Two-loop N -jettiness soft function for $pp \rightarrow 2j$ production

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We calculate the two-loop soft function for $pp \rightarrow jj$, $e^+e^- \rightarrow 4j$, or $ep \rightarrow 3j$ when the N -jettiness observable is measured. The result presented here makes up the necessary piece for realizing the full next-to-next-to-leading order predictions for these processes using the N -jettiness subtraction scheme.

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I. INTRODUCTION

The accumulation of the dataset and the continuously increasing precision of the experimental analyses bring the physics program at the Large Hadron Collider (LHC) and other experiments the ability to search for new physics from the small deviations of the data from the Standard Model predictions. In order to fulfill such searches, the precision of the theoretical predictions from Quantum Chromodynamics (QCD) has to match the small experimental errors to reliably interpret the collider data. Currently, the only theoretical tool, with no known substitutes, that enables us to make predictions for collider phenomenologies out of the first QCD principle is the perturbative calculation based on the systematic expansion of the strong coupling constant α_s or $\alpha_s L$ if large logarithmic correction L is present. In recent years, tremendous efforts have been made in realizing the theoretical predictions for the physical processes at the colliders with a full control of the final state kinematics beyond the next-to-leading order (NLO) accuracy. On the fixed order side, significant progress has been made in the past few years, which includes achieving the fully differential gluon-gluon fusion Higgs (H) production [1,2] and the single jet production in deep-inelastic scattering [3] at next-to-next-to-next-to-leading order (N^3 LO) in α_s and the next-to-next-to-leading order (NNLO) calculations of the single inclusive jet or dijet productions [4], $V/H + j$ processes [5–11], Higgs

production through the vector boson fusion [12,13], single top [14,15] and $t\bar{t}$ productions [16,17] at the LHC, the jet productions in deep-inelastic scattering [18,19], and the heavy flavor production in deep-inelastic neutrino scattering [20].

The rapid growth in the higher order predictions benefit from both the new higher loop calculations achieved and the theoretical schemes developed for dealing with the singularities in the real emissions. Recent achievements of the analytic two-loop five-point amplitude [21–23] highlight the current status of the loop calculations for the collider processes, and, meanwhile, new ideas keep on popping out [24–27]. To handle the infrared (IR) singularities in the real corrections, various schemes have been proposed and successfully applied to the LHC phenomenology, such as the local subtraction schemes including the antenna subtraction [28], the sector improved residue subtraction [29], and the nested soft-collinear subtraction [30], as well as the physical observable based global schemes like the q_T -subtraction [31], inclusive jet mass subtraction [32] and the N -jettiness subtraction [5,33]. Using the global schemes, people have realized many interesting higher order predictions for the physical processes at the hadron colliders including the numerically most challenging Higgs production at N^3 LO [1].

Among the global schemes, the N -jettiness scheme is the scheme being able to handle the IR singularities for generic jet processes at the hadron colliders. The scheme utilizes a threshold cutoff τ_{cut} in the N -jettiness event shape τ [34] to distinguish between the NNLO N jet (when $\tau < \tau_{\text{cut}}$) and the NLO $N + 1$ jet configurations ($\tau > \tau_{\text{cut}}$). Below the τ_{cut} , a Soft-Collinear-Effective-Theory [35–39] approach based on the factorization theorem [34],

$$d\sigma(\tau_{\text{cut}}) = BB \text{Tr}[S \cdot H] \prod_{i=1} J_i + \mathcal{O}(\tau_{\text{cut}}), \quad (1)$$

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is used to approximate the full QCD contribution. Here, B stands for the beam function due to the initial state energetic radiations, while J 's are the jet functions for the final state collinear radiations. The hard function H encodes the hard loop information. S is the soft function for soft radiations. The trace is over the relevant color space. All the ingredients in the factorization theorem are universal except for the hard function H which depends on the specific process under consideration. The difference between the factorization theorem and the full QCD is suppressed by power corrections in τ_{cut} , which have been studied extensively recently for the 0-jettiness case [40–44]. The universal beam and the jet functions are all known analytically to two-loops [45–48] and recently some of them are even calculated at three-loop order [49–51]. As for the quark beam function, its NNLO longitudinal polarized counterpart is also known in the literature [52]. The calculation of the soft function is complicated by its explicit dependence on the full N -jettiness measure; therefore, its analytic NNLO calculation is far out of reach, and people turn to numerical solutions. A generic framework was proposed in [53] for calculating such soft functions with arbitrary numbers of jets and the 1-jettiness soft functions in pp -collision were calculated using this scheme or a similar approach for both the massless jet [53,54] and the massive heavy flavor productions [55,56]. However, at NNLO, the most general calculations can only be covered when one considers the N -jettiness soft function with four external hard legs, which is missing in the original calculations. In this manuscript, we present such a calculation by extending the previously developed numerical method and show sufficient calculating details. We noticed a recent conference proceeding [57] on the 2-jettiness soft function using pySecDec. Our work uses a different and independent approach which can serve as a cross check of the calculations.

The rest of this work is organized as follows. We discuss the numerical setups and the phase space parametrization in detail in Sec. II. In Sec. III, we present our numerical results for the 2-jettiness observable in hadron-hadron collisions. Last we conclude in Sec. IV.

