Exact relativistic Green's functions for the time-independent potentials

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(Received 13 March 2019; published 29 May 2019)

The relativistic formalism of Green's functions is discussed in QCD and QED, where the relativistic Green's functions are constructed using the Schwinger proper time formalism and analyzed using the Fock–Feynman–Schwinger method. As a result, a simple and exact method is found for the relativistic systems, where the interaction can be written in a time-independent form. In this case one can write the relativistic Green's function as a one-dimensional integral of the corresponding nonrelativistic Green's function. The explicit example for the problem of a charge in the constant magnetic field is discussed in detail, and the exact agreement with the Schwinger relativistic form is demonstrated. A similar analysis is performed in the relativistic Coulomb problem, supporting the accuracy of the proposed relativistic formalism.

DOI: [10.1103/PhysRevD.99.096025](https://doi.org/10.1103/PhysRevD.99.096025)

I. INTRODUCTION

The problem of bound state wave functions and eigenvalues is central in the hadron physics and in QCD in general, and it is especially complicated due to the relativistic character of all interactions, provided by the field theory.

Historically the interaction generated by particle exchanges was considered as a basic and the corresponding fundamental equation, the Bethe-Salpeter equation [1–3] is used till now with the kernels, consisting of diagrams with combinations of particle exchanges, described by the corresponding propagators. In this way the Bethe-Salpeter equation corresponds to the summation of an infinite set of graphs of the ladder type, for a review see [4].

In the special case, when these propagators are treated instantaneously, e.g., for the virtual photon exchanges, one can use the instantaneous approximation for the relativistic Coulomb problem. A specific feature of the Salpeter equation [2,3] is that it can be made free of the aggravating relative time (or zero component of relative momentum) problem.

One special feature of QCD is that the basic interactions, e.g., confinement, are given by the vacuum background fields, and the latter are well described by the instantaneous interaction terms, generated by the average Wilson loops, to be called V_{inst} (see the Appendix A for the definition of V_{inst}). In addition there are other interaction terms,

corresponding to particle exchanges, which may be denoted as operators $\hat{V}_{\text{op}}$, and actually contributing additional terms of the wave functions, known as Fock column.

At this point one should stress the difference in the approach of the present paper and numerous investigations of the relativistic bound state problem in the literature [5–15], see [16] for a review.

In the standard approach the strong interaction is introduced via the Bethe-Salpeter (Salpeter) equation, or via Schwinger-Dyson equation, and one postulates the form of the interaction kernel, responsible for confinement trying to find the spectrum by solving integral equations in the momentum space.

In a more practical approach (see, e.g., [9]) one postulates the square root Hamiltonian with confinement and color Coulomb interaction and finds the hadron spectrum using several adjustable parameters.

In our approach one starts with the basic properties of the field theory in the nonperturbative vacuum.

In QCD the confinement is the central problem which generates masses of the 99% of the visible mass in the universe, and it is very important to take it into account from the very beginning. As it is known (see [17] and a recent review [18]), confinement is brought up by vacuum field configurations (it is not important here whether by field correlators or vacuum magnetic monopole-like configurations) in the Wilson loops and therefore confinement can be described by the instantaneous interactions in these loops.

It is important, that from the Wilson loops and vacuum background one obtains both confining linear potential and also color Coulomb interaction together with spin-dependent terms, so that the instantaneous part is the dominant one, and all other exchanges one can consider

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as a correction. Moreover, additional gluon exchanges should be treated as perturbative terms and not instantaneous, since they provide corrections with an account of additional gluons in the total wave function, and hence additional Fock components. In this way one can avoid the problem of double counting.

Another condition arises due to the fact, that the linearly rising confinement cannot be conveniently used in the momentum space, hence one needs relativistic bound state equations with the instantaneous interaction in the coordinate space, which is our main goal below.

The formalism discussed below was introduced 30 years ago [19–23] and was actively developed since then, however the rigorous derivation and comparison with exact examples were missing. The present paper is aimed to repair these defects.

The paper is organized as follows. In the next section we derive the proper time representation of the Green's function and the corresponding Hamiltonian form, where we show that for the time-independent interaction V_{inst} one obtains a simple expression for the Green's function, which is expressed via the 3d nonrelativisticlike form of Hamiltonian and via the corresponding Green's function, containing an integrable parameter.

In Sec. III we are checking this representation in the problem of relativistic Green's function of a charge in the constant magnetic field and demonstrate that the Schwinger expression for the Green's function [24] exactly coincides with ours.

In Sec. IV we discuss the relativistic Coulomb problem and in Sec. V the case of the relativistic harmonic oscillator and demonstrate the applicability of our approach. In Sec. VI we generalize the one-body problem in the external field to the two-body problem with instantaneous interaction, and obtain the equivalent relativistic Green's functions. We consider the problem of eigenfunctions and eigenvalues for our Green's functions, which have the form of one-dimensional integrals of quantum mechanical nonrelativistic Green's functions. It appears, that the resulting eigenvalues may approximately coincide with those of the “square root” (“spinless Salpeter equation”) [9] or the so-called einbein form of equation [19–22,25]. The character and accuracy of these approximations are discussed in detail. In the final section the main results of the paper are summarized and the possible areas of applications are discussed.

II. FORMULATION OF THE GREEN'S FUNCTION REPRESENTATION AND THE PROPER TIME HAMILTONIAN

We start with the scalar particle Green's function in the external field $A_\mu(z)$ and we write the corresponding proper time path-integral representation in the Euclidean space-time [19,26,27]

$$\begin{aligned} g(x, y) &= \left(\frac{1}{m^2 - D_\mu^2} \right)_{xy} = \int_0^\infty ds e^{-s(m^2 - D_\mu^2)} \\ &= \int_0^\infty ds (D^4 z)_{xy} e^{-K} W(x, y), \quad D_\mu = \partial_\mu - igA_\mu, \end{aligned} \quad (1)$$

where

$$\begin{aligned} (D^4 z)_{xy} &= \int \frac{d^4 p}{(2\pi)^4} \prod_k \frac{d^4 \Delta z(k)}{(4\pi\epsilon)^2} \\ &\times \exp \left[ip \left(\sum_k \Delta z(k) - (x - y) \right) \right], \quad N\epsilon = s \end{aligned} \quad (2)$$

$$K = m^2 s + \frac{1}{4} \int_0^s \left(\frac{dz_\mu}{d\tau} \right)^2 d\tau \quad (3)$$

and the Wilson line $W(x, y)$ is

$$W(x, y) = P_A \exp \left(ig \int_y^x A_\mu dz_\mu \right). \quad (4)$$

In what follows we shall consider the basic type of interactions, contained in $W(x, y)$, where there is no dependence on time, except that W depends on the points z of the trajectories $z(t_E)$, i.e., in (4) the fields $A_4(\mathbf{z})$, $\mathbf{A}(\mathbf{z})$ do not depend on time, but only on the 3d coordinates.

