N -level ghost spins and entanglement

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(Received 29 January 2019; published 7 May 2019)

Ghost spins, 2-level spinlike variables with indefinite norm have been studied in previous work. Here we explore various N -level generalizations of ghost spins. First we discuss a flavored generalization comprising N copies of the ghost-spin system, as well as certain ghost-spin chains which in the continuum limit lead to 2-dim bc -ghost conformal field theory with $O(N)$ flavor symmetry. Then we explore a symplectic generalization that involves antisymmetric inner products, and finally a ghost-spin system exhibiting N irreducible levels. We also study entanglement properties. In all these cases, we show the existence of positive norm “correlated ghost spin” states in two copies of ghost-spin ensembles obtained by entangling identical ghost spins from each copy: these exhibit positive entanglement entropy.

DOI: [10.1103/PhysRevD.99.106003](https://doi.org/10.1103/PhysRevD.99.106003)

I. INTRODUCTION

Ghost-spin systems [1–3] and their patterns of entanglement are interesting from multiple points of view. Some of these possible applications involve ghost sectors in theories with gauge symmetry, while others pertain to conjectures involving de Sitter physics and dS/conformal field theory (CFT) dualities [4–14] including higher spin versions. Ghost spins are 2-level spinlike variables but with indefinite norm [1]: thus in some ways, they are best regarded as simple quantum mechanical toy models for theories with negative norm states. Ghost-spin systems have interesting entanglement patterns by virtue of this indefinite norm. If one considers ghost spin systems with even number of ghost spins then it is possible to find subspaces in the Hilbert space where one obtains positive entanglement entropy for positive norm states [1,2]. The situation is not so fortuitous in the case of odd number of ghost spins. However, several interesting physical systems which contain indefinite norm states seem to admit even numbers of such states. Taking the study of the ghost spins further, it was shown in [3] that appropriate ghost-spin chains in the continuum limit give rise to the bc -ghost CFTs in two dimensions. This suggests that they may be regarded as microscopic building blocks for ghost-CFTs and perhaps more general nonunitary theories.

The explorations of the ghost-spin system so far have used 2-state spinlike variables with indefinite inner product. It is interesting to study generalizations of ghost spins with flavor degrees of freedom assigned to them. These flavor quantum numbers may be relevant for applications to non-Abelian gauge theories. However, the ghost spin system with flavors by themselves is an interesting set up worthy of exploration. Once we allow for flavor symmetry, there are multiple ways in which they can be incorporated in the ghost-spins framework. One way is to assign an index, say A , which takes N values, to a 2-component ghost spin with either $O(N)$ or $Sp(N)$ symmetry in the flavor index. The inner product in the flavor space is given by δ^{AB} for the $O(N)$ case and Ω^{AB} in the $Sp(N)$ case. More general inner products can also be analyzed.

In this paper we will explore these generalizations of the 2-state ghost-spin system to N -level systems. In Sec. II, we will begin with a brief recap of the known results followed by a list of the N -level generalizations of ghost spins that we discuss in this paper. We introduce the $O(N)$ and $Sp(N)$ inner products in the flavor indices as well as more general inner products J^{AB} which are symmetric but nonvanishing for $A \neq B$. In Sec. III, we look at the $O(N)$ generalization of the 2-level system where we denote the flavored ghost spins in terms of the 2-component spins carrying an additional index corresponding to the global $O(N)$ symmetry, $|\uparrow^A\rangle$. We write the indefinite inner product between spins by splitting it into indefinite product between $|\uparrow\rangle$ and $|\downarrow\rangle$ and a symmetric product δ_{AB} in the $O(N)$ index A . This as we will see is analogous to the bc -ghost system studied in the flavorless 2-level system. We then consider the ghost-spin chain with $O(N)$ flavor symmetry and show that it leads to the flavored bc -ghost CFTs (Sec. III A). We then consider

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correlated ghost-spin states in Sec. III B and analyze the entanglement pattern in them. We find that the results that we had obtained in the even ghost-spin system in the flavorless case carry over to the case with flavors. We generically find a subspace of correlated ghost-spin states in the Hilbert space with positive norm states having positive entanglement. Finally we briefly comment on more general inner products which lead to spin-glass type couplings in Sec. III C. We then turn our attention in Sec. IV to symplectic inner products between certain generalizations of ghost spins and study entanglement in correlated ghost-spin states. Finally in Sec. V we consider a generalization of 2-level ghost spins to N irreducible levels: this is slightly different from the flavored generalizations above. We study the entanglement pattern in correlated ghost-spin states here as well. In Sec. VI, we summarize our results and comment on their applications to the dS/CFT correspondence, in part reviewing the picture in [15] of dS_4 as approximately dual to a thermofield-double type entangled state between two copies of ghost-CFTs.

II. GHOST SPINS AND N -LEVEL GENERALIZATIONS

Before getting to N -level generalizations, we first briefly review some essential aspects of ghost spins. Ghost spins

were defined in [1] as simple toy quantum mechanical models for theories with negative norm states, abstracting from bc -ghost CFTs. These constructions were motivated by the studies [16,17] on certain complex extremal surfaces in de Sitter space with negative area in dS_4 which amount to analytic continuations of the Ryu-Takayanagi formulations of holographic entanglement entropy [18–21]. In contrast with a single spin which has $\langle \uparrow \uparrow \rangle = 1 = \langle \downarrow \downarrow \rangle$, a single ghost spin is defined as a 2-state spin variable with indefinite inner product

$$\langle \uparrow \downarrow \rangle = 1 = \langle \downarrow \uparrow \rangle, \quad \langle \uparrow \uparrow \rangle = 0 = \langle \downarrow \downarrow \rangle. \quad (2.1)$$

A general state $\psi^+|+\rangle + \psi^-|-\rangle$ thus has norm $|\psi^+|^2 - |\psi^-|^2$, which is not positive definite. By changing basis, the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$ have manifestly positive/negative norm, satisfying $\langle \pm | \pm \rangle = \pm 1$. We can then normalize general positive/negative norm states with norm ± 1 respectively. Consider now a state comprising two ghost spins: this has norm

$$|\psi\rangle = \psi^{\alpha\beta}|\alpha\beta\rangle: \quad \langle\psi|\psi\rangle = \gamma_{\alpha\kappa}\gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\lambda*}, \quad \gamma_{++} = 1, \quad \gamma_{--} = \langle - | - \rangle = -1, \quad (2.2)$$

where $\gamma_{\alpha\beta}$ is the indefinite metric. Thus although states $|-\rangle$ have negative norm, the state $|-\rangle|-\rangle$ has positive norm. The full density matrix is $\rho = |\psi\rangle\langle\psi| = \sum \psi^{\alpha\beta}\psi^{\kappa\lambda*}|\alpha\beta\rangle\langle\kappa\lambda|$. Tracing over one of the ghost spins leads to a reduced density matrix $(\rho_A)^{\alpha\kappa} = \gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\lambda*} = \gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\beta*}$,

$$\begin{aligned} (\rho_A)^{++} &= |\psi^{++}|^2 - |\psi^{+-}|^2, & (\rho_A)^{+-} &= \psi^{++}\psi^{-+*} - \psi^{+-}\psi^{--*}, \\ (\rho_A)^{-+} &= \psi^{-+}\psi^{++*} - \psi^{--}\psi^{+-*}, & (\rho_A)^{--} &= |\psi^{--}|^2 - |\psi^{+-}|^2, \end{aligned} \quad (2.3)$$

for the remaining ghost spin. Then $\text{tr}\rho_A = \gamma_{\alpha\kappa}(\rho_A)^{\alpha\kappa} = (\rho_A)^{++} - (\rho_A)^{--}$. Thus the reduced density matrix is normalized to have $\text{tr}\rho_A = \text{tr}\rho = \pm 1$ depending on whether the state (2.2) is positive or negative norm. The entanglement entropy calculated as the von Neumann entropy of ρ_A is $S_A = -\gamma_{\alpha\beta}(\rho_A \log \rho_A)^{\alpha\beta}$, perhaps best defined using a mixed-index reduced density matrix $(\rho_A)^\alpha{}_\kappa = \gamma_{\beta\lambda}(\rho_A)^{\alpha\beta}$. This can be illustrated via a simple family of states [1] with a diagonal reduced density matrix: setting $\psi^{-+*} = \psi^{+-}\psi^{--*}/\psi^{++}$ in the states (2.2) gives

$$\begin{aligned} (\rho_A)^{\alpha\beta}|\alpha\rangle\langle\beta| &= \pm x|+\rangle\langle+| \mp (1-x)|-\rangle\langle-|, & x &= \frac{|\psi^{++}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2} \quad [0 < x < 1], \\ (\rho_A)^\alpha{}_\kappa &= \gamma_{\alpha\beta}(\rho_A)^{\beta\kappa}: (\rho_A)^\alpha{}_+ = \pm x, & (\rho_A)^\alpha{}_- &= \pm(1-x), \end{aligned} \quad (2.4)$$

where the $\pm$ pertain to positive/negative norm states respectively (note that $\text{tr}\rho_A = (\rho_A)^\alpha{}_+ + (\rho_A)^\alpha{}_- = \pm 1$). The location of the negative eigenvalue is different for positive/negative norm states, leading to different results for the von Neumann entropy. Now $\log \rho_A$ simplifies to $(\log \rho_A)^\alpha{}_+ = \log(\pm x)$ and $(\log \rho_A)^\alpha{}_- = \log(\pm(1-x))$. The entanglement entropy defined as $S_A = -\gamma_{\alpha\beta}(\rho_A \log \rho_A)^{\alpha\beta}$ becomes $S_A = -(\rho_A)^\alpha{}_+(\log \rho_A)^\alpha{}_+ - (\rho_A)^\alpha{}_-(\log \rho_A)^\alpha{}_-$ and so

$$\langle\psi|\psi\rangle \geq 0: S_A = -(\pm x) \log(\pm x) - (\pm(1-x)) \log(\pm(1-x)). \quad (2.5)$$

For positive norm states, S_A is manifestly positive since $x < 1$, just as in an ordinary 2-spin system. Negative norm states give a negative real part for entanglement entropy since $x < 1$ and the logarithms are negative: further there is an imaginary part (the simplest branch has $\log(-1) = i\pi$).

Now consider restricting to the subspace

$$\begin{aligned} |\psi\rangle &= \psi^{++}|++\rangle + \psi^{--}|--\rangle \rightarrow \langle\psi|\psi\rangle \\ &= |\psi^{++}|^2 + |\psi^{--}|^2 > 0. \end{aligned} \quad (2.6)$$

These states of “correlated ghost spins” comprise entanglement between two copies of identical states: they can be seen to be strictly positive norm, with a positive reduced density matrix (2.3) and positive entanglement. In [15], a picture of de Sitter space as a thermofield double type state (with de Sitter entropy then emerging as the entanglement entropy) was discussed based on such correlated ghost-spin states in two copies of ghost-CFTs at the future and past boundaries of dS_4 in the static coordinatization. For ensembles with an even number of ghost spins, such correlated ghost-spin states always exist comprising positive norm subsectors, as argued in [2], where ensembles of ghost spins were developed further with regard to their entanglement properties. Odd ghost spins were found to behave differently: for instance, $|\psi\rangle = \psi^{++\dots}|++\dots\rangle + \psi^{--\dots} |--\dots\rangle$ has norm $\langle\psi|\psi\rangle = |\psi^{++\dots}|^2 + (-1)^n \times |\psi^{--\dots}|^2$ and mixed-index reduced density matrix components $(\rho_A)_+ = |\psi^{++\dots}|^2$, $(\rho_A)_- = (-1)^n |\psi^{--\dots}|^2$. This is not positive definite for n odd (even if $\langle\psi|\psi\rangle > 0$). Ensembles of ghost spins and spins were also found to exhibit interesting entanglement patterns.

In [3], certain 1-dim ghost-spin chains with specific nearest-neighbor interactions were found to yield bc -ghost CFTs in the continuum limit, i.e., these ghost-spin chains are in the same universality class as those ghost-CFTs. We will not review this here since this will effectively be encompassed in a related detailed description later.

N-level ghost-spins: In this paper, we generalize the 2-level ghost spin reviewed above to N -levels by considering various generalizations as outlined below:

- (i) $O(N)$ symmetry flavor generalization of the bc -ghost system:

$$\begin{aligned} \langle\downarrow^A|\uparrow^B\rangle &= \delta^{AB} = \langle\uparrow^A|\downarrow^B\rangle, \\ \langle\downarrow^A|\downarrow^B\rangle &= \langle\uparrow^A|\uparrow^B\rangle = 0, \quad A, B = 1, 2, \dots, N. \end{aligned} \quad (2.7)$$

These are essentially N copies of the 2-level ghost-spin system. It is then possible to find appropriate ghost-spin chains which lead to a generalization of the bc -ghost system but with internal $O(N)$ flavor indices. A simple generalization of this case involves the flavors having a spin-glass type interaction with coupling J_{AB} which is in general nonvanishing for $A \neq B$,

$$\begin{aligned} \langle\downarrow^A|\uparrow^B\rangle &= J^{AB} = \langle\uparrow^A|\downarrow^B\rangle; \\ \langle\downarrow^A|\downarrow^B\rangle &= \langle\uparrow^A|\uparrow^B\rangle = 0, \quad A, B = 1, 2, \dots, N. \end{aligned} \quad (2.8)$$

In flavor space, this thus encodes possibly nonlocal flavor couplings. Taking the J_{AB} matrix to be real and symmetric allows diagonalization and in that diagonal basis, this can be reduced to the above $O(N)$ flavored case.

