Generalized clockwork theory

Ido Ben-Dayan*

*Department of Physics, Ben-Gurion University of the Negev, P.O. Box 653, Be'er-Sheva 8410500, Israel
and Physics Department, Ariel University, Ariel*

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We generalize the clockwork theory in several directions. First, we consider beyond nearest neighbors interactions. Considering such interactions keeps a larger subgroup of the original $U(1)^{N+1}$ unbroken and can allow for different symmetry breaking patterns. We recover the original clockwork scenario in the presence of these additional interactions. In such case, the masses of the massive modes change, but a single massless mode remains intact. Such interactions are naturally interpreted as higher derivative terms from the point of view of extra dimensions. Second, we generalize the clockwork shift symmetry to non-Abelian global groups. Third, trivial embedding of the clockwork scenario in supergravity, yields an AdS minimum as big as the clockwork interaction. Specifically, the clockwork is connected to the cosmological constant. We analyze the different ways in which a Minkowski supersymmetric minimum can be constructed, and demonstrate simple supersymmetry (SUSY) breaking mechanisms that preserve or break the clockwork symmetry. We show that the clockwork direction is actually a special SUSY breaking direction, that does not require the inclusion of additional fields or parameters. Fourth, we review the extra-dimensional origin of the mechanism and interpretation, in the case of conformal coupling to gravity.

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I. INTRODUCTION

Clockwork theory (CW) has been proposed as a mechanism of generating light particles with suppressed interactions, while no small parameters exist in the UV theory, [1–3]. An earlier incarnation is actually in the context of having a super-Planckian axion decay constant in natural inflation models, in the case of many sites [4], or simple two sites case, [5–7]. Given that a large portion of contemporary theoretical physics involves generation of small/large numbers from $O(1)$ numbers in a “natural” way, several new applications of the idea have been suggested, like a clockwork WIMP [8], clockwork inflation [9], clockwork composite Higgs [10], photophilic QCD axion [11], and a solution to the hierarchy problem, [3].

The basic framework considers $N + 1$ complex scalar fields with global $U(1)^{N+1}$ symmetry. The symmetry is explicitly broken by “nearest neighbors” interaction to a single $U(1)_{\text{CW}}$. Thus, we have a “theory lattice,” where each scalar is sitting on a different site and interacts only with its nearest neighbors. In such case, there is a single massless mode, where its overlap with the j th site goes like

$\sim q^{-j}$, $q > 1$. Hence, if we couple “matter fields” to the N th site, the massless mode coupling to these matter fields is suppressed by q^{-N} . The idea can be implemented for scalars, fermions, gauge fields, and gravitons (at least at the linear level). In all cases a massless particle remains while the other N have a rather dense mass spectrum, all around the fundamental scale of the theory. In [3], it was suggested that such a framework may come from deconstructing a $5D$ scalar, potentially coming from “little string theory.” A recent analysis in [12] has shown that the CW phenomenon is purely Abelian, and cannot come from purely geometric extradimensional effects. However, see the response [13]. A rather interesting suggestion is discarding the use of elementary scalar fields and deriving a “clockworked” behavior from a sequence of strongly coupled sectors [14].

To avoid obscurity, our definition of the clockwork is the following: Considering a theory with $N + 1$ fields, charged under a $\mathcal{G}^{N+1}$ symmetry group with charge q , a “clockwork” term is turned on leaving a residual $\mathcal{G}$ symmetry and a massless mode, such that its overlap with the N massive modes behaves as q^{-j} for the j th field.

Beauty is in the eye of the beholder, and considering multiple fields to explain hierarchies is neither very elegant nor extremely new. The usefulness of CW lies in a natural generation of hierarchy, that goes as $\sim q^N$ rather than $\sim N$ in a theory whose fundamental parameters are of similar size. In this paper, we offer several simple generalizations and observations that may be useful for particle physics

*ido.bendayan@gmail.com

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phenomenology and cosmology. Both areas are riddled with hierarchy problems.

The purpose of these generalizations is twofold. First, understanding in what applications can we apply the CW mechanism in a useful way, that might be tested by observations. Second, what is the UV theory that generates CW behavior?

First, using effective field theory as a guiding principle, nothing forbids additional interactions of the CW mechanism beyond nearest neighbors interactions. We shall demonstrate that interactions of k neighbors results in a residual symmetry group of $U(1)^k$. The low energy theory will have k massless modes. These interactions will show up and modify the mass spectrum of the massive fields in the original CW scenario predicting a rather different phenomenology. For example, in the original CW scenario of complex scalar fields, the masses of the radial fields that get a vacuum expectation value (vev) $\sim f$ is parametrically larger than the massive axions $\sim \sqrt{\epsilon} f q$, where ϵ is a small coupling constant controlling the CW behavior and q some small integer. By going beyond nearest neighbors, the axions now have a mass of $\sim \sqrt{\epsilon} f q^k$, where k is the number of neighbors each site couples to. So in this setting, the axions can be more massive by a factor of q^{k-1} compared to the simple CW model. However, for consistency, these masses are still lighter than the masses of the radial modes. Additionally, the SM and its extensions have multiple (sometimes anomalous) $U(1)$ global symmetries. By going beyond nearest neighbors we can accommodate such symmetries in a straightforward way.

Considering the UV theory, we show that these beyond nearest neighbors interactions correspond to higher order derivatives in a 5D picture in Sec. III. In Sec. IV, we generalize the CW mechanism to non-Abelian global groups, and specifically to the $O(N)$ vector model.

Second, the CW has been realized in the supersymmetric context in [1]. The idea is to get CW with SUSY vacuum, from which one can start model building by breaking SUSY. We discuss the supersymmetric CW in Sec. V, and try to embed it in supergravity (SUGRA). The SUGRA behavior turns out to be very different than the global SUSY case. An immediate generalization to supergravity (SUGRA) with canonical Kähler potential, preserves the $U(1)_{\text{CW}}$, but gives vevs to additional fields, yielding an AdS supersymmetric vacuum. The AdS minimum is controlled by the CW coupling. The vacuum is lifted to a Minkowski SUSY vacuum by adding a constant term to the superpotential.

Alternatively, using a stabilizer field leads to N flat directions or CW that its energy scale is parametrically the cosmological constant. Instead, we offer a shift symmetric CW superpotential such that the resulting F-term scalar potential has the CW form with SUSY Minkowski minimum. We then demonstrate how to break SUSY to a dS without breaking $U(1)_{\text{CW}}$ by introducing a spurion

superfield. Furthermore, we show that breaking SUSY within the CW sector forces the massless CW mode to be the SUSY breaking direction. Then, no residual $U(1)_{\text{CW}}$ is left. Avoiding this conclusion requires the inclusion of additional parameters and is transparent once we move to the mass basis.

Third, to diminish the arbitrariness in invoking $N + 1$ scalar fields, one can view the CW as discretizing an extra dimension [3]. In the continuum limit, the theory behaves as a linear dilaton coming from little string theory, [15–17]. This allows the extra dimension to be warped. However, contrary to the Randall-Sundrum case where the warping is exponential, here the warping is polynomial. Thus, for the correct Planck mass, and a new physics scale at 10 TeV the proper size of the extra dimension is considerably larger, at the nanometer level [17]. Up to now, the residual CW $U(1)$ was explicitly broken by some operator, usually coupling the N th field to some different sector of the theory, for example the SM making the framework “technically natural.” It is interesting to consider whether the explicit breaking of the CW symmetry can arise naturally, rather than adding it by hand according to the problem one wishes to solve. In such case the theory will not be technically natural, but simply “natural” since all the couplings and scales will be determined by the UV theory. Therefore, in Sec. VI, after reviewing the 5D picture we conformally couple the 5D free scalar field to the Ricci scalar and/or add a mass term, that in 4D will provide an explicit breaking term to the CW symmetry. Upon discretization the coupling to the Ricci scalar makes the massless mode tachyonic, while the 5D mass term obviously gives a positive mass to the massless mode. We suggest a conformal coupling of the clockwork to a positively curved 5D manifold. This time, the conformal coupling to gravity gives a positive mass term to the massless CW mode. We find similar mass scales, but now the CW charges q become site-dependent, q_j . We analyze the case of site dependent charges in the Appendix. In both cases above, we assumed some other dynamics or energy sources, fixing the 5D metric.

The outcome of these simple generalizations is a mixed blessing. On the one hand, from a lattice point of view the CW idea can be generalized to any number of neighbors and to global non-Abelian symmetry groups. On the other hand, our investigation shows that CW is a rather delicate phenomena, and its embedding in SUGRA or beyond a free scalar field in 5D is problematic. The UV origin of CW theory is therefore obscure.

II. GENERALIZATION OF THE CLOCKWORK MECHANISM

A. Review of the clockwork mechanism

The original clockwork considers $N + 1$ complex scalars, ϕ_j , where $j = 0, 1, \dots, N$ with canonical kinetic terms, and a potential:

$$V(\phi_j) = \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^\dagger \phi_j + \frac{\lambda}{4} |\phi_j^\dagger \phi_j|^2 \right) + \sum_{j=0}^{N-1} (\epsilon \phi_j^\dagger \phi_{j+1}^3 + \text{H.c.}) \quad (2.1)$$

When $\epsilon \rightarrow 0$ we have a global $U(1)^{N+1}$ symmetry. Turning on $\epsilon \ll \lambda < 1$ breaks this symmetry to a $U(1)$. Under the remaining $U(1)$, the fields have charges 3^{-j} . We expand around the vacuum of the spontaneously broken theory $\langle |\phi_j|^2 \rangle = f^2 \equiv 2\tilde{m}^2/\lambda$, $\forall j$. Below the breaking scale $\sqrt{\lambda}f$, we have a theory of $N+1$ Goldstone bosons with $U_j = e^{i\pi_j(x)/f}$ and $j = 0, \dots, N$:

$$\mathcal{L} = -\sum_{j=0}^N f^2 \partial U_j^\dagger \partial U_j + \frac{m^2 f^2}{2} \sum_{j=0}^{N-1} (U_j^\dagger U_{j+1}^q + \text{H.c.}) \quad (2.2)$$

(2.1) corresponds to $q = 3$, but the U s are dimensionless, so we can consider any q , as is done in [3], and $m^2 = 2\epsilon f^2$. In general, one can assign different masses m_j and different charges q_j to each site. In terms of the low energy effective theory of the pions we have canonical kinetic terms and the following potential:

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-1} (\pi_j - q\pi_{j+1})^2 + \mathcal{O}(\pi^4) \\ = \frac{1}{2} \sum_{i,j=0}^N \pi_i M_{ij}^2 \pi_j + \mathcal{O}(\pi^4) \quad (2.3)$$

The theory is invariant under the shift symmetry $\pi_j \rightarrow \pi_j + r/q^j$, where $r \in \mathbb{R}$.