II. COMPUTATIONAL SETUP

In this section, we summarize the setups used in our calculation. We follow the general numerical framework proposed in [53] and show its capability in dealing with N -jettiness with four hard reference vectors in $pp \rightarrow jj$, $ep \rightarrow 3j$, and $0 \rightarrow 4j$. We note that the calculations with four reference vectors go through all possible configurations in calculating any N -jettiness soft functions at NNLO. We clarify some of the computational details of this framework.

For simplicity, we stick to the hadronic N -jettiness definition which reads

$$\tau = \sum_i \min(n_1 \cdot q_i, n_2 \cdot q_i, \dots, n_N \cdot q_i), \quad (2)$$

where n_i 's are the N lightlike hard reference vectors along the initial state hadronic beams or the final state jet axes, and q_i 's are the four momentum of the soft radiations. We note that here all the hard vector n_i 's are physical objects and are defined in 4-dimension while the soft momentum q_i 's can potentially be unresolvable and are therefore d -dimensional quantities, with $d = 4 - 2\epsilon$ in the dimensional regularization. The soft function with n real soft emissions then can be calculated by performing the phase space integration,

$$S(\tau) = \prod_i^n \int \frac{d^d q_i}{(2\pi)^{d-1}} \delta^+(q_i^2) \mathcal{S}(\{q_i\}) \Theta_{N,i}(\{q_i\}), \quad (3)$$

where we have introduced the N -jettiness measurement function,

$$\Theta_{N,i}(\{q_i\}) = \sum_{j=1}^N \delta\left(\tau - \sum_i^n n_{j_i} \cdot q_i\right) \prod_{k_i \neq j_i}^{N-1} \theta(n_{k_i} \cdot q_i - n_{j_i} \cdot q_i), \quad (4)$$

and $\mathcal{S}$ is the soft current matrix element for the soft radiations which, up to NNLO, has been known for a long time [58,59]. For N -jettiness with 3 reference vectors, to NNLO, the soft current can be expressed in terms of a sum of dipole contributions, while starting from four reference vectors, such as the case we study here, a new triple-pole structure involving 3 eikonal lines arises which is induced by the one-loop soft current. The triple-pole contribution, which will be calculated later in this work, is the only remaining piece for calculating the N -jettiness soft function with arbitrary external eikonal lines at $\mathcal{O}(\alpha_s^2)$. For completeness, we summarize the explicit form of the soft current $\mathcal{S}$ to $\mathcal{O}(\alpha_s^2)$ in the Appendix A 1.

In order to compute the soft function numerically, the key step is to find a suitable momentum parametrization to allow us to write the phase space integration in the form of

$$\begin{aligned} & \int_0^1 \prod_i dx_i x_i^{-1-a_i\epsilon} F(\{x_i\}) \\ &= \int_0^1 \prod_i dx_i \left[-\frac{1}{a_i\epsilon} \delta(x_i) + \left(\frac{1}{x_i}\right)_+ + \dots \right] F(\{x_i\}), \end{aligned} \quad (5)$$

which is ready for the Laurent expansion so as for us to identify all the ϵ -poles and evaluate the coefficients of the poles numerically. We thus now turn to the discussion of the phase space parametrization for single and double real emissions with four reference directions separately. We note that though we are focusing on the four reference vector case, the parametrization is general enough to be applicable directly to any N -jettiness computation at the

NNLO. In all cases below, we utilize the Sudakov decomposition to write an arbitrary lightlike four vector k^μ with respect to two of the reference vectors n_i and n_j as

$$k^\mu = \frac{n_j^\mu}{n_{ij}} k^+ + \frac{n_i^\mu}{n_{ij}} k^- + k_\perp^\mu, \quad \text{denoted as } k^\mu = (k^+, k^-, k_\perp^i), \quad (6)$$

with $k^+ = n_i \cdot k$, $k^- = n_j \cdot k$ and $k_\perp \cdot n_i = k_\perp \cdot n_j = 0$ and $|k_\perp|^2 = \frac{2}{n_{ij}} k^+ k^-$. Here we have introduced the notation $n_{ij} \equiv n_i \cdot n_j$.

A. Single real emission phase space parametrization

With 4 reference vectors, n_i , n_j , n_k , and n_l , at hand, for one real emission with momentum q_1^μ for computing the NLO soft function or the NNLO real-virtual contribution, we can parametrize all the involved vectors with respect to n_i and n_j , which are

$$\begin{aligned} n_k^\mu &= (n_k^+, n_k^-, 0, n_k^\perp; 0), \\ n_l^\mu &= (n_l^+, n_l^-, n_l^\perp s_{\phi_l}, n_l^\perp c_{\phi_l}; 0), \\ q_1^\mu &= (q_1^+, q_1^-, q_1^\perp s_{\phi_1} c_{\alpha_1}, q_1^\perp c_{\phi_1}; q_1^\perp s_{\phi_1} s_{\alpha_1}, 0), \end{aligned} \quad (7)$$

where we have used the freedom to choose the azimuthal angle of the vector n_k , and the additional ϵ -angles other than α_1 to be 0. Here $\alpha_1, \phi_l, \phi_1 \in [0, \pi]$ with

$$c_{\phi_l} = \frac{n_{ik} n_{jl} + n_{il} n_{jk} - n_{ij} n_{kl}}{2\sqrt{n_{ik} n_{il} n_{jk} n_{jl}}}. \quad (8)$$

Here the nonzero ϕ_l makes n_k and n_l span the azimuthal plane and therefore α_1 is necessary for parametrizing the d -dimensional momentum q_1 , since q_1 in general lies outside the plane n_k and n_l belong to. We note that since the reference vectors n_i 's are physical, for arbitrary numbers of the reference vectors, their “ $\perp$ ”-components will all be lying in one azimuthal plane, and thus Eq. (7) is the most general parametrization for computing the N -jettiness function with arbitrary N . For the processes similar to what we are considering here, the transverse components $n_k^\perp$ and $n_l^\perp$ have to align with each other ($\phi_l = 0$ or π) due to the momentum conservation, therefore one can simplify the parametrization by setting $\alpha_1 = 0$ which reduces to the parametrization used in [53,54] for computing 1-jettiness.

