \ell^4}{t^2}\left[\overline{\langle n_t^2\rangle} - \overline{\langle n_t\rangle^2}\right], \tag{3}$$

where we set $\langle s(t)\rangle = \ell^2/(2\kappa)\langle n_t\rangle$ and $\langle s(t)^2\rangle = \ell^4/(4\kappa^2)\langle n_t(n_t-1)\rangle \approx \ell^4/(4\kappa^2)\langle n_t^2\rangle$ [5]. We rewrite (3) in the form

$$\overline{\sigma_\Delta^2(t)} = \frac{\Delta^2 d^2\ell^4}{t^2}\left[\overline{\langle n_t^2\rangle} - \overline{\langle n_t\rangle}^2\right] - \frac{\Delta^2}{t^2}\left[\overline{m(t)^2} - \overline{m(t)}^2\right], \tag{4}$$

where we note that $m(t) = d\ell^2\langle n_t\rangle$, see Eq. (4) in the Supplemental Material. Notice that the first expression in square brackets denotes the disorder variance $\sigma_n^2(t)$ of the number of steps n_t to reach time t because $\overline{\langle n_t^k\rangle} = \overline{n_t^k}$ $(k = 1,2)$ is independent of the noise. The second term in square brackets denotes the disorder variance $\sigma_m^2(t)$ of the noise average mean square displacement $m(t)$, which we analyzed in the paper. Thus, we can restate (4) as

$$\overline{\sigma_\Delta^2(t)} = \frac{\Delta^2}{t^2}\left[d^2\ell^4\sigma_n^2(t) - \sigma_m^2(t)\right]. \tag{5}$$

This relation implies that at finite times $\sigma_n^2(t) \geqslant \sigma_m^2(t)/(d^2\ell^4) > 0$. Thus, in order to quantify $\overline{\sigma_\Delta^2(t)}$ for $d \geqslant 2$, we now focus on the variance $\sigma_n^2(t)$ of n_t.

We follow the methodology developed in our paper in order to determine explicit results for $\sigma_n^2(t)$. The disorder ensemble expectation $\overline{n_t}$ is encoded in $\overline{m}(t)$, see Eq. (31) in the Supplemental Material of our paper. The disorder ensemble expectation of

$\overline{n_t^2}$ is given by

$$\overline{n_t^2} = \sum_{n=0}^{\infty} n^2 \overline{\delta_{n,n_t}} = \sum_{n=0}^{\infty} n^2 \overline{\mathbb{I}(t_n \leqslant t < t_{n+1})}. \tag{6}$$

We evaluate the scaling of this sum by using expression (29) developed in the Supplemental Material of our paper for the average of the indicator function. For $d = 2$, we find that $\overline{n_t^2} \propto t^{2\beta} \ln(t)^{2-2\beta} \propto \overline{n_t}^2$. This implies that $\sigma_n^2(t) \propto t^{2\beta} \ln(t)^{2-2\beta}$ because at finite times $\sigma_n^2(t) > 0$. For $d > 2$, we obtain $\overline{n_t^2} \propto t^{2\beta} \propto \overline{n_t}^2$, which implies that $\sigma_n^2(t) \propto t^{2\beta}$. Thus, for $d \geqslant 2$, $\sigma_n^2(t) > \sigma_m^2(t)/(d^2\ell^4)$, compare to Eqs. (21) and (22) in our paper. This implies that

$$\overline{\sigma_\Delta^2(t)} = \frac{\Delta^2}{t^2} d^2 \ell^4 \sigma_n^2(t) + \cdots . \tag{7}$$

This gives for the ergodicity breaking parameter of Ref. [2],

$$\mathrm{EB} = \lim_{t\to\infty} \frac{\overline{\sigma_\Delta^2(t)}}{\overline{\langle m_\Delta(t)\rangle}^2} = \lim_{t\to\infty} \frac{\sigma_n^2(t)}{\overline{n_t}^2} \neq 0, \tag{8}$$

where we used $\overline{\langle m_\Delta(t)\rangle} = (\Delta/t) d\ell^2 \overline{n_t}$; EB is equal to a nonzero constant. This means that the mean square displacement exhibits weak ergodicity breaking in $d \geqslant 2$, which is consistent with Refs. [1–3]. Note that for $d < 2$, we find that $\sigma_n^2(t) \propto t^{4\beta/(2\beta - d\beta + d)} \propto \overline{m}(t)^2$, which indicates that $m(t)$ also shows weak ergodicity breaking in $d < 2$ in agreement with Refs. [3,4]. However, in this case we refrain from making a statement on ergodicity based on the disorder averages $\overline{\sigma_\Delta^2(t)}$ and $\overline{\langle m_\Delta(t)\rangle}$ because $m(t)$ and thus $\langle m_\Delta(t)\rangle$ are not self-averaging. In conclusion, unlike stated in our paper, the mean square displacement exhibits weak ergodicity breaking for $d \geqslant 2$.

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[5] Note that $s(t) = s_{n_t}$ in terms of the coarse grained particle trajectory (7) in the paper, whereas $s_n = \sum_{i=0}^{n-1} \hat{\tau}_i$ is Γ distributed with mean $n\ell^2/(2\kappa)$. Note also the $\hat{\tau}$ in Eq. (7) in the paper should read as $\hat{\tau}_n$.