The mass matrix M_π^2 is given by

$$M_\pi^2 = m^2 \begin{pmatrix} 1 & -q & 0 & \cdots & 0 \\ -q & 1+q^2 & -q & \cdots & 0 \\ 0 & -q & 1+q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+q^2 & -q \\ 0 & 0 & 0 & \cdots & -q & q^2 \end{pmatrix}. \quad (2.4)$$

Diagonalizing the mass matrix gives one massless mode and N massive modes with successive mass splittings:

$$m_{a_0}^2 = 0, \quad m_{a_k}^2 = m^2 \lambda_k, \\ \lambda_k = \left(1 + q^2 - 2q \cos \frac{k\pi}{N+1} \right), \quad k = 1, \dots, N \quad (2.5)$$

where the interaction basis π_j and mass basis a_j are related by:

$$\pi = \mathcal{O}a, \quad \mathcal{O}^T M_\pi^2 \mathcal{O} = \text{diag}(m_{a_0}^2, \dots, m_{a_N}^2) \quad (2.6)$$

The rotation matrix and normalization are given by:

$$\mathcal{O}_{j0} = \frac{\mathcal{N}_0}{q^j}, \quad \mathcal{O}_{jk} = \mathcal{N}_k \left[q \sin \frac{jk\pi}{N+1} - \sin \frac{(j+1)k\pi}{N+1} \right], \\ j = 0, \dots, N; \quad k = 1, \dots, N \quad (2.7)$$

$$\mathcal{N}_0 \equiv \sqrt{\frac{q^2 - 1}{q^2 - q^{-2N}}}, \quad \mathcal{N}_k \equiv \sqrt{\frac{2}{(N+1)\lambda_k}}. \quad (2.8)$$

$\mathcal{O}_{j0}$ is the amount the of the massless mode a_0 contained in each pion π_j . Because $\mathcal{O}_{j0} \sim q^{-j}$ it becomes smaller by a factor of q as we move away in j . Thus, the overlap with the last site, the N th one, is exponentially suppressed. By coupling a theory like the Standard Model to the N th site, we get an exponentially enhanced decay constant for the Goldstone interactions (with a_0), i.e., a scale exponentially larger than the scale of spontaneous symmetry breaking f :

$$\mathcal{L} = \frac{\pi_N}{16\pi^2 f} F_{\mu\nu} \tilde{F}^{\mu\nu} \Rightarrow \frac{1}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \left(\frac{a_0}{f_0} - \sum_{k=1}^N (-1)^k \frac{a_k}{f_k} \right), \\ f_0 = \frac{q^N f}{\mathcal{N}_0}, \quad f_k = \frac{f}{\mathcal{N}_k q \sin \frac{k\pi}{N+1}}. \quad (2.9)$$

To summarize, in the original CW model, before coupling to the sector to the SM, one has a residual $U(1)$, a massless Goldstone mode with eigenvector $\mathcal{O}_{j0} \sim q^{-j}$, massive axions with masses $m_{a_k}^2 \sim \epsilon f^2 q^2$ and massive radial modes with masses $m_{r_k}^2 \sim f^2$, (or $m_{r_k}^2 \sim f^2(1+\epsilon)$ to be precise), parametrically heavier than the axions. Both the masses of the radial modes and axions are densely spaced. The above analysis requires the hierarchy of couplings, $\epsilon \ll \lambda \ll 1$. We shall see that this hierarchy is relaxed once we generalize to beyond nearest neighbors.

B. Generalization of the clockwork mechanism beyond nearest neighbors

As an effective field theory, there is no reason to limit ourselves to nearest neighbors interactions as in (2.2), since many additional interactions still respect the $U(1)$ symmetry. One must take these interactions into account in a consistent way.¹ We expect these additional interactions to

¹If we wish to follow the original discussion of a renormalizable theory in (2.1), we can still consider tree level interactions of the sort $\phi_j^\dagger \phi_{j+1} \phi_{j+2}^2 + \text{H.c.}$ Upon diagonalization, we will still have two massless modes with the overlap of the last site behaving as $2^{-(N-1)}$ and $2^{-(N-2)}$.

modify the resulting phenomenology. Indeed, the coupling to beyond nearest neighbors changes the symmetry structure of the theory and its spectrum in an interesting manner.

Let us start by considering interactions between each site to the next to nearest neighbors. The potential with such couplings will look like:

$$\frac{m^2 f^2}{2} \sum_{j=0}^{N-2} (U_j^\dagger U_{j+1}^q U_{j+2}^p + \text{H.c.}) \quad (2.10)$$

Notice that since we couple each site to the two consecutive sites, we have to truncate the sum at $N - 2$ instead of $N - 1$. Considering again the pions gives:

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} (\pi_j - q\pi_{j+1} - p\pi_{j+2})^2 + \mathcal{O}(\pi^4). \quad (2.11)$$

Each term in the potential, is invariant under the transformation $\pi_j \rightarrow \pi_j + \alpha_j$ if:

$$\alpha_j = 2\pi\ell + q\alpha_{j+1} + p\alpha_{j+2} \quad (2.12)$$

for integer j , and integer ℓ . The original clockwork idea is the particular case of $p = 0$. Obviously, we have here a two dimensional space, spanned by α_{j+1} , α_{j+2} and as expected, we have $U(1)^2$ residual symmetry, instead of $U(1)$. We also have two massless modes. To recover the clockwork behavior we choose $q = \tilde{q}/2$ and $p = \tilde{q}^2/2$, then the potential will be made of terms $V \supset (\pi_j - \frac{\tilde{q}}{2}\pi_{j+1} - \frac{\tilde{q}^2}{2}\pi_{j+2})^2$. Let us drop the tildes. In such case the original clockwork symmetry is conserved: $\pi_j \rightarrow \pi_j + 1/q^j$, and there is an additional $U(1)$ symmetry.

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} \left(\pi_j - \frac{q}{2}\pi_{j+1} - \frac{q^2}{2}\pi_{j+2} \right)^2 + \mathcal{O}(\pi^4). \quad (2.13)$$

Diagonalizing the mass matrix will give two massless states, one of which is the clockwork with $\mathcal{O}_{j0} = \mathcal{N}_0/q^j$. A straightforward check shows that a second massless eigenstate is given by an alternating vector: $\mathcal{O}_{j1} \sim 2^j/(-q)^j$. However, such a vector is not orthogonal to the original clockwork mode. We orthogonalize the system using the Gram-Schmidt procedure that gives (no summation):

$$\mathcal{O}_{j1} = \frac{\mathcal{N}_1}{q^j} \left[(-2)^j - \frac{\sum_{i=0}^N (-2q^{-2})^i}{\sum_{i=0}^N q^{-2i}} \right]. \quad (2.14)$$

There are $N - 1$ massive states that can also be brought to the desired orthonormal form by the Gram-Schmidt procedure. Their mass will be dominated by q^4 terms rather than q^2 in the original clockwork. The exact expression for

the masses and eigenvectors can be obtained by recursion relations. However, it requires the analytical solution of a fourth order polynomial. While such a solution exists, it is not illuminating to write it down. Coupling more and more neighbors will generate a polynomial of degree larger than four, that does not have an analytical general solution. Even without an analytical expression for the massive modes, the massless mode and the essence of the CW mechanism exists with the q^{-j} overlap.

The generalization to any number of neighbors interactions is straightforward. Considering n nearest neighbors interactions, the Lagrangian will look like:

$$\mathcal{L} = -f^2 \sum_{j=0}^N \partial U_j^\dagger \partial U_j + \frac{m^2 f^2}{2} \sum_{j=0}^{N-n} (U_j^\dagger U_{j+1}^{q/n} \dots U_{j+n}^{q^n/n} + \text{H.c.}). \quad (2.15)$$

For n nearest neighbors interactions we will preserve n symmetries and the conserved symmetry group will be some $U(1)^n$. Such a generalization allows for various breaking patterns, not necessarily reaching the $U(1)$ of the clockwork type. If we wish to maintain the same clockwork behavior of $\alpha_{j+1}/\alpha_j \sim q^{-1}$, then coupling to further neighbors makes the other massless eigenstates expression cumbersome. It requires the simultaneous solution of:

$$\alpha_j = \sum_{k=1}^n \alpha_{j+k} \frac{q^{j+k}}{n}, \quad \frac{\alpha_{j+1}}{\alpha_j} = \text{const.} \quad (2.16)$$

where n is the number of neighbors that are coupled. Such recursive equations generate higher and higher polynomial equation for α_0 , α_1 , that for $n \geq 6$ do not have a general analytic solution.

Analyzing the spectrum of the theory, we see a qualitatively different behavior. For n nearest neighbors interactions, we have $U(1)^n$ symmetry, n massless modes, and a different tower of massive modes. While the radial modes maintain their mass spectrum of $m_{r_k}^2 \sim f^2$, the axions' mass spectrum behaves as $m_{a_k}^2 \sim \epsilon f^2 q^{2n}/n^2$. Thus, depending on q , n , the axions can be parametrically heavier than the simplest CW model.

Alternatively, if we wish to write the most general Lagrangian that preserves only the original clockwork symmetry, we can add all possible neighbors interactions of this type. Thus, the full clockwork Lagrangian is actually:

$$\mathcal{L} = -f^2 \sum_{j=0}^N \partial U_j^\dagger \partial U_j + \frac{m^2 f^2}{2} \sum_{k=1}^N \sum_{j=0}^{N-k} (U_j^\dagger U_{j+1}^{q/k} \dots U_{j+k}^{q^k/k} + \text{H.c.}). \quad (2.17)$$

In this Lagrangian all the $U(1)$ symmetries except the original clockwork are broken, and a massless mode still remains and the component at each successive site remains $\mathcal{O}_{j0} = \mathcal{N}_0/q^j$. The masses of the radial modes remain $m_{r_k}^2 \sim f^2$. The axions will be heavier, the dominant mass contribution $m_{a_k}^2 \sim \epsilon f^2 q^{2N}/N^2$, pushing their masses toward the radial modes. Such different mass splitting could have interesting phenomenological consequences, since it changes model building scenarios such as the photophilic QCD axion of [111].

The low energy Lagrangian (2.15) might seem questionable from a UV point of view, as it involves fractional charges. However, such form is quite abundant in type IIB string theory constructions [18–23]. For example, such potential arises from D1 or D3 instantons wrapped around cycles in the Calabi-Yau manifold. A superpotential of the form

$$W_{D3} = \sum_{\alpha} A_{\alpha} e^{-iJ_{\alpha}^{\beta} T_{\beta}} \quad (2.18)$$

where T_{β} are moduli fields, A_{α} are coefficients that generically depend (weakly) on other fields in the spectrum, so are approximately constant, and J_{α}^{β} is a constant matrix parametrizing the warping numbers of the D3 instantons. In many known scenarios [18–23], the entries of J_{α}^{β} are of the form $\frac{2\pi}{N}$ where N is a rank of a gauge group. Since $\frac{2\pi}{N}$ is an irrational number, it will give fractional charges. After stabilizing the imaginary part of T_{β} the outcome are CW terms $W \sim e^{-iJ_{\alpha}^{\beta} \Re T_{\beta}}$, such that J_{α}^{β} produces the CW form of (2.33). Since $|W|^2 \sim \cos(J_{\alpha}^{\beta} \Re T_{\beta})$, so J_{α}^{β} can be fractional and give fractional charges, and in particular q^n/n .

Another way to get fractional charges is by considering the 5D picture, for example in the case of a positively curved manifold in Sec. VI. The expression for the charge there is fractional and site dependent with $q_j = \frac{\cos(\sqrt{3}/2kja)}{\cos(\sqrt{3}/2k(j+1)a)}$, where a is the lattice spacing and k is the CW spring constant.

C. Clockwork theory with integer charges

Let us discuss a different possibility of getting a CW behavior beyond nearest neighbors, but with integer charges. Generically, coupling beyond nearest neighbors interaction implies a nonrenormalizable theory. Nonrenormalizability also occurs in the original CW for $q > 3$. Let us start by analyzing this case: Below the cutoff scale Λ , the general potential looks like:

$$V(\phi_j) = \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^{\dagger} \phi_j + \frac{\lambda}{4} |\phi_j^{\dagger} \phi_j|^2 \right) + \sum_{j=0}^{N-1} \left(\epsilon \frac{\phi_j^{\dagger} \phi_{j+1}^q}{\Lambda^{q-3}} + \text{H.c.} \right). \quad (2.19)$$

Let us check whether previous analysis with the separation between the radial and axial modes still holds. We would like to give each radial field an approximate vev as before of $\langle |\phi_j|^2 \rangle = f^2 \equiv 2\tilde{m}^2/\lambda$, $\forall j$, as well as $f < \Lambda$. For this to happen, we need:

$$\epsilon \frac{f^{1+q}}{\Lambda^{q-3}} q^2 \ll \lambda f^4 \Rightarrow \epsilon q^2 \ll \lambda \left(\frac{\Lambda}{f} \right)^{q-3}. \quad (2.20)$$

Thus, for $q > 3$, the desired hierarchy is easier to fulfill than the original CW. This will also be true when we couple beyond nearest neighbors. The masses of the radial modes are negligibly shifted to $m_{r_k}^2 \sim f^2(1 + \epsilon q^2 (f/\Lambda)^{q-3}) \sim f^2$.