Although, in general, n_i and n_j can be chosen arbitrarily for performing the Sudakov decomposition, in order to correctly isolate the ϵ -poles, it is more useful to choose n_i so that $n_i \cdot q$ contributes to τ as set by $\Theta_{N,i}$ while $n_j \cdot q$ appears as one of the singular poles in the denominator of the soft current matrix $\mathcal{S}$. With this parametrization, for nonzero α , the phase space integration for single emission can be modified to

$$\begin{aligned} &\int \frac{d^d q_1}{(2\pi)^{d-1}} \delta^+(q_1^2) F(q_1) \delta(\tau - n_i \cdot q_1) \theta(n_j \cdot q_1 - n_i \cdot q_1) \\ &= \frac{\pi}{4} \left(\frac{2}{n_{ij}}\right)^{1-\epsilon} \left(\frac{1}{2\pi}\right)^{3-2\epsilon} \tau^{-1-2\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \\ &\times \int_0^1 dx_1 dx_2 dx_3 x_1^{-1+\epsilon} \frac{-\epsilon}{x_3^{1+\epsilon}} [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon} \\ &\times [\tau^2 x_1^{-1}] [F(x_3) + F(1-x_3)], \end{aligned} \quad (9)$$

where we have made the variable changes

$$q_1^- = \frac{\tau}{x_1}, \quad c_{\phi_1} = \pi x_2, \quad c_{\alpha_1} = 1 - 2x_3. \quad (10)$$

It can also be shown that for $\alpha_1 = 0$, one can further modify the phase space integration to make $\phi_1 \in [0, 2\pi]$ when taking into account the properties of the soft current and the N -jettiness observable, as has been done in [53].

Equation (9) is our starting point for calculating the single emission contributions to the soft function numerically and we list in the Appendix A 2 all the final forms of the phase space integrations suitable for direct numerical evaluation of the one emission contributions.

B. Double real emission phase space parametrization

Now we sketch the parametrization for the double real emission with momenta q_1 and q_2 at NNLO. We want to add on top of Eq. (7) the momentum q_2 parametrization, and, meanwhile, require that the dot product $q_1^\perp \cdot q_2^\perp$ only relies on the azimuthal angle between these two vectors, so as to use the nonlinear transformation [53] in a later stage for the purpose of pole isolation. To find the particular parametrization of q_2 , we first rotate Eq. (7) around the $n_k^\perp$ axis by α_1 to eliminate the α_1 dependence in q_1^μ , then we rotate in the azimuthal plane by ϕ_1 to align the x -axis with $q_1^\perp$ to further remove the dependence of ϕ_1 in the parametrization of q_1^μ . These rotational operations lead to the momentum representation

$$\begin{aligned} q_1^\mu &= (q_1^+, q_1^-, 0, q_1^\perp; 0), \\ n_k^\mu &= (n_k^+, n_k^-, -n_k^\perp s_{\phi_1}, n_k^\perp c_{\phi_1}; 0), \\ n_l^\mu &= (n_l^+, n_l^-, n_l^\perp s_{\phi_l - \phi_1}^{\alpha_1}, n_l^\perp c_{\phi_l - \phi_1}^{\alpha_1}; -n_l^\perp s_{\phi_l} s_{\alpha_1}, 0), \end{aligned} \quad (11)$$

where we introduced the notation $c_{\phi_l - \phi_1}^{\alpha_1} \equiv c_{\phi_l} c_{\phi_1} + s_{\phi_l} s_{\phi_1} c_{\alpha_1}$. It is now straightforward to see that to satisfy our requirements, q_2^μ can be parametrized (in this frame after rotation) as

$$q_2^\mu = (q_2^+, q_2^-, q_2^\perp s_{\phi_2} c_{\alpha_2}, q_2^\perp c_{\phi_2}; q_2^\perp s_{\phi_2} s_{\alpha_2} c_\beta, q_2^\perp s_{\phi_2} s_{\alpha_2} s_\beta, 0), \quad (12)$$

so that $q_1^\perp \cdot q_2^\perp = -|q_1^\perp| |q_2^\perp| c_\beta$. The β angle here is necessary to guarantee that the ϵ -components of q_1 and q_2 are not always aligned with each other. Due to the Lorentz invariance of the individual emission phase space measure, the rotational operations used to derive the

FIG. 1. Coefficients of the $\tilde{L}^n$ terms for the $\mathcal{O}(\alpha_s)$ 2-jettiness soft function in $pp \rightarrow jj$ production, normalized to $\frac{\alpha_s}{4\pi} T_i \cdot T_j$.

parametrization of q_2 do not introduce any complexity to the Jacobian of the phase space integration. We also note that the parametrization derived here is the most general for computing the double real contributions to any N -jettiness soft functions. As for the processes we are interested in, in this work, using the same momentum conservation for the single real emission case, one can set α_1 and β to 0 to simplify the phase space integration.