This condition is satisfied by constant Coulomb and constant magnetic fields in QED. In the case of QCD we show in the Appendix A following [26,27], that both confinement and color Coulomb forces satisfy these criteria at least to the lowest order in $O(\alpha_s)$.

In this case one can rewrite (4) as

$$W(x, y) = P_A \exp \left(ig \int_{y_4}^{x_4} A_4(\mathbf{z}) dt_E + ig \int \mathbf{A}(\mathbf{z}) \frac{d\mathbf{z}}{dt_E} dt_E \right). \quad (5)$$

As it is clear in (1)–(4), the quantum evolution parameters are τ , and s , where $0 \leq \tau \leq s$, while $z_4(\tau)$ plays the same role as the spatial coordinates $\mathbf{z}(\tau)$, unless $W(x, y)$ introduces its own z_4 dependence. In our case, when $A_\mu(\mathbf{z})$ does depend on z_4 only via trajectories $\mathbf{z}(t_E)$, this does not happen and we are left with the only evolution parameter τ , s , that one can connect to the Euclidean time parameters t_E , $T = x_4 - y_4$ in the following way

$$d\tau = \frac{dt_E}{2\omega}, \quad s = \frac{T}{2\omega}, \quad T = x_4 - y_4, \quad (6)$$

and $g(x, y)$ acquires the form

$$g(x, y) = \sqrt{\frac{T}{8\pi}} \int \frac{d\omega}{\omega^{3/2}} (D^3 z)_{xy} e^{-K(\omega)} W(x, y), \quad (7)$$

$$K(\omega) = \int_0^T dt_E \left(\frac{\omega}{2} + \frac{m^2}{2\omega} + \frac{\omega}{2} \left(\frac{dz}{dt_e} \right)^2 \right). \quad (8)$$

As a next step one can introduce in the path integral (7) the Hamiltonian [27], obtaining in this way for $W \equiv 1$

$$\int (D^3z)_{\mathbf{xy}} e^{-K(\omega)} = \langle \mathbf{x} | e^{-H_0(\omega)T} | \mathbf{y} \rangle \equiv f_0(\mathbf{x}, \mathbf{y}) \quad (9)$$

with

$$H_0(\omega) = \frac{\mathbf{p}^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega},$$

$$f_0(\mathbf{x}, \mathbf{y}) = \left(\frac{\omega}{2\pi T} \right)^{3/2} \exp \left(-\frac{m^2 T}{2\omega} - \frac{(\mathbf{x} - \mathbf{y})^2}{2T} \omega - \frac{\omega T}{2} \right). \quad (10)$$

In the general case with the account of W one has

$$g(x, y) = \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} \langle \mathbf{x} | e^{-H(\omega)T} | \mathbf{y} \rangle, \quad (11)$$

where $H(\omega)$ using (5) can be written as

$$H(\omega) = \frac{(\mathbf{p} - e\mathbf{A})^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} + eA_0. \quad (12)$$

Using (1) and (9) one can write the free Green's function as

$$g_0(x, y) = \sqrt{\frac{T}{8\pi}} \int \frac{d\omega}{\omega^{3/2}} e^{-\frac{m^2 + \omega^2}{2\omega} T} \langle \mathbf{x} | e^{-\frac{\mathbf{p}^2}{2\omega} T} | \mathbf{y} \rangle$$

$$= \sqrt{\frac{T}{8\pi}} \int \frac{d\omega}{\omega^{3/2}} e^{-\frac{m^2 + \omega^2}{2\omega} T} g_{nr}^{(0)}(\mathbf{x}, \mathbf{y}, T), \quad (13)$$

where $g_{nr}^{(0)}$ is the nonrelativistic free Green's function for the mass equal to ω ,

$$g_{nr}^{(0)} = \langle \mathbf{x} | e^{-\frac{\mathbf{p}^2}{2\omega} T} | \mathbf{y} \rangle = \left(\frac{2\pi\omega}{T} \right)^{3/2} \exp \left(-\frac{(\mathbf{x} - \mathbf{y})^2 \omega}{2T} \right). \quad (14)$$

In the general case of an arbitrary static interaction one can take into account, that $\langle \mathbf{x} | e^{-HT} | \mathbf{y} \rangle$ is simply the 3d nonrelativistic Green's function $G_{nr}(\mathbf{x}, \mathbf{y}, T)$ for the given nonrelativistic Hamiltonian $H(\mathbf{p}, A_\mu)$, augmented by the term $\frac{m^2 + \omega^2}{2\omega}$, so that we obtain a remarkable expression for the relativistic Green's function of a particle in time-independent fields

$$g(x, y) = \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} G_{nr}(\mathbf{x}, \mathbf{y}, T). \quad (15)$$

We shall call this form the ω -representation (of a Green's function).¹ In the next section we shall check this relation in

¹Note, that in the early papers [19–21] the corresponding parameter was named μ .

the example of a particle in constant magnetic field. Here the relativistic $g(x, y)$ is given by Schwinger [24], while $G_{nr}(\mathbf{x}, \mathbf{y}, T)$ is known from textbooks on quantum mechanics [28].

Note that the term $\frac{1}{\omega^{3/2}}$ in (15) does not lead to divergence, since it is compensated by the vanishing of G_{nr} at small ω , which, e.g., can be seen in the exponent of (13). For the linear confinement G_{nr} contains also the term $\exp(-\frac{\text{const}}{\omega^{1/3}})$, see, e.g., [19,21].

In the QCD the situation with quark and gluon Green's functions is more complicated due to confinement, which requires, that all physical objects are gauge invariant and in SU(N) they obtain their masses due to confinement, while in QCD also quark masses contribute to the final result.

The most important requirement of confinement is that any Green's function of colored (nongauge invariant) system vanishes after vacuum averaging.

Therefore one cannot define the physical (i.e., gauge invariant and measurable) Green's function for a gluon or a quark.

Similarly any Hamiltonian (or any operator with measurable spectrum) in QCD must be gauge invariant, and, e.g., the often used operator $(m^2 + \hat{D}^2)$ is not gauge invariant and does not have physically measurable spectrum in the confining QCD vacuum. Therefore the gauge invariant Green's functions with physical spectrum can be formed using gauge invariant combinations, like $\langle \bar{q}\Gamma q(x) \bar{q}\Gamma q(y) \rangle$, $\langle \text{tr}(F_{\mu\nu} F_{\lambda\sigma}(x)) \text{tr} F_{\mu\nu} F_{\lambda\sigma}(y) \rangle$ etc., or taking one of q with infinite mass, one has gauge invariant Green's function of heavy-light mesons $G_{hl}(x, y) = \langle \bar{q}(x) \Phi(x, y) q(y) \rangle$. For gluons one may have a more complicated gauge invariant structure for the gluelump Green's functions $G_{gl}(x, y) = \langle \text{tr} F_{\mu\nu}(x) \Phi^{\text{adj}}(x, y) F_{\lambda\sigma}(y) \rangle$, where $\Phi(x, y)$ is the same as W , Eq. (4) for fundamental representation. In what follows we shall consider for simplicity first the case of the heavy-light meson, so that the local potentials $V_i(r)$ depend on the distance r between quark and the infinitely massive static antiquark.

We now turn to the case of QCD where the main interaction is due to vacuum fields, which produce both confining potential $V_{\text{conf}}(\mathbf{r})$ and the color Coulomb interaction $V_{\text{coul}}(\mathbf{r})$, as shown in the Appendix. In this case we consider a heavy-light system, where neglecting spin forces one can write the Hamiltonian $H(\omega)$ as

$$H_{HL}(\omega) = \frac{\mathbf{p}^2 + m^2 + \omega^2}{2\omega} + V_{\text{conf}}(\mathbf{r}) + V_{\text{coul}}(\mathbf{r}) \quad (16)$$

with

$$V_{\text{conf}}(\mathbf{r}) = \sigma r, \quad V_{\text{coul}}(\mathbf{r}) = -\frac{4}{3} \frac{\alpha_s(r)}{r}. \quad (17)$$

At this point one may have two different strategies to find relativistic eigenvalues and eigenfunctions from (15), (16)

using the corresponding nonrelativistic functions. In the first approach using the expansion

$$G_{nr}(\mathbf{x}, \mathbf{y}, \omega, T) = \sum_n \varphi_n^{(nr)}(\mathbf{x}, \omega) e^{-\varepsilon_n^{(nr)}(\omega)T} \varphi_n^{(nr)}(\mathbf{y}, \omega) \quad (18)$$

one has to integrate over $d\omega$ in (15) using the stationary point method, which yields

$$g_{rel}(x, y) \sim \sum \varphi_n^{(nr)}(\mathbf{x}, \omega_0) e^{-E_n(\omega_0)T} \varphi_n^{(nr)}(\mathbf{y}, \omega_0) \quad (19)$$

where

$$E_n(\omega_0, T) = \frac{m^2 + \omega_0^2}{2\omega_0} + \varepsilon_n^{(nr)}(\omega_0) \quad (20)$$

and ω_0 is to be found from the condition $\frac{\partial}{\partial \omega}(E_n(\omega))|_{\omega=\omega_0} = 0$. This type of approach was named “the einbein method” [19–21,23,25], and the corresponding eigenfunctions and eigenvalues were widely used in the hadron physics

Another approach exploits the form

$$G_{nr}(x, y, T, \omega) = \langle \mathbf{x} | e^{-H(\omega)T} | \mathbf{y} \rangle \quad (21)$$

and suggests to find the stationary point of the operator

$$H(\omega, T) = \frac{m^2 + \omega^2}{2\omega} + \frac{\mathbf{p}^2}{2\omega} + V(r). \quad (22)$$

In the case when the instantaneous interaction $V(r)$ does not depend on ω , one obtains a simple answer, usually called the spinless Salpeter or the square root kernel

$$H(\omega, T) = \sqrt{\mathbf{p}^2 + m^2} + V(r), \quad (23)$$

which is commonly used in relativized quark models [5–10], and is important in the dynamics of Regge trajectories [29–34].

As we shall see below, in general the “einbein” eigenvalues (20) and the “spinless Salpeter type” eigenvalues of (23) may not coincide and moreover the operator (23) may need renormalization [35,36] in the case when $V(r)$ is singular and may produce singular eigenfunctions. This happens, e.g., in the case of Coulomb interaction which we shall discuss below in Sec. IV.

III. RELATIVISTIC GREEN’S FUNCTION IN A CONSTANT MAGNETIC FIELD

We start with the relativistic Green’s function of a scalar particle in a constant magnetic field, which can be taken from Schwinger [24] in our notations as

$$G_S(x, y) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-im^2 s} \Phi(x, y) e^{-L(s)} \times \exp \left[\frac{i}{4} (x - y)_\mu C_{\mu\nu} (x - y)_\nu \right] \quad (24)$$

where $L(s) = \ln \left(\frac{\sin eBs}{eBs} \right)$,

$$C_{\mu\nu} = \frac{1}{s} (\delta_{\mu 3} \delta_{\nu 3} - \delta_{\mu 0} \delta_{\nu 0}) + eB \text{ctg}(eBs) (\delta_{\mu 1} \delta_{\nu 1} + \delta_{\mu 2} \delta_{\nu 2});$$

$$\Phi(x, y) = \exp \left(ie \int_y^x du_\mu A_\mu(u) \right).$$

The nonrelativistic Green’s function of the scalar particle with mass ω in the magnetic field B along the 3d axis is

$$G_{NR}(x, y, \omega) = \left(\frac{\omega}{2\pi it} \right)^{3/2} \left(\frac{eBt}{\sin \frac{eBt}{2\omega}} \right) \exp \frac{i\omega}{2t} (x_3 - y_3)^2 \times \exp \left\{ \frac{ieB}{4} \left[\text{ctg} \frac{eBt}{2\omega} ((x_1 - y_1)^2 + (x_2 - y_2)^2) \right]^4 \right\}, \quad (25)$$

where we have chosen the phase $\Phi \equiv 1$. One can check, that for $eB \rightarrow 0$ one obtains the free nonrelativistic Green’s function. Writing now the relativistic Green’s function as in (11) for $it = T$ in the Euclidean space-time, one has

$$G_{Rel}(x, y, T) = \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} e^{-\frac{m^2 + \omega^2}{2\omega} T} G_{NR}(x, y, T, \omega). \quad (26)$$

To reproduce the Schwinger result (24) one should write in (25) $\frac{\omega}{it} = \frac{\omega}{T} = \frac{1}{2s}$, and one obtains the identity of (26) and (24).

We do not consider here the spin degrees of freedom. Indeed, for the bispinor particle the resulting expression cannot be simply obtained in the form of (11), since these degrees of freedom are not present in the nonrelativistic Green’s function. However the simple magnetic dipole interaction, which appears for electron in the Schwinger expression as an extra factor $\exp [i \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu} s]$ can be easily reproduced if in the nonrelativistic Hamiltonian $H(\omega)$ (12) one adds a similar term $\Delta H(\omega) = \frac{e\sigma_{\mu\nu} F_{\mu\nu}}{2\omega} T$ to account for the spin interaction.

Finally, in QCD the numerator of the spinor or vector Green’s function is treated separately as shown in [27] and again is expressed via ω and field correlators, and the resulting expressions, e.g., for meson coupling constants, e.g., $f_\rho, f_D, f_{D_s}, f_B$ etc. are in very good agreement with lattice data and experiment [37].

We now come to the problem of eigenvalues. To this end we calculate the momentum transform of the Green’s function (24) and (26). From (24) one obtains [38]

$$\begin{aligned}
G_S(k) &= \int_0^\infty ds \exp \left[-ism^2 + is(k_0^2 - k_z^2) \right. \\
&\quad \left. - is\mathbf{k}_\perp^2 \frac{\tan(eBs)}{eBs} \right] \\
&= \int d^4(x-y) e^{ik(x-y)} G_S(x-y). \quad (27)
\end{aligned}$$

This can be expanded as a sum over Landau levels

$$G_S(k) = - \sum_{n=0}^\infty \frac{2(-1)^n e^{-\alpha_k} L_n(2\alpha_k)}{k_0^2 - k_z^2 - m^2 - (2n+1)eB + i\epsilon}, \quad (28)$$

where $\alpha_k = -\frac{k_\perp^2}{eB}$, and $L_n(x)$ is the Laguerre polynomial.

Another, and a more simple way to find eigenvalues of the relativistic Green's function is to write G_{NR} in (26) via the Hamiltonian in magnetic field, as in (11), (12)

$$G_{\text{NR}}(x, y, T, \omega) = \langle \mathbf{x} | e^{-H_{\text{mag}}(\omega)T} | \mathbf{y} \rangle \quad (29)$$

where H_{mag} using (12) can be written in a standard way

$$H_{\text{mag}}(\omega) = \frac{\mathbf{p}^2}{2\omega} + \frac{e^2(\mathbf{B} \times \mathbf{r})^2}{8\omega} - \frac{e\boldsymbol{\sigma}\mathbf{B}}{2\omega} + \frac{m^2 + \omega^2}{2\omega}, \quad (30)$$

where we have added the spin interaction term to apply our result also to a fermion.

One can immediately write the lowest eigenvalues of the Hamiltonian (30), known from nonrelativistic quantum mechanics [28]

$$\epsilon_n^{\text{non}}(\omega) = \frac{p_z^2 + eB(2n+1) - e\boldsymbol{\sigma}\mathbf{B}}{2\omega}. \quad (31)$$

Using (26) one can write $G_{\text{Rel}}(x, y, T)$ as follows

$$\begin{aligned}
G_{\text{Rel}}(x, y, T) &= \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} e^{-\frac{m^2 + \omega^2}{2\omega}T} \\
&\quad \times \sum_n \varphi_n(\omega, x) e^{-\epsilon_n^{\text{non}}(\omega)T} \varphi_n^+(\omega, y). \quad (32)
\end{aligned}$$

The resulting integral (32) can be estimated at large T using the stationary point method, which yields

$$\begin{aligned}
\frac{\partial}{\partial \omega} \left(\frac{m^2 + \omega^2 + \epsilon_n^{\text{non}}(\omega)}{2\omega} \right) \Big|_{\omega=\omega_0} &= 0, \\
\omega_0 &= \sqrt{m^2 + p_z^2 + eB(2n+1) - e\boldsymbol{\sigma}\mathbf{B}} \quad (33)
\end{aligned}$$

and the einbein eigenvalues are

$$\begin{aligned}
E_n &= \left(\frac{m^2 + \omega^2 + \epsilon_n^{\text{non}}(\omega)}{2\omega} \right) \Big|_{\omega=\omega_0} \\
&= \sqrt{m^2 + p_z^2 + eB(2n+1) - e\boldsymbol{\sigma}\mathbf{B}}. \quad (34)
\end{aligned}$$

This result exactly coincides with the eigenvalues of the relativistic Green's function (28) for scalar particle, when $e\boldsymbol{\sigma}\mathbf{B}$ is absent, and with the corresponding eigenvalues of a fermion with account of $\boldsymbol{\sigma}\mathbf{B}$.

Note, that the same eigenvalues obtain in the square-root (spinless Salpeter) method, when one first find the stationary point Hamiltonian $H_{\text{mag}}(\omega_0)$ from (30).

IV. THE CASE OF THE (COLOR) COULOMB INTERACTION

We consider a scalar particle in the Coulomb field $V_c(r) = -\frac{Z\alpha}{r}$, and using the Klein–Gordon equation following [39], one has

$$\left[\left(E + \frac{Z\alpha}{r} \right)^2 + \Delta - m^2 \right] \phi = 0 \quad (35)$$

which can be rewritten as

$$\left\{ -\frac{\partial^2}{\partial r^2} - \frac{2\partial}{r\partial r} + \frac{L^2 - Z^2\alpha^2}{r^2} - \frac{2Z\alpha E}{r} - (E^2 - m^2) \right\} \phi(r) = 0. \quad (36)$$

The resulting energy eigenvalues can be written as

$$E_{nl} = \frac{m}{\sqrt{1 + \frac{Z^2\alpha^2}{(n-\delta_l)^2}}}, \quad (37)$$

here δ_l is

$$\delta_l = l + 1/2 - \sqrt{(l + 1/2)^2 - Z^2\alpha^2}. \quad (38)$$

On the other hand, one can use as in (25) the non-relativistic Hamiltonian

$$H = \frac{\mathbf{p}^2}{2\omega} + V_c(r) + \frac{m^2 + \omega^2}{2\omega} \quad (39)$$

which yields eigenvalues

$$\epsilon_n(\omega) = -\frac{\omega(Z\alpha)^2}{2n^2} \quad (40)$$

and one can find the extremum in ω of the total eigenvalue

$$M_n(\omega) = \frac{m^2 + \omega^2}{2\omega} - \frac{\omega(Z\alpha)^2}{2n^2} \quad (41)$$

yielding

$$M_n(\omega_0) = m \sqrt{1 - \left(\frac{Z\alpha}{n}\right)^2} \quad (42)$$

which differs from (37), since we have treated in (39) the Coulomb field as a scalar, while in (35) its vector character is accounted for.

To take into account the vector character of the Coulomb interaction we shall use the “square root approach” to the full Hamiltonian (39)

$$\left. \frac{\partial H}{\partial \omega} \right|_{\omega=\omega_0} = 0, \quad \omega_0 = \sqrt{\mathbf{p}^2 + m^2} \quad (43)$$

which yields the square root Hamiltonian

$$H(\omega_0) = \sqrt{\mathbf{p}^2 + m^2} - \frac{Z\alpha}{r}; \quad H(\omega_0)\Psi_n = \tilde{E}_n \Psi_n \quad (44)$$

and the resulting equation for $\tilde{E}_n$ has the same form as in (35), when one takes the quadratic expression

$$\left(\sqrt{\mathbf{p}^2 + m^2}\right)^2 \Psi_n = \left(E_n + \frac{Z\alpha}{r}\right)^2 \Psi_n \quad (45)$$

with the same spectrum (37).

Turning now to the case of the relativistic fermion in the electromagnetic field, our Hamiltonian $H(\omega)$ has the form [35]

$$H(\omega) = \frac{(\mathbf{p} - e\mathbf{A})^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} + eA_0 - \frac{e\boldsymbol{\sigma}\mathbf{B}}{2\omega} - \frac{ie(\boldsymbol{\alpha}\mathbf{E})}{2\omega} \quad (46)$$

which simplifies when $\mathbf{A} = \mathbf{B} = 0$. Here $\boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$;

$$H_{II}(\omega) = \frac{\mathbf{p}^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} - \frac{Z\alpha}{r} - ie \frac{(\boldsymbol{\alpha}\mathbf{E})}{2\omega}. \quad (47)$$

Following the same way as in (43)–(45), but now with the additional last term in (47), one obtains the exact Dirac spectrum, as it was shown in [35],

$$M_n = \frac{m}{\sqrt{1 + \left(\frac{Z\alpha}{n-\delta_j}\right)^2}}, \quad n = 1, 2, \dots \quad (48)$$

where δ_j is the same as in (38) with the replacement $l \rightarrow j$. Note, that both forms (42) and (48) coincide to the lowest order $O((Z\alpha)^2)$.

V. THE CASE OF THE RELATIVISTIC HARMONIC OSCILLATOR INTERACTION

Here we consider the scalar particle with the harmonic oscillator interaction of the vector or scalar type. In the first case one has the Klein-Gordon equations similar to (35), but with the replacement $-\frac{Z\alpha}{r} \rightarrow cr^2$, $c > 0$, which yields

$$(\mathbf{p}^2 + m^2 - (cr^2)^2 + 2Ecr^2)\varphi = E^2\varphi. \quad (49)$$

This equation evidently has no lower bound for the eigenvalues and hence no discrete spectrum.

This conclusion agrees with the case of the Dirac equation with the harmonic oscillator interaction of the vector type, considered in [40], where it was shown both numerically and analytically that no discrete spectrum appears in this case.

Let us turn now to the case of the scalar harmonic oscillator interaction, in which case the wave equation has the form

$$[E^2 - \mathbf{p}^2 - (m^2 + cr^2)]\varphi = 0 \quad (50)$$

with the stable discrete spectrum

$$E_n = \sqrt{m^2 + \sqrt{4c}(n + 3/2)}. \quad (51)$$

This result again agrees qualitatively with the corresponding Dirac spectrum found in [40].

Another useful comparison can be made with the case of the scalar particle in the constant magnetic field, where the interaction term is also of the harmonic oscillator type; neglecting the spin term in (34) one obtains

$$E_n(\text{mag}) = \sqrt{m^2 + p_3^2 + eB(2n + 1)}. \quad (52)$$

One can see the same type of the spectrum in 2d, while in (52) additional free continuous spectrum is present in the third dimension.

VI. THE QUARK-ANTIQUARK RELATIVISTIC GREEN'S FUNCTIONS AND ENERGY EIGENVALUES

For the $\bar{q}q$ system the (relativistic) Green's function is obtained as a product of q and $\bar{q}$ Green's functions with the corresponding vertices,

$$G_{q\bar{q}}(x, y) = \left\langle \frac{(m_1 - \hat{D}_1)\Gamma(m_2 - \hat{D}_2)\Gamma}{(m_1^2 - \hat{D}_1^2)(m_2^2 - \hat{D}_2^2)} \right\rangle \quad (53)$$

and we note, that for the scalar quarks the Green's functions are the denominators in (53) without spin operator, i.e., using (11) one has the ω -representation for the $q\bar{q}$ Green's function (scalar quarks)

$$\begin{aligned} & \left\langle \frac{1}{(m_1^2 - D_\mu^2)(m_2^2 - \bar{D}_\mu^2)} \right\rangle_{xy} \\ &= \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \langle \mathbf{x}, \mathbf{x} | e^{-H(\omega_1, \omega_2, \mathbf{p}_1, \mathbf{p}_2)T} | \mathbf{y}, \mathbf{y} \rangle \end{aligned} \quad (54)$$

where

$$\begin{aligned} H(\omega_1, \omega_2, \mathbf{p}_1, \mathbf{p}_2) &= \frac{\mathbf{p}_1^2 + m_1^2 + \omega_1^2}{2\omega_1} + \frac{\mathbf{p}_2^2 + m_2^2 + \omega_2^2}{2\omega_2} \\ &+ V(\mathbf{r}_1 - \mathbf{r}_2). \end{aligned} \quad (55)$$

Now integrating over $d^3(\mathbf{x} - \mathbf{y})$ one obtains the Green's function and the Hamiltonian in the c.m. system $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = 0$, while taking the trace in the numerator of (53) one obtains finally the ω -representation for the $q\bar{q}$ c.m. Green's function [27]

$$\begin{aligned} G_{q\bar{q}}^{C.M.}(x, y) &= \int G_{q\bar{q}}(x, y) d^3(\mathbf{x} - \mathbf{y}) \\ &= \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \langle Y_\Gamma \rangle \\ &\times \langle 0 | e^{-H_{C.M.}(\omega_1, \omega_2, \mathbf{p})T} | 0 \rangle \end{aligned} \quad (56)$$

where

$$\langle Y_\Gamma \rangle = \frac{1}{4} \text{tr} \langle \Gamma(m_1 - \hat{D}_1) (\Gamma(m_2 - \hat{D}_2)) \rangle. \quad (57)$$

The spin d.o.f. enter in two ways: (1) in the numerator $\langle Y_\Gamma \rangle$ explicitly; (2) via the denominator, where spin-vacuum interaction $\langle \sigma F \sigma F \rangle$ ensures a strong self-energy correction

$$\begin{aligned} (m^2 - (\hat{D}_\mu \gamma_\mu)^2)^{-1} &= (m^2 - D_\mu^2 - g\sigma_{\mu\nu} F_{\mu\nu})^{-1} \\ &= G_0(1 + G_0 g\sigma F G_0 g\sigma F + \dots). \end{aligned} \quad (58)$$

The averaging over vacuum fields in the last term in (58) yields the self-energy correction to the Hamiltonian [41]

$$H_{C.M.} \rightarrow H_{C.M.} + \Delta M_{SE}, \quad \Delta M_{SE} = -\frac{b\sigma\eta_1}{\pi\omega_1} - \frac{b\sigma\eta_2}{\pi\omega_2}. \quad (59)$$

Here the factor “ b ” contains a renormalized contribution of field correlators we take it with corrections found in [42], as shown in Appendix B, and $\eta_i \equiv \eta(m_i)$ are calculated in [41], normalized as $\eta_i(0) = 1$. Note, that the effect of the (negative) ΔM_{SE} term is very important for the spectrum and as will be seen below, strongly decreases the mass. A similar role in the standard Salpeter approach (see, e.g., [9]) is played by an adjustable negative constant in the Hamiltonian, which however was found different for light

and heavy flavors and moreover violates the linear Regge trajectories. In our case the constant b is defined by field correlators D^E , D_1^E and does not depend on the quark masses, while the function $\eta(m_i)$ is well defined for all values of quark masses, see Appendix B for details.

As a result the total Hamiltonian with the color Coulomb, self-energy, and spin-spin interaction acquires the form [42,43], where the spin-spin term was obtained in the framework of the field correlator method in [44].

$$H = H_0 + V_{\text{conf}} + V_{\text{OGE}} + \Delta M_{SE} + \Delta M_{ss}, \quad (60)$$

where

$$H_0 = \sum_{i=1,2} \frac{m_i^2 + \omega_i^2}{2\omega_i} + \frac{\mathbf{p}^2}{2\tilde{\omega}}, \quad \tilde{\omega} = \frac{\omega_1\omega_2}{\omega_1 + \omega_2} \quad (61)$$

$$V_{\text{conf}} = \sigma r, \quad V_{\text{OGE}} = -\frac{4\alpha_s(r)}{3r},$$

$$\Delta M_{SE} = -\sum_i \frac{b\sigma\eta(\omega_i)}{\pi\omega_i} \quad (62)$$

$$\Delta M_{ss} = \frac{\sigma_1\sigma_2}{12\omega_1\omega_2} V_4(r), \quad V_4(r) = \frac{32\pi\alpha_s}{3} \delta^{(3)}(\mathbf{r}). \quad (63)$$

We start with the case of one light quark and a heavy antiquark. Neglecting the contribution of the latter, one obtains the one-particle problem in the linear potential, which we shall treat in our formalism and compare the spectrum to the Dirac equation with linear potential, and at the end discuss its relation to the heavy-light meson. In our present formalism we have two possibilities for the Hamiltonian

(1) The so-called einbein form [19,27]

$$\left(\frac{m_i^2 + \omega_i^2}{2\omega_i} + \frac{\mathbf{p}^2}{2\omega_i} + \sigma r + \Delta M_{SE} \right) \psi_n = M_n \psi_n. \quad (64)$$

(2) The square-root form (the spinless Salpeter equation)

$$\left(\sqrt{\mathbf{p}^2 + m_1^2} + \sigma r + \Delta M_{SE} \right) \psi_n = M'_n \psi_n. \quad (65)$$

In the case 1, taking into account, that eigenvalues ε_n of the equation

$$\left(\frac{\mathbf{p}^2}{2\omega} + \sigma r \right) \varphi_n = \varepsilon_n \varphi_n \quad (66)$$

are equal to $\varepsilon_n = \frac{\sigma^{2/3} a(n)}{(2\omega)^{1/3}}$, $a(0) = 2.338$, one obtains for $m = 0$

$$M_0 = 0.78 \text{ GeV}, \quad M'_0 = 0.72 \text{ GeV} (b = 2). \quad (67)$$

It is interesting to compare these values with the result for the ground state level of the Dirac equation with linear potential, which was done in [40],

$$M_0^D = 0.689 \text{ GeV}. \quad (68)$$

One can see a good agreement within 30 MeV of the spinless Salpeter result M'_0 with the Dirac eigenvalues (68).

One can realize, that both Eqs. (64), (65) contain only positive energy levels, and no contribution from negative energy levels is present in M_n , M'_n , whereas in M_n^D all 4 components of the wave function contribute.

Therefore in the heavy-light case one can see an additional (Dirac) eigenvalue suppression as compared to the standard self-energy mechanism.

We now turn back to the case of the $q\bar{q}$ system in the confining vacuum. We again consider the Hamiltonian (60) with the ΔM_{SE} , V_{OGE} , and ΔM_{ss} terms and take into account the color Coulomb interaction also in calculating ω , which is important for the lowest $q\bar{q}$ states with small masses m_q , $m_{\bar{q}}$, e.g., for the ρ meson.

In this case in the approach (2) (square root Hamiltonian) one obtains [43]

$$\bar{\omega} = \omega_{\text{Salp}} + \Delta\omega_{\text{coul}} = (0.335 + 0.054) \text{ GeV} = 0.39 \text{ GeV} \quad (69)$$

where $\Delta\omega_{\text{coul}}$ is the change of ω due to color Coulomb interaction.

The correction ΔM_{SE} and the eigenvalue of $H_0 + V_{\text{conf}}$ are

$$\Delta M_{\text{SE}} = -\frac{3\sigma}{\pi\bar{\omega}} \cong -0.415 \text{ GeV}, \quad M_0 = 1.339 \text{ GeV} \quad (70)$$

$$\langle V_{\text{OGE}} \rangle = -\frac{3}{4} \alpha_s \langle r^{-1} \rangle = -0.60 \langle r^{-1} \rangle = -0.218 \text{ GeV} \quad (71)$$

and the resulting spin-averaged mass is

$$\bar{M} = M_0 + \Delta M_{\text{SE}} + \langle V_{\text{OGE}} \rangle = 0.706 \text{ GeV}. \quad (72)$$

Finally one must take into account the hyperfine interaction, which yields

$$\Delta_{ss} = \frac{32\mathbf{s}_1\mathbf{s}_2\alpha_s\pi}{9\bar{\omega}^2} \varphi_n^2(0) \rightarrow \frac{8\alpha_s\pi}{9\bar{\omega}^2} \varphi_n^2(\omega), \quad (73)$$

where $\varphi_n^2(0)$ can be estimated as follows,

$$\varphi_n^2(0) = \frac{\bar{\omega}}{4\pi} \left(\sigma + \frac{4}{3} \alpha_s \left\langle \frac{1}{r^2} \right\rangle \right) \cong \frac{0.109}{4\pi} \text{ GeV}^3 \quad (74)$$

and as a result one obtains for ρ meson

$$\Delta_{ss}(\rho) = (0.048 \div 0.065) \text{ GeV}, \quad (75)$$

depending on whether one is using Coulomb-corrected (left) or noncorrected value of ω (right).

Finally one obtains

$$M_\rho(\text{square root Hamiltonian}) = (0.754 - 0.771) \text{ GeV} \quad (76)$$

to compare with the experimental value $M_\rho(\text{exp}) = 0.775 \text{ GeV}$.

In a similar way our ω technique allows to calculate all light meson masses, using only string tension σ , $\alpha_s(\Lambda_{\text{QCD}})$ and current quark masses, as shown in [19–21,23,29–32,43], without extra parameters. One should also stress that also orbital and radial linear Regge trajectories are obtained in the same formalism exploiting exact representation of the relativistic Green's functions, and the corresponding eigenvalues, see [43] for more results and references.

VII. CONCLUSION

We have derived above the exact relativistic Green's functions for the static (time-independent) interactions, which allow us to write down simple integral representation the ω -representation in terms of the nonrelativistic Green's function and Hamiltonian. These Green's functions and eigenvalues were compared to the known relativistic results of the constant magnetic field and Coulomb interaction also including spin, and the agreement was demonstrated.

In this way we can exploit the vacuum generated instantaneous interaction V_{conf} and V_{coul} in the ω -representation to obtain the basic results for masses and wave functions, leaving additional gluon and $q\bar{q}$ creation terms as a perturbation.

As a practical implementation the masses of heavy-light and light-light mesons have been considered and a close correspondence with experiment in the case of the ρ meson was shown in detail. It is important, that this method exploits only few basic parameters: σ , $\alpha_s(\Lambda_{\text{QCD}})$ and quark current masses, in contrast to other approaches with many additional parameters.

It is possible to make our theory even more fundamental, since one can express Λ_{QCD} via σ , as it was shown in [18], leaving σ and current quark masses as the only scale parameters in QCD.

At this point it is important to disclose the meaning of the term “exact” in the exact agreement of the results of our ω -representation and the Schwinger derivation, or Dirac spectrum. It is clear, that the fully exact result, e.g., in QCD

should contain contribution of additional $q\bar{q}$ pairs and any number of gluons interacting with initial number of particles, which create the so-called Fock column of wave functions, and this is not provided by the ω -representation or the Schwinger expression for the Green's function, where the electric charge enters only in the combination eB . Therefore one should associate the exact agreement with the coincidence of the corresponding Taylor series in powers of the external field.

The method, based on the ω -representations discussed above, was developing successfully during the last 30 years and has allowed to calculate analytically mesons, glueballs, hybrids, and baryons in terms of the minimal number of parameters σ , α_s , m_q , for the latest development see [43], see also [45–49] for similar methods and [50,51] for reviews.

ACKNOWLEDGMENTS

The author is grateful for help and useful advices to A. M. Badalian. Useful discussions with R. A. Abramchuk, M. A. Andreichikov and Z. V. Khaidukov are gratefully acknowledged. This work was done in the framework of the scientific program of the Russian Science Foundation, RNF, the Project No. 16-12-10414.

APPENDIX A

The starting point in the vacuum averaged Wilson loop, which we shall take simplicity in the **1, 4** plane and keeping the lowest, $\langle FF \rangle$ correlator, one has

$$\langle W(C) \rangle = \exp \left(-\frac{g^2}{2} \int d\sigma_{14}(u_1, u_4) d\sigma_{14}(v_1, v_4) \langle FF \rangle \right). \quad (\text{A1})$$

We shall show below, that it can be expressed via instantaneous potentials V_D , V_1 . Here $\langle FF \rangle$ is

$$\begin{aligned} g^2 \langle FF \rangle &= \frac{g^2}{N_c} \langle \text{tr}_f (F_{14}(u) \Phi(u, v) F_{14}(v) \Phi(v, u)) \rangle \\ &= D^E(u - v) + \frac{1}{2} \left(\frac{\partial}{\partial u_1} [(u_1 - v_1) D_1^E(u - v)] \right. \\ &\quad \left. + \frac{\partial}{\partial u_4} [(u_4 - v_4) D_1^E(u - v)] \right). \end{aligned} \quad (\text{A2})$$

Insertion of (A2) in (A1), and integrating

$$\int d\sigma_{14}(u_1, u_4) = \int_0^R du_1 \int_0^T du_4, \quad (\text{A3})$$

one obtains using the geometry in Fig. 1

$$\langle W(C) \rangle = \exp \left(- \int_0^T (V_D(R) + V_1(R)) dt \right) \quad (\text{A4})$$

FIG. 1. Calculation of the static potentials V_D , V_1 from field correlators.

where, denoting $w_1 = u_1 - v_1$, one has

$$V_D(R) = 2 \int_0^R (R - w_1) dw_1 \int_0^T dw_4 D^E(\sqrt{w_1^2 + w_4^2}) \quad (\text{A5})$$

$$V_1(R) = \int_0^R w_1 dw_1 \int_0^T dw_4 D_1^E(\sqrt{w_1^2 + w_4^2}) \quad (\text{A6})$$

as it was shown in [19], $D_1(x)$ at small x has the form

$$D_1(x) = \frac{4C_2\alpha_s}{\pi} \left\{ \frac{1}{x^4} + \frac{\pi^2 G_2}{24N_c} + \dots \right\} \quad (\text{A7})$$

and therefore it contributes to the color Coulomb interaction, $V_1^{\text{OGE}}(r) = -\frac{C_2\alpha_s(r)}{r}$, which is properly renormalized at small distances, see [52,53] for the analytic and lattice analysis,

It is important, that actually V_D , V_1 are nonlocal both in time and space, and the final local terms (A5), (A6) are obtained in the “local limit”, when the correlation length in D^E , D_1^E is going to zero, actually $D(w) = f(\frac{w}{\lambda})$, $\lambda \sim 0.1$ fm. Hence V_D , V_1 are local and instantaneous with this accuracy.

APPENDIX B

Following [41,54], one can use in (58) the expansion in powers of $(g\sigma_{\mu\nu}F_{\mu\nu})$ and take into account the vacuum averaging, which keeps nonzero the second order term, which be represented as

$$\Lambda = \int d^4(u - v) g^2 \langle \sigma F(u) \Phi(u, v) \sigma F(v) \Phi(v, u) \rangle G(u, v) \quad (\text{B1})$$

where $G(u, v) = (m_q^2 - D_{\mu}^2)_{u,v}^{-1}$ is the spinless quark Green's function.

One can use for the field correlator in (B1) the standard representation (A2) in terms of correlators $D^E(u - v)$ and $D_1^E(u - v)$, which have a small correlation length $\lambda \approx 0.2$ fm and decay exponentially at large distances, $D^E(w) = D^E(0) \exp(-|w|/\lambda)$. At the same time $G(u, v)$ at small distances $|u - v| \lesssim \lambda$ can be replaced by the corresponding free form

$$G(u, v) \approx G_0(u - v) = \frac{m_q K_1(m_q |w|)}{4\pi^2 |w|}, \quad w = u - v. \quad (\text{B2})$$

As a result Λ can be brought to the form

$$\Lambda = m_q(3D(0) + 3D_1(0))\varphi(m_q\lambda) \quad (\text{B3})$$

where

$$\varphi(m_q\lambda) = \int_0^\infty w^2 dw e^{-w/\lambda} K_1(m_q w) \quad (\text{B4})$$

and using the relation $\sigma = \frac{1}{2} \int D(x) d^2x = \frac{\pi D(0)}{\lambda^2}$, one obtains

$$\Delta M_{\text{SE}} = -\frac{\Lambda}{2\omega} = -\frac{3\sigma}{2\pi\omega} \eta(m_q\lambda)(1 + \xi). \quad (\text{B5})$$

Here $\eta(m_q\lambda) = \frac{m_q}{\lambda^2} \varphi(m_q\lambda)$; $\eta(0) = 1$, and $\xi = \frac{D_1^E(0)}{D^E(0)}$, denotes the contribution of the correlator $D_1^E(0)$, which is below $\frac{1}{3} D^E(0)$, according to the lattice measurements [55].

As a result the constant “b,” introduced in (59), according to (B5) is

$$b = \frac{3}{2}(1 + \xi), \quad 0 < \xi \leq 1/3 \quad (\text{B6})$$

where the parameter ξ is not known with good accuracy.

The function $\eta(m_q\lambda)$ is normalized as $\eta(0) = 1$, and it has asymptotics $\eta \approx \frac{2}{m_q^2 \lambda^2}$ at large m_q . E.g. for the b quark, $m_q \cong 5$ GeV one has $\eta(m_b\lambda) \cong 0.052$, which makes the contribution of ΔM_{SE} unimportant, $\Delta M_{\text{SE}}(m_b) \approx -0.89$ MeV.

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