Below the breaking scale $\sqrt{\lambda}f$, we have a theory of $N+1$ Goldstone bosons with $U_j = e^{i\pi_j(x)/f}$ and $j = 0, \dots, N$:

$$\mathcal{L} = - \sum_{j=0}^N f^2 \partial U_j^{\dagger} \partial U_j + \frac{m^2}{2} \frac{f^{q-1}}{\Lambda^{q-3}} \sum_{j=0}^{N-1} (U_j^{\dagger} U_{j+1}^q + \text{H.c.}) \quad (2.21)$$

with $m^2 = 2\epsilon f^2$. The CW massless mode will now have the following eigenvector:

$$\mathcal{O}_{j0} = \mathcal{N}_0 \left\{ 1, \frac{1}{q}, \frac{1}{q^2}, \dots, \frac{1}{q^N} \right\}. \quad (2.22)$$

The masses of the axial modes behave as $m_{a_k}^2 \sim \epsilon f^2 (f/\Lambda)^{q-3} q^2$. The exact diagonalization is straightforward. Apart from the massless CW mode, what kind of mass hierarchy can we get between the radial and axial modes? The radial modes have masses of $m_{r_k}^2 \sim f^2$ as long as $\epsilon q^2 \ll \lambda (f/\Lambda)^{q-3}$. The mass of the axions is $m_{a_k}^2 \sim \epsilon f^2 (f/\Lambda)^{q-3} q^2$. The “standard hierarchy” is then $m_{r_k}^2 > m_{a_k}^2$, which corresponds to $1 \gg \epsilon (f/\Lambda)^{q-3} q^2$ that is trivially satisfied since $1 > \lambda \gg \epsilon q^2 (f/\Lambda)^{q-3}$.

Let us generalize the CW nonrenormalizable model by considering interactions between each site to the next to nearest neighbors. The potential with such couplings will look like:

$$V(\phi_j) = \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^{\dagger} \phi_j + \frac{\lambda}{4} |\phi_j^{\dagger} \phi_j|^2 \right) + \sum_{j=0}^{N-2} \left(\epsilon \frac{\phi_j^{\dagger} \phi_{j+1}^q \phi_{j+2}^p}{\Lambda^{p+q-3}} + \text{H.c.} \right), \quad (2.23)$$

where p, q are integer charges. Considering again the pions gives:

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} (\pi_j - q\pi_{j+1} - p\pi_{j+2})^2 + \mathcal{O}(\pi^4). \quad (2.24)$$

Similarly to the fractional charges case, we have $U(1)^2$ residual symmetry, and two massless modes. Both massless modes will have “clockwork” behavior in the sense that the coupling to different sites is suppressed by powers of p and q , but with no elegant expression. For example for $N = 4$ the massless modes of next-to-nearest neighbors are

$$\begin{aligned}\mathcal{O}_{j0} &= \mathcal{N}_0 \left\{ 1, \frac{q}{p+q^2}, \frac{1}{p+q^2}, 0, \frac{1}{p(p+q^2)} \right\} \\ \mathcal{O}_{j1} &= \mathcal{N}_1 \left\{ 1, \frac{p+q^2}{q(2p+q^2)}, \frac{1}{2p+q^2}, \frac{1}{q(2p+q^2)}, 0 \right\}.\end{aligned}\quad (2.25)$$

The massive modes, are then dominated by $\sim \epsilon m^2(p^2 + q^2)$ rather than $\sim \epsilon m^2 q^2$.

To recover similar clockwork behavior we choose $q = \tilde{q}$ and $p = \tilde{q}^2$, then the potential will be made of terms $V \supset (\pi_j - \tilde{q}\pi_{j+1} - \tilde{q}^2\pi_{j+2})^2$. Let us drop the tildes.

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} (\pi_j - q\pi_{j+1} - q^2\pi_{j+2})^2 + \mathcal{O}(\pi^4). \quad (2.26)$$

The above potential is invariant under $\pi_i \rightarrow \pi_i + \left(\frac{-1 \pm \sqrt{5}}{2q}\right)^i$, that includes the golden ratio in the argument. Diagonalizing the mass matrix will give two massless states. We can always choose an ansatz for a massless eigenvector of the form

$$\begin{aligned}\mathcal{O}_{j0} &= \left\{ a_0, -\frac{a_1}{q}, \frac{a_2}{q^2}, -\frac{a_3}{q^3}, \dots, \frac{a_N}{q^N} \right\} \\ &\equiv \{a_j(-q)^j\}, \quad \text{no summation.}\end{aligned}\quad (2.27)$$

This generates a recursion relation between the different a_i . In the original CW, this relation is simply $a_j = -a_{j+1}$ and hence, once we fix $a_0 = 1$ all a_j s are fixed and we get $\mathcal{O}_{j0} = \mathcal{N}_0 \{1, \frac{1}{q}, \dots, \frac{1}{q^N}\}$. When we consider next-to-nearest neighbors, we have two free parameters a_0, a_1 , and the recursion relation is a Fibonacci sequence

$$\begin{aligned}a_{j+2} &= a_j + a_{j+1} \Rightarrow \mathcal{O}_{j0} \\ &= \mathcal{N}_0 \{(-q)^{-j} F_j\}, \quad \text{no summation.}\end{aligned}\quad (2.28)$$

F_j is the j th Fibonacci number dictated by a_0, a_1 . For instance $a_0 = a_1 = 1$ gives a slight modification of the original CW mode of $\mathcal{O}_{j0} = \mathcal{N}_0 \{1, -\frac{1}{q}, \frac{2}{q^2}, -\frac{3}{q^3}, \dots\}$. The second massless mode is dictated by a different choice of a_0, a_1 , that then has to undergo the Gram-Schmidt procedure in order to be orthogonal to the CW mode. The two different massless modes do scale with q^{-1}

between two successive sites, but there is a difference in the total power of q . For the CW mode the ratio $\mathcal{O}_{00}/\mathcal{O}_{N0} \sim q^N$, while for the second massless mode, one has $\mathcal{O}_{01}/\mathcal{O}_{N1} \sim q^{N-1}$. It is easy to understand this behavior by choosing two different zero modes. One corresponding to $a_0 = 1, a_1 = 0$ and the other with $a_0 = 0, a_1 = 1$. In such case, it is clear that one zero mode will have the highest ratio of q^N and the other with q^{N-1} . This behavior persists also after orthogonalization.

As a numerical example, let us consider coupling up to next-to-nearest neighbors and $N = 3$. The first massless mode is

$$\mathcal{O}_{j0} = \mathcal{N}_0 \left\{ 1, -\frac{1}{q}, \frac{2}{q^2}, -\frac{3}{q^3} \right\} \quad (2.29)$$

and the second orthogonal one is

$$\begin{aligned}\mathcal{O}_{j1} &= \mathcal{N}_1 \left\{ \frac{1}{q} + \frac{2}{q^3} + \frac{6}{q^5}, 1 + \frac{2}{q^4} + \frac{3}{q^6}, -\frac{1}{q} + \frac{1}{q^3} + \frac{3}{q^7}, \right. \\ &\quad \left. \frac{2}{q^2} - \frac{1}{q^4} + \frac{2}{q^6} \right\}.\end{aligned}\quad (2.30)$$

The two massive modes are

$$\mathcal{O}_{j2} = \frac{\mathcal{N}_2}{q^2} \{1, -(1+q), q(1-q), q^2\} \quad (2.31)$$

$$\mathcal{O}_{j3} = \frac{\mathcal{N}_3}{q^2} \{-1, -(1-q), q(1+q), q^2\} \quad (2.32)$$

with masses of $m_{2,3}^2 = m^2(1 \pm q + q^2 \mp q^3 + q^4)$.

As can be seen from the example, the massive modes behavior goes between unity and q^2 terms, compared to the original CW, where it consisted of unity and linear terms in q , (2.7). In general, there are $N - 1$ massive states that can also be brought to the desired orthonormal form by the Gram-Schmidt procedure. Their mass will be dominated by q^4 terms rather than q^2 in the original clockwork. The elements in eigenvectors will range between unity and q^2 . The case of the integer charges suffers from the same complexity as the fractional charges of the previous section, which is why we do not write down explicit analytical expressions, but only the leading behavior. Nevertheless, even without a compact analytic expression for the massive modes, the massless mode and the essence of the CW mechanism exists with the q^{-j} overlap.

The generalization to any number of neighbors interactions is straightforward. Considering n nearest neighbors interactions, the Lagrangian will look like:

$$\begin{aligned}\mathcal{L} &= -f^2 \sum_{j=0}^N \partial U_j^\dagger \partial U_j + \frac{m^2 f^2}{2} \left(\frac{f}{\Lambda}\right) \sum_{i=0}^n q^{i-4} \\ &\quad \times \sum_{j=0}^{N-n} (U_j^\dagger U_{j+1}^{q_1} \dots U_{j+n}^{q_n} + \text{H.c.}).\end{aligned}\quad (2.33)$$

For n nearest neighbors interactions we will preserve n symmetries and the conserved symmetry group will be some $U(1)^n$. This is because we have the shift symmetry as long as

$$\alpha_j = 2\pi\ell + \sum_{i=1}^n q_i \alpha_{j+i}. \quad (2.34)$$

Such a generalization allows for various breaking patterns, not necessarily reaching the $U(1)$ of the clockwork type. Once again the n massless modes will be suppressed various powers of the q_i s. If we wish to maintain the same clockwork behavior, we require $\alpha_{j+1}/\alpha_j \sim q^{-1}$, which enforces $q_i \sim q^i$. For n coupled nearest neighbors, the n massless modes are again spanned by an ansatz $\mathcal{O}_{j0} = \mathcal{N}_0 \{(-q)^{-j} a_j^{(n)}\}$ where $a_j^{(n)}$ is given by the recursion relation $a_j^{(n)} = a_{j-n}^{(n)} - (a_{j-(n-1)}^{(n)} + \dots + a_{j-1}^{(n)}) = a_{j-n}^{(n)} - F_j^{(n-1)}$, where $F_j^{(n)}$ being the n -step Fibonacci number, i.e., $F_j^{(n)} = \sum_{i=1}^n F_{j-i}^{(n)}$. The n different massless modes are determined by various possibilities of n seed numbers required for the recursion relations, $a_1^{(n)}, a_2^{(n)}, \dots$. These massless modes are not orthogonal and are brought to such form using the Gram-Schmidt procedure. The important point is that the CW overlap of q^{-j} is kept. Furthermore, the “maximal ratio” between the sites for n massless modes will be $q^N, q^{N-1}, \dots, q^{N-n}$.

Let us discuss the massive modes. The radial modes have the same mass of $m_{r_k}^2 \sim f^2$ as long as $\lambda \gg \epsilon(\frac{f}{\Lambda}) \sum_{i=0}^n q^{i-4} q^{2n}$. This is trivially satisfied. The axions massive modes will be dominated by $m_{a_k}^2 \sim \epsilon f^2 (\frac{f}{\Lambda}) \sum_{i=0}^n q^{i-4} q^{2n}$. So, depending on f, Λ, q , and n one can push the axial modes towards the radial ones, or make them lighter by increasing Λ .