Within this parametrization, the dot product $q_1 \cdot q_2$ takes the form

$$q_1 \cdot q_2 = \frac{1}{n_{ij}} \left[q_1^+ q_2^- + q_1^- q_2^+ - 2\sqrt{q_1^+ q_2^- q_1^- q_2^+} c_{\phi_2} \right], \quad (13)$$

which generates a linear singularity in the double real soft current if q_1 being collinear to q_2 is possible, which prevents us from extracting the ϵ -poles using the Laurent expansion in Eq. (5). To extract the poles, the angle c_{ϕ_2} is further parametrized as $c_{\phi_2} = 1 - 2\eta$, with the nonlinear parametrization

$$\eta = \frac{\left(\sqrt{q_1^+ q_2^-} - \sqrt{q_1^- q_2^+} \right)^2 (1 - x')}{\left(\sqrt{q_1^+ q_2^-} - \sqrt{q_1^- q_2^+} \right)^2 + 4\sqrt{q_1^+ q_2^- q_1^- q_2^+} x'}, \quad (14)$$

$$x' = \sin^2\left(\frac{\pi}{2}x\right),$$

if the soft momenta q_1 and q_2 can be collinear to each other, and

$$\eta = x', \quad x' = \sin^2\left(\frac{\pi}{2}x\right), \quad (15)$$

if q_1 and q_2 can in no way be collinear as constrained by the jettiness measurement, for instance when $\tau = n_i \cdot q_1 + n_j \cdot q_2$.

Last we comment on the choice of n_i and n_j in the Sudakov decomposition. The same logic follows the single emission case. The reference vectors n_i and n_j are chosen if $n_i \cdot q_1$ and $n_j \cdot q_2$ contribute to τ , i.e., $\tau = n_i \cdot q_1 + n_j \cdot q_2$. Otherwise, if both $n_i \cdot q_1$ and $n_i \cdot q_2$ contribute to the jettiness observable, then n_i will be naturally picked as one reference axis for the decomposition, and, for convenience, n_j is chosen in the way that the double real soft current is singular as $n_j \cdot q_1 \rightarrow 0$ and/or $n_j \cdot q_2 \rightarrow 0$. Additional variable changes are needed to map all variables onto the regime $[0, 1]$ suitable for doing the Laurent expansion. The mapping is standard in the sector decomposition calculations and has been detailed in [54,56]. After the mapping, in the cases where one encounters the fractional power of the form $s^{-\frac{1}{2}+\epsilon}$ with $s \in [0, 1]$, a further variable transform $s = \sin^2(\frac{\pi}{2}x)$, with $x \in [0, 1]$, is then used.

FIG. 2. Coefficients of the $\tilde{L}^n$ with $n = 3, 2, 1, 0$ in $S_{12}^{(2),N_F}$ (left panel) and $S_{13}^{(2),N_F}$ (right panel).

III. NUMERICAL RESULTS

In this section, we present our main results. We highlight our results using the 2-jettiness soft function to $\mathcal{O}(\alpha_s^2)$ in $pp \rightarrow jj$. Other processes such as $e^+e^- \rightarrow 4j$ can be easily obtained by changing properly the values of λ_{ij} in Eq. (A4), without doing any additional integrations. The 2-jettiness soft function depends on four hard directions, which we denote as n_1, n_2, n_3 , and n_4 . We align n_1 and n_2 with the incoming beam axes in the $\pm z$ -directions, and let n_3 and n_4 lie along the outgoing jet directions. For the purpose of presenting, though not at all required in our calculation, we let the incoming partons carry exactly the same energy in the collision and show the 2-jettiness soft function as a function of n_{13} and we present the results in the Laplacian space $\int_0^\infty d\tau e^{-\tau z} \sigma(\tau)$.

We start from the NLO soft function. At NLO, the renormalized 2-jettiness soft function can be written as a sum of dipole terms

$$S^{(1)} = \frac{\alpha_s}{4\pi} \sum_{i \neq j} T_i \cdot T_j \left(4C_{ij}^{(1,-1)} \tilde{L}^2 + 4C_{ij}^{(1,0)} \tilde{L} + \left[2C_{ij}^{(1,1)} + \frac{2}{3} \pi^2 C_{ij}^{(1,-1)} \right] \right) \equiv \sum_{i \neq j} T_i \cdot T_j S_{ij}^{(1)}, \quad (16)$$

where $C_{ij}^{(1,i)}$ are the coefficients of the ϵ^i -poles returned by our calculation using the method for single emission as sketched before. The coefficients of the logarithmic terms $\tilde{L}^2$ and $\tilde{L}$ can be predicted by solving the renormalization group equation (RGE) of the soft function and, therefore, serve as checks of our numerical approach.

In Fig. 1, we show the NLO coefficients of the $\tilde{L}^n$ ($n = 2, 1, 0$) terms for dipole $S_{12}^{(1)}$ and $S_{13}^{(1)}$ as functions of n_{13} . These are the only independent terms in the special kinematics we have chosen. In both cases, we have normalized the results to $\frac{\alpha_s}{4\pi}$. The red, blue, and black dots represent the coefficients for $\tilde{L}^2, \tilde{L}^1$, and $\tilde{L}^0$ from our numerical setups, respectively. The solid lines are obtained by solving the RGE analytically. For the purpose of comparing RGE with our numerical calculations, we show the ratios of the numerical results to the RGE predictions¹ and the numerical error estimation in the lower panels for the $\tilde{L}^2$ and $\tilde{L}^1$ coefficients. From the plot, we can see that the difference between the analytic results and our numerical predictions are at the per-mil level and

¹For $i, j = 1, 2$, since the coefficient of the $\tilde{L}^1$ term vanishes, we show instead the difference between our numerical evaluation and the RGE result.

FIG. 3. Coefficients of the $\tilde{L}^n$ with $n = 3, 2, 1, 0$ in $S_{12}^{(2),C_A}$ (left panel) and $S_{13}^{(2),C_A}$ (right panel).

within the numerical error estimations. Hence, perfect agreements between these two calculations are found, which implies the validity of the computational framework at this order. We also present the normalized numerical uncertainties for the L^0 terms in the bottom panels of Fig. 1, which are also found to be small and at the subpercentage level.

At NNLO, other than the Abelian terms which can be obtained by exponentiating the NLO results, the soft function receives non-Abelian contributions from the

$q\bar{q}$ -emission, the gg -emission, the real-virtual corrections, and the renormalization. The real-virtual corrections can be further broken down to the dipole and the triple-pole contributions, and the rest are of the dipole forms. The summation of all the dipole contributions can be organized by the color factors C_A and $N_F T_R$, while the triple-pole real-virtual corrections can be massaged into one single color term $f_{abc} T_1^a T_2^b T_3^c$ with the help of the color charge conservation $\sum T_i = 0$. The final results of the NNLO soft function can therefore be written as

$$S^{(2)} = \frac{1}{4} \sum_{i \neq j, k \neq l} \{T_i \cdot T_j, T_k \cdot T_l\} S_{ij}^{(1)} S_{kl}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\sum_{i \neq j} T_i \cdot T_j [C_A S_{ij}^{(2),C_A} + N_F T_R S_{ij}^{(2),N_F}] + f_{abc} T_1^a T_2^b T_3^c S_{123}^{(2)} \right), \quad (17)$$

where all the NNLO dipoles $S_{ij}^{(2)}$ and triple-pole $S_{123}^{(2)}$ in the non-Abelian terms can be written as a series in terms of the logarithms $\tilde{L}$. And, once again, all the coefficients of those logarithmic terms $\tilde{L}^n$ with $n = 3, 2, 1$ can be predicted by RGE with the knowledge of the full NLO results obtained before.

The coefficients of the $N_F T_R$ terms for the dipoles $S_{12}^{(2)}$ and $S_{13}^{(2)}$ are shown in Fig. 2, which include the contributions from the $q\bar{q}$ emission and the N_F term in the renormalization.

The ratios to the RGE predictions along with the error estimations are given in the lower panels for the $\tilde{L}^3$, $\tilde{L}^2$, and $\tilde{L}^1$ coefficients. The differences are all at the subpercentage level and well covered by the numerical uncertainties. Perfect agreements are thus established between the numerical calculations of the $\tilde{L}^n$ coefficients for $n = 3, 2, 1$, as shown in colored dots and the RGE predictions in solid lines, which validates not only the NNLO computations of the N_F contributions but also the correctness of the nonlogarithmic

FIG. 4. Coefficients of the $\tilde{L}^n$ with $n = 3, 2, 1, 0$ in $S_{123}^{(2)}$ (left panel) and the ϵ -pole terms (right panel).

term in the NLO calculations. The NNLO nonlogarithmic prediction (in black dots) displayed here is not presented in the original NNLO N -jettiness soft paper [53] and is one of our main results of this work. The associated numerical errors are also shown in the bottom panels of Fig. 2 and found to be well under control. We note that the NNLO results here and below are also presented in a conference proceeding [57] in a different way with a different approach. A direct comparison with the results therein is currently not possible since no much numerical details other than final result plots are given, but we do find qualitative agreements when reading the plots shown in [57].

The same situation happens to the C_A term, as displayed in Fig. 3, in which we also compared $\tilde{L}^n$ coefficients from the numerical calculation against the results from the RGE to find no difference. We note that the appearing large error for the $\tilde{L}^1$ coefficient in the $i, j = 1, 3$ ratio plot is solely due to the reason that the coefficients are close to vanishing for the configurations we considered.

The nonlogarithmic coefficients are represented by the black dots.

Last, we give the prediction for the triple-pole term $S_{123}^{(2)}$, which arises firstly for four hard reference directions. Just like the dipole contributions, the logarithms in the triple-pole are still predictable from the RGE. The analytic

predictions are once again found to agree with our numerical results, as clearly seen in the left panel of Fig. 4. Other than that, we also compare the ϵ -pole terms without the $\tilde{L}$'s from our numerical calculations with the ones predicted by the RGEs and find satisfactory agreements. The results for the nonlogarithmic term (left panel in Fig. 4), as well as the term entirely unpredictable from RGE (right panel, normalized further to -2π), are indicated in black dots, which finalizes all the NNLO contributions to the 2-jettiness for the $pp \rightarrow jj$ process.

IV. CONCLUSIONS

In this work, we present the calculation of the N -jettiness soft function with four hard reference directions using the computational framework proposed in [53] using the approach of sector decomposition. We generalize the parametrizations used for the 1-jettiness calculation to the arbitrary N -jettiness case. We specifically calculated the NNLO N -jettiness soft function with four external hard legs, where the last new piece at this order, the triple-pole configuration, first arises. We managed to isolate all the singularities from the soft function integrands and reduce the computation to a set of numerical integrals which are ready to evaluate. We checked the numerically computed

logarithmic terms in the soft function against the predictions from the RGEs and found perfect agreements. The non-logarithmic terms which can not be obtained through the known RGEs are calculated directly in this work. We note that although in this work we present results for the four external hard leg case, the parametrization introduced is in its most general form and is applicable to arbitrary numbers of hard external legs, due to the fact that at NNLO we at most have 2 soft emissions being unresolvable and defined in d -dimension. When we go to higher hard multiplicities, only additional measurement functions originating from the definition of the N -jettiness observable are needed to be inserted into the phase space integration, but this does not give rise to any new computing structures and introduces no further complexity. We also note that the calculations of this work are performed on a small workstation with 40 CPU cores. To achieve the accuracy shown in this manuscript, roughly 45 sec is needed for one fixed value of n_{ij} . Given that the relatively stable tree level phase space evaluations compared with the double real phase space in a NNLO calculation, though not the most economic, we believe that the numerical evaluation of the soft function can be done on-the-fly in an event-by-event basis on super computing clusters. One can also do a two parameter fitting (e.g., in terms of n_{13} and n_{14} in $p_1 p_2 \rightarrow j_3 j_4$ collisions) of the soft function for practical applications. All in all, we expect the result obtained in this manuscript to find its quick applications in both the fixed NNLO calculations and achieve the parton shower matching [60] for the relevant processes.

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APPENDIX: INGREDIENTS FOR NUMERIC EVALUATION

1. Soft current

For completeness, we list all the soft current matrix elements here, which were first derived in the seminal works [58] and [59] by Catani and Grazzini.

The NLO soft current with one soft emission q is given by

$$\mathcal{S}^{(1)} = -(4\pi\alpha_s\mu^{2\epsilon}) \times 2 \times \sum_{i \neq j} T_i \cdot T_j \mathcal{S}_{ij}^{(1)}(q), \quad (\text{A1})$$

with the NLO dipole given by

$$\mathcal{S}_{ij}^{(1)}(q) = \frac{n_{ij}}{2n_i \cdot q n_j \cdot q}. \quad (\text{A2})$$

The NNLO real-virtual correction can be written as the sum of the dipole $\mathcal{S}_{\text{RV,dipole}}^{(2)}$ and triple-pole $\mathcal{S}_{\text{RV,tri}}^{(2)}$ contributions, with the dipole contribution being

$$\mathcal{S}_{\text{RV,dipole}}^{(2)} = \frac{(4\pi\alpha_s\mu^{2\epsilon})^2}{4\pi^2} \times \frac{(4\pi)^\epsilon}{\epsilon^2} \frac{\Gamma^4(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)} \times C_A \sum_{i \neq j} T_i \cdot T_j [\mathcal{S}_{ij}^{(1)}(q)]^{1+\epsilon}, \quad (\text{A3})$$

and the triple-pole term

$$\mathcal{S}_{\text{RV,tri}}^{(2)} = \frac{(4\pi\alpha_s\mu^{2\epsilon})^2}{4\pi^2} \times \frac{(4\pi)^\epsilon}{\epsilon^2} \frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} 2\sin(\pi\epsilon) \times \sum_{i \neq j \neq k} f_{abc} T_k^a T_i^b T_j^c (\lambda_{ij} - \lambda_{iq} - \lambda_{jq}) \mathcal{S}_{ki}(q) [\mathcal{S}_{ij}(q)]^\epsilon, \quad (\text{A4})$$

where $\lambda_{ij} = 1$ if i and j are both incoming or outgoing, otherwise $\lambda_{ij} = 0$.

The NNLO real-real emission is given by

$$\mathcal{S}_{q\bar{q}}^{(2)} = -\frac{1}{2} (4\pi\alpha_s\mu^{2\epsilon})^2 T_R N_F \sum_{i \neq j} T_i \cdot T_j \mathcal{J}_{ij}, \quad (\text{A5})$$

from the $q\bar{q}$ emission. To get this form we have used the color charge conservation $\sum_i T_i = 0$. Here,

$$\mathcal{J}_{ij} = \mathcal{I}_{ii} + \mathcal{I}_{jj} - 2\mathcal{I}_{ij}, \quad (\text{A6})$$

and

$$\mathcal{I}_{ij} = -\frac{2n_{ij} q_1 \cdot q_2 + n_i \cdot (q_1 - q_2) n_j \cdot (q_1 - q_2)}{2(q_1 \cdot q_2)^2 n_i \cdot (q_1 + q_2) n_j \cdot (q_1 + q_2)}, \quad (\text{A7})$$

where q_1 and q_2 are the momenta carried by the real emissions. We denote the term proportional to $2n_{ij} q_1 \cdot q_2$ as $\mathcal{I}_{ij}^{\text{II}}$ and the rest $\mathcal{I}_{ij}^{\text{I}}$.

And the gg emission gives the non-Abelian contribution

$$\mathcal{S}_{gg,\text{non-abe}}^{(2)} = -\frac{1}{2} (4\pi\alpha_s\mu^{2\epsilon})^2 C_A \sum_{i \neq j} T_i \cdot T_j \mathcal{T}_{ij}, \quad (\text{A8})$$

with

$$\mathcal{T}_{ij} = 2\mathcal{S}_{ij} - \mathcal{S}_{ii} - \mathcal{S}_{jj}, \quad (\text{A9})$$

where

$$\mathcal{S}_{ij} = \mathcal{S}_{ij}^{\text{s.o.}} - \frac{1}{2} \frac{n_i \cdot q_1 n_j \cdot q_2 + n_i \cdot q_2 n_j \cdot q_1}{n_i \cdot (q_1 + q_2) n_j \cdot (q_1 + q_2)} \mathcal{S}_{ij}^{\text{s.o.}} + (1-\epsilon) \mathcal{I}_{ij}^{\text{I}} + 2\mathcal{I}_{ij}^{\text{II}}, \quad (\text{A10})$$

and the current in the strongly ordering limit

$$\mathcal{S}_{ij}^{\text{s.o.}} = \frac{n_{ij}}{q_1 \cdot q_2} \left(\frac{1}{n_i \cdot q_1 n_j \cdot q_2} + \frac{1}{n_j \cdot q_1 n_i \cdot q_2} \right) - \frac{n_{ij}^2}{n_i \cdot q_1 n_i \cdot q_2 n_j \cdot q_1 n_j \cdot q_2}. \quad (\text{A11})$$

The Abelian piece for the gg double real emission is

$$\mathcal{S}_{gg, \text{Abelian}}^{(2)} = 4 \times (4\pi\alpha_s \mu^{2\epsilon})^2 \times \frac{1}{2} \sum_{i \neq j, k \neq l} \{T_i \cdot T_j, T_k \cdot T_l\} \mathcal{S}_{ij}^{(1)}(q_1) \mathcal{S}_{kl}^{(1)}(q_2). \quad (\text{A12})$$

2. Numerical integrations for single emission

Here we list all final numerical integrations for evaluating single emission contributions. We note again that though we only show the N -jettiness with four hard directions n_i, n_j, n_k , and n_l , the parametrization presented here is general enough for any N -jettiness soft functions. To extend to the case with more reference vectors, one simply insert more θ functions originating from the N -jettiness measurements. We use the notation $q||n_i$ to denote the separation of q and n_i , which is the smallest and thus $\tau = n_i \cdot q$.

a. Final forms for numerical integration at NLO

At NLO, the contribution can be written as a sum of dipoles. We assume that the soft momentum q is emitted from dipole ij . For the integrations shown below, we have extracted out an overall factor $-16\pi^2 (\frac{\alpha_s}{2\pi}) (\frac{e^{\gamma_E}}{4\pi})^\epsilon T_i \cdot T_j$.

(i) *Case 1: $q_1||n_i$.*

For $q_1||n_i$, the final form suitable for numerical integration is

$$I_{ij}^{i,(1)} = \frac{\pi}{4} \left(\frac{n_{ij}}{2} \right)^\epsilon \left(\frac{1}{2\pi} \right)^{3-2\epsilon} \tau^{-1-2\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{-1+\epsilon} \frac{-\epsilon}{x_3^{1+\epsilon}} [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon} \times [F_i^{ij}(x_3) + F_i^{ij}(1-x_3)], \quad (\text{A13})$$

where we have defined

$$F_i^{ij}(x_3) = \theta(A_{ij,k}(x_1, c_{\phi_1}) - x_1) \theta \times (A_{ij,l}(x_1, c_{\phi_1-\phi_1}) - x_1), \quad (\text{A14})$$

and

$$A_{ij,k}(x, c_\phi) = \frac{n_{ik}}{n_{ij}} + \frac{n_{jk}}{n_{ij}} x - \sqrt{\frac{2x}{n_{ij}}} n_k^\perp c_\phi. \quad (\text{A15})$$

As for the case in which $q_1||n_j$, the result can be obtained by switching i and j in Eq. (A13).

(ii) *Case 2: $q_1||n_k$.*

While for $q_1||n_k$, we choose n_k and n_i to be the reference vectors, to have $q_1^+ = n_k \cdot q_1, q_1^- = n_i \cdot q_1$. Following the same variable changes shown in Eq. (10), we get the final form for the numerical evaluation

$$I_{ij}^{k,(1)} = \frac{\pi}{4} \frac{n_{ij}}{n_{ik}} \left(\frac{n_{ik}}{2} \right)^\epsilon \left(\frac{1}{2\pi} \right)^{3-2\epsilon} \tau^{-1-2\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^\epsilon \frac{-\epsilon}{x_3^{1+\epsilon}} \frac{[2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{A_{ki,j}(x_1, c_{\phi_1})} \times [F_k^{ij}(x_3) + F_k^{ij}(1-x_3)], \quad (\text{A16})$$

where

$$F_k^{ij}(x_3) = \theta(A_{ki,j}(x_1, c_{\phi_1}) - x_1) \theta(A_{ki,l}(x_1, c_{\phi_1-\phi_1}) - x_1). \quad (\text{A17})$$

Again, we can switch k and l to obtain the $q_1||n_l$ contribution.

b. Final forms for 2-parton correlated real-virtual contribution

For the 2-parton correlated real-virtual contribution, we have for our numerical evaluation:

(i) *Case 1: $q_1||n_i$.*

$$I_{ij, \text{RV}}^{i,(2)} = \frac{\pi}{4} \left(\frac{n_{ij}}{2} \right)^{2\epsilon} \left(\frac{1}{2\pi} \right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{-1+2\epsilon} \frac{-\epsilon}{x_3^{1+\epsilon}} [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon} \times [F_i^{ij}(x_3) + F_i^{ij}(1-x_3)]. \quad (\text{A18})$$

(ii) *Case 2: $q_1||n_k$.*

$$I_{ij, \text{RV}}^{k,(2)} = \frac{\pi}{4} \frac{n_{ij}}{n_{ik}} \left(\frac{n_{ij} n_{ik}}{4} \right)^\epsilon \left(\frac{1}{2\pi} \right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{3\epsilon} \frac{-\epsilon}{x_3^{1+\epsilon}} \frac{[2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{[A_{ki,j}(x_1, c_{\phi_1})]^{1+\epsilon}} \times [F_k^{ij}(x_3) + F_k^{ij}(1-x_3)]. \quad (\text{A19})$$

Here we have normalized to

$$16\pi^2 \left(\frac{\alpha_s}{2\pi} \right)^2 \left(\frac{e^{\gamma_E}}{4\pi} \right)^{2\epsilon} \frac{(4\pi)^\epsilon}{\epsilon^2} \frac{\Gamma^4(1-\epsilon) \Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon) \Gamma(1+2\epsilon)} C_A T_i \cdot T_j. \quad (\text{A20})$$

The $q||n_j$ and $q||n_l$ cases can be easily obtained by switching properly the indices.

c. Final forms for 3-parton correlated real-virtual contribution

Now we turn to the triple-pole case in the real-virtual correction, in which the 3-parton correlated emission contributes. This configuration first arises in this case with four external legs. We assume the soft emission is from triple-pole ijk , and we normalize our results to

$$32\pi^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \frac{(4\pi)^\epsilon \Gamma^3(1-\epsilon) \Gamma^2(1+\epsilon)}{\epsilon^2 \Gamma(1-2\epsilon)} \times \sin(\pi\epsilon) f_{abc} T_k^a T_i^b T_j^c (\lambda_{ij} - \lambda_{iq} - \lambda_{jq}). \quad (\text{A21})$$

(i) *Case 1: $q_1||n_i$.*

$$I_{ijk}^{i,(2)} = \frac{\pi n_{ik}}{4 n_{ij}} \left(\frac{n_{ij}}{2}\right)^{2\epsilon} \left(\frac{1}{2\pi}\right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{-1+2\epsilon} \frac{-\epsilon [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{x_3^{1+\epsilon} A_{ij,k}(x_1, c_{\phi_1})} \times [F_i^{ij}(x_3) + F_i^{ij}(1-x_3)], \quad (\text{A22})$$

where F_i^{ij} has been defined before.

(ii) *Case 2: $q_1||n_j$.*

We let $q_1^+ = n_j \cdot q_1$, $q_1^- = n_i \cdot q_1$ to find

$$I_{ijk}^{j,(2)} = \frac{\pi n_{ik}}{4 n_{ij}} \left(\frac{n_{ij}}{2}\right)^{2\epsilon} \left(\frac{1}{2\pi}\right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{2\epsilon} \frac{-\epsilon [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{x_3^{1+\epsilon} A_{ji,k}(x_1, c_{\phi_1})} \times [F_j^{ij}(x_3) + F_j^{ij}(1-x_3)]. \quad (\text{A23})$$

(iii) *Case 3: $q_1||n_k$.*

We let $q_1^+ = n_k \cdot q_1$ and $q_1^- = n_l \cdot q_1$ to get

$$I_{ijk}^{k,(2)} = \frac{\pi n_{ik}}{4 n_{kl}} \left(\frac{n_{ij} n_{kl}}{4}\right)^\epsilon \left(\frac{1}{2\pi}\right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{-1+3\epsilon} \frac{-\epsilon [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{x_3^{1+\epsilon} [A_{kl,i}(x_1, c_{\phi_1})]^{1+\epsilon}} \times [F_k^{ij}(x_3) + F_k^{ij}(1-x_3)]. \quad (\text{A24})$$

Here we redefine $F_k^{ij}(x_3)$ as

$$F_k^{ij}(x_3) = \left[\frac{1}{A_{kl,j}(x_1, c_{\phi_j-\phi_1})} \right]^\epsilon \theta(A_{kl,i}(x_1, c_{\phi_1}) - x_1) \times \theta(A_{kl,j}(x_1, c_{\phi_j-\phi_1}) - x_1). \quad (\text{A25})$$

(iv) *Case 4: $q_1||n_l$.*

We let $q_1^+ = n_l \cdot q_1$ and $q_1^- = n_k \cdot q_1$ to have

$$I_{ijk}^{l,(2)} = \frac{\pi n_{ik}}{4 n_{kl}} \left(\frac{n_{ij} n_{kl}}{4}\right)^\epsilon \left(\frac{1}{2\pi}\right)^{3-2\epsilon} \tau^{-1-4\epsilon} \frac{2\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \times \int_0^1 dx_1 dx_2 dx_3 x_1^{3\epsilon} \frac{-\epsilon [2^2 s_{\phi_1}^2 (1-x_3)]^{-\epsilon}}{x_3^{1+\epsilon} [A_{lk,i}(x_1, c_{\phi_1})]^{1+\epsilon}} \times [F_l^{ij}(x_3) + F_l^{ij}(1-x_3)], \quad (\text{A26})$$

where

$$F_l^{ij}(x_3) = \left[\frac{1}{A_{lk,j}(x_1, c_{\phi_j-\phi_1})} \right]^\epsilon \theta(A_{lk,i}(x_1, c_{\phi_1}) - x_1) \times \theta(A_{lk,j}(x_1, c_{\phi_j-\phi_1}) - x_1). \quad (\text{A27})$$

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