- (ii) N -levels with symplecticlike structure:

$$\begin{aligned} \langle\uparrow^A|\downarrow^B\rangle &= i\Omega^{AB}, \quad \langle\downarrow^A|\uparrow^B\rangle = i\Omega^{AB}, \\ \langle\uparrow^A|\uparrow^B\rangle &= 0 = \langle\downarrow^A|\downarrow^B\rangle, \quad A, B = 1, \dots, 2N. \end{aligned} \quad (2.9)$$

These have a symplectic structure built into the inner product, which was in part motivated by 3-dim ghost-CFTs of symplectic fermions [22,23] that have been discussed in the conjectured duals to higher spin dS_4 [7].

- (iii) N irreducible levels, i.e., we generalize the two states $|\uparrow\rangle, |\downarrow\rangle$ to $|e_1\rangle, \dots, |e_N\rangle$ such that

$$\begin{aligned} \langle e_i|e_i\rangle &= 0; \quad \langle e_i|e_j\rangle = 1 \quad \text{for } i \neq j, \\ i, j &= 1, 2, \dots, N. \end{aligned} \quad (2.10)$$

This case is slightly different from the previous cases in that the elemental ghost spins are not 2-level anymore (with flavor indices), but irreducibly N -level.

In all these cases representing N -level generalizations of ghost-spin ensembles, we will argue that correlated ghost-spin states exist comprising a uniformly positive norm subspace of states with the interpretation of entanglement between two copies of the state space. This will be the main point of the paper.

It is possible to find operator realizations consistent with some of these inner products above. The first case essentially comprises N copies of the bc -operator algebra,

$$\{\sigma_b^A, \sigma_c^B\} = \delta^{AB}. \quad (2.11)$$

We can then define ghost-spin-chain Hamiltonians with nearest neighbor hopping type interactions but with the flavors decoupled,

$$H = \sum_n (\sigma_{bn}^A \sigma_{c(n+1)}^B + \sigma_{b(n+1)}^A \sigma_{cn}^B) \delta_{AB} \xrightarrow{?} \int b^A \partial c^A. \quad (2.12)$$

Based on the fact that the continuum limit for the single flavor case is the familiar bc -ghost CFT [3], the continuum limit can be argued to be flavored generalizations of bc -ghost CFTs, with the flavor contractions exhibiting $O(N)$ symmetry. We will discuss this in detail in Sec. III. The symplectic inner products above are consistent with the operator algebra

$$\{\sigma_b^A, \sigma_c^B\} = i\Omega_{AB}, \quad (2.13)$$

as we will discuss in Sec. V. The continuum limit is less clear in this case, although there are indications that these may be related to logarithmic CFTs [24–30].

III. N -LEVEL GHOST SPINS WITH $O(N)$ FLAVOR SYMMETRY

In this section, we consider an N -level generalization of ghost-spins with $O(N)$ symmetry among the N internal flavor indices, defined by (2.7), i.e., we have the elemental inner products

$$\begin{aligned} \langle \uparrow^A | \downarrow^B \rangle &= \delta^{AB} = \langle \downarrow^A | \uparrow^B \rangle, & \langle \uparrow^A | \uparrow^B \rangle &= 0 = \langle \downarrow^A | \downarrow^B \rangle, & A, B &= 1, 2, \dots, N, \\ |\pm^A\rangle &= \frac{1}{\sqrt{2}}(|\uparrow^A\rangle \pm |\downarrow^A\rangle), & \langle \pm^A | \pm^B \rangle &= \pm \delta^{AB}, & \langle \pm^A | \mp^B \rangle &= 0. \end{aligned} \quad (3.1)$$

This is essentially N copies of the 2-level ghost spin reviewed in Sec. II. In the second line, we have defined a convenient basis where the inner product is diagonal: this makes manifest the negative norm basis states.

Consider first a single ghost spin with N flavors. The general configuration is defined by specifying the simultaneous configurations for each of the N flavors so the general state comprising various basis states $|s_i\rangle$ is

$$|\psi\rangle = \psi^{s_i} |s_i\rangle, \quad |s_i\rangle \equiv \left\{ \begin{array}{c} \uparrow^1 \\ \uparrow^2 \\ \uparrow^3 \\ \vdots \end{array} \right\rangle, \quad \left\{ \begin{array}{c} \downarrow^1 \\ \downarrow^2 \\ \uparrow^3 \\ \vdots \end{array} \right\rangle, \quad \left\{ \begin{array}{c} \uparrow^1 \\ \downarrow^2 \\ \uparrow^3 \\ \vdots \end{array} \right\rangle, \quad \dots \right\}, \quad (3.2)$$

i.e., in the first basis state, the first flavor is $\uparrow^1$, the second is $\uparrow^2$, third being $\uparrow^3$ and so on, and likewise for the other basis states. It is important to note that the $|s_i\rangle$ are really direct product states over the various flavor components: although we have written them as column vectors for convenience of notation (especially in light of the discussion later on multiple ghost spins), the inner products between these states is not a dot product between two column vectors. Instead we define the inner products between the configurations $|s_i\rangle$ as

$$\langle s_i | s_j \rangle = \frac{1}{N!} \sum \epsilon_{A_1 A_2 \dots A_N} \epsilon_{B_1 B_2 \dots B_N} \langle s_i^{A_1} | s_j^{B_1} \rangle \langle s_i^{A_2} | s_j^{B_2} \rangle \dots \langle s_i^{A_N} | s_j^{B_N} \rangle, \quad (3.3)$$

where $i, j = 1, 2, \dots, 2^N$ label the configurations, $A_1, B_1, \dots = 1, 2, \dots, N$ label the flavors and $\epsilon_{A_1 A_2 \dots A_N}$ is the totally symmetric tensor with $\epsilon_{12 \dots N} = 1$ and $\epsilon_{A_1 A_2 \dots A_N}$ vanishes if any two labels are the same. In other words, $\epsilon_{A_1 A_2 \dots A_N}$ is simply a bookkeeping device for ensuring that each elemental state $|s_j^A\rangle$ in $|s_j\rangle$ is paired with another corresponding elemental state in $\langle s_i|$.

To illustrate how this works, let us consider a simple example of a single ghost spin with $N = 2$ flavors: the distinct configurations of this system are described by the basis states

$$|s_1\rangle = |\uparrow_1^1\rangle, \quad |s_2\rangle = |\downarrow_1^1\rangle, \quad |s_3\rangle = |\uparrow_2^1\rangle, \quad |s_4\rangle = |\downarrow_2^1\rangle. \quad (3.4)$$

Then the inner product (3.3) simplifies to

$$\langle s_i | s_j \rangle = \frac{1}{2!} \sum \epsilon_{A_1 A_2} \epsilon_{B_1 B_2} \langle s_i^{A_1} | s_j^{B_1} \rangle \langle s_i^{A_2} | s_j^{B_2} \rangle = \langle s_i^1 | s_j^1 \rangle \langle s_i^2 | s_j^2 \rangle + \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle, \quad (3.5)$$

where we have used $\epsilon_{12} = 1 = \epsilon_{21}$. Using the elemental inner products in (3.1) gives

$$\langle s_i | s_j \rangle = \langle s_i^1 | s_j^1 \rangle \langle s_i^2 | s_j^2 \rangle. \quad (3.6)$$

Writing this out explicitly, we have

$$\begin{aligned} \langle s_1 | s_4 \rangle &= \langle \uparrow^1 | \downarrow^1 \rangle \langle \uparrow^2 | \downarrow^2 \rangle = 1, & \langle s_2 | s_3 \rangle &= \langle \downarrow^1 | \downarrow^1 \rangle \langle \downarrow^2 | \uparrow^2 \rangle = 1, \\ \langle s_3 | s_2 \rangle &= \langle \downarrow^1 | \uparrow^1 \rangle \langle \uparrow^2 | \downarrow^2 \rangle = 1, & \langle s_4 | s_1 \rangle &= \langle \downarrow^1 | \uparrow^1 \rangle \langle \downarrow^2 | \uparrow^2 \rangle = 1. \end{aligned} \quad (3.7)$$

The other inner products vanish. Based on these inner products for the basis states, we can write the norm for the generic state as

$$|\psi\rangle = c_i |s_i\rangle \Rightarrow \langle\psi|\psi\rangle = (c_i^* \langle s_i|) \cdot (c_i |s_i\rangle) = c_1^* c_4 + c_4^* c_1 + c_2^* c_3 + c_3^* c_2. \quad (3.8)$$

Appropriate pairs of states can be used to define a new basis of positive and negative norm states: $s_1 \pm s_4$, $s_2 \pm s_3$ and so on have norm ± 2 respectively. Likewise for $N = 4$ flavors, the $2^4 = 16$ configurations $|s_i\rangle$ are¹

$$|s_1\rangle = \left| \begin{array}{c} \uparrow^1 \\ \uparrow^2 \\ \uparrow^3 \\ \uparrow^4 \end{array} \right\rangle, \quad |s_2\rangle = \left| \begin{array}{c} \uparrow^1 \\ \uparrow^2 \\ \uparrow^3 \\ \downarrow^4 \end{array} \right\rangle, \quad |s_3\rangle = \left| \begin{array}{c} \uparrow^1 \\ \uparrow^2 \\ \downarrow^3 \\ \uparrow^4 \end{array} \right\rangle, \quad \dots, \quad |s_{15}\rangle = \left| \begin{array}{c} \downarrow^1 \\ \downarrow^2 \\ \downarrow^3 \\ \uparrow^4 \end{array} \right\rangle, \quad |s_{16}\rangle = \left| \begin{array}{c} \downarrow^1 \\ \downarrow^2 \\ \downarrow^3 \\ \downarrow^4 \end{array} \right\rangle \quad (3.9)$$

and the inner product (3.3) is

$$\langle s_i | s_j \rangle = \frac{1}{4!} \sum \epsilon_{A_1 A_2 A_3 A_4} \epsilon_{B_1 B_2 B_3 B_4} \langle s_i^{A_1} | s_j^{B_1} \rangle \langle s_i^{A_2} | s_j^{B_2} \rangle \langle s_i^{A_3} | s_j^{B_3} \rangle \langle s_i^{A_4} | s_j^{B_4} \rangle. \quad (3.10)$$

Using (3.1), this simplifies to

$$\begin{aligned} \langle s_i | s_j \rangle &= \frac{1}{4!} \sum_{A_1, A_2, A_3, A_4} \epsilon_{A_1 A_2 A_3 A_4}^2 \langle s_i^{A_1} | s_j^{A_1} \rangle \langle s_i^{A_2} | s_j^{A_2} \rangle \langle s_i^{A_3} | s_j^{A_3} \rangle \langle s_i^{A_4} | s_j^{A_4} \rangle \\ &= \langle s_i^1 | s_j^1 \rangle \langle s_i^2 | s_j^2 \rangle \langle s_i^3 | s_j^3 \rangle \langle s_i^4 | s_j^4 \rangle, \end{aligned} \quad (3.11)$$

noting that there are $4!$ nonzero $\epsilon_{A_1 A_2 A_3 A_4}$ components, which can succinctly be written as

$$\langle s_i | s_{17-i} \rangle = 1, \quad (3.12)$$

with other inner products vanishing. Similarly, for general N , the inner product (3.3) is

$$\langle s_i | s_j \rangle = \frac{1}{N!} \sum_{A_1, \dots, A_N} \epsilon_{A_1 \dots A_N}^2 \langle s_i^{A_1} | s_j^{A_1} \rangle \dots \langle s_i^{A_N} | s_j^{A_N} \rangle = \langle s_i^1 | s_j^1 \rangle \dots \langle s_i^N | s_j^N \rangle = \prod_A \langle s_i^A | s_j^A \rangle. \quad (3.13)$$

Thus two generic states have inner product

$$\langle \psi_1 | \psi_2 \rangle = (\psi_1^{s_j})^* \psi_2^{s_j} \langle s_j | s_i \rangle = (\psi_1^{s_j})^* \psi_2^{s_j} \prod_A \langle s_i^A | s_j^A \rangle \quad (3.14)$$

A. Ghost-spin chain for the bc -CFT with $O(N)$ symmetry

We want to now study infinite ghost-spin chains along the lines of those in [3], but with additional flavor structure respecting the $O(N)$ flavor symmetry we have been discussing so far. In effect, this amounts to N flavor copies of ghost spins at each lattice site. Thus we define two species of N -component commuting spin variables, σ_{bn}^A and σ_{cn}^A at each lattice site n satisfying

¹These states can be written in terms of a basis which is manifestly $O(N)$ invariant. Namely

$$|t_0\rangle = \prod_{i=1}^N |\uparrow^i\rangle, \quad |t_{\text{vec}}\rangle = \prod_{j \neq i, i=1}^N |\uparrow^i\rangle |\downarrow^j\rangle, \quad \forall j, |t_{\text{adj}}\rangle = \prod_{j \neq k \neq i, i=1}^N |\uparrow^i\rangle |\downarrow^j\rangle, \quad \forall j, k, \dots$$