Except the case of next to nearest neighbors mentioned in the footnote of $V \sim (\phi_j^\dagger \phi_{j+1} \phi_{j+2}^2 + \text{H.c.})$, (2.33) does not come from some renormalizable simple tree-level theory. Lifting (2.33) to the UV theory is of the following form:

$$V(\phi_j) = \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^\dagger \phi_j + \frac{\lambda}{4} |\phi_j^\dagger \phi_j|^2 \right) + \epsilon \sum_{j=0}^{N-n} \left(\frac{\phi_j^\dagger \prod_{i=1}^n \phi_{j+i}^{q_i}}{\Lambda \sum q_i^{-3}} + \text{H.c.} \right). \quad (2.35)$$

Once again the CW behavior will force $q_i \sim q^i$.

²To be precise the axion masses will always behave as $m_{a_k}^2 \sim \epsilon f^2 (\frac{f}{\Lambda}) \sum_{i=0}^n q^{i-4} [(1 + q^{2n}) + \sum_{k=1}^{2n-1} \alpha_k q^k]$. So one can always express the mass as $m_{a_k}^2 = m^2 ((q-1)^{2n} + \sum_{k=1}^{2n-1} \beta_k q^k)$. This property is crucial in proving the mass gap when we consider an extra dimensional interpretation.

We are now in the position to add all the interactions with any number of neighbors and preserving a single CW $U(1)$ for integer charges, as we should have done in the first place from the EFT point of view. Unlike the charge assignment of q^n/n , here we cannot simply add up previous solutions with any number of neighbors, because generically, not a single $U(1)$ symmetry will be conserved. For a conservation of the $U(1)_{\text{CW}}$, we need to add up charges in a way that does not violate the symmetry. This can be done if we arrange the beyond nearest neighbors interactions in suitable pairs, for $N+1$ even:

$$V = \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^\dagger \phi_j + \frac{\lambda}{4} |\phi_j^\dagger \phi_j|^2 \right) + \epsilon \left(\sum_{j=0}^{N-1} c_1 \phi_j^\dagger \phi_{j+1}^q + \sum_{j=0}^{N-3} c_3 \phi_j^\dagger \phi_{j+1}^q (\phi_{j+2}^\dagger \phi_{j+3}^q)^{\beta_{33}} + \dots + c_N \phi_0^\dagger \phi_1^q (\phi_2^\dagger \phi_3^q)^{\beta_{3N}} \dots (\phi_{N-1}^\dagger \phi_N^q)^{\beta_{NN}} + \text{H.c.} \right) \quad (2.36)$$

where the c_j fix the correct mass dimensions with relevant powers of Λ , and the different β correspond to some integer power. If all $\beta_{ij} = 1$ then we reproduce the original massless CW mode. The massive modes will slightly change but still scale as $m_{a_k}^2 \sim q^2$, while if we choose β_{ij} such that each neighbor has a consecutively higher power of q , such as $\beta_{33} = q^2$, i.e., $\supset \sum_{j=0}^{N-3} c_3 \phi_j^\dagger \phi_{j+1}^q \phi_{j+2}^\dagger \phi_{j+3}^{q^3}$, then the massive modes will scale as $m^2 \sim q^{2N}$. Of course there are many intermediate possibilities.

Alternatively, we need a theory with different charges. Nearest neighbors CW with different charges is discussed in the Appendix. For all possible number of neighbors, the full low energy CW potential will now look like:

$$V = \frac{m^2 f^2}{2} \sum_{k=1}^N \sum_{j=0}^{N-k} (c_k U_j^\dagger \prod_{i=1}^k U_{j+i}^{q_{ki}} + \text{H.c.}) \quad (2.37)$$

$$= \frac{m^2 f^2}{2} \left(\sum_{j=0}^{N-1} c_1 U_j^\dagger U_{j+1}^{q_{11}} + \sum_{j=0}^{N-2} c_2 U_j^\dagger U_{j+1}^{q_{21}} U_{j+2}^{q_{22}} + \dots + c_N U_0^\dagger U_1^{q_{N1}} U_2^{q_{N2}} \dots U_N^{q_{NN}} + \text{H.c.} \right) \quad (2.38)$$

where $c_k \equiv (f/\Lambda)^{\sum_{i=1}^k q_{ki}-3}$ is the appropriate power of f/Λ . This is the most general assignment of charges possible. To keep a single CW massless mode, such that $\pi_j \rightarrow \pi_j + \alpha_j$ we need the simultaneous solution of N equations $\alpha_j = 2\pi\ell + \sum_{i=1}^k q_{ki} \alpha_{j+i}$ for $k = 1$ to N . These are N equations, but we have $N(N+1)/2$ charges. So it is very easy to fulfill such constraint. A specific example, with a “single” charge q is as follows:

$$\begin{aligned}
V = & \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^\dagger \phi_j + \frac{\lambda}{4} |\phi_j^\dagger \phi_j|^2 \right) \\
& + \epsilon \left(\sum_{j=0}^{N-1} c_1 \phi_j^\dagger \phi_{j+1}^q + \sum_{j=0}^{N-2} c_2 \phi_j^\dagger \phi_{j+1}^{q-1} \phi_{j+2}^q \right. \\
& + \sum_{j=0}^{N-3} c_3 \phi_j^\dagger \phi_{j+1}^{q-1} \phi_{j+2}^{q-1} \phi_{j+3}^q + \cdots \\
& \left. + c_N \phi_0^\dagger \phi_1^{q-1} \phi_2^{q-1} \cdots \phi_N^q + \text{H.c.} \right) \quad (2.39)
\end{aligned}$$

where the c_j fix the correct mass dimensions, with $c_j = \Lambda^{-(jq-(j-1)-3)} = \Lambda^{-(j(q-1)-2)}$.

The outcome in all of the above cases, is a potential preserving a single CW $U(1)$, and the massless mode has exactly the same profile as the original CW, $\mathcal{O}_{j0} = \{1, 1/q, \dots, 1/q^N\}$. So we have constructed a CW model with all possible neighbor interactions.

III. EXTRA DIMENSION INTERPRETATION

A. Alternating sign

Considering the CW theory from a 5D point of view, we cannot reproduce either the full (2.1) or (2.2). We can however get the mass matrix (2.4) by considering a massless free scalar field in 5D, [3]. We would like to give the generalized clockwork mechanism an extra dimensional interpretation. We therefore introduce alternating signs to the potential and the symmetry. This can be achieved either by a field redefinition $\phi_i \rightarrow (-1)^i \phi_i$ or by considering a “negative” charge $q < 0$. In both cases this is a redefinition that does not change the physics. However, after redefinition, discretizing the extra dimension, higher derivatives in 5D correspond to couplings beyond nearest neighbors in a straightforward manner. The symmetry and potential are now:

$$\pi_j \rightarrow \pi_j + \frac{1}{(-q)^j} \quad (3.1)$$

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} \left(\pi_j + \frac{q}{2} \pi_{j+1} - \frac{q^2}{2} \pi_{j+2} \right)^2 + \mathcal{O}(\pi^4). \quad (3.2)$$

Diagonalizing the mass matrix will give two massless states, one of which is the clockwork with $\mathcal{O}_{j0} = \mathcal{N}_0 / (-q)^j$ and $N-1$ massive states, again with the masses distributed with $\Delta m/m_a \sim 1$. The second massless state is simply taking $q \rightarrow -q$ in (2.14), $\mathcal{O}_{j1} = \frac{\mathcal{N}_1}{(-q)^j} [(-2)^j - \frac{\sum_{i=0}^N (-2q^{-2})^i}{\sum_{i=0}^N q^{-2i}}]$. This can again be generalized to any number of neighbors n yielding:

$$V(\pi_j) = \sum_{j=0}^{N-n} \left(\pi_j - \sum_{k=1}^n \frac{(-q)^k}{n} \pi_{j+k} \right)^2 + \mathcal{O}(\pi^4). \quad (3.3)$$

Lifting back to the U_j s is trivial:

$$\mathcal{L} = \sum_{j=0}^N \partial U_j^\dagger \partial U_j - \sum_{j=0}^{N-n} (U_j^\dagger \Pi_{k=1}^n U_{j+k}^{(-q)^k/n} + \text{H.c.}). \quad (3.4)$$

Again, for n nearest neighbors interaction, we have a residual $U(1)^n$ symmetry, and we can further break it with different breaking patterns. If we wish to maintain only the clockwork shift symmetry, we can again sum all neighbors interactions:

$$\mathcal{L} = \sum_{j=0}^N \partial U_j^\dagger \partial U_j - \sum_{n=1}^N \sum_{j=0}^{N-n} (U_j^\dagger \Pi_{k=1}^n U_{j+k}^{(-q)^k/n} + \text{H.c.}). \quad (3.5)$$

The generalization provided here and in the previous section can be applied in a straightforward manner to gauge bosons and gravitons at the linear level, as was done in [3].

B. Higher derivatives in the extra dimension

Coupling to next to nearest neighbors is a nonlocal interaction, and in the continuum language where $N \rightarrow \infty$, we expect higher order derivatives. Indeed, considering a scalar in 5D, coupling to two consecutive neighbors will correspond to a $(\partial_y^2 \phi)^2$, and coupling to n neighbors to $(\partial_y^n \phi)^2$, where y is the extra dimension.

To see this, let us rewrite a more general form of the clockwork Lagrangian with two nearest neighbors interaction:

$$V(\pi_j) = \frac{m^2}{2} \sum_{j=0}^{N-2} (\pi_j - \beta_1 q \pi_{j+1} + \beta_2 q^2 \pi_{j+2})^2 + \mathcal{O}(\pi^4) \quad (3.6)$$

$$\pi_i \rightarrow \pi_i + \frac{\alpha_j}{(-q)^j} \quad (3.7)$$

$$\Rightarrow \alpha_j = \beta_1 \alpha_{j+1} - \beta_2 \alpha_{j+2}. \quad (3.8)$$

If the last equation is fulfilled, we have a shift symmetry, similar to clockwork, but with somewhat different transformation laws. Similarly, the massless eigenvector behaves like $(-q)^{-j}$ with $\mathcal{O}(1)$ coefficient. Consider a compact extra dimension $-\pi R \leq y \leq \pi R$, and identifying $-y$ with y . Starting from a five dimensional free scalar field with higher derivatives in the extra dimension, we get³:

$$ds^2 = X(|y|)(-dt^2 + d\vec{x}^2) + Y(|y|)dy^2 \quad (3.9)$$

³An equivalent result is obtained by considering, $\mathcal{S} = 2 \int d^4x \int_0^{\pi R} dy \sqrt{-g} \{ (-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi) - \frac{1}{2\Lambda^2} (\sqrt{-g_{yy}} g^{yy} \partial_y^2 \phi)^2 \}$. In such case one has the “correct” two powers of the inverse metric, but the action is still not Lorentz invariant in the full 5D spacetime, and considering a Lorentz invariant action will include higher derivative terms in the 4D equations of motion, as well as introducing site dependence to the clockwork parameters.

$$\mathcal{S} = 2 \int d^4x \int_0^{\pi R} dy \sqrt{-g} \left\{ \left(-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \tilde{\phi} \right) - \frac{1}{2\Lambda^2} g^{yy} (\partial_y^2 \tilde{\phi})^2 \right\}. \quad (3.10)$$

Dimensional analysis requires that the higher derivative term will be suppressed by some dimensionful parameter,

like Λ^{-2} . Due to the explicit appearance of only ∂_y^2 Lorentz invariance is broken in the extra dimension. Writing a Lorentz covariant action, for instance $\mathcal{S} \supset (\Box_5 \phi)^2$ will result in higher derivative terms from the 4D spacetime part in the equations of motion, as well as more complicated coupling between 4D spacetime derivatives and the extra dimension. Performing a field redefinition to get canonical kinetic terms we get:

$$\begin{aligned} \mathcal{S} &= - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 + \frac{1}{\Lambda^2} \frac{X^2}{Y^{1/2}} \left(\partial_y^2 \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 \right\} \\ &= - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 \right\} \\ &\quad - \int_0^{\pi R} dy \int d^4x \frac{X^2}{\Lambda^2 Y^{1/2}} \left[\frac{\phi''}{X^{1/2} Y^{1/4}} - 2\phi' \frac{(X^{1/2} Y^{1/4})'}{(X^{1/2} Y^{1/4})^2} + \phi \left(-\frac{(X^{1/2} Y^{1/4})'}{(X^{1/2} Y^{1/4})^2} \right)' \right]^2 \end{aligned} \quad (3.11)$$

where prime denotes a derivative with respect to y . Let us discretize the extra dimension with lattice spacing a , such that $\pi R = Na$ and use the notation $F_j = F(ja)$, where $F = X, Y, \phi$ and j runs from zero to N . The original clockwork scenario comprises of the first terms in the action, provided that we identify:

$$m_j^2 = \frac{N^2 X_j}{\pi^2 R^2 Y_j}, \quad q_j = \frac{X_j^{1/2} Y_j^{1/4}}{X_{j+1}^{1/2} Y_{j+1}^{1/4}}. \quad (3.12)$$

Discretizing the new term gives:

$$\mathcal{S} \supset - \sum_{j=0}^{N-2} \int d^4x \frac{N^4 X_j}{\pi^4 \Lambda^2 R^4 Y_j} (2/q_j^2 - v_{j+2})^2 \left\{ \frac{\phi_{j+2}}{(2/q_j^2 - v_{j+2})} - \frac{2/q_j}{(2/q_j^2 - v_{j+2})} \phi_{j+1} + \phi_j \right\}^2 \quad (3.13)$$

where $v_{j+2} \equiv \frac{X_{j+2}^{1/2} Y_{j+2}^{1/4}}{X_j^{1/2} Y_j^{1/4}}$. Requiring the deconstruction to be independent of the specific site, enforces $v_{j+2} = q_j^{-2}$, thence:

$$\mathcal{S} \supset - \int d^4x \sum_{j=0}^{N-2} \frac{N^4 X_j}{\pi^4 \Lambda^2 R^4 Y_j q_j^4} \{ q_j^2 \phi_{j+2} - 2q_j \phi_{j+1} + \phi_j \}^2. \quad (3.14)$$

So to recover (3.6), we demand $\beta_1 = 2, \beta_2 = 1$. The “mass” parameter here is different than the original clockwork scenario, $(M^{(2)})_j^2 = m_j^2 (1 + \frac{N^2}{\pi^2 R^2 \Lambda^2})$ but it is just an overall shift. The massless mode remains massless. Notice that no new charges have been introduced, and therefore the solutions for X, Y , giving site independent charges q and masses m for the nearest neighbors interaction are similar to the original proposal [3]⁴:

$$X_j \propto Y_j \propto e^{\frac{-4k\pi R j}{3N}}, \quad q = e^{k\pi R/N}, \quad m^2 = \frac{N^2}{\pi^2 R^2}. \quad (3.15)$$

In the original CW, plugging (3.15) into (2.5), we get that we can take the continuum limit of $N \rightarrow \infty$:

$$\begin{aligned} m_{a_j}^2 &= m^2 \left(1 + q^2 - 2q \cos \frac{j\pi}{N+1} \right), \quad j = 1, \dots, N \\ m_{a_j}^2 &\simeq \frac{N^2}{\pi^2 R^2} \left((q-1)^2 + q \left(\frac{j\pi}{N+1} \right)^2 + \mathcal{O}(N^{-3}) \right) \simeq k^2 + \frac{j^2}{R^2} + \mathcal{O}(N^{-1}). \end{aligned} \quad (3.16)$$

⁴In the continuum, this corresponds to $X(|y|) = Y(|y|) = e^{\frac{-4k|y|}{3}}$.

The next to nearest neighbors interaction gets a leading mass term of the form:

$$M^2 = \frac{N^4}{\pi^4 R^4 \Lambda^2}. \quad (3.17)$$

So we need to check whether the continuum limit works as well. Both with integer and noninteger powers, we have shown that the axion masses are dominated by $m_{a_k}^2 \sim \epsilon f^2 \left(\frac{f}{\Lambda}\right) \sum_{i=0}^n q^{i-4} (1 + q^{2n}) = m^2 ((q-1)^{2n} + \sum_{k=1}^{2n-1} \beta_k q^k)$, where n is the number of nearest neighbors. In the case at hand we discuss next to nearest neighbors so $n = 2$. The leading behavior will be

$$m_{a_j}^2 \sim M^2 (q-1)^4 \sim \frac{N^4}{\pi^4 R^4 \Lambda^2} \left(\frac{k^4 \pi^4 R^4}{N^4} \right) \sim \frac{k^4}{\Lambda^2} + \dots \quad (3.18)$$

giving a finite result. The mass gap, is determined by the clockwork spring k and the energy scale Λ . A general formula for the β_k coefficients is too complicated to write down, but given the power of q , it will be subleading. Based on dimensional grounds and numerical examples, we expect it to contribute $m_{a_j}^2 \sim \frac{k^4}{\Lambda^2} + c \frac{k^2 j^2}{\Lambda^2 R^2} \dots$ where c is some $\mathcal{O}(1)$ coefficient.

One may wonder whether other interesting solutions to (3.10), exist rather than trying to reproduce the original clockwork.

IV. $O(N)$ CLOCKWORK

The clockwork mechanism can be implemented for multiple copies of $O(N)$ models and within a single $O(N)$ model as well. Consider $M+1$ copies of $O(N)$ models:

$$\mathcal{L} = \sum_{j=0}^M -\frac{1}{2} (\partial_\mu \vec{\phi})^2 + \epsilon \sum_{j=0}^{M-1} \left[\frac{\tilde{\mu}^2}{2} (\vec{\phi}_j + q \vec{\phi}_{j+1})^2 + \frac{\tilde{g}}{4N} (\vec{\phi}_j + q \vec{\phi}_{j+1})^4 \right]. \quad (4.1)$$

For $\epsilon \rightarrow 0$ we have a global $O(N)^{M+1}$ symmetry as well as shift symmetry for every vector $(\mathbb{R}^N)^{M+1}$. Turning on ϵ breaks the symmetry down to a single $O(N)$, since all vectors have to be rotated by the same orthogonal matrix. However, there is still a shift symmetry of $\vec{\phi}_j \rightarrow \vec{\phi}_j + \frac{\vec{c}}{(-q)^j}$, where $\vec{c}$ is a constant vector. So the full symmetry group is now $O(N) \ltimes \mathbb{R}^N$. The eigenvector corresponding to the conserved $O(N)$ remains massless, and this vector will have the same q^{-j} suppression in overlap with the different $\vec{\phi}_j$.

A. Linear sigma model

The original clockwork discusses the breaking of $U(1)^N$ after spontaneous symmetry breaking. We now generalize it

to $M+1$ copies of $O(N)$ models in the broken phase. Consider $M+1$ copies of $O(N)$ models:

$$\mathcal{L} = \sum_{j=0}^M \left(-\frac{1}{2} (\partial_\mu \vec{\phi})^2 - \frac{\mu^2}{2} (\vec{\phi}_j)^2 + \frac{\lambda}{4} (\vec{\phi}_j^2)^2 \right) + \epsilon \sum_{j=0}^{M-1} \left[\frac{\tilde{\mu}^2}{2} (\vec{\phi}_j - q \vec{\phi}_{j+1})^2 + \frac{\tilde{g}}{4N} (\vec{\phi}_j - q \vec{\phi}_{j+1})^4 \right]. \quad (4.2)$$

Notice that here, the shift symmetry from the previous paragraph is gone. We use the conventional parametrization where the N th field gets a vev v_i :

$$\vec{\phi}_i = (\pi_{ik}, v_i + \sigma_i), \quad v_i = \frac{\mu}{\sqrt{\lambda}}. \quad (4.3)$$

For simplicity, we took all the vevs to be the same. It is a trivial generalization to consider for each model a different μ_i, λ_i such that the vevs will be different. In this case, when $\epsilon \rightarrow 0$ we have $M+1$ copies of $O(N)$ models in their broken phase, such that there is a global $O(N-1)^{M+1}$ symmetry, and there are $(M+1) \times (N-1)$ massless Goldstone bosons. These are the π_{ik} . Turning on the clockwork terms induces a coupling that will break the symmetry explicitly to a single $O(N-1)$. However, the analysis of the massless modes and their localization is not as trivial. The CW term induces a nonzero vev for the σ_i fields and ruins the localization. To avoid this, we need a vanishing vev for these fields also after turning on the CW term. This is doable. Consider the following potential:

$$V = \sum_{j=0}^M \left(-\frac{\mu_0^2}{2q^{2j}} (\vec{\phi}_j)^2 + \frac{\lambda}{4} (\vec{\phi}_j^2)^2 \right) + \epsilon \sum_{j=0}^{M-1} \left[\frac{\tilde{\mu}^2}{2} (\vec{\phi}_j - q \vec{\phi}_{j+1})^2 + \frac{\tilde{g}}{4N} (\vec{\phi}_j - q \vec{\phi}_{j+1})^4 \right]. \quad (4.4)$$

Notice that here the quadratic term is different for each $O(N)$ vector, with $\mu_j = \frac{\mu_0}{q^j}$.⁵ Obviously this is less robust than (4.2), where in principle every $O(N)$ vector could have any quadratic term. Using again the conventional parametrization

$$\vec{\phi}_i = (\pi_{ik}, v_i + \sigma_i), \quad v_i = \frac{\mu_i}{\sqrt{\lambda}} = \frac{\mu_0}{q^i \sqrt{\lambda}}, \quad (4.5)$$

ensures $\langle \sigma_j \rangle = 0$ even in the presence of the CW term. To demonstrate this, let us reinsert the conventional parametrization into the potential:

⁵We could have constructed the desired behavior by having $\lambda_j = \lambda_0 q_j^2$ or any other combination that gives $v_j = v_0/q^j$.

$$\begin{aligned}
V = & \sum_{j=0}^M \left[-\frac{\mu_0^2}{2q^{2j}} (\pi_{ij}\pi_{ij} + (v_j + \sigma_j)^2) + \frac{\lambda}{4} (2(v_j + \sigma_j)^2 \pi_{ij}\pi_{ij} + (v_j + \sigma_j)^4 + \mathcal{O}(\pi^4)) \right] \\
& + \epsilon \sum_{j=0}^{M-1} \left[\frac{\tilde{\mu}^2}{2} \{ (\pi_{ij} - q\pi_{ij+1})^2 + (v_j + \sigma_j - qv_{j+1} - q\sigma_{j+1})^2 \} \right. \\
& \left. + \frac{\tilde{g}}{4N} \{ 2(\pi_{ij} - q\pi_{ij+1})^2 (v_j + \sigma_j - qv_{j+1} - q\sigma_{j+1})^2 + (v_j + \sigma_j - qv_{j+1} - q\sigma_{j+1})^4 + \mathcal{O}(\pi^4) \} \right]. \quad (4.6)
\end{aligned}$$

Since $v_j = v_0/q^j$ all the vev terms in the CW term vanish and we are left with:

$$\begin{aligned}
V = & \sum_{j=0}^M \left[-\frac{\mu_0^2}{2q^{2j}} (\pi_{ij}\pi_{ij} + (v_j + \sigma_j)^2) + \frac{\lambda}{4} (2(v_j + \sigma_j)^2 \pi_{ij}\pi_{ij} + (v_j + \sigma_j)^4 + \mathcal{O}(\pi^4)) \right] \\
& + \epsilon \sum_{j=0}^{M-1} \left[\frac{\tilde{\mu}^2}{2} \{ (\pi_{ij} - q\pi_{ij+1})^2 + (\sigma_j - q\sigma_{j+1})^2 \} + \frac{\tilde{g}}{4N} \{ 2(\pi_{ij} - q\pi_{ij+1})^2 (\sigma_j - q\sigma_{j+1})^2 + (\sigma_j - q\sigma_{j+1})^4 + \mathcal{O}(\pi^4) \} \right]. \quad (4.7)
\end{aligned}$$

The CW term is positive semidefinite and obviously $\langle \pi_{ij} \rangle = \langle \sigma_j \rangle = 0$, $\forall i, j$ is the minimum of the CW term⁶ and therefore the full potential. There are no new tadpole contributions coming from the CW term. This is because we can write the quadratic CW term (of both π s and σ s) as follows:

$$\begin{pmatrix} \vec{\phi}_0 & \cdots & \vec{\phi}_M \end{pmatrix}^T \times \begin{pmatrix} 1 & -q & 0 & \cdots & 0 \\ -q & 1+q^2 & -q & 0 & \cdots \\ 0 & -q & 1+q^2 & -q & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -q & 1+q^2 & -q \\ \cdots & \cdots & \cdots & -q & q^2 \end{pmatrix} \begin{pmatrix} \vec{\phi}_0 & \cdots & \vec{\phi}_M \end{pmatrix} \quad (4.8)$$

where each entry in the matrix corresponds to a vector of length N . The σ are all massive fields since they have an additional contribution from the standard $O(N)$ model terms. Finally, we have a single $O(N-1)$ and a single linear combination of π_{ik} massless, again with overlap suppressed with the standard q^{-j} . The eigenvalues of such a matrix will be the same as the clockwork, with a massless $O(N-1)$ vector. As a numerical example, consider the case of $N=4$, $M=2$. We shall have a block diagonal mass matrix for the π s, where each block is

$$\begin{pmatrix} 1 & -q & 0 \\ -q & 1+q^2 & -q \\ 0 & -q & q^2 \end{pmatrix} \quad (4.9)$$

giving us an $O(3)$ CW massless vector.

Establishing the CW behavior for a global $O(N)$ symmetry, one can still have exponential seclusion of

the massless mode, unlike the gauge symmetry result reported in [12]. Considering, for instance,

$$\mathcal{L} \supset -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{(\vec{\phi}_N)^2}{8\pi f^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (4.10)$$

the massless mode coupling will behave as

$$\mathcal{L} \supset \frac{a_0^2}{8\pi f_0^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad f_0 = q^N f. \quad (4.11)$$

V. CLOCKWORK SUGRA

A. Canonical Kähler potential

The clockwork mechanism in global SUSY has been suggested in [1]. Considering $3(N+1)$ chiral superfields $S_j, \Phi_j, \tilde{\Phi}_j$. For $q=2$ one can write down the renormalizable superpotential:

⁶It is also clear that $\langle \pi_{ij} \rangle = \langle \sigma_j \rangle = 0$, $\forall i, j$ is at least an extremum regardless of the signs of the different terms in the potential.

$$W = \sum_{j=0}^N \lambda S_j (\Phi_j \tilde{\Phi}_j - v^2) + \epsilon \sum_{j=0}^{N-1} (\Phi_j \tilde{\Phi}_{j+1}^2 + \tilde{\Phi}_j \Phi_{j+1}^2). \quad (5.1)$$

Taking $\epsilon \rightarrow 0$ reveals a $U(1)^{N+1}$ global symmetry. Turning on ϵ breaks these symmetries into a single $U(1)$ with hierarchical charges with S_j being neutral, Φ_j a charge of 2^{-j} and $\tilde{\Phi}_j$ with charge $-(2)^{-j}$. It is interesting to note, that taking $\lambda \rightarrow 0$, produces *two* $U(1)$ symmetries, unlike the nonsupersymmetric case, where there is only a single $U(1)$ [11]. The requirement for a SUSY minimum $W_i = 0$ for all chiral superfields gives the vev $\Phi_j \tilde{\Phi}_j = v^2$ as well as nonzero S_j , see below. SUSY is conserved. We then have $2(N+1)$ massive chiral superfields and $N+1$ massless ones. The low energy theory below the scale λv can then be parameterized as $\Phi_j = v e^{\Pi_j/v}$, $\tilde{\Phi}_j = v e^{-\Pi_j/v}$, yielding:

$$W = 2\epsilon v^3 \sum_{j=0}^{N-1} \cosh\left(\frac{\Pi_j - 2\Pi_{j+1}}{v}\right). \quad (5.2)$$

Notice, that we still have a Minkowski SUSY minimum for the low energy theory as well.

However, turning on gravity, we shall see that this construction is insufficient. We encounter the well-known problem of getting AdS SUSY vacuum. To get a viable phenomenology, we must uplift this minimum into a SUSY breaking Minkowski or dS minima. As such, adding a constant term in the superpotential W to uplift the vacuum and building a CW model on top of that is simply a test of the uplift term. We therefore seek alternative constructions of the clockwork within SUGRA.

The simplest generalization to SUGRA is straightforward. Throughout this section, we consider the Planck mass to be unity. Consider a canonical Kähler potential $K = \sum_{j=0}^N |\Phi_j|^2 + |\tilde{\Phi}_j|^2 + |S_j|^2$. The Kähler potential K is invariant under the same $U(1)^{N+1}$ as the superpotential for $\epsilon \rightarrow 0$ case, as well as additional $U(1)$ for each chiral superfield, to a total of $U(1)^{3(N+1)}$. The F-term scalar potential reads:

$$V = e^K (D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - 3|W|^2). \quad (5.3)$$

The requirement for a supersymmetric minimum is now changed to $D_i W = 0$ for all chiral superfields (no summation):

$$D_{S_j} W = \lambda (\Phi_j \tilde{\Phi}_j - v^2) + \bar{S}_j W \quad (5.4)$$

$$D_{\Phi_j} W = \lambda S_j \tilde{\Phi}_j + \epsilon (\tilde{\Phi}_{j+1}^2 + 2\Phi_j \tilde{\Phi}_{j-1}) + \bar{\Phi}_j W. \quad (5.5)$$

K , W are invariant on interchanging $\tilde{\Phi}_j$ and Φ_j , so we just consider the fields without the tilde. Considering the case of

$\epsilon \rightarrow 0$, it is clear that the only supersymmetric minimum is that of global SUSY with $W_i = W = 0$ at the minimum and $\langle S_j \rangle = 0$, $\langle \Phi_j \tilde{\Phi}_j \rangle = v^2$. We now need to check whether the different terms in the scalar potential are still invariant under the clockwork $U(1)$. Since the Kähler potential K and the superpotential W are invariant, and the Kähler metric is a unit metric, we just need to check the Kähler derivative. The S_j Kähler derivative is invariant. Regarding the Φ_j :

$$\begin{aligned} \Phi_i &\rightarrow e^{i\alpha_i} \Phi_i, & \tilde{\Phi}_i &\rightarrow e^{-i\alpha_i} \tilde{\Phi}_i, & \alpha_i &= 2\alpha_{i+1}, \\ D_{\Phi_i} W &\rightarrow e^{-i\alpha_i} D_{\Phi_i} W \Rightarrow |D_i W|^2 \rightarrow |D_i W|^2. \end{aligned} \quad (5.6)$$

Thus, the entire scalar potential and kinetic terms are invariant under the clockwork symmetry. Let us now consider the $\epsilon \neq 0$ case in (5.1). First, let us look for a global SUSY minimum, i.e., $W = W_i = 0$. From (5.4) we get $\langle \Phi_j \tilde{\Phi}_j \rangle = v^2$. Substituting this vev into (5.5) shows that vanishing S_j is not a solution anymore. Rather, $\langle S_0 \rangle = -\epsilon v/\lambda$, $\langle S_N \rangle = -2\epsilon v/\lambda$, $\langle S_j \rangle = -3\epsilon v/\lambda$, $\forall j \neq 0, N$. Substituting these vevs into W we get $W \neq 0$, so we have a contradiction. However, this can readily be fixed. Consider adding a constant term to the superpotential:

$$W = w_0 + \sum_{j=0}^N \lambda S_j (\Phi_j \tilde{\Phi}_j - v^2) + \epsilon \sum_{j=0}^{N-1} (\Phi_j \tilde{\Phi}_{j+1}^2 + \tilde{\Phi}_j \Phi_{j+1}^2). \quad (5.7)$$

The purpose of the additional constant w_0 is to make sure that $W = 0$ at the supersymmetric minimum, thus ensuring a global SUSY solution. Demanding $W_i = W = 0$ for the SUSY minimum for all chiral superfields gives $\Phi_i = \tilde{\Phi}_i = v$. The S_i now receive nonzero vevs:

$$S_0 = -\frac{\epsilon v}{\lambda}, \quad S_N = -\frac{2\epsilon v}{\lambda}, \quad S_i = -\frac{3\epsilon v}{\lambda} \quad \forall i \neq 0, N. \quad (5.8)$$

This fixes the constant w_0 to be $w_0 = -2N\epsilon v^3$. Thus, we have trivially generalized the CW scenario to SUGRA at the price of adding an arbitrary tuned constant w_0 to the superpotential W . This constant w_0 is parametrically the same as the CW term.

Pursuing such phenomenology is viable, but hinges on the exact value of w_0 , whatever its origin is [24,25]. It makes sense to consider alternatives that do not rely on w_0 , which we turn to next.

B. Shift symmetric Kähler potential

Shift symmetries in the Kähler potential are abundant in SUGRA constructions, especially in the inflationary literature, [26–28]. A possible construction is by using a so-called “stabilizer” field, eloquently explained in [27]. Consider a superpotential and Kähler potential of the form:

$$W = Sf(\Phi_i), \quad K = S\bar{S} + \sum_i \frac{1}{2}(\Phi_i + \bar{\Phi}_i)^2. \quad (5.9)$$

In such case, we have a Minkowski SUSY vacuum at, the kinetic terms of the fields are canonical and the potential at $S = \Re(\Phi_i) = 0$ is simply:

$$V = f^2(\Im\Phi_i). \quad (5.10)$$

Hence, by choosing the function f to have only arguments of the form $f = f(\Phi_i - q\Phi_{i+1})$, we can have potentials that manifestly have the shift symmetry of $\Phi_i \rightarrow \Phi_i + c/q^i$.

The structure of the vacuum in this case is rather generic. Assuming a Minkowski vacuum, $V|_0 = 0 \Rightarrow f|_0 = 0$. In such a case, at the extremum $V_i = 2ff_i = 0$, and thus the mass matrix at the vacuum becomes:

$$V_{ij}|_0 = 2(ff_{ij} + f_i f_j)|_0 = 2f_i f_j|_0. \quad (5.11)$$

In such case we do not have one flat direction and a single massless modes, but the usual CW shift symmetry and N massless modes. There is no hierarchy generated between a single massless mode and N massive states. We can uplift these N massless modes by explicitly breaking SUSY using another CW coupling:

$$V(\Im\Phi_i) = f^2(\Im\Phi_i - q\Im\Phi_{i+1}) + \epsilon(\Im\Phi_i - p\Im\Phi_{i+1})^2 \quad (5.12)$$

where p is the new CW charge, and $\epsilon \ll 1$ is a small SUSY breaking parameter giving us finally, an embedding of CW in SUGRA. This is of course a fine-tuned construction.

Alternatively, we can have a small CC, such that $f|_0 = W_0$.⁷ Obviously here, we do not have a SUSY Minkowski minimum (with $S = f(\Phi_i)_0 = 0$), but rather a dS SUSY breaking one, (with $S = 0$, $f(\Phi_i) \neq 0$) In such case:

$$\begin{aligned} V|_0 &= W_0^2, \\ V_i|_0 &= 2ff_i|_0 = 0 \Rightarrow f_i|_0 = 0, \\ V_{ij}|_0 &= 2W_0 f_{ij}|_0. \end{aligned} \quad (5.13)$$

Arranging f_{ij} to have positive semidefinite mass matrix is easy, for instance, expanding around the minimum, we can simply take $f = W_0 + \sum \alpha(\Phi_i - q\Phi_{i+1})^2$, so we have a massless CW mode and N massive states, according to (2.5)

$$\begin{aligned} m_{a_0}^2 &= 0, \quad m_{a_k}^2 = 2W_0\alpha \left(1 + q^2 - 2q \cos \frac{k\pi}{N+1}\right), \\ k &= 1, \dots, N. \end{aligned} \quad (5.14)$$

⁷Notice that here, W_0 has mass dimension two, while in the previous subsection, w_0 had mass dimension three.

However, as we can see the mass is related to the CC $\Lambda^4 \sim W_0^2$. So the fundamental scale of the CW sector is parametrically connected to the CC.

C. Shift symmetric superpotential

Whereas in VA w_0 was a parameter ensuring a Minkowski SUSY minimum, in the shift symmetric Kähler potential case, it resulted in an actual CC. Here we try a different approach and for $N + 1$ chiral superfields consider a manifestly symmetric superpotential of the form:

$$W = \sum_{i=0}^N m(\Phi_i - q\Phi_i)^2. \quad (5.15)$$

The superpotential is invariant under the shift symmetry $\Phi_i \rightarrow \Phi_i + \alpha_i/q^i$. A Minkowski vacua with $W = W_i = 0$ at the minimum exist for vevs $\langle \Phi_i \rangle = q\langle \Phi_{i+1} \rangle$, with some $\langle \Phi_0 \rangle \equiv v$.⁸ For any Kähler potential, we shall have the following potential and derivatives [25]:

$$V = e^K(D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - 3|W|^2), \quad (5.16)$$

$$\partial_k V = e^K(D_k D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - 2D_k W \bar{W}), \quad (5.17)$$

$$\nabla_l \partial_k V = e^K(D_l D_k D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - D_l D_k W \bar{W}), \quad (5.18)$$

$$\begin{aligned} \nabla_l \partial_k V &= e^K(-R_{k\bar{l}i\bar{m}} D_n W D_{\bar{j}} \bar{W} K^{i\bar{j}} K^{n\bar{m}} \\ &\quad + K_{k\bar{l}} D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - D_k W D_{\bar{l}} \bar{W} \\ &\quad + D_k D_i W D_{\bar{l}} \bar{W} K^{i\bar{j}} - 2K_{k\bar{l}} W \bar{W}). \end{aligned} \quad (5.19)$$

In the above ∂_i denotes differentiation with respect to a chiral scalar ϕ^i , $K_i = \partial_i K$ etc., and

$$\begin{aligned} D_i X_j &= \nabla_i X_j + K_i X_j, \\ \nabla_i X_j &= \partial_i X_j - \Gamma_{ij}^k X_k, \\ \Gamma_{ij}^k &= K^{k\bar{l}} \partial_i K_{j\bar{l}}, \\ R_{i\bar{j}k\bar{l}} &= K_{m\bar{l}} \partial_{\bar{j}} \Gamma_{ik}^m. \end{aligned} \quad (5.20)$$

Evaluating these quantities at the Minkowski SUSY vacuum for any Kähler potential gives:

$$V = \partial_k V = \nabla_l \partial_k V = W = W_i = 0 \quad (5.21)$$

and the only nonzero term is

$$\nabla_l \partial_k V|_0 = e^K K^{i\bar{j}} W_{ik} \bar{W}_{j\bar{l}}|_0 \quad (5.22)$$

where W_{ij} is a matrix exactly of the form (2.4), with m as the mass parameter instead of m^2 . Hence, for canonical Kähler, the mass matrix for the scalars will be:

⁸Any $\langle \Phi_0 \rangle \equiv v$ including a vanishing v is a solution, so we have a flat direction with degenerate Minkowski vacua.

$$m_{i\bar{j}}^2 = e^{K|_0} |m|^2 \left| \begin{pmatrix} 1 & -q & 0 & \cdots & 0 \\ -q & 1+q^2 & -q & \cdots & 0 \\ 0 & -q & 1+q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+q^2 & -q \\ 0 & 0 & 0 & \cdots & -q & q^2 \end{pmatrix} \right|^2 \quad (5.23)$$

with guaranteed clockworking, a single massless complex superfield $m_{a_0}^2 = 0$ and N parametrically heavy ones. The masses will be the square of the original CW masses for real fields, $m_{a_k}^2 = e^{K|_0} |m|^2 (1 + q^2 - 2q \cos \frac{k\pi}{N+1})^2$. The main difference compared to the previous section, was the existence of a scale m instead of the stabilizer field S .

For other Kähler manifolds, one has to look more carefully at the mass matrix. In the most general case, the canonically normalized mass matrix is [25]

$$\mathcal{M} = \begin{pmatrix} K^{i\bar{m}} N_{\bar{m}j} & K^{i\bar{m}} N_{\bar{m}\bar{j}} \\ K^{\bar{i}m} N_{mj} & K^{\bar{i}m} N_{m\bar{j}} \end{pmatrix} \quad (5.24)$$

with

$$\begin{aligned} N_{i\bar{j}} &= \nabla_i \partial_{\bar{j}} V, \\ N_{ij} &= \nabla_i \partial_j V - \Gamma_{ij}^k \partial_k V. \end{aligned} \quad (5.25)$$

As before $N_{ij} = 0$ at the vacuum, while $N_{i\bar{j}} = m_{i\bar{j}}^2$ from (5.23). Thus, for canonical Kähler, we get the clockwork. For a diagonal Kähler each mode Φ_i will get multiplied by the corresponding inverse Kähler term $K^{\bar{i}i}|_0$ (no summation), and the massless mode will have different weighting of each field. Finally for a general Kähler potential, we are still guaranteed a single complex massless mode, since the Kähler metric $K_{i\bar{j}}$ is invertible, but whether the exponential suppression exists and the corresponding overlap of each site depends on the specific Kähler potential considered.

To summarize, embedding the CW mechanism in SUGRA is a problematic issue. According to VA, VB we either (a) must add some constant as large as the CW term to “fix” the CC, or (b) get many flat directions and no clockworking. One has to further introduce explicit SUSY breaking CW to generate the desired CW hierarchy.

To avoid such a conclusion, we must manifestly build the CW symmetry into the superpotential W along with a scale m that does not come from a vev of another chiral superfield. In such case, we get the correct CW mass matrix for the case of canonical Kähler potential.

D. SUSY breaking

As everyone knows, SUSY is broken in nature, so the analysis will not be complete without incorporating a

SUSY breaking mechanism. Specifically, it is important to analyze the connection between SUSY breaking and the CW symmetry. Can we have one without the other? An immediate SUSY breaking mechanism, is again provided by w_0 of Sec. VA. If we do not tune the constant to be $|w_0| = 2N\epsilon v^3$, but somewhat smaller, then we will not get an exact cancellation of the CC, and SUSY will be broken exactly by this residual.

1. SUSY breaking with CW symmetry conservation using a spurion

A more interesting case is reconsidering the shift symmetric superpotential $W = \sum_{i=0}^N m(\Phi_i - q\Phi_{i+1})^2$. SUSY breaking can be achieved in a straightforward way by spurion analysis with a canonical Kähler. Consider SUSY breaking, while the CW symmetry is left untouched:

$$\begin{aligned} K &= \sum_{i=0}^N \Phi_i \bar{\Phi}_i + S \bar{S}, \\ W &= \sum_{i=0}^N m(\Phi_i - q\Phi_{i+1})^2 + \delta m^2 S, \end{aligned} \quad (5.26)$$

where $\delta m \ll m$. All the Φ_i fields still fulfill $D_i W = 0$ if $\langle \Phi_i \rangle = q \langle \Phi_{i+1} \rangle$, for any $\langle \Phi_0 \rangle \equiv v$. However, due to the appearance of the spurion, an actual minimum occurs only if all the vevs of all fields including S vanish. Then SUSY is spontaneously broken, with $W|_{\text{vac}} = W_{\Phi_i}|_{\text{vac}} = 0$ and $W_S = \delta m^2$. In the vacuum we shall have

$$V = e^K |W_S|^2 \quad \text{and} \quad V_{\text{vac}} = |\delta m|^4 > 0. \quad (5.27)$$

It is obvious that the CW shift symmetry of W is unharmed. The SUSY breaking scale is controlled by δm that is much smaller than the rest of the scales in the problem. From this point, SUGRA model building with collider signature analysis can ensue in a standard way. For example soft SUSY breaking terms can break the CW symmetry and lead to viable models.

2. SUSY breaking with CW breaking

Let us now try to induce SUSY breaking using the CW fields themselves and breaking the CW symmetry at the

same time. For a viable SUSY breaking with a positive CC, we shall see that the SUSY breaking direction is special and this direction is exactly the massless eigenmode of the clockwork.

Let us assume a quadratic Kähler, i.e., $K = \sum_{i=0}^N \Phi_i \bar{\Phi}_i$ or $K = \sum_{i=0}^N \frac{(\Phi_i + \bar{\Phi}_i)^2}{2}$ or any mixture of such potentials, as long as $K_{i\bar{j}} = \delta_{ij}$, with the following superpotential, written around the desired SUSY breaking minimum

$$W = \sum_{i=0}^N m(\Phi_i - q\Phi_{i+1})^2 + \sum_{i=0}^N c_i \Phi_i. \quad (5.28)$$

For $c_i \rightarrow 0$, both SUSY and the CW shift symmetry of the superpotential are conserved. For $c_i \neq 0$ the CW shift symmetry is broken. Let us analyze the SUSY breaking. Obviously, at $\Phi_i = 0$, we shall have $W|_0 = 0$ and $W_i|_0 = c_i$. Thus $D_i W|_0 = c_i$ and all fields collectively participate in SUSY breaking. To find a minimum, we refer again to (5.17):

$$\partial_k V = e^K (D_k D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} - 2 D_k W \bar{W}) = 0. \quad (5.29)$$

The second term always vanishes since $W|_0 = 0$, so we are left with:

$$0 = D_k D_i W D_{\bar{j}} \bar{W} K^{i\bar{j}} = D_k D_i W c_i \\ = (W_{ki} + K_{ki} W + K_i W_k) c_i = W_{ki} c_i. \quad (5.30)$$

Hence, we got an eigenvalue problem with eigenvalue zero, of the matrix W_{ij} which is exactly (2.4) (only with m instead of m^2):

$$(M_\pi^2)_{ij} c_j = 0. \quad (5.31)$$

The solution of the zero eigenvalue is exactly the CW vector, $c_i = v^2/q^i$ where v is some constant. Thus, we found that for a canonical Kähler with the CW superpotential, there is a unique SUSY breaking direction, that is the CW direction:

$$W = \sum_{i=0}^N m(\Phi_i - q\Phi_{i+1})^2 + \sum_{i=0}^N \frac{v^2}{q^i} \Phi_i. \quad (5.32)$$

Any single or several $c_i \Phi_i$ will not yield a good SUSY breaking minimum, as it will not fulfill (5.17), or (5.31). I.e., this model has the simplest realization of SUSY and CW symmetry being broken at a single stroke and from within the CW gears, rather than creating a hierarchy of symmetry breaking, or adding fields “external” to the CW Lagrangian. This is a unique breaking pattern, that limits model building possibilities. In such case, it is actually easy to write W in the mass basis. Denoting the rotated superfields in the mass basis by $\tilde{a}_i$, the superpotential is simply:

$$W = \tilde{v}^2 \tilde{a}_0 + \sum_{i=1}^N m_i \tilde{a}_i^2 \quad (5.33)$$

where m_i are the masses written in (2.5), with m instead of m^2 and $\tilde{v}^2 = v^2/\mathcal{N}_0$, using a proper normalization of $\tilde{a}_0$. It is clear that all $\tilde{a}_{i \neq 0}$ will sit at their supersymmetric minimum at $\tilde{a}_i = 0$, while only the CW mode will break SUSY, as well as the flat direction of $\tilde{a}_0$ which one has for $v \rightarrow 0$. The minimum of the scalar potential is now at $V|_0 = v^4 \sum_{i=0}^N q^{-2i} = \tilde{v}^4 > 0$. From here, model building continues in a standard way.

3. SUSY breaking in the mass basis

Once we move to the mass basis, it is clear how to construct spontaneously broken SUSY using the CW fields, while preserving the CW symmetry. Note that a canonical Kähler of the form $K = \sum_{i=0}^N \Phi_i \bar{\Phi}_i$ becomes $K = \sum_{i=0}^N \tilde{a}_i \bar{\tilde{a}}_i$. All one has to do is choose one of the $\tilde{a}_{i \neq 0}$ to be the spurion:

$$W = \delta m^2 (\tilde{a}_i - c) + \sum_{i=1}^N m_i \tilde{a}_i^2. \quad (5.34)$$

For proper choices of δm^2 , c , one has a SUSY breaking solution. As we can see, this requires a tuned constant $w_0 \equiv -\delta m^2 c$. The massless CW mode $\tilde{a}_0$ still does not appear in W , that still possesses a CW shift symmetry.

To summarize, we see that either SUSY is broken by some spurion outside the CW sector where the CW symmetry is conserved, or the CW mechanism, dictates the SUSY breaking direction. Without w_0 , the SUSY breaking direction is exactly the original massless CW mode, that plays the role of the spurion, leaving no residual CW symmetry. With w_0 , SUSY breaking can occur along any of the $\tilde{a}_i$ directions and the CW shift symmetry is conserved.

VI. CONFORMAL COUPLING AND MASS TERMS IN THE EXTRA DIMENSION

It is interesting to contemplate the possibility of some generic explicit breaking of the residual $U(1)$ symmetry, beyond the direct coupling to the specific sector we are interested in. If so, we do not only have technical naturalness, but can potentially have a “natural” UV theory that includes the breaking of the CW symmetry. Such a possibility could come out directly from the extra dimensional interpretation. Considering again the extra dimensional picture, [3] reproduced the mass matrix of the 4D CW and suggested that the continuum is coming from a linear dilaton model.

Because the main aim of this paper is to generalize CW theory, it makes sense to consider how additional terms in 5D affect the CW structure. Since, we can at most recover

the mass matrix of CW, we limit ourselves to terms that are at most quadratic in the 5D scalar field. A free massless scalar field in 5D generated the CW mass matrix (2.4). We therefore expect the additional quadratic terms to generate an explicit breaking of the CW symmetry as desired. We explore the effect of a conformal coupling to gravity, as the most minimal variation, where conformal symmetry, or to be precise, local Weyl symmetry, $g_{MN} \rightarrow e^{2\omega(x)} g_{MN}$, $\phi \rightarrow e^{-3/2\omega(x)} \phi$ is maintained.⁹ Deconstruction will introduce a length scale, the lattice spacing, and will break the conformal symmetry, and generate a 4D mass term. We then consider explicitly adding mass terms in the extra dimension. Finally, we deviate from the metric suggested in [3], and consider positively curved 5D manifolds.

The 5D dimensional action of a scalar field conformally coupled to gravity is

$$\mathcal{S} = -2 \int_0^{\pi R} dy \int d^4x \sqrt{-g_5} \left[g^{MN} \partial_M \phi \partial_N \phi + \frac{\xi}{2} \phi^2 \mathcal{R}_5 \right] \quad (6.1)$$

where $\xi = (D-2)/(2(D-1)) = 3/8$. As shown in [29], the conformally coupled scalar will have the profile $\phi = X^{-3/4}(y) B(x^\mu)$, for $X(y) = Y(y)$ in the line element as written in (3.9). Let us write the 5D metric proposal suggested in [3].¹⁰

$$ds^2 = e^{\frac{4k|y|}{3}} (\eta_{\mu\nu} dx^\mu dx^\nu + dy^2) = \left(\frac{z}{z_0} \right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \quad (6.2)$$

where $z_0 \leq z \leq z_\pi$. Substituting the CW metric (6.2) the following Ricci curvature and volume element are

$$\mathcal{R}_5 = -\frac{16}{3} \left(\frac{kz_0}{z} \right)^2, \quad (6.3)$$

$$\sqrt{-g_5} = \left(\frac{z}{z_0} \right)^4. \quad (6.4)$$

Notice that the numerical coefficient in $\mathcal{R}_5 = -16/3 = -2/\xi$. The action after canonically normalizing the 4D field is

$$\mathcal{S} = - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 + \xi X \mathcal{R}_5 \phi^2 \right\}. \quad (6.5)$$

⁹Some analysis along these lines with the Randall-Sundrum (RS) metric has been carried out in [29].

¹⁰Due to the conformal coupling to gravity, adding a bulk CC and using a linear dilaton as in [3] will not generate the CW metric, or also later (6.14). Since we are interested with the 4D physics, we remain agnostic about the UV completion that generates the action (6.1).

In any choice of coordinates, substituting (6.2) gives a constant negative mass term due to the conformal coupling:

$$\mathcal{S} = - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 - 2k^2 \phi^2 \right\}. \quad (6.6)$$

In general, such a term should not necessarily worry us, as this is a negatively curved space-time. However, upon discretization, the 4D metric is Minkowski. In such case we shall still get the same expressions for the charges q_j as in (3.12). But, the masses $m_{a_j}^2$ in (3.12), are shifted by a negative constant $-2k^2$.

$$\mathcal{S} = \int d^4x \left[\sum_{j=0}^N (\partial_\mu \phi_j)^2 + \sum_{j=0}^{N-1} m_j^2 (\phi_j - q_j \phi_{j+1})^2 - 2k^2 \phi_j^2 \right] \quad (6.7)$$

$$m_j^2 \equiv \frac{N^2 X_j}{\pi^2 R^2 Y_j}, \quad q_j \equiv \frac{X_j^{1/2} Y_j^{1/4}}{X_{j+1}^{1/2} Y_{j+1}^{1/4}}. \quad (6.8)$$

Hence, in the 4D picture, all $m_{a_j}^2$ will be shifted by a negative constant. In particular, the massless mode becomes tachyonic, with mass $m_{a_0}^2 = -2k^2$.¹¹ The tachyon arises due to the negative curvature of the 5D manifold. In RS, we get a similar negative term, but it will be site dependent, since:

$$\xi X \mathcal{R}_5 \phi^2 = -\frac{15}{2} \hat{k}^2 e^{2\hat{k}z} \phi^2. \quad (6.9)$$

Of course one can consider nonminimal coupling to the 5D Ricci scalar that is not conformal and with the opposite sign, such that it does give a positive mass term.¹² But it is not clear what we gain from such a coupling that is not already encapsulated in simply adding a mass term to the 5D scalar, since the conformal symmetry is lost.

Hence, let us consider adding an explicit 5D mass term $M^2 \phi^2/2$. This of course breaks the conformal symmetry. In such case, after canonically normalizing the 4D kinetic term we will have:

$$\mathcal{S} = - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 + X(M^2 + \xi \mathcal{R}_5) \phi^2 \right\} \quad (6.10)$$

¹¹In principle, there can be additional negative mass modes depending on k and R , the size of the extra dimension.

¹²It may be that such a change of sign simply means moving the tachyon to the gravitational part of the lagrangian. Let us ignore that for the moment.

Since in RS the Ricci scalar is constant, the tachyon can be exactly cancelled, or become massive with $M^2 \geq -\xi\mathcal{R}_5 = \frac{15}{2}\hat{k}^2$ upon discretization. Hence, we achieve an explicit breaking at the price of adding another mass scale M to the game. Of course, once we add an explicit 5D mass term, there is no motivation to consider the conformal coupling as the conformal symmetry is lost.

Back to the CW case with minimal coupling to gravity let us consider the 5D mass term. The contribution of the 5D mass term will be site dependent, according to the value of X_j . Generically, there will be no massless mode. To maintain the massless mode, with minimal coupling to gravity, one has to choose specifically that $M^2 = \alpha(q)m_j^2 = \alpha(q)(N/\pi R)^2$, where $\alpha(q)$ is some function of q that guarantees the vanishing of the determinant. For example, in the case of $N = 2$, we get $\alpha = q^{-4/3}(1 + q^{2/3})$. With the conformal coupling to gravity, the parameter α becomes dependent on k as well in some complicated expression, $\alpha = \alpha(q, k)$.

To summarize, two possible ways to uplift the massless mode from 5D are conformal coupling to gravity or adding a 5D mass term. The 5D mass term generates site-dependent mass terms in 4D, but with a judicious choice of parameters, a massless mode can be maintained. The conformal coupling to gravity makes the CW massless mode tachyonic after discretization. Adding a 5D mass term can remove the tachyonic instability, restoring zero mass for specific values of the parameters q, k , or generate a positive 4D mass term.

A. Positively curved 5D manifold

The tachyonic generation of mass is due to the negatively curved nature of the 5D manifold. Considering a positively curved 5D spacetime with constant positive curvature, will give a positive mass term to the massless mode while still maintaining conformal symmetry in 5D. Considering again the general ansatz [3]:

$$\begin{aligned} ds^2 &= X(y)\eta_{\mu\nu}dx^\mu dx^\nu + Y(y)dy^2 \\ &\equiv e^{2f(y)}(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2) \\ &= e^{2g(z)}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \end{aligned} \quad (6.11)$$

with the choice $X(y) = Y(y) \equiv e^{2f(y)}$. The action after canonically normalizing the 4D field is

$$\begin{aligned} \mathcal{S} = - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2}Y^{1/4}} \right)^2 \right. \\ \left. + \xi X \mathcal{R}_5 \phi^2 \right\}. \end{aligned} \quad (6.12)$$

Conformal symmetry is guaranteed if $\xi = 3/8$. We would like to get $X\mathcal{R}_5 \equiv e^{2f(y)}\mathcal{R}_5 \equiv e^{2g(z)}\mathcal{R}_5 = k^2$, for some positive real k , ensuring that the gravity induced mass

term upon discretization is positive. Working with the y coordinates, we demand:

$$X\mathcal{R}_5 = -4[2f''(y) + 3f'(y)^2] = 4k^2 \quad (6.13)$$

$$\Rightarrow X = \cos^{4/3} \left(\frac{\sqrt{3}}{2} ky \right). \quad (6.14)$$

The metric is well-defined, provided that $k < 1/\sqrt{3}R$, where πR is the size of the extra dimension.¹³

$$\begin{aligned} \mathcal{S} = - \int_0^{\pi R} dy \int d^4x \left\{ (\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2}Y^{1/4}} \right)^2 \right. \\ \left. + 4\xi k^2 \phi^2 \right\} \end{aligned} \quad (6.15)$$

The discretization proceeds in the same manner as before. For minimal coupling to gravity $\xi \rightarrow 0$, we have a massless mode, the mass parameter is unchanged but the charges now become site-dependent:

$$\begin{aligned} m_j^2 &= \frac{N^2 X_j}{\pi^2 R^2 Y_j}, \quad q_j = \frac{X_j^{1/2} