Birkhoff's theorem in Hořava gravity

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We study Birkhoff's theorem, which states the absence of time-dependent and spherically symmetric vacuum solutions in four-dimensional Hořava gravity, which has been proposed as a renormalizable quantum gravity without the ghost problem. We prove that the theorem is still valid for the usual types of solutions which admit the general relativity limit in the low-energy (IR) region. However, for the unusual types of solutions, the theorem can be violated in the high-energy (UV) region, due to nonlinear effects. This implies that the scalar graviton can emerge as the result of nonlinear UV effects but is decoupled in the IR regime. An important implication of the nonlinear, UV scalar graviton in big bang cosmology is also discussed.

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I. INTRODUCTION

Birkhoff's theorem, which states the absence of time-dependent, spherically symmetric vacuum solutions, is an important consequence of general relativity (GR) [1,2]. This implies that a spherically symmetric solution must be static, given by the Schwarzschild solution. Furthermore, it also implies the absence of gravitational radiation for pulsating or collapsing, spherically symmetric bodies, which can be stated as the absence of spin-0, or “scalar gravitons” in modern terms. Birkhoff's theorem is also quite important for the study of cosmological as well as astrophysical problems [3].

On the other hand, it is also well known that GR would not be a UV complete theory due to the lack of renormalizability. Several years ago, Hořava proposed a renormalizable, higher-derivative gravity theory, without the ghost problem in the usual covariant higher-curvature gravities, by considering different scaling dimensions for space and time [4]. So, it would be an important question whether Birkhoff's theorem can still be valid or needs to be modified for this quantum gravity model. Actually, since there have been long-standing debates about the scalar graviton mode and the recovery of GR in IR, the study of Birkhoff's theorem is also directly related

to the fundamental question about the consistency of Hořava gravity [5–7].

To this end, we start by considering the Arnowitt-Deser-Misner decomposition of the metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad (1)$$

and the Hořava gravity action,¹ which is power-counting renormalizable in four dimensions [4],

$$S = \int d\eta d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \mathcal{V}[g_{ij}] \right\},$$

$$-\mathcal{V}[g_{ij}] = \frac{\kappa^2 \mu^2 [(\Lambda_W - \omega)R - 3\Lambda_W^2]}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2$$

$$- \frac{\kappa^2}{2\nu^4} \left(C_{ij} - \frac{\mu\nu^2}{2} R_{ij} \right) \left(C^{ij} - \frac{\mu\nu^2}{2} R^{ij} \right), \quad (2)$$

where $K_{ij} = (2N)^{-1}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature; $C^{ij} = \epsilon^{ik\ell} \nabla_k (R^j_\ell - R \delta^j_\ell/4)$ is the Cotton tensor; κ , λ , ν , μ , Λ_W are coupling parameters; and ω is an IR-modification parameter which breaks softly the detailed balance condition in IR [4,9,10] so that Newton's gravity or the GR limit exists, without changing the improved UV behaviors. [$\epsilon^{ik\ell}$ is the Levi-Civita symbol, and R_{ij} and R are the three-dimensional (Euclidean) Ricci tensor and scalar, respectively.]

In addition to these standard terms of the action, for completeness, we will also consider the extension terms which depend on the proper acceleration $a_i = \partial_i \ln N$ [7],

¹For $\lambda = 1/3$, a separate consideration with different coefficients is needed [8]. We will briefly mention the results for this case later.

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which have been introduced to avoid the problems raised in [5,6]. Since the key role of the extension terms is in IR, we will consider the modification of $\mathcal{V}[g_{ij}]$ by

$$\delta V[g_{ij}, a_i] = -\frac{\sigma}{2} a_i a^i, \quad (3)$$

which is the only relevant term in IR [7].

In order to study Birkhoff's theorem, let us consider the spherically symmetric, time-dependent metric with the ansatz,

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

Then, the Hamiltonian and momentum constraints are reduced to

$$\begin{aligned} \mathcal{H} = & -\frac{2(1-\lambda)}{\kappa^2} \dot{\beta}^2 e^{-2\alpha} + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left[\frac{2(\Lambda_W - \omega) e^{-2\beta}}{r^2} (2r\beta' + e^{2\beta} - 1) - 3\Lambda_W^2 \right] \\ & + \frac{\kappa^2 \mu^2 e^{-4\beta}}{8(1-3\lambda)r^4} \{-2(1-\lambda)r^2\beta'^2 - 4\lambda r\beta'(e^{2\beta} - 1) - (1-2\lambda)(e^{2\beta} - 1)^2\} \\ & - \sigma e^{-2\beta} \left(\frac{1}{2} \alpha'^2 + \alpha'' + \frac{2}{r} \alpha' - \alpha' \beta' \right) = 0, \end{aligned} \quad (5)$$

$$\mathcal{H}^r = \frac{e^{-(\alpha+2\beta)}}{r} [(2 - r(1-\lambda)\alpha')\dot{\beta} + r(1-\lambda)\dot{\beta}'] = 0, \quad (6)$$

and the field equations of g^{ij} are reduced to a single equation (see the Appendix for details),

$$\begin{aligned} E_\beta = & \frac{2(1-\lambda)}{\kappa^2} (\ddot{\beta} - \dot{\alpha}\dot{\beta}) e^{2(\beta-\alpha)} + \frac{\kappa^2 \mu^2 (\Lambda_W - \omega)}{4(1-3\lambda)r} (\alpha' + \beta') \\ & - \frac{\sigma}{2} \left(\alpha'' + \frac{2}{r} \alpha' - \alpha' \beta' \right) \\ & + \frac{\kappa^2 \mu^2 e^{-2\beta}}{4(1-3\lambda)r^4} \{(1-\lambda)[r^2(\beta'^2 - \beta'' - \alpha'\beta') + (e^{2\beta} - 1)] \\ & - \lambda r(\alpha' + \beta')(e^{2\beta} - 1)\} = 0. \end{aligned} \quad (7)$$

In the GR case, where $\lambda = 1$, $\sigma = 0$, and the higher-derivative terms $\{\dots\}$ terms in (5) and (7) are absent, it is easy to see that one can obtain the unique solution $\dot{\beta} = 0$ from $\mathcal{H}_i = 0$, and $\dot{\alpha}' = 0$ from $\dot{\mathcal{H}} = 0$, which tells the time independence of metric (4) and so proves Birkhoff's theorem: $\dot{\alpha}' = 0$ implies $\alpha(t, r) = a(t) + b(r)$ but $a(t)$ can be removed by redefining the time coordinate t .

Generally, however, there could exist solutions which may break the theorem due to either the IR Lorentz violation from $\lambda \neq 1$ or $\sigma \neq 0$, or the UV Lorentz violation from higher-derivative terms. We accordingly classify the solutions largely by the time dependence of β , i.e., $\dot{\beta} = 0$ or $\dot{\beta} \neq 0$.

A. Case $\dot{\beta} = 0$

For $\lambda = 1$, this is the only possible solution of the momentum constraint (6), as in the GR case, though not the unique solution for $\lambda \neq 1$. However, for either $\lambda = 1$ or $\lambda \neq 1$, one can prove $\dot{\alpha}' = 0$, i.e., admitting Birkhoff's

theorem, for generic values of coupling parameters, except for one particular set of parameters which relates UV and IR. To prove this, we consider the time derivatives of (5) and (7), which reduce to

$$\dot{\mathcal{H}} = -\sigma e^{-2\beta} \left[\left(\alpha' - \beta' + \frac{2}{r} \right) \dot{\alpha}' + \dot{\alpha}'' \right] = 0, \quad (8)$$

$$\begin{aligned} \dot{E}_\beta = & \frac{\kappa^2 \mu^2 e^{-2\beta}}{4(1-3\lambda)r^4} [-\lambda r(e^{2\beta} - 1) - (1-\lambda)r^2\beta'] \\ & + (\Lambda_W - \omega)r^3 e^{2\beta} \dot{\alpha}' - \frac{\sigma}{2} \left[\left(-\beta' + \frac{2}{r} \right) \dot{\alpha}' + \dot{\alpha}'' \right] \\ = & \left\{ \frac{\sigma}{2} \alpha' + \frac{\kappa^2 \mu^2}{4(1-3\lambda)r^4} [-\lambda r(1 - e^{-2\beta}) - (1-\lambda)r^2 e^{-2\beta} \beta'] \right. \\ & \left. + (\Lambda_W - \omega)r^3 \right\} \dot{\alpha}' = 0, \end{aligned} \quad (9)$$

where we have used (8) in the last step of (9).

In the presence of the extension term (3), i.e., $\sigma \neq 0$, one finds that $\dot{\alpha}' = 0$ is the only possible solution so that Birkhoff's theorem is satisfied: for the case of $\{\dots\} = 0$ in (9), $\alpha(t, r)$ can be integrated as $\alpha(t, r) = a(t) + b(r)$ so that $\dot{\alpha}' = 0$ is satisfied again.

On the other hand, in the absence of the extension term, i.e., $\sigma = 0$, (9) gives the usual solution $\dot{\alpha}' = 0$, or an unusual solution for $\beta(t, r)$,

$$\beta(t, r) = -\ln \sqrt{1 + (\omega - \Lambda_W)r^2 + Cr^{\frac{2\lambda}{\lambda-1}}}, \quad (10)$$

which makes $\{\dots\} = 0$ in (9), even without knowing $\dot{\alpha}'$, where C is an integration constant which corresponds to the mass for the static case. In general, the second, unusual solution (10) is not compatible with the Hamiltonian constraint (5), which reduces to

$$\mathcal{H} = \frac{\kappa^2 \mu^2}{8(1-3\lambda)(1-\lambda)r^4} [-(1-3\lambda)C^2 r^{\frac{4\lambda}{\lambda-1}} + 3\omega(\omega - 2\Lambda_W)(1-\lambda)r^4] = 0. \quad (11)$$

However, there exists one exceptional, compatible solution when the conditions

$$C = 0, \quad \omega(\omega - 2\Lambda_W) = 0 \quad (12)$$

are satisfied.²

There are two possible solutions for the second condition in (12), i.e., $\omega = 0$ or $\omega = 2\Lambda_W$. The first case, $\omega = 0$, corresponds to the solution without the IR-modification term [11]. The second case, $\omega = 2\Lambda_W$, is the corresponding new solution but, this time, with the IR modification. A curious property of these solutions is that $\alpha(t, r)$ is not constrained by the equations of motions so that $\alpha(t, r)$ can be an arbitrary function of space and time. It is this latter property that allows the time dependence in the metric and so could violate Birkhoff's theorem, even though $\dot{\beta} = 0$ from (10) and (12). However, we note that these solutions do not have the GR limit, $\lambda \rightarrow 1$, $\mu \rightarrow 0$, $\omega \rightarrow \infty$, $\Lambda_W \rightarrow \infty$, with $\mu^2\omega$, $\mu^2\Lambda_W \sim \text{fixed}$ [12], so that Birkhoff's theorem could be violated but only in the non-GR branch.³ Another peculiar property of the solution for $\omega = \omega(\Lambda_W)$ is that it requires some correlations between IR and UV terms so that only their combined equations have the solution, though separate ones do not.

B. Case $\dot{\beta} \neq 0$

For $\lambda \neq 1$, in addition to the usual solution $\dot{\beta} = 0$, one may also consider the case of $\dot{\beta} \neq 0$ generally, which breaks Birkhoff's theorem manifestly from the momentum constraint (6). In order to see whether this is really a feasible solution or not, we need to see whether it is compatible with the Hamiltonian constraints and other field equations. Actually, in this case, $\alpha(t, r)$ can be determined as

$$\alpha'(t, r) = \frac{2}{r(1-\lambda)} + \frac{\dot{\beta}'}{\dot{\beta}} \quad (13)$$

from (6) and can be integrated as

$$\alpha(t, r) = \ln(\dot{\beta}(t, r)r^{\frac{2}{1-\lambda}}) + a(t), \quad (14)$$

²Note that, with the conditions in (12), the solution (10) is valid for arbitrary λ , including the $\lambda = 1$ case: actually, with $C = 0$, the solution corresponds to the zero-mass limit of the $\lambda = 1$ static black hole solution in [10].

³The solution for $\omega = 0$ seems to be due to the UV detailed balance condition since it does not exist for more generic UV actions [11]. However, the solution for $\omega = 2\Lambda_W$ seems to be more generic since one can also obtain a similar solution with the modified choice of $\omega = \omega(\Lambda_W)$ for more generic UV terms.

with one undetermined, time-dependent function $a(t)$. Then, after some computations with this general solution (14), one can find that the Hamiltonian constraint (5) reduces to, for the standard case of $\sigma = 0$,

$$\begin{aligned} & \frac{16(1-\lambda)(1-3\lambda)}{\kappa^4 \mu^2} r^{\frac{4}{\lambda-1}} e^{-2a(t)} \\ &= \frac{2(\Lambda_W - \omega)}{r^2} (-rf' + 1 - f) - 3\Lambda_W^2 \\ &+ \frac{1}{r^4} \left\{ -\frac{1}{2}(1-\lambda)r^2 f'^2 + 2\lambda r f'(1-f) - (1-2\lambda)(1-f)^2 \right\}, \end{aligned} \quad (15)$$

where we have introduced $f(t, r) \equiv e^{-2\beta(t, r)}$ for convenience. Then, (15) and (7) become the equations for $\beta(t, r)$ and $a(t)$. Once we get the solutions for (15), one can prove or disprove Birkhoff's theorem by checking whether or not it can still satisfy the remaining field equation (7). To this end, instead of getting the explicit solutions for the full equations with both the UV and IR parts, which is a formidable task for our case of $\lambda \neq 1$, we consider the limiting solutions for the UV and IR equations separately.

First, for the Hamiltonian equation (15) with only the UV terms ($\{\dots\}$ terms in the right-hand side), which corresponds to the limiting case of $\omega, \Lambda_W \rightarrow 0$ of the full equation (15), one can solve the equation explicitly and find that there are two general solutions but only one solution is compatible with the field equation (7), which is given by⁴

$$f(t, r) = 1 \pm \frac{4|\lambda - 1|r^{\frac{2\lambda}{\lambda-1}}}{\epsilon \sqrt{-\kappa^4 \mu^2}} e^{-a(t)}, \quad (16)$$

where $\epsilon \equiv \text{sign}(3\lambda - 1)$. This solution is peculiar in that $a(t)$ is unconstrained and there is no integration constant, and this is due to a factorization of an algebraic equation, called Abel's equation, in the UV limit of $\omega, \Lambda_W \rightarrow 0$ [14]. However, from our starting assumption, we find $\dot{\beta} \propto \dot{a} \neq 0$; i.e., $a(t)$ cannot be trivially a constant so that the solution manifestly violates Birkhoff's theorem in the UV regime.⁵

On the other hand, from the relation (13) or (14), $\alpha(t, r)$ is given by

$$\alpha(t, r) = \ln \left[\frac{2|\lambda - 1|r^2 \dot{a}(t)}{[\pm \epsilon \sqrt{-\kappa^4 \mu^2} + 4|\lambda - 1|r^{\frac{2\lambda}{\lambda-1}} e^{-a(t)}]} \right]. \quad (17)$$

⁴In the IR region of large r , the solution approaches $f = e^{-2\beta} = 1$ for $\lambda < 1$, which is consistent with the case of $\lambda = 1$ in Sec. I A. In other words, there is no discontinuity at the limit $\lambda \rightarrow 1_-$ in IR, which may be comparable to the Vainshtein mechanism for the massless limit of massive gravity [13].

⁵For the case of $\lambda = 1/3$, where a separate analysis is needed, we find that there is no corresponding time-dependent solution in UV [15]. This is the only result which is qualitatively different from the case of $\lambda \neq 1/3$ and it would probably be due to the additional (anisotropic) Weyl symmetry in UV [4,8].

Moreover, it is interesting to note that the solutions (16) and (17) exist only for *de Sitter* branch with a positive cosmological constant $\Lambda \propto -\mu^2 > 0$, like our current accelerating Universe [16].

Second, for the Hamiltonian equation (15) with only the IR terms (the first two terms in the right-hand side), one can also get the explicit solution but find that it is not compatible with the field equation (7) either. So, even for the $\lambda \neq 1$ case with an IR Lorentz violation, there is no time-dependent, i.e., Birkhoff's theorem violating, solution in the IR regime.

Finally, with the extension terms in (3), the Hamiltonian equation (15) has additional contributions

$$-\frac{8(1-3\lambda)\sigma}{\kappa^2\mu^2}f\left[U' + \frac{1}{2}(U^2 + UJ) + \frac{2}{r}\left(\frac{2-\lambda}{1-\lambda}\right)U + \frac{J}{r(1-\lambda)} + \frac{2}{r^2}\frac{(2-\lambda)}{(1-\lambda)^2}\right], \quad (18)$$

where we have introduced $U(t, r) \equiv (\ln \dot{\beta})'$, $J(t, r) \equiv -2\beta'$, so that the original partial-differential-equation problem is reduced to an ordinary-differential-equation problem with respect to r , with a fixed time t . With the extension terms, solving even the IR equation of (15), which actually shows the key role of the extension terms, is a difficult task and its general solution is not available. However, there exists a simple situation that allows an exact solution, which is now compatible with Eq. (7) [we will omit the detailed derivation, which is quite cumbersome],

$$\alpha(t, r) = \ln\left(\frac{8\dot{C}_1(t)}{r\sqrt{-\kappa^4\mu^2\omega}}\right), \quad \beta(t, r) = \frac{C_1(t)}{r^2} + b(r),$$

with $\lambda = -1$, $\Lambda_W = 0$, $\sigma = -\mu^2\kappa^2\omega/8$, $a(t) = -\ln\sqrt{-\kappa^4\mu^2\omega/64}$, and two arbitrary functions $C_1(t)$ and $b(r)$. From our stating assumption of $\dot{\beta} \neq 0$, $C_1(t)$ cannot be just a constant so that this solution violates Birkhoff's theorem manifestly even in IR. This situation is quite different from the case of $\dot{\beta} = 0$ and the other cases of

$\dot{\beta} \neq 0$ with $\sigma = 0$, where the violations of Birkhoff's theorem may occur as the UV effects (or combined UV/IR effects for the case of $\dot{\beta} = 0$). For other values of parameters, we have obtained the (time-dependent) solutions numerically, by solving about $\beta(t, r)$ and $\dot{\beta}(t, r)$ using Mathematica (Fig. 1).

In conclusion, we have proved that, for the standard form of Hořava gravity, Birkhoff's theorem is satisfied in IR for the usual types of solutions which admit the GR limit but can be violated in UV for the unusual types of solutions. In relation to the gravitational radiations, this implies that the scalar gravitons could exist as the results of UV effects but be decoupled in the IR regime. Here, it is important to note that we have considered the problems with the full nonlinearity by obtaining the exact (time-dependent) solutions for the nonlinear equations of motion. This result is consistent with the (fully nonlinear) constraint analysis [17], which has been thought to be inconsistent [18,19], but in contrast to the absence of the scalar gravitons for the whole UV and IR energy ranges in the linear perturbation analyses [20–24]. This implies a remarkable fact that the UV emergence of time-dependent solutions, i.e., violation of Birkhoff's theorem, or equivalently scalar gravitons, is the nonlinear effect in UV. Actually, if we consider small $a(t)$, the UV time-dependent solution (16) and (17) can be expanded as

$$f^{-1} = e^{2\beta} = (1 + \zeta^{-1})(1 + a(t)\zeta^{-1} + \mathcal{O}(a(t)^2)), \quad (19)$$

$$N^2 = e^{2\alpha} = \frac{1}{4}r^{\frac{4}{\lambda-1}}\dot{a}^2(t)\zeta^{-2}[1 + 2a(t)\zeta^{-1} + \mathcal{O}(a(t)^2)], \quad (20)$$

where $\zeta = 1 \pm \frac{\epsilon\sqrt{-\kappa^4\mu^2}}{4|\lambda-1|}r^{\frac{2\lambda}{\lambda-1}}$. This shows explicitly that, when we consider small ζ^{-1} as well, $a(t)$ does not appear at the leading, linear orders but emerges only at the subleading, i.e., nonlinear orders, consistently with the constraint analysis [17]. [Note that, at the leading order, the $\dot{a}^2(t)$ factor in (20) can be removed by redefining the time as $dt \rightarrow dt' = dt/\dot{a}(t)$.]

FIG. 1. Plots of numerical solutions for $f(t, r)^{-1} = e^{2\beta(t, r)}$ (left), $\dot{\beta}(t, r)$ (right) vs r for varying σ , at $t = t_0$. Here, we have considered $\lambda = 0.35$, $\Lambda_W = 0$, $\omega = 0.225$, $\mu = 3$, $\kappa = 1$, $a(t_0) = e^3$, $\dot{a}(t_0) = 0$. These show two different branches of solutions with different asymptotes, $f = 1$ for $\sigma > 1$ (upper curves) or $f = \infty$ for $\sigma \leq 1$ (lower curves).

On the other hand, for the extended Hořava gravity with the term of (3), Birkhoff's theorem is still satisfied for $\beta = 0$, i.e., $\dot{\alpha}' = 0$, so that there are no time-dependent solutions for the full theory with both the UV and IR terms. However, we have shown an explicit time-dependent solution which violates the theorem in IR for $\dot{\beta} \neq 0, \lambda \neq 1$. This is consistent with the perturbative [7] as well as nonperturbative analyses [25] but this seems to be potentially problematic since it implies the existence of gravitational radiations even for pulsating or collapsing, spherically symmetric bodies in IR, which have not been detected yet; even more, it does not reproduce the GR or Newton's gravity limit in IR, which has been well tested [26].

On the contrary, the existence of a scalar graviton mode, which seems to be represented by a scalar function $a(t)$, indicating the instantaneous mode, in the general solution (14), would have an important role in cosmology. Usually, we need (at least one) primordial scalar matter field, with an instantaneous background, in order to accommodate the observed (nearly scale invariant) scalar power spectrum in CMB data within the inflationary theory [27]. But, now the

scalar degree of freedom which is inherent in the nonlinear UV regime of Hořava gravity could have a role similar to that of the primordial scalar field in the early Universe. It would be an outstanding question whether Hořava gravity can provide a consistent framework for the big bang cosmology without introducing the artificial primordial scalar field and inflationary scenario [24].

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APPENDIX: FULL EQUATIONS OF MOTION

The Hamiltonian and momentum constraints, following from the variations of the action (2) with the extension term (3) for N and N^i , respectively, are given by

$$\mathcal{H} \equiv -\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2[(\Lambda_W - \omega)R - 3\Lambda_W^2]}{8(1-3\lambda)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2\nu^4}Z_{ij}Z^{ij} + \sigma\left(\frac{1}{2}\frac{\nabla_i N \nabla^i N}{N^2} - \frac{\nabla_k \nabla^k N}{N}\right) = 0, \quad (\text{A1})$$

$$\mathcal{H}^i \equiv \nabla_k(K^{ki} - \lambda K g^{ki}) = 0, \quad (\text{A2})$$

where

$$Z_{ij} \equiv C_{ij} - \frac{\mu\nu^2}{2}R_{ij}. \quad (\text{A3})$$

The equations of motion from variation of δg^{ij} are given by [11,28,29]

$$E_{ij} \equiv \frac{2}{\kappa^2}E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2}E_{ij}^{(2)} + \frac{\kappa^2\mu^2(\Lambda_W - \omega)}{8(1-3\lambda)}E_{ij}^{(3)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}E_{ij}^{(4)} - \frac{\mu\kappa^2}{4\nu^2}E_{ij}^{(5)} - \frac{\kappa^2}{2\nu^4}E_{ij}^{(6)} + \frac{\sigma}{2}E_{ij}^{(7)} = 0, \quad (\text{A4})$$

where

$$\begin{aligned} E_{ij}^{(1)} &= N_i \nabla_k K^k_j + N_j \nabla_k K^k_i - K^k_i \nabla_j N_k - K^k_j \nabla_i N_k - N^k \nabla_k K_{ij} - 2NK_{ik}K_j^k - \frac{1}{2}NK^{k\ell}K_{k\ell}g_{ij} + NKK_{ij} + \dot{K}_{ij}, \\ E_{ij}^{(2)} &= \frac{1}{2}NK^2g_{ij} + N_i \partial_j K + N_j \partial_i K - N^k (\partial_k K)g_{ij} + \dot{K}g_{ij}, \\ E_{ij}^{(3)} &= N \left(R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\frac{\Lambda_W^2}{\Lambda_W - \omega}g_{ij} \right) - (\nabla_i \nabla_j - g_{ij}\nabla_k \nabla^k)N, \\ E_{ij}^{(4)} &= NR \left(2R_{ij} - \frac{1}{2}Rg_{ij} \right) - 2(\nabla_i \nabla_j - g_{ij}\nabla_k \nabla^k)(NR), \\ E_{ij}^{(5)} &= \nabla_k [\nabla_j (NZ^k_i) + \nabla_i (NZ^k_j)] - \nabla_k \nabla^k (NZ_{ij}) - \nabla_k \nabla_\ell (NZ^{k\ell})g_{ij}, \end{aligned}$$

$$\begin{aligned}
E_{ij}^{(6)} = & -\frac{1}{2}NZ_{k\ell}Z^{k\ell}g_{ij} + 2NZ_{ik}Z_j^k - N(Z_{ik}C_i^k + Z_{jk}C_j^k) + NZ_{k\ell}C^{k\ell}g_{ij} - \frac{1}{2}\nabla_k[N\epsilon^{mk\ell}(Z_{mi}R_{j\ell} + Z_{mj}R_{i\ell})] \\
& + \frac{1}{2}R^n_\ell\nabla_n[N\epsilon^{mk\ell}(Z_{mi}g_{kj} + Z_{mj}g_{ki})] - \frac{1}{2}\nabla_n[NZ_m^n\epsilon^{mk\ell}(g_{ki}R_{j\ell} + g_{kj}R_{i\ell})] - \frac{1}{2}\nabla_n\nabla^n\nabla_k[N\epsilon^{mk\ell}(Z_{mi}g_{j\ell} + Z_{mj}g_{i\ell})] \\
& + \frac{1}{2}\nabla_n[\nabla_i\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{j\ell} + \nabla_j\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{i\ell}] \\
& + \frac{1}{2}\nabla_\ell[\nabla_i\nabla_k(NZ_{mj}\epsilon^{mk\ell}) + \nabla_j\nabla_k(NZ_{mi}\epsilon^{mk\ell})] - \nabla_n\nabla_\ell\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{ij}, \\
E_{ij}^{(7)} = & \frac{1}{N}\left(-\frac{1}{2}g_{ij}\nabla_iN\nabla^iN + \nabla_iN\nabla_jN\right).
\end{aligned}$$

In general, from the spherical symmetry, there are two nonvanishing field equations $E_{rr}, E_{\theta\theta} = E_{\phi\phi}/(r^2 \sin \theta)$ but one finds that there is only one independent equation due to a (remarkable) relation $2e^\beta E_{\theta\theta} - 2r^2 e^{-\beta} E_{rr} - 2(e^{-\beta})' r^3 E_{rr} - r^3 e^{-\beta} (E_{rr})' + 2r^3 \frac{\partial}{\partial t}(e^{3\beta} \mathcal{H}^r)/\kappa^2 - r^3 e^\beta (e^\alpha)' \mathcal{H}/2 = 0$.

Also, E_β in (7) is given by

$$E_\beta = e^{-\alpha} E_{rr} + \frac{1}{2} e^{2\beta} \mathcal{H}. \quad (\text{A5})$$

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