Where, 0 is the trivial representation, vec corresponds to the vector representation and adj is the adjoint(antisymmetric) representation of $O(N)$. To see the relation between the $|t\rangle$ and the $|s\rangle$ bases, consider for example, the states in (3.9): these get organized into representations of $O(4)$ as **1**, **4**, **6**, **4**, **1** with the states $|s_1\rangle$ and $|s_{16}\rangle$ forming two singlets, $|s_2\rangle$ to $|s_5\rangle$ ($|s_{12}\rangle$ to $|s_{15}\rangle$) belonging to the vector representation and $|s_6\rangle$ to $|s_{11}\rangle$ to the adjoint representations of $O(4)$. We will, however, continue using the basis given in (3.2) for convenience.

$$\{\sigma_{bn}^A, \sigma_{cn}^B\} = \delta^{AB}, \quad [\sigma_{bn}^A, \sigma_{bn'}^B] = [\sigma_{cn}^A, \sigma_{cn'}^B] = [\sigma_{bn}^A, \sigma_{cn'}^B] = 0. \quad (3.15)$$

These commuting spin-variables are Hermitian $\sigma_b^{\dagger A} = \sigma_b^A$, $\sigma_c^{\dagger A} = \sigma_c^A$ and their action on ghost-spin states is

$$\sigma_b^A |\downarrow^B\rangle = 0, \quad \sigma_b^A |\uparrow^B\rangle = \delta^{AB} |\downarrow^B\rangle, \quad \sigma_c^A |\downarrow^B\rangle = \delta^{AB} |\uparrow^B\rangle, \quad \sigma_c^A |\uparrow^B\rangle = 0, \quad (3.16)$$

where there is no summation over the flavor index B . For multiple ghost-spin states in the chain, the σ_{bn}^A operator acts to lower the A^{th} -flavor state within the ghost-spin configuration at site n , and likewise σ_{cn}^A is the corresponding raising operator. To illustrate this explicitly, consider two ghost spins with $N = 2$ flavors: then

$$\sigma_{b1}^2 |\uparrow_2^1\rangle |\downarrow_2^1\rangle = |\downarrow_2^1\rangle |\downarrow_2^1\rangle, \quad \sigma_{c2}^1 |\uparrow_2^1\rangle |\downarrow_2^1\rangle = |\uparrow_2^1\rangle |\uparrow_2^1\rangle, \quad \sigma_{b1}^2 \sigma_{c2}^1 |\uparrow_2^1\rangle |\downarrow_2^1\rangle = |\downarrow_2^1\rangle |\uparrow_2^1\rangle, \dots \quad (3.17)$$

The ghost-spin chain is then defined by a Hamiltonian encoding interactions between the ghost spins at various lattice sites. Since the flavors do not mix, the interaction Hamiltonian here is simply a straightforward generalization involving a decoupled sum over various flavors of the single flavor one in [3]. So consider a 1-dim ghost-spin chain with a “hopping” type interaction Hamiltonian

$$H = J \sum_n \sum_{A=1}^N (\sigma_{bn}^A \sigma_{c(n+1)}^A + \sigma_{bn}^A \sigma_{c(n-1)}^A) = J \sum_n \sum_{A=1}^N (\sigma_{bn}^A \sigma_{c(n+1)}^A + \sigma_{b(n+1)}^A \sigma_{cn}^A), \quad (3.18)$$

where $n, n+1, n-1$ label nearest label lattice sites in the chain. The action on a nearest neighbor pair of ghost spins at lattice sites $(n, n+1)$ is given as

$$\begin{aligned} \sigma_{bn}^A \sigma_{c(n+1)}^A : \left(\begin{array}{c} \vdots \\ \dots \otimes |\uparrow^A\rangle_n \otimes |\downarrow^A\rangle_{n+1} \otimes \dots \\ \vdots \end{array} \right) &\rightarrow J \left(\begin{array}{c} \vdots \\ \dots \otimes |\downarrow^A\rangle_n \otimes |\uparrow^A\rangle_{n+1} \otimes \dots \\ \vdots \end{array} \right), \\ \sigma_{b(n+1)}^A \sigma_{cn}^A : \left(\begin{array}{c} \vdots \\ \dots \otimes |\downarrow^A\rangle_n \otimes |\uparrow^A\rangle_{n+1} \otimes \dots \\ \vdots \end{array} \right) &\rightarrow J \left(\begin{array}{c} \vdots \\ \dots \otimes |\uparrow^A\rangle_n \otimes |\downarrow^A\rangle_{n+1} \otimes \dots \\ \vdots \end{array} \right). \end{aligned} \quad (3.19)$$

The N flavors are decoupled and the interaction between ghost spins at two neighboring lattice sites is through the same flavor at the two sites. Thus we can follow the analysis in [3] flavor-by-flavor.

Toward constructing the continuum limit of the ghost-spin chain (3.18), we note that the $\sigma_{b,c}^A$ operators commute at neighboring lattice sites as in the single flavor case. Thus we define two species of N -component fermionic operators satisfying the anticommutation relations

$$\{a_{bi}^A, a_{cj}^B\} = \delta_{ij} \delta^{AB}, \quad \{a_{bi}^A, a_{bj}^B\} = \{a_{ci}^A, a_{cj}^B\} = 0, \quad (3.20)$$

which anticommute at different lattice sites i, j also. The action of these fermionic operators on ghost-spin states is

$$\begin{aligned} a_b^A |\downarrow^B\rangle &= 0, & a_b^A |\uparrow^B\rangle &= \delta^{AB} |\downarrow^B\rangle, & a_c^A |\downarrow^B\rangle &= \delta^{AB} |\uparrow^B\rangle, & a_c^A |\uparrow^B\rangle &= 0, \\ \langle \downarrow^B | a_b^A &= 0, & \langle \uparrow^B | a_b^A &= \delta^{AB} \langle \downarrow^B |, & \langle \downarrow^B | a_c^A &= \delta^{AB} \langle \uparrow^B |, & \langle \uparrow^B | a_c^A &= 0. \end{aligned} \quad (3.21)$$

This is obtained by constructing a flavored generalization of the Jordan-Wigner transformation in [3] for the commuting spin variables (σ_b^A, σ_c^A) as

$$\begin{aligned} \sigma_{b1}^A &= a_{b1}^A, & \sigma_{c1}^A &= a_{c1}^A, & \sigma_{b2}^A &= i(1 - 2a_{c1}^A a_{b1}^A) a_{b2}^A, & \sigma_{c2}^A &= -i(1 - 2a_{c1}^A a_{b1}^A) a_{c2}^A, & \dots, \\ \sigma_{bn}^A &= i(1 - 2a_{c1}^A a_{b1}^A) i(1 - 2a_{c2}^A a_{b2}^A) \dots i(1 - 2a_{c(n-1)}^A a_{b(n-1)}^A) a_{bn}^A, \\ \sigma_{cn}^A &= (-i)(1 - 2a_{c1}^A a_{b1}^A) (-i)(1 - 2a_{c2}^A a_{b2}^A) \dots (-i)(1 - 2a_{c(n-1)}^A a_{b(n-1)}^A) a_{cn}^A, & \dots \end{aligned} \quad (3.22)$$

for each flavor index A independently (i.e., the transformations are decoupled for distinct flavors). The inverse transformations for the fermionic ghost-spin variables (a_b^A, a_c^A) are

$$\begin{aligned} a_{b1}^A &= \sigma_{b1}^A, & a_{c1}^A &= \sigma_{c1}^A, & a_{b2}^A &= i(1 - 2\sigma_{c1}^A \sigma_{b1}^A) \sigma_{b2}^A, & a_{c2}^A &= -i(1 - 2\sigma_{c1}^A \sigma_{b1}^A) \sigma_{c2}^A, & \dots, \\ a_{bn}^A &= i(1 - 2\sigma_{c1}^A \sigma_{b1}^A) i(1 - 2\sigma_{c2}^A \sigma_{b2}^A) \dots i(1 - 2\sigma_{c(n-1)}^A \sigma_{b(n-1)}^A) \sigma_{bn}^A, \\ a_{cn}^A &= (-i)(1 - 2\sigma_{c1}^A \sigma_{b1}^A) (-i)(1 - 2\sigma_{c2}^A \sigma_{b2}^A) \dots (-i)(1 - 2\sigma_{c(n-1)}^A \sigma_{b(n-1)}^A) \sigma_{cn}^A, & \dots \end{aligned} \quad (3.23)$$

for each flavor index A independently. The factor $(1 - 2\sigma_{ci}^A \sigma_{bi}^A)$ is -1 or $+1$ depending on whether the i th location is occupied by ($\uparrow^A$) or not ($\downarrow^A$), which means $(1 - 2\sigma_{ci}^A \sigma_{bi}^A)^2 = 1$. Using

$$[\pm i(1 - 2\sigma_{ci}^A \sigma_{bi}^A)]^\dagger = \pm i(1 - 2\sigma_{ci}^A \sigma_{bi}^A), \quad (3.24)$$

we can check that the operators a_{bn}^A, a_{cn}^A are Hermitian. Now substituting the Jordan-Wigner transformation (3.22) in the ghost-spin Hamiltonian in the commuting spin variables (3.18) gives

$$\begin{aligned} H &= J \sum_n \sum_{A=1}^N (\sigma_{bn}^A \sigma_{c(n+1)}^A + \sigma_{bn}^A \sigma_{c(n-1)}^A), \\ &= J \sum_n \sum_{A=1}^N (i^{n-1} [1]^A [2]^A \dots [n-1]^A a_{bn}^A (-i)^n [1]^A [2]^A \dots [n]^A a_{c(n+1)}^A \\ &\quad + i^{n-1} [1]^A [2]^A \dots [n-1]^A a_{bn}^A (-i)^{n-2} [1]^A [2]^A \dots [n-2]^A a_{c(n-1)}^A), \end{aligned} \quad (3.25)$$

where $[k]^A = (1 - 2a_{ck}^A a_{bk}^A)$. Commuting the various $[k]^A$ factors gives

$$\begin{aligned} H &= J \sum_n \sum_{A=1}^N ((-i) a_{bn}^A (1 - 2a_{cn}^A a_{bn}^A) a_{c(n+1)}^A + i(1 - 2a_{c(n-1)}^A a_{b(n-1)}^A) a_{bn}^A a_{c(n-1)}^A), \\ &= iJ \sum_n \sum_{A=1}^N a_{bn}^A (a_{c(n+1)}^A - a_{c(n-1)}^A). \end{aligned} \quad (3.26)$$

We see that this Hamiltonian for the 1-dimensional chain of N -level ghost spins with $O(N)$ symmetry breaks up as a decoupled sum of N copies of the Hamiltonian for 2-level ghost-spins in [3]. Then following the analysis there for each flavor independently and taking the continuum limit, we can show that we obtain N copies of bc -ghost CFTs with $O(N)$ flavor symmetry. This finally gives

$$H = \sum_{A=1}^N \sum_k k b_{-k}^A c_k^A = \sum_{A=1}^N \sum_{k>0} k (b_{-k}^A c_k^A + c_{-k}^A b_k^A), \quad (3.27)$$

which is essentially the operator L_0 for a bc -ghost CFT enjoying $O(N)$ -flavor symmetry, with action

$$S = \int d^2z \sum_{A=1}^N b^A \partial c^A, \quad (3.28)$$

and a corresponding antiholomorphic part. Further details are similar to [3], except with multiple flavors.

B. Correlated ghost-spin states and entanglement

We now return to ghost-spin ensembles and their entanglement properties. Along the lines in (3.2) for enumerating states in the $\uparrow, \downarrow$ -basis, we can clearly use the $|\pm^i\rangle$ -basis to define the basis states $|s_i\rangle$ there: the advantage in the $|\pm^i\rangle$ -basis is that positive/negative norm states are easier to identify manifestly. For a single ghost spin with $N = 2$ flavors, we have then

$$|s_1\rangle = |_{+2}^{+1}\rangle, \quad |s_2\rangle = |_{-2}^{+1}\rangle, \quad |s_3\rangle = |_{+2}^{-1}\rangle, \quad |s_4\rangle = |_{-2}^{-1}\rangle. \quad (3.29)$$

We remind the reader that although we are using column vectors for notational convenience, these are really direct product states: the inner product (3.3) here gives

$$\begin{aligned}\langle s_1|s_1\rangle &= \langle +^1|+^1\rangle\langle +^2|+^2\rangle = 1, & \langle s_4|s_4\rangle &= \langle -^1|-^1\rangle\langle -^2|-^2\rangle = 1, \\ \langle s_2|s_2\rangle &= \langle +^1|+^1\rangle\langle -^2|-^2\rangle = -1, & \langle s_3|s_3\rangle &= \langle -^1|-^1\rangle\langle +^2|+^2\rangle = -1,\end{aligned}\quad (3.30)$$

and this is an orthonormal basis of positive and negative norm states. A generic state then has norm

$$|\psi\rangle = c_i|s_i\rangle \Rightarrow \langle\psi|\psi\rangle = |c_1|^2 + |c_4|^2 - |c_2|^2 - |c_3|^2. \quad (3.31)$$

Thus states made from $|s_2\rangle, |s_3\rangle$ alone have negative norm. It is straightforward to write down similar basis states for arbitrary N flavors. For N flavors, there are 2^N basis states $|s_i\rangle$. The inner products are

$$\langle s_i|s_i\rangle = \prod_A \langle s_i^A|s_i^A\rangle. \quad (3.32)$$

Since this is a diagonal basis now, we have $\langle s_i|s_i\rangle = \pm 1$ respectively when there is an even or odd number of $|{-}^A\rangle$ elemental ghost spins in $|s_i\rangle$. This is exemplified in the $N = 2$ case (3.30) above.

Now let us consider two ghost spins. The states can be made from the 2^{2N} basis states $|s_i\rangle|s_j\rangle$ obtained by tensor products of the single ghost-spin states. The inner products between them are

$$\langle (s_k|s_l) \cdot (|s_i|s_j) \rangle = \langle s_k|s_i\rangle\langle s_l|s_j\rangle \quad (3.33)$$

Then the general state and its norm are

$$\begin{aligned}|\psi\rangle &= \psi^{s_i s_j} |s_i\rangle |s_j\rangle, \\ \langle\psi|\psi\rangle &= (\psi^{s_k s_l})^* \psi^{s_i s_j} \prod_A \langle s_k^A|s_i^A\rangle \prod_B \langle s_l^B|s_j^B\rangle,\end{aligned}\quad (3.34)$$

with two products over the flavor components of the two basis states. Tracing over say the second ghost spin in this state leads to a subsystem comprising the single remaining ghost spin, with reduced density matrix defined as

$$(\rho_A)^{s_k s_i} = (\psi^{s_k s_l})^* \psi^{s_i s_j} \langle s_l|s_j\rangle = (\psi^{s_k s_l})^* \psi^{s_i s_j} \prod_B \langle s_l^B|s_j^B\rangle. \quad (3.35)$$

The entanglement entropy of this reduced density matrix can then be calculated using the formulation in [1–3]: we will see this below. Restricting attention for simplicity to $N = 2$ flavors, we can use the four basis states (3.29). Then the 2 ghost-spin states can be described using the 16 basis states $|s_{i,j}\rangle \equiv |s_i\rangle|s_j\rangle$, or more explicitly,

$$\begin{aligned}&|_{+2}^{+1}\rangle|_{+2}^{+1}\rangle, \quad |_{+2}^{+1}\rangle|_{-2}^{+1}\rangle, \quad |_{+2}^{+1}\rangle|_{+2}^{-1}\rangle, \quad |_{+2}^{+1}\rangle|_{-2}^{-1}\rangle, \quad |_{-2}^{+1}\rangle|_{+2}^{+1}\rangle, \quad |_{-2}^{+1}\rangle|_{-2}^{+1}\rangle, \quad |_{-2}^{+1}\rangle|_{+2}^{-1}\rangle, \quad |_{-2}^{+1}\rangle|_{-2}^{-1}\rangle, \\ &|_{+2}^{-1}\rangle|_{+2}^{+1}\rangle, \quad |_{+2}^{-1}\rangle|_{-2}^{+1}\rangle, \quad |_{+2}^{-1}\rangle|_{+2}^{-1}\rangle, \quad |_{+2}^{-1}\rangle|_{-2}^{-1}\rangle, \quad |_{-2}^{-1}\rangle|_{+2}^{+1}\rangle, \quad |_{-2}^{-1}\rangle|_{-2}^{+1}\rangle, \quad |_{-2}^{-1}\rangle|_{+2}^{-1}\rangle, \quad |_{-2}^{-1}\rangle|_{-2}^{-1}\rangle,\end{aligned}\quad (3.36)$$

The inner products (3.33) can then be seen to give the norms, e.g.,

$$\langle s_{i,i}|s_{i,i}\rangle = (\langle s_i|s_i\rangle)^2 = 1, \quad \langle s_{1,2}|s_{1,2}\rangle = \langle s_1|s_1\rangle\langle s_2|s_2\rangle = -1, \quad \langle s_{2,3}|s_{2,3}\rangle = \langle s_2|s_2\rangle\langle s_3|s_3\rangle = 1, \dots \quad (3.37)$$

and so on, using (3.30). It is clear again that the norms are again ± 1 depending on whether the state $|s_{i,j}\rangle$ contains an even or odd number of $|{-}^A\rangle$ elemental states. The general state has norm

$$|\psi\rangle = \sum \psi^{s_i s_j} |s_i\rangle |s_j\rangle \quad \langle\psi|\psi\rangle = (\psi^{s_k s_l})^* \psi^{s_i s_j} \langle s_k|s_i\rangle\langle s_l|s_j\rangle = |\psi^{s_i s_j}|^2 \langle s_{i,j}|s_{i,j}\rangle. \quad (3.38)$$

This is a sum over $|\psi^{s_i s_j}|^2$ weighted by ± 1 depending on the sign of the norm of $|s_{i,j}\rangle$. A particularly interesting subset of states are what we call “correlated states,” generalizing the discussion in [2]. These are of the form

$$|\psi^{\text{corr}}\rangle = \sum \psi^{s_i s_j} |s_i\rangle |s_j\rangle \langle\psi^{\text{corr}}|\psi^{\text{corr}}\rangle = \sum_{i=1}^4 |\psi^{s_i s_j}|^2 > 0, \quad (3.39)$$

and are necessarily positive norm, even though some of the individual basis states are negative norm. The basis states $|s_i\rangle|s_j\rangle$ here are of the form

$$|_{+2}^{+1}\rangle|_{+2}^{+1}\rangle, \quad |_{-2}^{+1}\rangle|_{-2}^{+1}\rangle, \quad |_{+2}^{-1}\rangle|_{+2}^{-1}\rangle, \quad |_{-2}^{-1}\rangle|_{-2}^{-1}\rangle, \quad (3.40)$$

and we see explicitly that this subspace of correlated states is obtained by entangling some configuration for the first ghost spin with an identical configuration for the second ghost spin. Thus we have only 4 states which span the correlated ghost-spin subspace. It is clear that these are necessarily positive norm since there is an even number of minus ghost spins (any odd number in each column is doubled). Note that this is a smaller subspace than that comprising all positive norm states which simply need to have an even number of minus signs: e.g., the basis state $|_{+2}^{+1}\rangle|_{-2}^{-1}\rangle$ is positive norm but the two ghost spins have different configurations. This can be generalized to two ghost spins with N flavors in a straightforward manner: the general state is again of the form (3.39) but with the $|s_i\rangle$ encoding N flavor ghost-spin configurations. There are 2^N basis states $|s_i\rangle$ so this subspace of correlated states is 2^N -dimensional, somewhat smaller than the 2^{2N} -dimensional space of all states.

These correlated ghost-spin states entangle identical ghost spins between the two sets of ghost spins. These states necessarily encode positive entanglement since any sublinear combination of the norm is still positive definite (one way to see this is to note that this can be mapped to an auxiliary system of ordinary spins, which has no minus signs and is entirely positive norm). More explicitly, for a state of the form $|\psi\rangle = \psi^{s_I, s_I} |s_I\rangle |s_I\rangle + \psi^{s_J, s_J} |s_J\rangle |s_J\rangle$ made of the states $|s_{I,I}\rangle, |s_{J,J}\rangle$, the reduced density matrix (3.35) can be taken to construct a mixed-index reduced density matrix as in [1–3], which then makes explicit the contraction structure with respect to the ghost-spin inner product metric (incorporating the signs for negative norm states). To be explicit, consider $|\psi\rangle = \psi^{s_1, s_1} |s_1\rangle |s_1\rangle + \psi^{s_2, s_2} |s_2\rangle |s_2\rangle$ noting that $|s_1\rangle$ and $|s_2\rangle$ are positive and negative norm, respectively. Then

$$\begin{aligned} (\rho_A)_{s_1}^{s_1} &= (\rho_A)^{s_1, s_1} = |\psi^{s_1, s_1}|^2, \\ (\rho_A)_{s_2}^{s_2} &= -(\rho_A)^{s_2, s_2} = |\psi^{s_2, s_2}|^2; \quad \text{tr} \rho_A = \langle \psi | \psi \rangle = 1. \end{aligned} \quad (3.41)$$

The mixed-index reduced density matrix here is $(\rho_A)_{s_j}^{s_i} = \gamma_{s_j, s_k} (\rho_A)^{s_i, s_k} = \langle s_j | s_k \rangle (\rho_A)^{s_i, s_k}$, where the metric $\gamma_{s_i, s_j} = \langle s_i | s_j \rangle$ is defined by the inner products (3.30). This description can be generalized to all such states, and indeed to all 2-ghost-spin states (3.38): in this case, it can be shown along the lines of the single flavor case that more general positive norm subsectors exist. In general however, the state space has many branches of negative norm states, and the reduced density matrix in general has negative eigenvalues with a complex entanglement entropy correspondingly (as was already the case in the single flavor case).

These states can be generalized to any even number of ghost spins. For odd numbers of ghost spins however, this

structure does not prevail: there are states that are positive norm but the reduced density matrix continues to have negative eigenvalues so that the entanglement entropy is not positive.

It is now interesting to consider two copies of ghost-spin chains and consider correlated ghost-spin states representing entanglement between the two chains. This is motivated by the discussion and picture in [15] of dS_4 as dual to a thermofield-double type entangled state in two copies $\text{CFT}_F \times \text{CFT}_P$ of the ghost-CFT at I^+ and I^- (reviewed in the Discussion in Sec. VI). So let us consider $\mathcal{GC}_1 \times \mathcal{GC}_2$ where each $\mathcal{GC}$ represents a ghost-spin chain whose continuum limit gives a bc -ghost CFT with flavor symmetry as in Sec. III A. Configurations of each $\mathcal{GC}$ can be represented schematically by

$$|\sigma\rangle \equiv (\dots |s_n\rangle |s_{n+1}\rangle \dots) \quad (3.42)$$

Then correlated entangled states in $\mathcal{GC}_1 \times \mathcal{GC}_2$ can be represented as

$$|\psi\rangle = \psi^{\sigma, \sigma} |\sigma\rangle |\sigma\rangle, \quad \langle \psi | \psi \rangle = \sum_{|\sigma\rangle} |\psi^{\sigma, \sigma}|^2 > 0. \quad (3.43)$$

Now the states $|\sigma\rangle$ include the ground state as well as excited states. If we restrict to the ground states alone, then since the flavors are all decoupled from each other, the ground states $|\sigma\rangle$ comprise a 2^N -dimensional subspace noting that each $|s_n\rangle$ at lattice site n in $|\sigma\rangle$ has 2^N possibilities. Thus tracing over the second ghost-spin chain copy $\mathcal{GC}_2$, we obtain the entanglement entropy of $|\psi\rangle$ restricting to ground states $|\sigma_g\rangle$ as

$$\begin{aligned} S_A &= - \sum_{i=1}^{2^N} |\psi^{\sigma_g, \sigma_g}|^2 \log |\psi^{\sigma_g, \sigma_g}|^2 \rightarrow -2^N \frac{1}{2^N} \log \frac{1}{2^N} \\ &= N \log 2. \end{aligned} \quad (3.44)$$

We have used $\sum_g |\psi^{\sigma_g, \sigma_g}|^2 = 1$ from normalization and imposed maximal entanglement, which equates all the coefficients giving $|\psi^{\sigma_g, \sigma_g}|^2 = \frac{1}{2^N}$. Thus the entanglement entropy scale as the number of flavors N .

C. Symmetric, spin-glass type, inner products

Here we briefly mention a generalization of the $O(N)$ flavored case but with the various flavors talking to each other, with a spin glass type coupling. We define the elemental inner products (2.8) using a symmetric form J^{AB} and take the inner products between the configurations $|s_i\rangle$ to be (3.3).

Let us consider first $N = 2$ flavors: then the states are as in (3.4) and the inner products are

$$\langle s_i | s_j \rangle = \langle s_i^1 | s_j^1 \rangle \langle s_i^2 | s_j^2 \rangle + \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle. \quad (3.45)$$

Using the elemental inner products (2.8), the nonzero inner products are

$$\begin{aligned}\langle s_1|s_4\rangle &= \langle \uparrow^1|\downarrow^1\rangle\langle \uparrow^2|\downarrow^2\rangle + \langle \uparrow^1|\downarrow^2\rangle\langle \uparrow^2|\downarrow^1\rangle = J^{11}J^{22} + J^{12}J^{21}, \\ \langle s_4|s_1\rangle &= \langle \downarrow^1|\uparrow^1\rangle\langle \downarrow^2|\uparrow^2\rangle + \langle \downarrow^1|\uparrow^2\rangle\langle \downarrow^2|\uparrow^1\rangle = J^{11}J^{22} + J^{12}J^{21}, \\ \langle s_2|s_2\rangle &= \langle \uparrow^1|\downarrow^2\rangle\langle \downarrow^2|\uparrow^1\rangle = J^{12}J^{21}, \quad \langle s_2|s_3\rangle = \langle \uparrow^1|\downarrow^1\rangle\langle \downarrow^2|\uparrow^2\rangle = J^{11}J^{22}, \\ \langle s_3|s_2\rangle &= \langle \downarrow^1|\uparrow^1\rangle\langle \uparrow^2|\downarrow^2\rangle = J^{11}J^{22}, \quad \langle s_3|s_3\rangle = \langle \downarrow^1|\uparrow^2\rangle\langle \uparrow^2|\downarrow^1\rangle = J^{12}J^{21}.\end{aligned}\quad (3.46)$$

The metric in the space of $|s_i\rangle$'s is real, symmetric and its determinant is $(\det J)(J^{11}J^{22} + J^{12}J^{21})^3$, where $\det J = J^{11}J^{22} - J^{12}J^{21}$. For an orthogonal matrix J^{AB} , $\det J \neq 0$ and the metric is nonsingular only if $J^{11}J^{22} + J^{12}J^{21} \neq 0$.

