Note on the Rényi entropy of 2D perturbed fermions

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(Received 5 March 2019; published 20 May 2019)

In this paper, we study the Rényi entropy of 2D massless free fermions perturbed by the $T\bar{T}$ term and the $J\bar{T}$ term. In particular, we compute the correlators in the first-order perturbation. Three cases—the vacuum state of an infinite size system with $T\bar{T}$ perturbation, the excited states of a finite size system with $T\bar{T}$ perturbation, and the vacuum state of an infinite size system with $J\bar{T}$ perturbation—are analyzed. We use the bosonization approach to study the perturbative expansions of Rényi entropy. In the bosonization language, the twist operator is known explicitly, with which the computation of correlators in the perturbative expansion can be simply performed. Moreover, we show that the $T\bar{T}$ and $J\bar{T}$ terms have a simple form and are similar to each other. For the first and third cases, we reproduce the known results for Rényi entropy using the bosonization method, while for the second case, we obtain new results for the excited states.

DOI: [10.1103/PhysRevD.99.106008](https://doi.org/10.1103/PhysRevD.99.106008)

I. INTRODUCTION

Quantum entanglement is a very important notion in characterizing the correlation between different subsystems in the quantum system, and it has become a topic of great interest in both high-energy physics and condensed matter physics during recent years. For a quantum system, the entanglement entropy S_A corresponding to a subsystem A is defined via the von Neumann entropy

$$S_A = -\text{Tr} \rho_A \ln \rho_A, \quad (1)$$

where ρ_A is the reduced density matrix of subsystem A , which is defined by tracing out the degrees of freedom in the complement of A , namely, $\bar{A}$, as

$$\rho_A = \text{Tr}_{\bar{A}} \rho. \quad (2)$$

The von Neumann entropy is not a unique way to describe quantum entanglement; it has been generalized into the so-called Rényi entropy as

$$S_A^{(n)} = -\frac{1}{n-1} \ln \text{Tr} \rho_A^n \quad (3)$$

with real parameter $n \geq 0$. The von Neumann entropy can be obtained by taking the limit $n \rightarrow 1$, namely,

$$S_A = \lim_{n \rightarrow 1} S_A^{(n)}. \quad (4)$$

Besides as a measure of entanglement, the Rényi entropy is important in its own right. For example, if $S_A^{(n)}$ is known for all n , then the spectral of the reduced density matrix ρ_A is known.

Nevertheless, the entanglement entropy or Rényi entropy is very difficult to calculate for the general quantum field theory (QFT). Different methods have been developed to compute them for a variety of systems among which free field theories or theories with additional symmetry, e.g., the conformal symmetry, are mostly concentrated. The early studies on entanglement entropy were done in the context of black hole physics with the aim to explain the possible microscopic origin of black hole Bekenstein-Hawking area entropy, where the entanglement entropy of a free boson was calculated by discretizing the field and the area law of entanglement entropy was found [1,2]. Later, the entanglement entropy for the 2D conformal field theory (CFT) was derived in Refs. [3,4] by employing the powerful conformal symmetry, and the analytic expression of entanglement entropy was obtained, in which a universal term was shown to depend on the central charge c of the CFT and the size of the subsystem. Meanwhile, the replica trick was also proposed to evaluate the entanglement entropy in Refs. [3,4]. For the 2D CFT, another important tool, the twist operator, was introduced in Ref. [5] to calculate the entanglement entropy and Rényi entropy. The twist operator is conformal primary with conformal dimension depending on central charge c ; this property allows people to obtain analytic results regarding the entanglement entropy and Rényi entropy, especially in the large c limit,

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in which cases the CFT might have classical gravity dual. Beside utilizing the twist operator, some other methods are also developed to calculate the entanglement entropy, such as the heat kernel method [6], the method involving the modular Hamiltonian [7,8], and so on.

In many cases, entanglement entropy and Rényi entropy are considered in a vacuum state. Thus, it is interesting to extend the study into finite temperature or excited states, either from the CFT side [9] or from the holographic perspective, e.g., the first-order excited states [10–12] and the second-order excited states [13], or theories away from the conformal point [14,15]. In this paper, we will employ the twist operator and bosonization approaches [16,17] to study the perturbation of Rényi entropy for 2D free fermions perturbed by two particular perturbations, i.e., the $T\bar{T}$ deformation as well as the $J\bar{T}$ deformation. These kind of deformations in the 2D CFT have attracted much attention recently, since they have remarkable properties from the viewpoints of both field theory and holography. Below, we will review some basic aspects of the $T\bar{T}$ -deformed QFT.

For any 2D Euclidean QFT with stress energy tensor $T_{\mu\nu}$, one can deform the theory by a composite operator $T\bar{T}(z)$, which is defined by examining a special operator-product expansion (OPE) below as [18]

$$\lim_{z \rightarrow z'} (T(z)\bar{T}(z') - \Theta(z)\bar{\Theta}(z')) \\ = \mathcal{O}_{T\bar{T}}(z') + \sum_i F_i(z - z') \partial_{z'} \mathcal{O}_i(z'), \quad (5)$$

where $T = -2\pi T_{zz}$, $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$, $\Theta = 2\pi T_{z\bar{z}} = 2\pi T_{\bar{z}z}$. This OPE is remarkable by virtue of the fact that the second term on the right-hand side of Eq. (5) involves only derivatives of operators, which leads one to define the operators $T\bar{T}(z) \equiv \mathcal{O}_{T\bar{T}}(z)$ by modulo those derivative terms. The deformed action $S(\lambda)$ is parametrized by a coupling constant λ :

$$\int d^2z T\bar{T}(z) = \frac{\partial S(\lambda)}{\partial \lambda}. \quad (6)$$

Note that the stress tensor defined by using $S(\lambda)$ also depends on λ implicitly. Actually, for the 2D integrable QFT, the $T\bar{T}$ deformation is a special case of an infinite set of integrable deformations, which means the deformed theory is still integrable in the sense that there are still an infinite number of local integrals of motion [19,20].

It is known that $T\bar{T}$ deformation has good properties from the point of view of the field theory. Also it will be useful in the study of the AdS/CFT correspondence [21]. When generalizing the holographic duality into more general cases, it is natural to consider deformations either from the bulk side or from the boundary CFT side. Many different kinds of deformations have been investigated so far. Recently, it was suggested that gravity with a finite cutoff in

the bulk is dual to the $T\bar{T}$ -deformed CFT [22]. From the renormalization group point of view, since the $T\bar{T}$ deformation is irrelevant, the UV properties of the field theory will be influenced, which indicates that the bulk geometry remains AdS in the interior while it changes asymptotically near the boundary. The suggestion in Ref. [22] was supported by the nontrivial facts that the energy spectrum of the deformed field theory are consistent with the prediction from the gravity side, and the propagating speed of small perturbations of the stress tensor is the same as that of metric perturbations with Dirichlet boundary conditions on the cutoff surface. Later on, further evidence from different aspects has been found in Refs. [23–27]. Especially the holographic entanglement entropy [28,29] was shown to match with the calculation from the dual $T\bar{T}$ -deformed CFT side [30,31] (see also [32–34]).

Motivated by the remarkable properties of $T\bar{T}$ perturbation, we will investigate the $T\bar{T}$ perturbation for free fermions by calculating the Rényi entropy for a vacuum state and, in particular, for excited states. We will employ the bosonization approach as mentioned before. As we will show, for free fermions this approach can also be applied to the $J\bar{T}$ perturbation, which is a generalization of $T\bar{T}$ deformation. The $J\bar{T}$ -deformed CFT also has an interesting holographic dual; for related discussions, see, e.g., Refs. [35–39].

In the remaining parts of the paper, we first introduce the bosonization method for calculating the Rényi entropy of free fermions in Sec. II. Then, in Sec. III, we consider the first-order perturbation for the Rényi entropy under the $T\bar{T}$ and the $J\bar{T}$ deformations, including both the vacuum state and excited states. Finally, we give a conclusion and discussions in Sec. IV.

II. THE BOSONIZATION APPROACH

In this section, we will introduce the basics of the approach to compute the Rényi entropy by using twist operators in the 2D CFT and the bosonization of free fermions [16,17]. Let the subsystem A be an interval $(0, x_0)$ in an infinite line and the whole system staying in the vacuum state $|0\rangle$; then the reduced density matrix of the subsystem A is

$$\rho_A = \text{Tr}_{\bar{A}} |0\rangle\langle 0|. \quad (7)$$

In the language of the path integral, the vacuum state can be expressed as

$$\psi[\phi_0(x)] = \int_{\phi(\tau=0,x)=\phi_0(x)} [d\phi] e^{-S_E[\phi]}, \quad (8)$$

where $S_E[\phi]$ is the Euclidean action of the field $\phi(x)$, and the path integral is performed in the lower half plane, i.e., $\tau < 0$, $-\infty < x < \infty$. It then follows that

$$\text{Tr} \rho_A^n = \frac{Z_n}{Z_1^n}, \quad Z_n = \int [d\phi]_{R_n} e^{-S_E[\phi]}, \quad (9)$$

where the integral domain is the n -sheeted Riemann surface denoted as R_n , with a branch cut along the interval A . On the other hand, one can view the path integral in Eq. (9) on R_n as n independent fields on a single sheet Riemann surface R_1 :

$$Z_n = \int [d\phi]_{R_1} e^{-S_E[\phi_1] - \dots - S_E[\phi_n]}. \quad (10)$$

In addition, the n fields should fulfill boundary conditions $\phi_i(x, 0^+) = \phi_{i+1}(0^-, x, \cdot)$, $x \in [0, x_0]$. By the cyclic Z_n symmetry, the boundary conditions can be implemented by introducing the twist operators and antitwist operators S_n and $\bar{S}_n$ sitting at two ending points $x_1 = (0, 0)$ and $x_2 = (0, x_0)$, respectively, as

$$S_n : \phi_i \rightarrow \phi_{i+1}, \quad \bar{S}_n : \phi_{i+1} \rightarrow \phi_i. \quad (11)$$

The (anti)twist operator is primary with conformal dimension $(\Delta_n, \bar{\Delta}_n)$:

$$\Delta_n = \bar{\Delta}_n = \frac{c}{24} \left(n - \frac{1}{n} \right). \quad (12)$$

With the twist operator inserted into Eq. (10), one obtains

$$\text{Tr} \rho_A^n = \langle S_n(0) \bar{S}_n(x_0) \rangle_{R_1} \sim \frac{1}{|x_0|^{4\Delta_n}}. \quad (13)$$

For general 2D CFTs, the explicit form of twist operators is unknown. However, the situation is better for free fermions in which the twist operator can be analytically expressed explicitly. Below, we will take this advantage to study the Rényi entropy for free fermions. To be more precisely, we consider the Dirac fermions whose action takes the form

$$S_0 = \int d^2z (\psi^* \partial \psi + \bar{\psi}^* \partial \bar{\psi}) \quad (14)$$

with ψ and $\bar{\psi}$ being the chiral and antichiral components of Dirac fermion, respectively:

$$\Psi(z, \bar{z}) = \begin{pmatrix} \psi(z) \\ \bar{\psi}(\bar{z}) \end{pmatrix}. \quad (15)$$

In order to use the replica trick to calculate the Rényi entropy for fermions, one first defines n copies of fields ψ_i , $i = 1, 2, \dots, n$, on which the twist operator can act. Second, consider the action of twist operators on fermions as the following:

$$S_n : \psi_i \rightarrow \psi_{i+1}, \quad \bar{S}_n : \bar{\psi}_{i+1} \rightarrow \bar{\psi}_i, \quad (16)$$

and the condition $\psi_{n+1} = (-1)^{n-1} \psi_1$ should be satisfied, which is modified slightly from Eq. (11) introduced previously. Furthermore, this action can be expressed more clearly in the matrix notation as [16,17]

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \dots \\ \vdots & 0 & 1 & \\ & \dots & & \\ & \dots & 0 & 1 \\ (-1)^{n-1} & & & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}. \quad (17)$$

This matrix can be diagonalized, and the corresponding eigenvectors are

$$\tilde{\psi}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^n \psi_j e^{2\pi i j k / n} \times \left(k = -\frac{n-1}{2}, -\frac{n-1}{2} + 1, \dots, \frac{n-1}{2} \right), \quad (18)$$

and the corresponding eigenvalues are $e^{-i2\pi k/n}$. Then the twist operator can be written as a production:

$$S_n = \prod_k s_k, \quad \bar{S}_n = \prod_k \bar{s}_k, \quad (19)$$

where for each eigenvector the action is

$$s_k : \tilde{\psi}_k \rightarrow e^{-i2\pi k/n} \tilde{\psi}_k. \quad (20)$$

The explicit form of s_k can be obtained via the bosonization method, which is [40]

$$s_k(z, \bar{z}) = e^{\frac{ik}{n}(H_k(z) - \bar{H}_k(\bar{z}))}, \quad \bar{s}_k(z, \bar{z}) = e^{-\frac{ik}{n}(H_k(z) - \bar{H}_k(\bar{z}))}. \quad (21)$$

Here $H_k(z)$ and its antichiral counterparts are scalar fields having the following correlator:

$$\langle H_{k_1}(z_1) H_{k_2}(z_2) \rangle = -\delta_{k_1 k_2} \log(z_1 - z_2). \quad (22)$$

Note that fields with different index k_i are uncorrelated. In addition to the twist operator, the fermion fields in the diagonal representation are bosonized as

$$\tilde{\psi}_k(z) = e^{iH_k(z)}, \quad \tilde{\psi}_k^*(z) = e^{-iH_k(z)}, \quad (23)$$

$$\bar{\tilde{\psi}}_k(z) = e^{-i\bar{H}_k(z)}, \quad \bar{\tilde{\psi}}_k^*(z) = e^{i\bar{H}_k(z)}. \quad (24)$$

From the bosonization representation, one can easily read the conformal dimension for each operator. For the fermion this is

$$\Delta(\tilde{\psi}_k) = \frac{1}{2}, \quad \bar{\Delta}(\tilde{\psi}_k) = 0, \quad (25)$$

$$\Delta(\tilde{\bar{\psi}}_k) = 0, \quad \bar{\Delta}(\tilde{\bar{\psi}}_k) = \frac{1}{2}. \quad (26)$$

And for the twist operator

$$\Delta(s_k) = \bar{\Delta}(s_k) = \frac{k^2}{2n^2}, \quad (27)$$

which satisfy

$$\Delta(S_n) = \sum_k \Delta(s_k). \quad (28)$$

III. RÉNYI ENTROPY IN DEFORMED CFT

In this section, we will consider the perturbations of Rényi entropy for free fermions when the $T\bar{T}$ and $T\bar{J}$ terms are presented.

A. $T\bar{T}$ perturbation for a vacuum state

The action of free fermions deformed by the $T\bar{T}$ term is

$$S_1 = S_0 + \lambda \int d^2z T\bar{T}, \quad (29)$$

where λ is a coupling constant and T and $\bar{T}$ are stress tensors related to the unperturbed conformal field theory S_0 . To calculate the Rényi entropy, from Eq. (9) we have

$$\text{Tr} \rho_A^n = \frac{Z_n}{Z_1^n}, \quad Z_n = \int_{R_n} [d\phi] e^{-S_0 - \lambda \int_{R_n} d^2z T\bar{T}}. \quad (30)$$

At the first-order perturbation in λ , we have

$$\text{Tr} \rho_A^n = \frac{Z_{(0)n}}{Z_{(0)1}^n} \left(1 - \lambda \int_{R_n} d^2z \langle T\bar{T} \rangle_{R_n} + n\lambda \int_{R_1} d^2z \langle T\bar{T} \rangle_{R_1} \right) \quad (31)$$

with

$$\begin{aligned} \int_{R_n} \langle T\bar{T} \rangle_{R_n} &= \frac{\int_{R_n} [d\phi] (\int_{R_n} d^2z T\bar{T}(z)) e^{-S_0}}{\int_{R_n} [d\phi] e^{-S_0}} \\ &= \frac{\sum_{i=1}^n \langle (\int_{R_1} d^2z T_i \bar{T}_i(z)) S_n(0) \bar{S}_n(x_0) \rangle_{R_1}}{\langle S_n(0) \bar{S}_n(x_0) \rangle_{R_1}}, \end{aligned} \quad (32)$$

where in the last step the path integral on R_n is equivalently performed on R_1 via introducing n copies of the original field and twist operator as discussed in the previous section. Note that the index i in $T_i \bar{T}_i$ is the replica index, and $T_i \bar{T}_i$ for each i is constructed from the i th copy of n fields. With Eq. (31), it is straightforward to obtain the variation of Rényi entropy to the first order in λ :

$$\delta S_A^{(n)} = \frac{\lambda}{n-1} \left(\frac{\sum_{i=1}^n \langle (\int_{R_1} d^2z T_i \bar{T}_i(z)) S_n(0) \bar{S}_n(x_0) \rangle_{R_1}}{\langle S_n(0) \bar{S}_n(x_0) \rangle_{R_1}} - n \int_{R_1} d^2z \langle T\bar{T} \rangle_{R_1} \right). \quad (33)$$

Next, we will calculate $\delta S_A^{(n)}$ for free fermions on the plane. In this case, the stress tensor for the action in Eq. (14) can be expressed as

$$T = \psi^* \partial \psi, \quad \bar{T} = \bar{\psi}^* \bar{\partial} \bar{\psi}. \quad (34)$$

Note that the last term in Eq. (33) vanishes, since T and $\bar{T}$ are factorized on the plane, and on plane one has

$$\langle T \rangle_{R_1} = \langle \bar{T} \rangle_{R_1} = 0. \quad (35)$$

Therefore, we need to calculate the three-point correlation function from the first term of Eq. (33). This was computed in Ref. [32] by using the Ward identity. Here we will apply the bosonization method to compute this correlator. First, we transform the perturbation into the diagonal representation by using

$$\sum_{i=1}^n T_i \bar{T}_i = \frac{1}{n} \sum_{k_1, k_2, k_3, k_4} \tilde{\psi}_{k_1}^* \partial \tilde{\psi}_{k_2} \tilde{\psi}_{k_3}^* \bar{\partial} \tilde{\psi}_{k_4} \delta_{k_1 - k_2 + k_3 - k_4, nm}. \quad (36)$$

Here $m = 0, \pm 1$, since $-2(n-1) \leq k_1 - k_2 + k_3 - k_4 \leq 2(n-1)$, and thus $(k_1 - k_2 + k_3 - k_4)$ can equal to $\pm n$ or zero provided $n > 1$. Furthermore, only terms with $k_1 = k_2$ and $k_3 = k_4$ will contribute to the three-point correlation function in Eq. (33), which can be seen by considering the following correlation function (assuming $k_1 \neq k_2$):

$$\langle \tilde{\psi}_{k_1}^*(z) \partial_z \tilde{\psi}_{k_2}(z) S_n(0) \bar{S}_n(x_0) \rangle. \quad (37)$$

Note that the OPE of $\tilde{\psi}_{k_1}^*$ with $\tilde{\psi}_{k_2}$ is regular, since they are independent fields. Thus, by substituting Eqs. (19), (21), and (23) into Eq. (37) and using the point-splitting method, one obtains

$$\begin{aligned} &\lim_{z_1 \rightarrow z} \partial_z \langle \tilde{\psi}_{k_1}^*(z_1) \tilde{\psi}_{k_2}(z) S_n(0) \bar{S}_n(x_0) \rangle \\ &= \lim_{z_1 \rightarrow z} \partial_z \langle e^{-iH_{k_1}(z_1)} e^{iH_{k_2}(z)} \prod_{k_3} : e^{\frac{ik_3}{n}(H_{k_3}(0) - \bar{H}_{k_3}(\bar{0}))} \\ &: \prod_{k_4} : e^{\frac{ik_4}{n}(H_{k_4}(x_0) - \bar{H}_{k_4}(\bar{x}_0))} : \rangle. \end{aligned} \quad (38)$$

When $k_1 \neq k_2$, the above correlator should vanish; otherwise, the neutrality condition for the vertex operator is violated. Following the same arguments, one can also find that $k_3 = k_4$ is necessary for the $T\bar{T}$ term to make a contribution to the correlation function. Therefore, one obtains

$$\begin{aligned} \sum_{i=1}^n T_i \bar{T}_i &= \frac{1}{n} \sum_{k_1, k_2, k_3, k_4} \tilde{\psi}_{k_1}^* \partial \tilde{\psi}_{k_2} \tilde{\psi}_{k_3}^* \bar{\partial} \tilde{\psi}_{k_4} \delta_{k_1-k_2+k_3-k_4, nm} \\ &\rightarrow \frac{1}{n} \sum_{k_1, k_2} \tilde{\psi}_{k_1}^* \partial \tilde{\psi}_{k_1} \tilde{\psi}_{k_2}^* \bar{\partial} \tilde{\psi}_{k_2} = \frac{1}{n} T_{\text{tot}} \bar{T}_{\text{tot}}, \end{aligned} \quad (39)$$

where the arrow “ $\rightarrow$ ” means the relation holds in the correlator and T_{tot} (similar for $\bar{T}_{\text{tot}}$) is the total stress tensor for n copies of fields:

$$T_{\text{tot}} = \sum_i T_i = \sum_k \tilde{\psi}_k^* \partial \tilde{\psi}_k. \quad (40)$$

This composite operator is singular and requires some normal ordering prescription; we adopt the following normal ordering:

$$\begin{aligned} T(z) &:: \psi^*(z) \partial_z \psi(z) := \lim_{z_1 \rightarrow z} \left(\frac{1}{2} (\psi^*(z_1) \partial_z \psi(z) \right. \\ &\quad \left. - \partial_{z_1} \psi^*(z_1) \psi(z)) - \frac{1}{(z_1 - z)^2} \right). \end{aligned} \quad (41)$$

In the bosonization language, the first term in the above expression is¹

$$\begin{aligned} \lim_{z_1 \rightarrow z} \psi^*(z_1) \partial_z \psi(z) &= \lim_{z_1 \rightarrow z} \partial_z [e^{-iH(z_1)} e^{iH(z)}] \\ &= \lim_{z_1 \rightarrow z} \partial_z \left(\frac{e^{-iH(z_1) + iH(z)}}{z_1 - z} \right) \\ &= \lim_{z_1 \rightarrow z} \left(\frac{e^{-iH(z_1) + iH(z)}}{(z_1 - z)^2} \right. \\ &\quad \left. + \frac{e^{-iH(z_1) + iH(z)} i \partial_z H(z)}{z_1 - z} \right) \\ &= \frac{1}{\epsilon^2} + \frac{1}{2} H'(z)^2 - \frac{i}{2} H''(z) \end{aligned} \quad (44)$$

with $\epsilon = z_1 - z$. Similarly, the second term is

¹Here the following formula for vertex operators [41] is used:

$$\begin{aligned} :e^{A_1}::e^{A_2}::\dots::e^{A_n} &::= e^{A_1+A_2+\dots+A_n} \\ &::= e^{A_1+A_2+\dots+A_n} : \exp \left(\sum_{i < j} \langle A_i A_j \rangle \right), \end{aligned} \quad (42)$$

where $::$ denote the normal ordering and $\langle \dots \rangle$ is the vacuum expectation value. In the main text, we suppressed the $::$ for the vertex operator for simplicity. From the above formula, it follows that on plane one has

$$\begin{aligned} :e^{iaH(z)}::e^{-iaH(w)} &::= :e^{iaH(z)-iaH(w)}: e^{a^2 \langle H(z)H(w) \rangle} \\ &= \frac{:e^{iaH(z)-iaH(w)}:}{z-w}. \end{aligned} \quad (43)$$

$$\lim_{z_1 \rightarrow z} \partial_{z_1} \psi^*(z_1) \psi(z) = -\frac{1}{\epsilon^2} - \frac{1}{2} H'(z)^2 - \frac{i}{2} H''(z). \quad (45)$$

Consequently, we obtain a simple form for the stress tensor

$$T_{\text{tot}}(z) = \sum_k :(\partial H_k(z))^2:, \quad (46)$$

and a similar expression for the antichiral part is

$$\bar{T}_{\text{tot}}(\bar{z}) = \sum_k :(\bar{\partial} \bar{H}_k(\bar{z}))^2:. \quad (47)$$

With those results in hand, we can calculate the three-point correlation function which contributes to the first-order perturbation for Rényi entropy in Eq. (33) as follows:

$$\begin{aligned} \sum_l \langle T_l(z) \bar{T}_l(\bar{z}) S_n(0) \bar{S}_n(x_0) \rangle \\ = \frac{1}{n} \sum_{k_1, k_2} \langle :(\partial H_{k_1})^2(z) : :(\bar{\partial} \bar{H}_{k_2})^2(\bar{z}) : \prod_{k_3} e^{\frac{ik_3}{n} (H_{k_3}(0) - \bar{H}_{k_3}(\bar{0}))} \\ \times \prod_{k_4} e^{-\frac{ik_4}{n} (H_{k_4}(x_0) - \bar{H}_{k_4}(\bar{x}_0))} \rangle. \end{aligned} \quad (48)$$

First, considering the chiral part which is denoted as A [terms involved $H(z)$, not $\bar{H}(\bar{z})$] in the correlator,

$$\begin{aligned} A &\equiv \sum_{k_1} \langle :(\partial H_{k_1}(z))^2 : \prod_{k_2} e^{\frac{ik_2}{n} H_{k_2}(0)} \prod_{k_3} e^{-\frac{ik_3}{n} H_{k_3}(x_0)} \rangle \\ &= \sum_{k_1} \langle :(\partial H_{k_1}(z))^2 : e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} \rangle \\ &\quad \times \prod_{k \neq k_1} \langle e^{\frac{ik}{n} H_k(0)} e^{-\frac{ik}{n} H_k(x_0)} \rangle \\ &= \sum_{k_1} \langle :(\partial H_{k_1}(z))^2 : e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} \rangle \prod_{k \neq k_1} \left(-\frac{1}{x_0} \right)^{\frac{k^2}{n^2}} \\ &= \sum_{k_1} \langle :(\partial H_{k_1}(z))^2 : e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} \rangle \prod_k \left(-\frac{1}{x_0} \right)^{\frac{k^2}{n^2}}, \end{aligned} \quad (49)$$

where Eq. (42) is used in the last step. Using the Wick theorem, the correlator in the last line can be calculated as

$$\begin{aligned} \partial H_{k_1}(z) &: e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x)} : \\ &= \frac{ik_1}{n} \partial_z [\langle H_{k_1}(z) (H_{k_1}(0) - H_{k_1}(x_0)) \rangle] : e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} : \\ &\quad + \text{normal ordering} \\ &= \frac{ik_1}{n} \frac{x_0}{z(z-x_0)} : e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} : + \text{normal ordering}. \end{aligned} \quad (50)$$

By iterating this procedure twice, it then follows that

$$\begin{aligned}
& :(\partial H_{k_1}(z))^2::e^{\frac{ik_1}{n}H_{k_1}(0)-\frac{ik_1}{n}H_{k_1}(x_0)}: \\
& = -\frac{k_1^2}{n^2} \frac{x_0^2}{z^2(z-x_0)^2} :e^{\frac{ik_1}{n}H_{k_1}(0)-\frac{ik_1}{n}H_{k_1}(x_0)}: + \text{normal ordering}.
\end{aligned} \tag{51}$$

Finally, substituting Eq. (51) into Eq. (49) yields

$$\begin{aligned}
A &= \prod_k \left(-\frac{1}{x_0} \right)^{\frac{k^2}{n^2}} \sum_{k_1} \left(-\frac{k_1^2}{n^2} \right) \frac{x_0^2}{z^2(z-x_0)^2} \\
&= -2\Delta_n \frac{x_0^2}{z^2(z-x_0)^2} \left(-\frac{1}{x_0} \right)^{2\Delta_n}.
\end{aligned} \tag{52}$$

Following the same steps as derived above for the chiral part of the correlator, one can obtain the result for the antichiral part denoted as $\bar{A}$, which turns out to be

$$\begin{aligned}
\bar{A} &\equiv \sum_{k_1} \left\langle :(\bar{\partial} \bar{H}(\bar{z}))^2: \prod_{k_2} e^{-\frac{ik_2}{n}\bar{H}_{k_2}(\bar{0})} \prod_{k_3} e^{\frac{ik_3}{n}\bar{H}_{k_3}(\bar{x}_0)} \right\rangle \\
&= -2\Delta_n \frac{\bar{x}_0^2}{(\bar{z}-\bar{x}_0)^2\bar{z}^2} \left(-\frac{1}{\bar{x}_0} \right)^{2\Delta_n}.
\end{aligned} \tag{53}$$

Combining the chiral and antichiral parts, we obtain the final result

$$\sum_l \langle T_l(z) \bar{T}_l(\bar{z}) S_n(0) \bar{S}_n(x_0) \rangle = \frac{1}{n} \frac{4\Delta_n^2}{|z-x_0|^4 |z|^4 |x_0|^{4\Delta_n-4}}, \tag{54}$$

which has the correct form for the three-point correlation function in the CFT. The integral of this function can be worked out by using Feynman parametrization and dimensional regularization²:

$$\begin{aligned}
&\int d^2z \sum_l \langle T_l(z) \bar{T}_l(\bar{z}) S_n(0) \bar{S}_n(x_0) \rangle \\
&= -\frac{64\Delta_n^2}{n\varepsilon} \frac{1}{|x_0|^{4\Delta_n+2}} + \text{finite terms},
\end{aligned} \tag{56}$$

where ε is a dimensionless cutoff. Substituting into Eq. (33) yields

²Here the following integral is used [23]:

$$\begin{aligned}
\int \frac{d^2x}{[(x-x_1)^2(x-x_2)^2]^2} &= -\frac{16\pi}{\varepsilon} \frac{1}{(x_1-x_2)^6} + \frac{4\pi}{(x_1-x_2)^6} \\
&\times (2\ln[\pi(x_1-x_2)^2] - 5 + 2\gamma) + O(\varepsilon),
\end{aligned} \tag{55}$$

where ε is the dimensionless cutoff introduced in dimensional regularization and γ is the Euler constant.

$$\delta S_A^{(n)} = -\frac{16\pi\lambda(n-1)}{n^3\varepsilon} \frac{1}{|x_0|^2} + \text{finite terms}. \tag{57}$$

Note that the finite terms vanish as $n \rightarrow 1$. Thus, the first-order entanglement entropy $\delta S_A^{(n=1)}$ equals zero, which is similar to the results in Ref. [31], where the first-order entanglement entropy is finite for the finite temperature case and vanishes for a finite size system. Meanwhile, for both cases in Ref. [31], the first-order Rényi entropy is divergent.³

Until now, we have discussed the case where the system is of infinite range and stays in its vacuum state. Along the same lines, one can consider the system with a finite size or finite temperature, in which the correlators on the cylinder are involved. We will not discuss these two cases explicitly in the present paper. Instead, we would like to consider a related situation involving excited states with a finite size by applying the bosonization approach.

B. $T\bar{T}$ perturbation for an excited state

In this subsection, we will analyze the Rényi entropy of excited states for a system with a finite size. For a finite size and zero temperature, the Rényi entropy of the general CFT with $T\bar{T}$ deformation has been considered in Ref. [31] as mentioned above. The case we considered here is concentrated on free fermions. As in the previous subsection, we will show that the bosonization method is still applicable.

Following Ref. [44], the excited states on the cylinder can be prepared as

$$|\Upsilon\rangle = \lim_{\xi, \bar{\xi} \rightarrow -\infty} \Upsilon(\xi, \bar{\xi})|0\rangle, \tag{59}$$

where $\xi = t + i\sigma(\sigma \sim \sigma + 2\pi, -\infty < t < \infty)$ is the coordinate on the infinite cylinder and $\Upsilon(\xi, \bar{\xi})$ is a local operator inserted at an infinite past to create an incoming excited state. Let the subsystem A be the region $\alpha < \sigma < \beta$, $t = 0$, and then the corresponding reduced density matrix is

$$\rho_A = \text{Tr}_{\bar{A}} |\Upsilon\rangle\langle\Upsilon|. \tag{60}$$

³It is interesting to note that, for a massive 2D fermion field whose mass term can be treated as a relevant perturbation, the entanglement entropy (EE) is [42]

$$S = \frac{1}{3} \ln \frac{L}{\varepsilon} - \frac{1}{6} (mL \ln mL)^2 + O((mL)^2 \log mL). \tag{58}$$

In this case, the divergence does not modify by the mass term. However, in higher dimension $d > 2$, when mass is turned on, the EE contains a new divergent term $\sim m^{d-2} \ln m\varepsilon$ [6,43]. Thus, the relevant mass perturbation modified the divergent behavior of EE in the massless theory.

As before, to compute the Rényi entropy, first, we have

$$\text{Tr} \rho_A^n = \lim_{\xi, \bar{\xi} \rightarrow -\infty} \frac{\int_{C_n} D\phi \prod_{i=1}^n \Upsilon_i^\dagger(-\xi) \Upsilon_i(\xi) e^{-S_1}}{[\int_{C_1} D\phi \Upsilon^\dagger(-\xi) \Upsilon(\xi) e^{-S_1}]^n}, \quad (61)$$

where S_1 is the perturbed action (29) and C_n is n copies of the cylinder pasted along the interval A in a cyclic order. Then keeping the first order of λ in the variance of the Rényi entropy between the perturbed and unperturbed excited states,

$$\delta S_A^{(n)} = \frac{\lambda}{n-1} \lim_{\xi, \bar{\xi} \rightarrow -\infty} \left(\lambda \frac{\int_{C_n} d^2 z \langle T \bar{T}(z) \prod_{i=1}^n \Upsilon_i^\dagger(-\xi) \Upsilon_i(\xi) \rangle_{C_n}}{\langle \prod_{i=1}^n \Upsilon_i^\dagger(-\xi) \Upsilon_i(\xi) \rangle_{C_n}} - n \lambda \frac{\int_{C_1} d^2 z \langle T \bar{T}(z) \Upsilon^\dagger(-\xi) \Upsilon(\xi) \rangle_{C_1}}{\langle \Upsilon^\dagger(-\xi) \Upsilon(\xi) \rangle_{C_1}} \right), \quad (62)$$

which reduces to Eq. (33) when Υ is the identity operator.

In the following, we will choose $\Upsilon(z) = \psi(z)$, in which case the bosonization method can be used and we can directly bosonize the field $\psi(z)$. The vacuum state may also be excited by other operators constructed from the fermion fields $\psi(z)$, e.g., the operator $\bar{\Psi}\Psi = \bar{\psi}^* \psi + \psi^* \bar{\psi}$ as discussed in Refs. [45,46]. In addition, for free fermions, $\delta S_A^{(n)}$ with all integer n can, in principle, be calculated; for simplicity, we consider only the second Rényi entropy ($n=2$) and will show that the bosonization formula is still applicable and can simplify the calculation.

To compute $\delta S_A^{(n=2)}$, let us map the n copies of cylinder C_n to n copies of planes denoted as R_n via the exponential map [44]

$$z = e^\xi. \quad (63)$$

Here, R_n , $v = e^{i\beta}$ is pasted along the interval on a unit circle with ending points $u = e^{i\alpha}$ on each plane, which is different from the situations in the previous section. Under the exponential map, the stress tensor of the field is transformed via⁴

$$T(\xi) = z^2 T^{(p)}(z) - \frac{c}{24},$$

$$\psi(\xi) = \left(\frac{\partial z}{\partial \xi} \right)^{\frac{1}{2}} \psi^{(p)}(z) = z^{\frac{1}{2}} \psi^{(p)}(z), \quad (66)$$

where $c=1$ is the central charge of the underlying Dirac fermion and the index (p) denotes the quantities defined on the plane. The antichiral counterparts have similar expressions. As for the Rényi entropy, notice that the denominator in the first term of Eq. (62) is the most complicated correlator in $\delta S_A^{(n)}$ which can be rewritten as

$$\lim_{\xi, \bar{\xi} \rightarrow -\infty} \int_{C_n} d^2 \xi \left\langle T \bar{T}(\xi) \prod_{i=1}^n \psi_i^\dagger(-\xi) \psi_i(\xi) \right\rangle_{C_n}$$

$$= \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \int_{R_n} \frac{d^2 z}{z \bar{z}} \left\langle \left(z^2 T^{(p)}(z) - \frac{c}{24} \right) \right.$$

$$\left. \times \left(\bar{z}^2 \bar{T}^{(p)}(\bar{z}) - \frac{c}{24} \right) \prod_{i=1}^n \psi_i^{(p)*}(w_2) \psi_i^{(p)}(w_1) \right\rangle_{R_n}, \quad (67)$$

where $w_1 = 1/w_2$ and the prefactor w_2^n comes from the definition Eq. (65). Expanding the integrand and focusing first on the correlator involving both $T^{(p)}$ and $\bar{T}^{(p)}$, the remaining correlators in the integrand can be evaluated in a similar manner:

$$\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \int_{R_n} d^2 z |z|^2 \left\langle T^{(p)} \bar{T}^{(p)}(z) \prod_{i=1}^n \psi_i^{(p)*}(w_1) \psi_i^{(p)}(w_2) \right\rangle_{R_n}$$

$$= \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \int_{R_1} d^2 z |z|^2 \sum_{i=1}^n \left\langle T_i^{(p)} \bar{T}_i^{(p)}(z) \prod_{i=1}^n \psi_i^{(p)*}(w_1) \psi_i^{(p)}(w_2) S_n(u) \bar{S}_n(v) \right\rangle_{R_1}, \quad (68)$$

⁴The Hermitian conjugate has a nontrivial action on coordinates on the cylinder with Euclidean time t as $t \rightarrow -t$, $\sigma \rightarrow \sigma$. By an exponential map, this leads to coordinates on the plane transforming under the Hermitian conjugate as $z \rightarrow \frac{1}{z^*}$. Therefore,

$$[\psi(\xi)]^\dagger = [z^{\frac{1}{2}} \psi^{(p)}(z)]^\dagger = \left(\frac{1}{z^*} \right)^{\frac{3}{2}} \psi^{(p)*} \left(\frac{1}{z^*} \right), \quad (64)$$

where in the last step we have used the definition

$$[\psi^{(p)}(z)]^\dagger = \left(\frac{1}{z^*} \right) \psi^{(p)*} \left(\frac{1}{z^*} \right). \quad (65)$$

where the correlator evaluated on n -sheet plane R_n is written equivalently on a single plane R_1 with a twist operator inserted. Note that the index i in the first line represents the i th sheet where the operator is inserted in, while the index i in the second line means the i th copy of n fields [we will omit the index (p) hereafter, since only quantities on the plane are needed in what follows]. Now the correlator in the second line of the above equation is similar to the one in Eq. (54) and can also be evaluated by using the bosonization method. After transforming the fermion field into a diagonal representation by employing Eq. (18), we obtain (for $n = 2$)

$$\begin{aligned}\psi_1(z) &= -\frac{i}{\sqrt{2}}(\tilde{\psi}_+(z) - \tilde{\psi}_-(z)), \\ \psi_2(z) &= -\frac{1}{\sqrt{2}}(\tilde{\psi}_+(z) + \tilde{\psi}_-(z))\end{aligned}\quad (69)$$

with the abbreviated notation $\tilde{\psi}_\pm \equiv \tilde{\psi}_{\pm 1/2}$. Subsequently, the product of fermion fields in the last line of Eq. (68) becomes

$$\begin{aligned}\prod_{i=1}^2 \psi_i(z) &= \frac{i}{2}(\tilde{\psi}_+^2(z) - \tilde{\psi}_-^2(z) + 2\tilde{\psi}_+(z)\tilde{\psi}_-(z)) \\ &= i\tilde{\psi}_+(z)\tilde{\psi}_-(z),\end{aligned}\quad (70)$$

where in the first step the anticommutative property for fermionic fields is utilized and in the second step the expression is simplified because the terms $\tilde{\psi}_-^2(z)$ and $\tilde{\psi}_+^2(z)$ vanish, which can be seen from the vanishing OPE as z approaching w :

$$\begin{aligned}e^{iH_\pm(z)}e^{iH_\pm(w)} &\sim e^{i(H_\pm(z)+H_\pm(w))}(z-w), \\ \tilde{\psi}_\pm(z) &= e^{iH_\pm(z)}.\end{aligned}\quad (71)$$

Thus, in terms of boson fields $H_\pm(z)$ we obtain

$$\prod_{i=1}^2 \psi_i(w_1) = ie^{iH_+(w_1)}e^{iH_-(w_1)}\quad (72)$$

and for Hermitian conjugate fields

$$\prod_{i=1}^2 \psi_i^*(w_2) = -ie^{-iH_+(w_2)}e^{-iH_-(w_2)}.\quad (73)$$

In the present case, the $T\bar{T}$ term in the last line of Eq. (68) still satisfies Eq. (39) for the same reason as has been discussed there. Thus, the calculation can be simplified by applying the bosonized form of the stress tensor in Eq. (46). Combining the above expressions with twist operators in Eq. (21), we can now compute the correlator in Eq. (68) as

$$\begin{aligned}\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^2 \sum_{j=1}^{n=2} \left\langle T_j \bar{T}_j(z) \prod_{i=1}^{n=2} \psi_i^*(w_2) \psi_i(w_1) S_n(u) \bar{S}_n(v) \right\rangle_{R_1} \\ \equiv \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^2 B \bar{B},\end{aligned}\quad (74)$$

where the chiral and antichiral parts have been denoted as B and $\bar{B}$, respectively. The antichiral part $\bar{B}$ has the same form as that of Eq. (54), since the excitation of chiral fields $\psi(z)$ will affect only the chiral part of the correlator while keeping the antichiral part unchanged. The explicit expression of $\bar{B}$ is

$$\bar{B} = -\frac{1}{8} \left(\frac{\bar{u} - \bar{v}}{(\bar{z} - \bar{u})(\bar{z} - \bar{v})} \right)^2 \left(\frac{1}{(\bar{u} - \bar{v})} \right)^{\frac{1}{8}},\quad (75)$$

and the chiral part B can be written as

$$B \equiv \sum_k \left\langle (\partial H_k(z))^2 : e^{-iH_+(w_2)} e^{-iH_-(w_2)} e^{iH_+(w_1)} e^{iH_-(w_1)} \prod_{k_1} e^{\frac{ik_1}{n} H_{k_1}(u)} \prod_{k_1} e^{-\frac{ik_2}{n} H_{k_2}(v)} \right\rangle,\quad (76)$$

which can be further decomposed into the sum of the following two terms:

$$B_1 \equiv \langle (\partial H_+(z))^2 e^{-iH_+(w_2)} e^{iH_+(w_1)} e^{\frac{i}{4}H_+(u)} e^{-\frac{i}{4}H_+(v)} \rangle \langle e^{-iH_-(w_2)} e^{iH_-(w_1)} e^{-\frac{i}{4}H_-(u)} e^{\frac{i}{4}H_-(v)} \rangle,\quad (77)$$

$$B_2 \equiv \langle e^{-iH_+(w_2)} e^{iH_+(w_1)} e^{\frac{i}{4}H_+(u)} e^{-\frac{i}{4}H_+(v)} \rangle \langle (\partial H_-(z))^2 e^{-iH_-(w_2)} e^{iH_-(w_1)} e^{-\frac{i}{4}H_-(u)} e^{\frac{i}{4}H_-(v)} \rangle.\quad (78)$$

By using a similar normal ordering as in Eq. (51), a straightforward calculation yields

$$B_1 = -\left(\frac{w_1 - w_2}{(z - w_1)(z - w_2)} + \frac{u - v}{4(z - u)(z - v)} \right)^2 \left(\frac{1}{w_2 - w_1} \right)^2 \left(\frac{1}{u - v} \right)^{\frac{1}{8}},\quad (79)$$

$$B_2 = -\left(\frac{w_1 - w_2}{(z - w_1)(z - w_2)} - \frac{u - v}{4(z - u)(z - v)} \right)^2 \left(\frac{1}{w_2 - w_1} \right)^2 \left(\frac{1}{u - v} \right)^{\frac{1}{8}}.\quad (80)$$

Consequently, the chiral part is

$$B = B_1 + B_2 = -2 \left(\frac{1}{u-v} \right)^{\frac{1}{8}} \left(\frac{1}{w_2 - w_1} \right)^2 \left[\left(\frac{w_1 - w_2}{(z - w_1)(z - w_2)} \right)^2 + \left(\frac{u - v}{4(z - u)(z - v)} \right)^2 \right]. \quad (81)$$

Taking the limit $w_1 \rightarrow 0$, $w_2 \rightarrow \infty$, we have

$$\lim_{\substack{w_1 \rightarrow \infty \\ w_2 \rightarrow 0}} w_2^2 B = -2 \left(\frac{1}{u-v} \right)^{\frac{1}{8}} \left[\frac{1}{z^2} + \left(\frac{u-v}{4(z-u)(z-v)} \right)^2 \right], \quad (82)$$

where the second term is just the Hermitian conjugate of $\bar{B}$ in Eq. (75) and is presented whether the whole state is vacuum or excited, while the first term appears due to the excitation. So far, we have obtained the correlator in Eq. (68) which belongs to one of four correlators in Eq. (67). The remaining three correlators can be calculated following the same procedures; we just list their results as follows (for $n = 2$):

$$\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \sum_{j=1}^n \left\langle T_j \prod_{i=1}^n \psi_i^*(w_1) \psi_i(w_2) S_n(u) \bar{S}_n(v) \right\rangle_{R_1} = \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} \frac{w_2^2 B}{(\bar{u} - \bar{v})^{\frac{1}{8}}}, \quad (83)$$

$$\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \sum_{j=1}^n \left\langle \bar{T}_j \prod_{i=1}^n \psi_i^*(w_1) \psi_i(w_2) S_n(u) \bar{S}_n(v) \right\rangle_{R_1} = \frac{\bar{B}}{(u - v)^{\frac{1}{8}}}, \quad (84)$$

$$\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow 0}} w_2^n \left\langle \prod_{i=1}^n \psi_i^*(w_1) \psi_i(w_2) S_n(u) \bar{S}_n(v) \right\rangle_{R_1} = \frac{1}{|u - v|^{\frac{1}{4}}}. \quad (85)$$

Next, let us consider the denominator of the first term of $\delta S_A^{(n)}$ in Eq. (62). By doing similar calculations as above, the results for $n = 2$ turn out to be

$$\lim_{\xi, \bar{\xi} \rightarrow -\infty} \left\langle \prod_{i=1}^n \Upsilon_i^\dagger(-\xi) \Upsilon_i(\xi) \right\rangle_{C_n} = \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2^n \left\langle \prod_{i=1}^n \psi_i^*(w_1) \psi_i(w_2) S_n(u) \bar{S}_n(v) \right\rangle_{R_1} = \frac{1}{|u - v|^{\frac{1}{4}}}, \quad (86)$$

which is just equal to Eq. (85). The remaining correlators in $\delta S_A^{(n)}$, i.e., the second term in Eq. (62), are evaluated on a single cylinder C_1 ; thus, no twist operators are needed. To compute these correlators, we first map the cylinder C_1 to plane R_1 by the exponential map Eq. (63), and then we directly make a bosonization for these fields without performing the Fourier transformation, since there is only one copy of the field. After performing these procedures, we find the denominator is equal to one, and the nominator is

$$\begin{aligned} & \lim_{\xi, \bar{\xi} \rightarrow -\infty} \int_{C_1} d^2 \zeta \langle T \bar{T}(\zeta) \psi^\dagger(-\xi) \psi(\xi) \rangle_{C_1} \\ &= \lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2 \int_{R_1} \frac{d^2 z}{z \bar{z}} \left\langle \left(z^2 T(z) - \frac{c}{24} \right) \left(\bar{z}^2 \bar{T}(\bar{z}) - \frac{c}{24} \right) \psi^*(w_2) \psi(w_1) \right\rangle_{R_1} \end{aligned} \quad (87)$$

with the correlator computed by the bosonization method:

$$\lim_{\substack{w_1 \rightarrow 0 \\ w_2 \rightarrow \infty}} w_2 \left\langle \left(z^2 T(z) - \frac{c}{24} \right) \left(\bar{z}^2 \bar{T}(\bar{z}) - \frac{c}{24} \right) \psi^*(w_2) \psi(w_1) \right\rangle_{R_1} = \frac{cz^2}{24} + \left(\frac{c}{24} \right)^2. \quad (88)$$

Then the integral in Eq. (87) is

$$\lim_{\xi, \bar{\xi} \rightarrow -\infty} \int_{C_1} d^2 \zeta \langle T \bar{T}(\zeta) \psi^\dagger(-\xi) \psi(\xi) \rangle_{C_1} = \int_{R_1} \frac{d^2 z}{z \bar{z}} \left(\frac{cz^2}{24} + \left(\frac{c}{24} \right)^2 \right) = 2\pi \left(\frac{c}{24} \right)^2 \ln r|_\epsilon^\Lambda, \quad (89)$$

where we used polar coordinate $z = re^{i\theta}$, with which the term proportional to c can be readily calculated and equals zero. Here, Σ and ϵ are the IR and UV cutoff of r , respectively. As for the integral (67), we are not able to evaluate this integral exactly so far. The integrand is divergent at some points, and thus some regularization might be needed; for discussions of some similar integrals, refer to, e.g., Ref. [47]. We would like to come back to this problem in the future.

C. $J\bar{T}$ perturbation

In this subsection, we will consider another kind of perturbation which is called the $J\bar{T}$ perturbation. We take the same setup for the present case as in Sec. III A; i.e., the whole system is in its vacuum state living on an infinite line, and the subsystem is a finite interval.

As described in Ref. [37] for free fermions, the $J\bar{T}$ perturbation can be constructed as follows. Let us consider the action Eq. (14), which possesses internal symmetries as well as conformal symmetry. One of the internal symmetries is the $U(1)$ symmetry of ψ : $\psi \rightarrow e^{ia}\psi$, $\psi^* \rightarrow e^{-ia}\psi^*$, while keeping the $\bar{\psi}$ unchanged. The current associated with this symmetry is

$$J = J_z = \psi^* \psi. \quad (90)$$

Combining this current with the antichiral part of the stress tensor

$$\bar{T} = \bar{\psi}^* \bar{\partial} \bar{\psi}, \quad (91)$$

one obtains the $J\bar{T}$ term for the unperturbed action S_0 . Then the perturbed action can be expressed as

$$S = S_0 + \lambda \int d^2 z J \bar{T}. \quad (92)$$

In order to compute the first-order perturbation of the Rényi entropy, one just needs to replace the $T\bar{T}$ term in Eq. (33) with the $J\bar{T}$ term; thus, the resulting correlator to be calculated is

$$\sum_l \langle J_l(z) \bar{T}_l(\bar{z}) S_n(0) \bar{S}_n(x_0) \rangle. \quad (93)$$

To proceed, one can also employ the bosonization method to evaluate this correlator. As we have done before, we first transform the perturbation term to a diagonal representation:

$$\begin{aligned} \sum_{l=1}^n J_l \bar{T}_l(z) &= \frac{1}{n} \sum_{k_1, k_2, k_3, k_4} \tilde{\psi}_{k_1}^* \tilde{\psi}_{k_2} \tilde{\psi}_{k_3}^* \bar{\partial} \tilde{\psi}_{k_4} \delta_{k_1 - k_2 + k_3 - k_4, mn}, \\ m &= \pm 1, 0 \\ &\rightarrow \frac{1}{n} \sum_{k_1, k_3} \tilde{\psi}_{k_1}^* \tilde{\psi}_{k_1} \tilde{\psi}_{k_3}^* \bar{\partial} \tilde{\psi}_{k_3} \equiv \frac{1}{n} J_{\text{tot}} \bar{T}_{\text{tot}}, \end{aligned} \quad (94)$$

which is similar to Eq. (39). The arrow $\rightarrow$ means that the replacement holds in the correlator. Here we have introduced the notation

$$J_{\text{tot}} = \sum_l J_l = \sum_k \tilde{\psi}_k^* \tilde{\psi}_k, \quad (95)$$

which can be bosonized as

$$\begin{aligned} \tilde{\psi}_k^* \tilde{\psi}_k(z) &= \lim_{\epsilon \rightarrow 0} \left(e^{-iH_k(z+\epsilon)} e^{iH_k(z)} - \frac{1}{\epsilon} \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{e^{-iH_k(z+\epsilon) + iH_k(z)} - 1}{\epsilon} = -i\partial H_k(z). \end{aligned} \quad (96)$$

Therefore, in the language of bosonization, the perturbation term takes the form

$$\sum_{l=1}^n J_l \bar{T}_l(z) = -\frac{i}{n} \sum_{k_1, k_2} \partial H_{k_1}(z) : (\bar{\partial} \bar{H}_{k_2})^2(\bar{z}) :. \quad (97)$$

which has the similar form as the case of $T\bar{T}$ perturbation.

Consider the first-order perturbation

$$\begin{aligned} &\sum_l \langle J_l(z) \bar{T}_l(\bar{z}) S_n(0) \bar{S}_n(x_0) \rangle \\ &= \frac{-i}{n} \sum_{k_1, k_2} \left\langle \partial H_{k_1}(z) : (\bar{\partial} \bar{H}_{k_2})^2(\bar{z}) : \prod_{k_3} e^{\frac{ik_3}{n} (H_{k_3}(0) - \bar{H}_{k_3}(\bar{0}))} \right. \\ &\quad \left. \times \prod_{k_4} e^{-\frac{ik_4}{n} (H_{k_4}(x_0) - \bar{H}_{k_4}(\bar{x}_0))} \right\rangle, \end{aligned} \quad (98)$$

which can be factorized into the chiral part and the antichiral part. The antichiral part is the same as the $T\bar{T}$ case as discussed in Sec. III A, and the chiral part turn out to be zero, which can be seen as follows:

$$\begin{aligned} &\sum_{k_1} \left\langle \partial H_{k_1}(z) \prod_{k_2} e^{\frac{ik_2}{n} H_{k_2}(0)} \prod_{k_3} e^{-\frac{ik_3}{n} H_{k_3}(x_0)} \right\rangle \\ &= \sum_{k_1} \left\langle \partial H_{k_1}(z) e^{\frac{ik_1}{n} H_{k_1}(0)} e^{-\frac{ik_1}{n} H_{k_1}(x_0)} \right\rangle \prod_{k \neq k_1} \langle e^{\frac{ik}{n} H_k(x)} e^{-\frac{ik}{n} H_k(0)} \rangle \\ &= -\sum_{k_1} \frac{ik_1}{n} \frac{x_0}{z(z-x_0)} \prod_k \left(-\frac{1}{x_0} \right)^{\frac{k^2}{n^2}} = 0, \end{aligned} \quad (99)$$

where in the last step the summation over k_1 cancels to zero. This can be understood as follows: Since J is a primary operator, the first-order contribution to Rényi entropy vanishes if the perturbation operator is primary. This is in contrast with $T\bar{T}$ perturbation, where T is not primary, and thus the first-order contribution to Rényi entropy is nonzero. Therefore, to see the nonvanishing contribution to Rényi entropy for $J\bar{T}$ perturbation, one should go to a higher order perturbation.

IV. CONCLUSION

In this paper, we investigated the $T\bar{T}$ as well as the $J\bar{T}$ perturbation of 2D massless Dirac fermions which is a free CFT. We mainly studied the correlator functions that appeared in the perturbation of Rényi entropy by using the bosonization approach for three cases. In the first case, we considered the Rényi entropy to the first order of a vacuum state on a plane and showed that the $T\bar{T}$ term can be written in a simple form in terms of the bosonic fields as presented in Eqs. (46) and (47). By using this bosonic form of the stress tensor, we reproduced the three-point correlator that appeared in the first-order variation of the Rényi entropy, which has been previously computed via the Ward identity in Ref. [32]. The advantage of the bosonization method is that it allows us to handle more complicated correlators in a simple way, while the Ward identity is insufficient to obtain the correlators, e.g., the second case we considered. In this case, we generalized the above discussion for a vacuum state to excited states on a cylinder, where the states are excited by the chiral operator $\psi(z)$. We explicitly analyzed the correlators at the first-order perturbation of the second Rényi entropy $\delta S_A^{(n=2)}$. Although for excited states the correlators are more complicated than the ones for the vacuum state, we showed that the bosonization method is still useful, and the corresponding correlators can be simply evaluated using this method. In the last case, we considered the $J\bar{T}$ perturbation for a vacuum state on the plane. The bosonized form of the $J\bar{T}$ term presented in Eq. (94) exhibits a similar form as the $T\bar{T}$ term. Then,

applying the bosonization approach, we showed that the Rényi entropy vanishes at the first-order perturbation, which is consistent with the fact that for a vacuum state the first-order perturbation of single interval Rényi entropy caused by primary operators always vanishes.

In the discussion for the Rényi entropy, we analyzed only the cases for the single interval subsystem; it is also tractable to generalize the situation into multi-interval cases, e.g., studying the perturbation of mutual information for free fermions which involves a two-interval subsystem [48]. Another direction is to study the effect of $T\bar{T}$ when extended symmetry is present, e.g., turning on the chemical potential for spin-3 operator W_3 , which can be written as $(\partial\phi)^3$ in the bosonization language. In addition, it is also interesting to consider a finite temperature and finite size system. This system for a free fermion without perturbation has been discussed in Refs. [49,50].

ACKNOWLEDGMENTS

We thank Soumangsu Chakraborty for helpful discussions. This work was supported by the National Natural Science Foundation of China (No. 11675272), the Open Project Program of State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, China (No. Y5KF161CJ1), the Fundamental Research Funds for the Central Universities, and the Project funded by China Postdoctoral Science Foundation (No. 2019M653137).

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