We will not dwell more on this, although this may be worth investigating further.

IV. SYMPLECTIC INNER PRODUCTS

We want to study ‘‘symplectically flavored’’ ghost spins. We introduce the symplectic structure by defining the elemental inner products with an antisymmetric matrix:

$$\langle \uparrow^A|\downarrow^B\rangle = i\Omega^{AB}, \quad \langle \downarrow^A|\uparrow^B\rangle = i\Omega^{AB}, \quad \langle \uparrow^A|\uparrow^B\rangle = 0 = \langle \downarrow^A|\downarrow^B\rangle; \quad A, B = 1, \dots, 2N, \quad (4.1)$$

where Ω^{AB} is a symplectic form, which is antisymmetric, i.e., $\Omega^{AB} = -\Omega^{BA}$. We will take the only nonzero elements as

$$\Omega^{12} = 1 = -\Omega^{21}, \quad \Omega^{34} = 1 = -\Omega^{43}, \quad \dots, \quad \Omega^{2N-1, 2N} = 1 = -\Omega^{2N, 2N-1}. \quad (4.2)$$

For a single symplectically flavored ghost-spin there are 2^{2N} distinct configurations comprising the basis states $|s_1\rangle, \dots, |s_{2^{2N}}\rangle$ and a generic state is

$$|\psi\rangle = \psi^{s_i}|s_i\rangle \Rightarrow \langle\psi|\psi\rangle = (\psi^{s_j})^* \psi^{s_i} \langle s_j|s_i\rangle. \quad (4.3)$$

We define inner products $\langle s_j|s_i\rangle$ between the basis states as

$$\langle s_i|s_j\rangle = \frac{1}{(2N)!} \sum \epsilon_{A_1 A_2 \dots A_{2N}} \epsilon_{B_1 B_2 \dots B_{2N}} \langle s_i^{A_1}|s_j^{B_1}\rangle \langle s_i^{A_2}|s_j^{B_2}\rangle \dots \langle s_i^{A_{2N}}|s_j^{B_{2N}}\rangle, \quad (4.4)$$

where $i, j = 1, 2, \dots, 2^{2N}$ label the configurations, $A_1, B_1, \dots = 1, 2, \dots, 2N$ label the flavors and $\epsilon_{A_1 A_2 \dots A_{2N}}$ is the totally symmetric tensor with $\epsilon_{123\dots 2N} = 1$ and $\epsilon_{A_1 A_2 \dots A_{2N}}$ vanishes if any two labels are the same. Thus as in (3.3), $\epsilon_{A_1 A_2 \dots A_{2N}}$ ensures that each elemental state $|s_j^A\rangle$ in $|s_j\rangle$ is paired with another corresponding elemental state in $\langle s_i|$.

Let us consider first a single ghost-spin with 2 flavors ($N = 1$): then the distinct configurations comprise the four basis states

$$|s_1\rangle = |\uparrow^1_1\rangle, \quad |s_2\rangle = |\uparrow^1_2\rangle, \quad |s_3\rangle = |\downarrow^1_1\rangle, \quad |s_4\rangle = |\downarrow^1_2\rangle, \quad (4.5)$$

i.e., in $|s_1\rangle$, the first flavor is $\uparrow^1$ and the second flavor is $\uparrow^2$, and likewise for $|s_2\rangle, |s_3\rangle, |s_4\rangle$. The nonzero elemental inner products in (4.1) are

$$\langle \uparrow^1|\downarrow^2\rangle = i = \langle \downarrow^1|\uparrow^2\rangle, \quad \langle \uparrow^2|\downarrow^1\rangle = -i = \langle \downarrow^2|\uparrow^1\rangle \quad (4.6)$$

and the inner products (4.4) between the configurations simplify to

$$\langle s_i|s_j\rangle = \langle s_i^1|s_j^2\rangle \langle s_i^2|s_j^1\rangle. \quad (4.7)$$

Then nonzero inner products for the configurations (3.4) are

$$\begin{aligned}\langle s_1|s_4\rangle &= \langle \uparrow^1|\downarrow^2\rangle\langle \uparrow^2|\downarrow^1\rangle = 1, & \langle s_4|s_1\rangle &= \langle \downarrow^1|\uparrow^2\rangle\langle \downarrow^2|\uparrow^1\rangle = 1, \\ \langle s_2|s_2\rangle &= \langle \uparrow^1|\downarrow^2\rangle\langle \downarrow^2|\uparrow^1\rangle = 1, & \langle s_3|s_3\rangle &= \langle \downarrow^1|\uparrow^2\rangle\langle \uparrow^2|\downarrow^1\rangle = 1.\end{aligned}\quad (4.8)$$

These give a real, symmetric and nonsingular metric in the space of configurations $|s_i\rangle$. Based on these inner products, we can write the norm for the generic state as

$$|\psi\rangle = c_i |s_i\rangle: \langle\psi|\psi\rangle = c_1^* c_4 + c_4^* c_1 + |c_2|^2 + |c_3|^2. \quad (4.9)$$

Likewise for a single ghost-spin with 4 flavors ($N = 2$), there are 16 distinct configurations comprising of basis states (3.9). The nonzero elemental inner products with $\Omega^{12} = 1$ and $\Omega^{34} = 1$ are

$$\begin{aligned} \langle\uparrow^1|\downarrow^2\rangle &= i = \langle\downarrow^1|\uparrow^2\rangle, & \langle\uparrow^2|\downarrow^1\rangle &= -i = \langle\downarrow^2|\uparrow^1\rangle, \\ \langle\uparrow^3|\downarrow^4\rangle &= i = \langle\downarrow^3|\uparrow^4\rangle, & \langle\uparrow^4|\downarrow^3\rangle &= -i = \langle\downarrow^4|\uparrow^3\rangle. \end{aligned} \quad (4.10)$$

Since these are the only nonzero elemental inner products, the inner products (4.4) for the basis states (3.9) reduce to

$$\begin{aligned} \langle s_i | s_j \rangle &= \frac{1}{4!} \sum \epsilon_{A_1 A_2 A_3 A_4} \epsilon_{\tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_4} \langle s_i^{A_1} | s_j^{\tilde{A}_1} \rangle \langle s_i^{A_2} | s_j^{\tilde{A}_2} \rangle \\ &\quad \times \langle s_i^{A_3} | s_j^{\tilde{A}_3} \rangle \langle s_i^{A_4} | s_j^{\tilde{A}_4} \rangle, \end{aligned} \quad (4.11)$$

where $\tilde{A}_k = |\Omega^{\tilde{A}_k A_k}| A_k$, e.g., for $A_1 = 1, \tilde{A}_1 = 2$, for $A_2 = 4, \tilde{A}_2 = 3$, etc. As there are only $4!$ such nonzero terms in the above inner product, it becomes

$$\begin{aligned} \langle s_i | s_j \rangle &= \epsilon_{1234} \epsilon_{2143} \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle \langle s_i^3 | s_j^4 \rangle \langle s_i^4 | s_j^3 \rangle \\ &= \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle \langle s_i^3 | s_j^4 \rangle \langle s_i^4 | s_j^3 \rangle. \end{aligned} \quad (4.12)$$

The nonzero inner products between $|s_i\rangle$'s computed using this formula can be written compactly as

$$\langle s_i | \tilde{s}_j \rangle = 1, \quad (4.13)$$

where $|\tilde{s}_j\rangle$ is defined such that if the A th flavor entry in $|s_i\rangle$ is $\uparrow^A$ (or $\downarrow^A$) then the $\tilde{A}$ -th flavor entry in $|\tilde{s}_j\rangle$ is $\downarrow^{\tilde{A}}$ (or $\uparrow^{\tilde{A}}$) for $\tilde{A} = |\Omega^{\tilde{A} A}| A$. We see that the metric $\langle s_i | s_j \rangle$ is real, symmetric and nonsingular.

We can generalize this to $2N$ flavors, where the nonzero elemental inner products are

$$\begin{aligned} \langle\uparrow^1|\downarrow^2\rangle &= i, & \langle\uparrow^3|\downarrow^4\rangle &= i, & \dots, & \langle\uparrow^{2N-1}|\downarrow^{2N}\rangle &= i, \\ \langle\downarrow^1|\uparrow^2\rangle &= i, & \langle\downarrow^3|\uparrow^4\rangle &= i, & \dots, & \langle\downarrow^{2N-1}|\uparrow^{2N}\rangle &= i, \\ \langle\uparrow^2|\downarrow^1\rangle &= -i, & \langle\uparrow^4|\downarrow^3\rangle &= -i, & \dots, & \langle\uparrow^{2N}|\downarrow^{2N-1}\rangle &= -i, \\ \langle\downarrow^2|\uparrow^1\rangle &= -i, & \langle\downarrow^4|\uparrow^3\rangle &= -i, & \dots, & \langle\downarrow^{2N}|\uparrow^{2N-1}\rangle &= -i. \end{aligned} \quad (4.14)$$

Then the inner products (4.4) become

$$\begin{aligned} \langle s_i | s_j \rangle &= \frac{1}{(2N)!} \sum \epsilon_{A_1 A_2 \dots A_{2N}} \epsilon_{\tilde{A}_1 \tilde{A}_2 \dots \tilde{A}_{2N}} \langle s_i^{A_1} | s_j^{\tilde{A}_1} \rangle \langle s_i^{A_2} | s_j^{\tilde{A}_2} \rangle \dots \langle s_i^{A_{2N}} | s_j^{\tilde{A}_{2N}} \rangle, \\ &= \epsilon_{1234 \dots 2N-1 2N} \epsilon_{2143 \dots 2N 2N-1} \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle \dots \langle s_i^{2N-1} | s_j^{2N} \rangle \langle s_i^{2N} | s_j^{2N-1} \rangle, \\ &= \langle s_i^1 | s_j^2 \rangle \langle s_i^2 | s_j^1 \rangle \dots \langle s_i^{2N-1} | s_j^{2N} \rangle \langle s_i^{2N} | s_j^{2N-1} \rangle, \end{aligned} \quad (4.15)$$

where $\tilde{A}_k = |\Omega^{\tilde{A}_k A_k}| A_k$. Using the elemental inner products, we see that the nonzero inner products are $\langle s_i | \tilde{s}_j \rangle = 1$, with $|\tilde{s}_j\rangle$ as defined earlier.

Thus the norm of a generic state $|\psi\rangle = \psi^{s_i} |s_i\rangle$ is

$$\langle\psi|\psi\rangle = (\psi^{s_j})^* \psi^{s_i} \langle s_j | s_i \rangle = (\psi^{s_j})^* \psi^{s_i} \langle s_j^1 | s_i^2 \rangle \langle s_j^2 | s_i^1 \rangle \dots \langle s_j^{2N-1} | s_i^{2N} \rangle \langle s_j^{2N} | s_i^{2N-1} \rangle. \quad (4.16)$$

Along the same lines, the general inner product between any two states is

$$\langle\psi_1|\psi_2\rangle = (\psi_1^{s_j})^* \psi_2^{s_i} \langle s_j | s_i \rangle = (\psi_1^{s_j})^* \psi_2^{s_i} \langle s_j^1 | s_i^2 \rangle \langle s_j^2 | s_i^1 \rangle \dots \langle s_j^{2N-1} | s_i^{2N} \rangle \langle s_j^{2N} | s_i^{2N-1} \rangle. \quad (4.17)$$

Correlated ghost-spin states: We want to now construct correlated ghost-spin states that are positive norm and positive entanglement, along the lines of the discussion for $O(N)$ flavored cases in Sec. III B. This is most transparent in a diagonal basis where positive and negative norm states are manifest. For concreteness, let us consider the basis states (4.5) for a single ghost-spin with 2 flavors, i.e., $N = 1$. The norm (4.9) for a generic state can be recast using a diagonal basis $|s_\pm\rangle, |s_2\rangle, |s_3\rangle$ as

$$|s_\pm\rangle = \frac{1}{\sqrt{2}}(|s_1\rangle \pm |s_4\rangle): \quad \langle\psi|\psi\rangle = |c_2|^2 + |c_3|^2 + |c_+|^2 - |c_-|^2. \quad (4.18)$$

Thus there are 3 basis states with positive norm and one with negative norm. This can be carried out for more flavors as well. For $N = 2$ for instance, we have 16 basis states which can be recast as 10 positive norm and 6 negative norm states, using (4.13): besides the states with $\langle s_i | s_i \rangle \neq 0$, there are states with off-diagonal inner products like $|s_{1,4}\rangle$ above whose linear combinations then add to the set of diagonal positive norm states. Note that the numbers of positive and negative norm basis states are not equal. With general N , i.e., $A, B = 1, \dots, 2N$, it can be seen that there are 2^{2N} basis states in all: of these there are $\frac{2^{2N}+2^N}{2}$ positive norm states and $\frac{2^{2N}-2^N}{2}$ negative norm states (this is easily verified for $N = 1, 2$ above). For large N , we see that the number of positive and negative norm states become asymptotically equal.

In terms of such a diagonal basis, we can consider 2 ghost-spins and explicitly construct correlated ghost-spin states similar structurally to (3.39) in the $O(N)$ flavored case. Let us label these diagonal basis states for a single ghost spin as $|s_i\rangle$ (which should not be confused with the earlier nondiagonal $|s_i\rangle$ basis). Then the 2-ghost-spin states can be made from the 2^{4N} basis states $|s_i\rangle|s_j\rangle$ obtained by tensor products of the single ghost-spin states. The general state and its norm are then similar in structure to (3.38) for the $O(N)$ case, Sec. III B. Correlated ghost-spin states can then be constructed as in (3.39) giving $|\psi^{\text{corr}}\rangle = \sum \psi^{s_i, s_j} |s_i\rangle|s_j\rangle$: it can be seen that these are positive norm and positive entanglement as in (3.39), (3.41). This subspace has dimension 2^{2N} , the number of basis states. Since the details here are very similar to that in Sec. III B, we will not describe them further here.

It is worth noting that the symplectic invariance is at the level of the elemental ghost-spin basis states $\{|\uparrow^A\rangle, |\downarrow^A\rangle\}$: generic basis states $|v_i\rangle = v_i^{\uparrow^A} |\uparrow^A\rangle + v_i^{\downarrow^A} |\downarrow^A\rangle$, have inner product

$$\begin{aligned} \langle v_1 | v_2 \rangle &= (v_1^{\uparrow^A})^* v_2^{\downarrow^B} \langle \uparrow^A | \downarrow^B \rangle + (v_1^{\downarrow^A})^* v_2^{\uparrow^B} \langle \downarrow^A | \uparrow^B \rangle \\ &= i[(v_1^{\uparrow^A})^* v_2^{\downarrow^B} \Omega^{AB} + (v_1^{\downarrow^A})^* v_2^{\uparrow^B} \Omega^{AB}], \end{aligned} \quad (4.19)$$

which is invariant under symplectic transformations. To see this explicitly, consider two flavors ($N = 1$) for simplicity. Then a symplectic transformation by a real pseudo-orthogonal matrix $R \in Sp(2)$ is $R^T \Omega R = \Omega$, $R \Omega R^T = \Omega$, $R^{-1} = -\Omega R^T \Omega$, where Ω is the symplectic form with $\Omega^{12} = 1 = -\Omega^{21}$, and $\Omega^{-1} = -\Omega$. Thus $(v_1^{\uparrow^A})^* v_2^{\downarrow^B} \Omega^{AB}$ and $(v_1^{\downarrow^A})^* v_2^{\uparrow^B} \Omega^{AB}$ are invariant under $|\uparrow^A\rangle \rightarrow R^{AB} |\uparrow^B\rangle$, $|\downarrow^A\rangle \rightarrow R^{AB} |\downarrow^B\rangle$, i.e.,

$$\begin{aligned} (\tilde{v}_1^{\uparrow^A})^* \Omega^{AB} \tilde{v}_2^{\downarrow^B} &= (v_1^{\uparrow^C})^* R^{CA} \Omega^{AB} R^{BD} v_2^{\downarrow^D} = (v_1^{\uparrow^C})^* \Omega^{CD} v_2^{\downarrow^D}, \\ (\tilde{v}_1^{\downarrow^A})^* \Omega^{AB} \tilde{v}_2^{\uparrow^B} &= (v_1^{\downarrow^C})^* R^{CA} \Omega^{AB} R^{BD} v_2^{\uparrow^D} = (v_1^{\downarrow^C})^* \Omega^{CD} v_2^{\uparrow^D}, \end{aligned} \quad (4.20)$$

where we have used $R^{CA} \Omega^{AB} R^{BD} = R^T \Omega R = \Omega$.

The elemental inner products (4.6) are consistent with (and motivated by) an operator algebra along with states, defined as (these arise in theories of symplectic fermions [25])

$$\begin{aligned} \{\sigma_b^A, \sigma_c^B\} &= i\Omega^{AB} \hat{K}; & |\uparrow^A\rangle &= \sigma_c^A |\downarrow\rangle, & \langle \downarrow^A | &= \langle \uparrow | \sigma_b^A, \\ |\downarrow^A\rangle &= \sigma_b^A |\uparrow\rangle, & \langle \uparrow^A | &= \langle \downarrow | \sigma_c^A, \end{aligned} \quad (4.21)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are ghost-spin states with $\langle \uparrow | \downarrow \rangle = 1 = \langle \downarrow | \uparrow \rangle$. The hermiticity of $\{\sigma_b^A, \sigma_c^B\}$ for Hermitian σ_b^A , σ_c^B and real Ω^{AB} gives $\hat{K}^\dagger = -\hat{K}$ i.e., $\hat{K}$ is anti-Hermitian. This anti-Hermitian operator leads

$$(\langle \uparrow | \hat{K} | \downarrow \rangle)^\dagger = \langle \downarrow | \hat{K}^\dagger | \uparrow \rangle = -\langle \downarrow | \hat{K} | \uparrow \rangle, \quad (4.22)$$

which implies that for $\langle \uparrow | \hat{K} | \downarrow \rangle = 1$, $\langle \downarrow | \hat{K} | \uparrow \rangle = -1$. Using these we get the elemental inner products as

$$\begin{aligned} \langle \uparrow^A | \downarrow^B \rangle &= \langle \downarrow | \sigma_c^A \sigma_b^B | \uparrow \rangle = i\Omega^{BA} \langle \downarrow | \hat{K} | \uparrow \rangle = i\Omega^{AB}, \\ \langle \downarrow^A | \uparrow^B \rangle &= \langle \uparrow | \sigma_b^A \sigma_c^B | \downarrow \rangle = i\Omega^{AB} \langle \uparrow | \hat{K} | \downarrow \rangle = i\Omega^{AB}. \end{aligned} \quad (4.23)$$

It is then possible to construct ghost-spin chains with nearest neighbor interactions between operators at neighboring lattice sites, somewhat similar to the ghost-spin chain for the bc -ghost CFTs. However the continuum limit is less clear in this case, in part due to technical difficulties such as the construction of the Jordan-Wigner transformation to obtain fermionic versions of the $\sigma_{b,c}$ operators above which anticommute with each other (the $\sigma_{b,c}$ are bosonic spinlike operators commuting at neighboring lattice sites while anticommuting at the same site). Note however that the case with $N = 1$ has structure similar to that appearing in the theory of anticommuting scalars: this is a logarithmic CFT in 2-dimensions [24–30]. So perhaps the continuum limit here gives symplectic fermions $\int \Omega_{AB} \partial \phi^A \partial \phi^B$: we hope to explore this further.

V. N IRREDUCIBLE LEVELS

In this section, we consider a generalization of ghost spins that consists of N irreducible levels, defined as

$$\langle e_i | e_i \rangle = 0, \quad \langle e_i | e_j \rangle = 1 \quad \forall i \neq j; \quad i, j = 1, 2, \dots, N. \quad (5.1)$$

For $N = 2$, the basis states $|e_1\rangle, |e_2\rangle$ are identical to the $|\uparrow\rangle, |\downarrow\rangle$ basis states, and this system reduces to the 2-level ghost spin reviewed in Sec. II. Flavored generalizations can be constructed by adding additional flavor indices to these, along the lines we have described for 2-level ghost spins in the previous sections: we will not do so here however.

Using the inner products above, it is clear that there are various negative norm states here as well: e.g., $|e_i\rangle - |e_j\rangle$ has norm -2 . Using a diagonal basis helps as in the 2-level case to identify positive and negative norm states clearly. This can be done using the transformations in Appendix A: we can choose an orthonormal basis where the basis states and their inner products are

$$\begin{aligned} |\alpha\rangle &\equiv \{|+\rangle, |2\rangle, \dots, |N\rangle\}; \\ \langle\alpha|\beta\rangle &= \eta_{\alpha\beta}; \quad \eta_{++} = 1, \quad \eta_{22} = \eta_{33} = \dots = \eta_{NN} = -1, \quad \eta_{\alpha\beta} = 0 \quad \forall \alpha \neq \beta, \\ \text{i.e. } \langle++\rangle &= 1, \quad \langle\alpha|\alpha\rangle = -1, \quad \alpha = 2, \dots, N. \end{aligned} \quad (5.2)$$

Then the generic state and its norm in both bases are

$$|\psi\rangle = \psi^i |e_i\rangle; \quad \langle\psi|\psi\rangle = (\psi^i)^* \psi^j \langle e_i | e_j \rangle = \sum_{i \neq j} (\psi^i)^* \psi^j, \quad (5.3)$$

$$|\psi\rangle = \psi^\alpha |\alpha\rangle; \quad \langle\psi|\psi\rangle = (\psi^\alpha)^* \psi^\beta \langle\alpha|\beta\rangle = (\psi^\alpha)^* \psi^\beta \eta_{\alpha\beta} = |\psi^+|^2 - \sum_{\alpha=2}^N |\psi^\alpha|^2. \quad (5.4)$$

To illustrate this, let us consider $N = 3$. The generic state and its norm are

$$\begin{aligned} |\psi\rangle &= \psi^1 |e_1\rangle + \psi^2 |e_2\rangle + \psi^3 |e_3\rangle = \psi^+ |+\rangle + \psi^2 |2\rangle + \psi^3 |3\rangle, \\ \langle\psi|\psi\rangle &= (\psi^1)^* \psi^2 + (\psi^2)^* \psi^1 + (\psi^1)^* \psi^3 + (\psi^3)^* \psi^1 + (\psi^2)^* \psi^3 + (\psi^3)^* \psi^2 \\ &= |\psi^+|^2 - |\psi^2|^2 - |\psi^3|^2. \end{aligned} \quad (5.5)$$

In some sense, this is a ghost-spin generalization of the N -level spins that arise in the Heisenberg spin chain: perhaps appropriate interaction Hamiltonians for ghost-spin chains on a 1-dim lattice can be studied along those lines.

Correlated ghost spins and entanglement: We want to construct correlated ghost-spin states analogous to the discussion in Sec. III B. So consider a system of two ghost spins with N irreducible levels. The orthonormal basis for this system is

$$|u_A u_B\rangle \equiv |\alpha\rangle |\beta\rangle \equiv |\alpha\beta\rangle \quad \forall \alpha, \beta = +, 2, \dots, N, \quad (5.6)$$

where each $|\alpha\rangle$ is a single ghost-spin basis state in (5.2). A generic state $|\psi\rangle = \psi^{\alpha\beta} |\alpha\beta\rangle$ has a norm $\langle\psi|\psi\rangle = \eta_{\alpha\kappa} \eta_{\beta\lambda} (\psi^{\alpha\beta})^* \psi^{\kappa\lambda}$, which can be expanded as

$$\langle\psi|\psi\rangle = \left(\sum_{\alpha} |\psi^{\alpha\alpha}|^2 + \sum_{\alpha, \beta \neq +} |\psi^{\alpha\beta}|^2 \right) - \left(\sum_{\alpha \neq +} (|\psi^{+\alpha}|^2 + |\psi^{\alpha+}|^2) \right). \quad (5.7)$$

We see a manifest division between the positive and negative norm subspaces. For $N = 3$, we can see this explicitly as

$$\langle\psi|\psi\rangle = (|\psi^{++}|^2 + |\psi^{22}|^2 + |\psi^{33}|^2 + |\psi^{23}|^2 + |\psi^{32}|^2) - (|\psi^{+2}|^2 + |\psi^{2+}|^2 + |\psi^{+3}|^2 + |\psi^{3+}|^2). \quad (5.8)$$

For general N , by tracing over the second ghost spin, the reduced density matrix is

$$\rho_A = (\rho_A)^{\alpha\kappa} |\alpha\rangle \langle\kappa|; \quad (\rho_A)^{\alpha\kappa} = \psi^{\alpha\beta} (\psi^{\kappa\beta})^* \eta_{\beta\beta}. \quad (5.9)$$

The mixed index reduced density matrix is $(\rho_A)_{\beta}^{\alpha} = \eta_{\beta\kappa} (\rho_A)^{\kappa\alpha} = \eta_{\beta\kappa} \eta_{\lambda\lambda} \psi^{\kappa\lambda} (\psi^{\alpha\lambda})^*$.

Correlated ghost-spins: From the norm above we see that the states $|++\rangle, |22\rangle, \dots, |NN\rangle$ span the subspace of correlated ghost-spin states, where a generic correlated ghost-spin state is

$$|\psi\rangle = \psi^{\alpha\alpha} |\alpha\alpha\rangle; \quad \langle\psi|\psi\rangle = |\psi^{++}|^2 + |\psi^{22}|^2 + \dots + |\psi^{NN}|^2. \quad (5.10)$$

Entanglement pattern in a general state : Consider a slightly more general state

$$|\psi\rangle = \sum_{\alpha=+}^N \psi^{\alpha\alpha} |\alpha\alpha\rangle + \sum_{\beta=2}^N (\psi^{+\beta} |+\beta\rangle + \psi^{\beta+} |\beta+\rangle), \quad (5.11)$$

whose norm is

$$\langle\psi|\psi\rangle = \sum_{\alpha=+}^N |\psi^{\alpha\alpha}|^2 - \sum_{\beta=2}^N (|\psi^{+\beta}|^2 + |\psi^{\beta+}|^2). \quad (5.12)$$

The off-diagonal components of the reduced density matrix are

$$\begin{aligned} (\rho_A)^{+\alpha} &= \psi^{++}\psi^{\alpha+*} - \psi^{+\alpha}\psi^{\alpha\alpha*}, \quad \forall \alpha = 2, \dots, N, \quad \alpha \neq +, \\ (\rho_A)^{\alpha\beta} &= \psi^{\alpha+}\psi^{\beta+*}, \quad \forall \alpha, \beta = 2, \dots, N, \quad \alpha \neq +, \quad \beta \neq +. \end{aligned} \quad (5.13)$$

From $(\rho_A)^{\alpha\beta} = 0$, we see that only one of $\psi^{\alpha+}$ is nonzero, i.e., $\psi^{2+} \neq 0$, $\psi^{\alpha+} = 0$, $\alpha = 3, \dots, N$. Then $(\rho_A)^{+\alpha} = 0$ gives $\psi^{+2} \neq 0$ and $\psi^{+\alpha} = 0$, $\alpha = 3, \dots, N$.

So we consider the state

$$|\psi\rangle = \psi^{++} |++\rangle + \psi^{22} |22\rangle + \dots + \psi^{NN} |NN\rangle + \psi^{+2} |+2\rangle + \psi^{2+} |2+\rangle, \quad (5.14)$$

whose norm is

$$\langle\psi|\psi\rangle = |\psi^{++}|^2 + \dots + |\psi^{NN}|^2 - |\psi^{+2}|^2 - |\psi^{2+}|^2. \quad (5.15)$$

The nonzero components of the reduced density matrix are

$$\begin{aligned} (\rho_A)^{++} &= |\psi^{++}|^2 - |\psi^{+2}|^2, & (\rho_A)^{22} &= |\psi^{2+}|^2 - |\psi^{22}|^2, \\ (\rho_A)^{+2} &= \psi^{++}\psi^{2+*} - \psi^{+2}\psi^{22*}, & (\rho_A)^{\alpha\alpha} &= -|\psi^{\alpha\alpha}|^2, \quad \alpha = 3, \dots, N. \end{aligned} \quad (5.16)$$

Choosing $\psi^{2+*} = \frac{\psi^{+2}\psi^{22*}}{\psi^{++}}$ and defining $x \equiv |\psi^{++}|^2 - |\psi^{+2}|^2$ and $r \equiv \frac{|\psi^{22}|^2}{|\psi^{++}|^2} > 0$, the mixed-index components of ρ_A are

$$(\rho_A)^+_+ = x, \quad (\rho_A)^2_2 = xr, \quad (\rho_A)^{\alpha}_{\alpha} = |\psi^{\alpha\alpha}|^2, \quad \alpha = 3, \dots, N \quad (5.17)$$

and

$$\langle\psi|\psi\rangle = x + xr + |\psi^{33}|^2 + \dots + |\psi^{NN}|^2 = \pm 1. \quad (5.18)$$

Now depending on if x is positive or negative we have the following three cases.

- (i) If $x > 0$, $|\psi\rangle$ has necessarily positive norm and $\langle\psi|\psi\rangle = 1$ implies $0 < (\rho_A)^{\alpha}_{\alpha} < 1$ for all $\alpha = +, 2, \dots, N$ giving $S_A > 0$.
- (ii) If $x < 0$, the norm of $|\psi\rangle$ can be positive or negative.
- (i) For positive norm, i.e., $\langle\psi|\psi\rangle = -|x| - |x|r + \sum_{\alpha=3}^N |\psi^{\alpha\alpha}|^2 = 1$, we get

$$\begin{aligned} S_A &= |x| \log |x| + |x|r \log |x|r - |\psi^{33}|^2 \log(|\psi^{33}|^2) - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \\ &= |x| \log |x| + |x|r \log |x|r - \left(1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \log \left(1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \\ &\quad - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r). \end{aligned} \quad (5.19)$$

We see that $\text{Im}(S_A)$ is not constant and $\text{Re}(S_A) < 0$ when

$$|x| \log |x| + |x|r \log |x|r < \left(1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \log \left(1 + |x| + |x|r + \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) + \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2). \quad (5.20)$$

(ii) For a negative norm state $|\psi\rangle$, i.e., $\langle\psi|\psi\rangle = -|x| - |x|r + \sum_{\alpha=3}^N |\psi^{\alpha\alpha}|^2 = -1$, we get

$$\begin{aligned} S_A &= |x| \log |x| + |x|r \log |x|r - |\psi^{33}|^2 \log(|\psi^{33}|^2) - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \\ &= |x| \log |x| + |x|r \log |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \\ &\quad - \left(-1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \log \left(-1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \end{aligned} \quad (5.21)$$

$\text{Im}(S_A)$ is constant if $|x| + |x|r = c$, where c is a constant and $c > 1$ (from the norm). Then $\text{Re}(S_A)$ becomes

$$\begin{aligned} \text{Re}(S_A) &= |x| \log |x| + (c - |x|) \log(c - |x|) - \left(c - 1 - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \log \left(c - 1 - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \\ &\quad - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2). \end{aligned} \quad (5.22)$$

We see that $\text{Re}(S_A) < 0$ for those values of $|x|$, $c > 1$, $\psi^{\alpha\alpha}$ which satisfy

$$|x| \log |x| + (c - |x|) \log(c - |x|) < \left(c - 1 - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \log \left(c - 1 - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) + \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2). \quad (5.23)$$

VI. DISCUSSION

We have constructed N -level generalizations of the 2-level ghost spins in [1–3]. These include (i) a flavored generalization comprising N copies of the ghost-spin system and corresponding ghost-spin chains which lead to 2-dim bc -ghost CFTs with $O(N)$ flavor symmetry, (ii) a spin-glass type coupling in flavor space, (iii) a symplectic generalization involving antisymmetric inner products between the elemental ghost spins, and (iv) an irreducible ghost-spin system with N internal levels. We have studied entanglement properties in these cases: among other things, these show the existence of positive norm states in two copies of ghost-spin ensembles obtained by entangling identical ghost spins from each copy: these are akin to the correlated ghost-spin states in [2,3], and exhibit positive entanglement.

We now describe briefly some of the motivations from dS/CFT, in particular [15], for the studies here. Generalizations of gauge/gravity duality for de Sitter space or dS/CFT involve conjectured dual hypothetical Euclidean nonunitary CFTs living on the future boundary $\mathcal{I}^+$ [4–6]. Using the dictionary $\Psi_{\text{dS}} = Z_{\text{CFT}}$ [6], where Ψ_{dS} is the late-time Hartle-Hawking wave function of the universe with appropriate boundary conditions and Z_{CFT} the dual CFT partition function, the dual CFT $_d$ energy-momentum tensor correlators reveal central charge coefficients $\mathcal{C}_d \sim i^{1-d} \frac{d^{d-1}}{G_{d+1}}$ in dS $_{d+1}$ (effectively analytic continuations from AdS/CFT). This is real and negative in dS $_4$, with $\mathcal{C}_3 \sim -\frac{R_{\text{dS}}^2}{G_4}$ so that dS $_4$ /CFT $_3$ is reminiscent of ghostlike nonunitary theories. Bulk expectation values are of the form $\langle\phi_k\phi_{k'}\rangle \sim \int D\phi\phi_k\phi_{k'}|\Psi_{\text{dS}}|^2$. This involves the probability

$|\Psi_{\text{dS}}|^2 = \Psi_{\text{dS}}^* \Psi_{\text{dS}}$, which suggests that bulk de Sitter physics involves two copies of the dual CFT— $\text{CFT}_F \times \text{CFT}_P$ on the future and past boundaries. This is unlike in AdS/CFT where $Z_{\text{bulk}} = Z_{\text{bdry}}$ implies boundary correlators can be obtained as a limit of bulk ones. In the dS case, while $Z_{\text{CFT}} = \Psi_{\text{dS}}$ of a single dual CFT copy at I^+ can be used to obtain boundary correlators (e.g., $\langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z_{\text{CFT}}}{\delta \varphi_k^0 \delta \varphi_{k'}^0}$ for operators O_k dual to modes φ_k), bulk observables require $|\Psi_{\text{dS}}|^2$, the bulk probability, and so two copies of the dual Z_{CFT} . This dovetails with the structure of extremal surfaces and entanglement as we see below.

In AdS, surfaces anchored at one end of a subsystem dip into the bulk radial direction and then begin to return to the boundary at turning points. In dS, the boundary at I^+ is spacelike and surfaces dip into the time direction which ends up making their structure quite different, as studied in [16]. For instance, considering the dS Poincaré slicing $ds^2 = \frac{R_{\text{dS}}^2}{\tau^2} (-d\tau^2 + dx_i^2)$, a strip subsystem on some boundary Euclidean time $w = \text{const}$ slice of I^+ with width along x gives a bulk extremal surface $x(\tau)$ described by $\dot{x}^2 \equiv (\frac{dx}{d\tau})^2 = \frac{B^2 \tau^{2d-2}}{1+B^2 \tau^{2d-2}} (B^2 > 0)$. w, x can be taken as any of the x_i (so the boundary Euclidean time slice is not sacrosanct). Compared with the AdS case, the denominator here crucially has a relative minus sign. Thus there is no *real* turning point here where the surface starting at I^+ begins to turn back toward I^+ : this requires $|\dot{x}| \rightarrow \infty$ while here $|\dot{x}| \leq 1$. There are also complex extremal surfaces however, which exhibit turning points: these end up amounting to analytic continuation from the AdS Ryu-Takayanagi surfaces. While their interpretation is not entirely conclusive, in dS₄ they have *negative* area, consistent with the negative central charge in dS₄/CFT₃ as mentioned above.

Since surfaces starting at I^+ do not turn back, it is then interesting to ask if they could instead stretch all the way to the past boundary I^- . In [15], connected codim-2 extremal surfaces in the static patch coordinatization of de Sitter space were found stretching from I^+ to I^- passing through the vicinity of the bifurcation region with divergent area $\frac{l^2}{4G_4} \frac{1}{\epsilon}$, where $\epsilon = \frac{c}{\tau}$ is the dimensionless ultraviolet cutoff and the coefficient scales as de Sitter entropy. To elaborate a little, the static patch coordinatization can be recast as $ds^2 = \frac{l^2}{\tau^2} (-\frac{d\tau^2}{1-\tau^2} + (1-\tau^2)dw^2 + d\Omega_{d-1}^2)$, with the future/past universes F/P parametrized by $0 \leq \tau \leq 1$ with horizons at $\tau = 1$, while the Northern/Southern diamonds N/S have $1 < \tau \leq \infty$. The boundaries at $\tau = 0$ are now of the form $R_w \times S^{d-1}$, resembling the Poincaré slicing locally. Setting up the extremization for codim-2 surfaces on boundary Euclidean time slices can be carried out: on S^{d-1} equatorial planes for instance we obtain $\dot{w}^2 = \frac{B^2 \tau^{2d-2}}{1-\tau^2 + B^2 \tau^{2d-2}}$. The minus sign here, reflecting the horizons, makes the structure of these surfaces interesting, drawing parallels with the AdS extremization (we refer to [15] for

further details). The limit $B \rightarrow 0$ gives surfaces passing through the vicinity of the bifurcation region as stated above, with the width Δw approaching all of $I^\pm$. These connected surfaces stretching between $I^\pm$ are akin to rotated versions of the connected surfaces of Hartman, Maldacena [31] in the AdS black hole. This led to the speculation there that dS₄ is approximately dual to an entangled thermofield-double type state of the form

$$|\psi\rangle = \sum \psi^{i_n^F, i_n^P} |i_n^F\rangle |i_n^P\rangle \quad (6.1)$$

akin to the thermofield double [32] dual to AdS black holes. Here $\psi^{i_n^F, i_n^P}$ are coefficients entangling a generic ghost-spin $|i_n^F\rangle$ from CFT_F at I^+ with an identical one $|i_n^P\rangle$ from CFT_P at I^- . The constituent states are schematically continuum versions of N level ghost spins, with N related to dS₄ entropy $\frac{l^2}{4G_4}$. Since bulk time evolution maps configurations at I^- to those at I^+ [5], we have the schematic map $|i_n^P\rangle \rightarrow S[i_n^P, i_n^F] |i_n^F\rangle \equiv |i_n^F\rangle$ where $S[i_n^P, i_n^F]$ is the operator representing bulk time evolution (note that this is a bulk object that is to be distinguished from operators in the CFT representing boundary Euclidean time evolution). If states $|i^P\rangle$ at I^- map faithfully and completely to states $|i^F\rangle$ at I^+ , then S is expected to be a unitary operator (this is also vindicated by the fact that exchanging $I^\pm$ is a bulk symmetry). This suggests that the entangled states (6.1) are unitarily equivalent to similar maximally entangled states $|\psi\rangle = \sum \psi^{i_n^F, i_n^F} |i_n^F\rangle |i_n^F\rangle$ in two CFT_F copies of the ghost-CFT solely at I^+ . The state (6.1) is akin to a correlated ghost-spin state with an even number of ghost spins, as discussed in [2,3]. It necessarily has positive norm $\sum_{i_n^F, i_n^P} \gamma_{i_n^F, i_n^P}^F \gamma_{i_n^F, i_n^P}^P \psi^{i_n^F, i_n^P} (\psi^{i_n^F, i_n^P})^* \rightarrow \sum_{i_n} |\psi^{i_n^F, i_n^F}|^2$, since we are entangling identical states i_n^F and i_n^P : thus it has positive entanglement, as in [2,3]. Since each constituent state $|i_n^{F,P}\rangle$ is N -level, i.e., with N internal degrees of freedom, the entanglement entropy scales as $N \sim \frac{l^2}{G_4}$. The toy models in Sec. III B of correlated ghost-spin states (3.43) and their entanglement entropy (3.44) are of this form, written explicitly. The state (6.1) is akin to the thermofield double dual to the eternal AdS black hole [32]. This suggests the speculation that 4-dim de Sitter space is perhaps approximately dual to CFT_{F} \times \text{CFT}_P in the entangled state (6.1) and the generalized entanglement entropy of the latter scales as de Sitter entropy. (See [33] for another approach to de Sitter entropy based on the dS/dS correspondence [34].)}

The investigations in this paper on N -level generalizations of ghost spins are geared toward constructing microscopic ghost-spin states that reflect the N -level internal structure which might ultimately give rise in appropriate continuum limits to theories such as the $Sp(N)$ ghost-CFT dual to higher spin dS₄ (see also the recent work [14]). As we have seen, the N -level generalizations here do admit

positive norm subsectors of the form of the correlated ghost-spin states indicated in (6.1).

As mentioned in the Introduction, the ghost-spin system has possible applications in gauge theories. The continuum limit of a d -dim ghost-spin system (as in [3] for the 2-dim case) with flavor quantum numbers may be relevant for studying entanglement in gauge theories in a covariant setting. A better understanding of the ghost-spin system and its coupling to ordinary spin systems as in [2,3] generalized to d -dimensions would be an ideal sandbox for understanding covariant formulations of subregion entanglement in gauge theories.

We have been thinking of ghost-spins as microscopic building blocks for ghostlike CFTs, and perhaps more general nonunitary CFTs. The discussions in this paper on ghost-spin chains have recovered 2-dim bc -ghost CFTs with flavor symmetries. The obvious generalization to 3 dimensions of the 2-dim case discussed in Sec. III (and in [3] for the single flavor case) has nearest neighbor hopping-type interactions of the elemental form $h \sim \sum_{nn'} \sigma_{b,\vec{n}}^A \sigma_{c,\vec{n}'}^A$. This contains three σ_b and three σ_c operators at each lattice site, with possible flavor indices reflecting internal flavor symmetries. It would be interesting to study such 3-dim ghost-spin chains toward obtaining 3-dim ghost-CFTs in the continuum limit: so far, we have encountered conceptual difficulties as well as technical ones. We hope to report on this in the future.

ACKNOWLEDGMENTS

It is a pleasure to thank Philip Argyres, Jan de Boer, R. Loganayagam, Juan Maldacena, Rob Myers, Eva Silverstein and Tadashi Takayanagi for useful conversations on [15] and some of the content in this paper. While this work was in progress, K. K. thanks the organizers of the PiTP school “From Qubits to spacetime,” IAS, Princeton; K. N. thanks the organizers of “Strings 2018,” Okinawa, and the longterm Yukawa memorial workshop “New Frontiers in String Theory,” Yukawa Institute, Kyoto, Japan. The research of K. K. and K. N. was supported in part by the International Centre for Theoretical Sciences (ICTS) during a visit for participating in the program—AdS/CFT at 20 and Beyond (Code: ICTS/adscft20/05). We thank the organizers of the Indian Strings Meeting ISM2018, IISER Trivandrum, for hospitality as this work was being finalized. This work is partially supported by a grant to CMI from the Infosys Foundation.

APPENDIX: ORTHONORMAL BASIS FOR N-LEVEL IRREDUCIBLE GHOST SPIN

We give the transformations which transform the defining basis for N -level irreducible ghost spin to an orthonormal basis. For even N -level ghost spin, the transformations to orthonormal basis are

$$\begin{aligned}
 |+\rangle &= \frac{\sum_{i=1}^N |e_i\rangle}{\sqrt{N^2 - N}}, & |2\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle), & \dots, & \left|\frac{N}{2} + 1\right\rangle = \frac{1}{\sqrt{2}}(|e_{N-1}\rangle - |e_N\rangle), \\
 \left|\frac{N}{2} + 2\right\rangle &= \frac{1}{2}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle), & \left|\frac{N}{2} + 3\right\rangle &= \frac{1}{\sqrt{12}}\left(\sum_{i=1}^4 |e_i\rangle - 2(|e_5\rangle + |e_6\rangle)\right), \\
 &\vdots \\
 |N-1\rangle &= \frac{1}{\sqrt{\frac{(N-1)(N-3)}{2}}}\left(\sum_{i=1}^{N-4} |e_i\rangle - \left(\frac{N-1}{2} - 1\right)(|e_{N-3}\rangle + |e_{N-2}\rangle)\right), \\
 |N\rangle &= \frac{1}{\sqrt{\frac{N(N-2)}{2}}}\left(\sum_{i=1}^{N-2} |e_i\rangle - \left(\frac{N}{2} - 1\right)(|e_{N-1}\rangle + |e_N\rangle)\right).
 \end{aligned} \tag{A1}$$

For odd $N + 1$ -level ghost spin (where N is even), the transformations to orthonormal basis as

$$\begin{aligned}
 |+\rangle &= \frac{\sum_{i=1}^{N+1} |e_i\rangle}{\sqrt{N(N+1)}}, & |2\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle), & \dots, & \left|\frac{N}{2} + 1\right\rangle = \frac{1}{\sqrt{2}}(|e_{N-1}\rangle - |e_N\rangle), \\
 \left|\frac{N}{2} + 2\right\rangle &= \frac{1}{2}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle), & \left|\frac{N}{2} + 3\right\rangle &= \frac{1}{\sqrt{12}}\left(\sum_{i=1}^4 |e_i\rangle - 2(|e_5\rangle + |e_6\rangle)\right), \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
|N-1\rangle &= \frac{1}{\sqrt{\frac{(N-1)(N-3)}{2}}} \left(\sum_{i=1}^{N-4} |e_i\rangle - \left(\frac{N-1}{2} - 1 \right) (|e_{N-3}\rangle + |e_{N-2}\rangle) \right), \\
|N\rangle &= \frac{1}{\sqrt{\frac{N(N-2)}{2}}} \left(\sum_{i=1}^{N-2} |e_i\rangle - \left(\frac{N}{2} - 1 \right) (|e_{N-1}\rangle + |e_N\rangle) \right), \\
|N+1\rangle &= \frac{1}{\sqrt{(N+1)^2 - (N+1)}} \left(\sum_{i=1}^N |e_i\rangle - N|e_{N+1}\rangle \right).
\end{aligned} \tag{A2}$$

To illustrate these transformation, we write them explicitly for $N = 3$ and $N = 4$. For a 3-level ghost spin, the orthonormal basis are

$$|+\rangle = \frac{1}{\sqrt{6}}(|e_1\rangle + |e_2\rangle + |e_3\rangle), \quad |2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle), \quad |3\rangle = \frac{1}{\sqrt{6}}(|e_1\rangle + |e_2\rangle - 2|e_3\rangle) \tag{A3}$$

and for $N = 4$ the orthonormal basis are

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{12}}(|e_1\rangle + |e_2\rangle + |e_3\rangle + |e_4\rangle), & |2\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle), \\
|3\rangle &= \frac{1}{\sqrt{2}}(|e_3\rangle - |e_4\rangle), & |4\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle).
\end{aligned} \tag{A4}$$

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- [1] K. Narayan, On dS_4 extremal surfaces and entanglement entropy in some ghost CFTs, *Phys. Rev. D* **94**, 046001 (2016).
- [2] D. P. Jatkar and K. Narayan, Entangled spins and ghost-spins, *Nucl. Phys. B* **922**, 319 (2017).
- [3] D. P. Jatkar and K. Narayan, Ghost-spin chains, entanglement and bc -ghost CFTs, *Phys. Rev. D* **96**, 106015 (2017).
- [4] A. Strominger, The dS / CFT correspondence, *J. High Energy Phys.* **10** (2001) 034.
- [5] E. Witten, Quantum gravity in de Sitter space, *arXiv:hep-th/0106109*.
- [6] J. M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, *J. High Energy Phys.* **05** (2003) 013.
- [7] D. Anninos, T. Hartman, and A. Strominger, Higher spin realization of the dS /CFT correspondence, *Classical Quantum Gravity* **34**, 015009 (2017).
- [8] R. Bousso, A. Maloney, and A. Strominger, Conformal vacua and entropy in de Sitter space, *Phys. Rev. D* **65**, 104039 (2002).
- [9] V. Balasubramanian, J. de Boer, and D. Minic, Notes on de Sitter space and holography, *Classical Quantum Gravity* **19**, 5655 (2002); Exploring de Sitter space and holography, *Ann. Phys. (Amsterdam)* **303**, 59 (2003).
- [10] D. Harlow and D. Stanford, Operator dictionaries and wave functions in AdS /CFT and dS /CFT, *arXiv:1104.2621*.
- [11] G. S. Ng and A. Strominger, State/operator correspondence in higher-spin dS /CFT, *Classical Quantum Gravity* **30**, 104002 (2013).
- [12] D. Das, S. R. Das, A. Jevicki, and Q. Ye, Bi-local construction of $Sp(2N)/dS$ higher spin correspondence, *J. High Energy Phys.* **01** (2013) 107.
- [13] D. Anninos, F. Denef, and D. Harlow, The wave function of Vasiliev's universe—A few slices thereof, *Phys. Rev. D* **88**, 084049 (2013).
- [14] D. Anninos, F. Denef, R. Monten, and Z. Sun, Higher spin de Sitter Hilbert space, *arXiv:1711.10037*.
- [15] K. Narayan, On extremal surfaces and de Sitter entropy, *Phys. Lett. B* **779**, 214 (2018).
- [16] K. Narayan, de Sitter extremal surfaces, *Phys. Rev. D* **91**, 126011 (2015).
- [17] K. Narayan, de Sitter space and extremal surfaces for spheres, *Phys. Lett. B* **753**, 308 (2016).
- [18] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from AdS /CFT, *Phys. Rev. Lett.* **96**, 181602 (2006).
- [19] S. Ryu and T. Takayanagi, Aspects of holographic entanglement entropy, *J. High Energy Phys.* **08** (2006) 045.
- [20] V. E. Hubeny, M. Rangamani, and T. Takayanagi, A covariant holographic entanglement entropy proposal, *J. High Energy Phys.* **07** (2007) 062.
- [21] M. Rangamani and T. Takayanagi, Holographic entanglement entropy, *Lect. Notes Phys.* **931**, 1 (2017).

- [22] A. LeClair, Quantum critical spin liquids, the 3D Ising model, and conformal field theory in $2 + 1$ dimensions, [arXiv:cond-mat/0610639](#).
- [23] A. LeClair and M. Neubert, Semi-Lorentz invariance, unitarity, and critical exponents of symplectic fermion models, *J. High Energy Phys.* **10** (2007) 027.
- [24] V. Gurarie, Logarithmic operators in conformal field theory, *Nucl. Phys.* **B410**, 535 (1993).
- [25] H. G. Kausch, Curiosities at $c = -2$, [arXiv:hep-th/9510149](#).
- [26] V. Gurarie, M. Flohr, and C. Nayak, The Haldane-Rezayi quantum Hall state and conformal field theory, *Nucl. Phys.* **B498**, 513 (1997).
- [27] H. G. Kausch, Symplectic fermions, *Nucl. Phys.* **B583**, 513 (2000).
- [28] M. Flohr, Bits and pieces in logarithmic conformal field theory, *Int. J. Mod. Phys. A* **18**, 4497 (2003).
- [29] M. Krohn and M. Flohr, Ghost systems revisited: Modified Virasoro generators and logarithmic conformal field theories, *J. High Energy Phys.* **01** (2003) 020.
- [30] V. Gurarie, Logarithmic operators and logarithmic conformal field theories, *J. Phys. A* **46**, 494003 (2013).
- [31] T. Hartman and J. Maldacena, Time evolution of entanglement entropy from black hole interiors, *J. High Energy Phys.* **05** (2013) 014.
- [32] J. M. Maldacena, Eternal black holes in anti-de Sitter, *J. High Energy Phys.* **04** (2003) 021.
- [33] X. Dong, E. Silverstein, and G. Torroba, De Sitter holography and entanglement entropy, *J. High Energy Phys.* **07** (2018) 050.
- [34] M. Alishahiha, A. Karch, E. Silverstein, and D. Tong, The dS/dS correspondence, *AIP Conf. Proc.* **743**, 393 (2005).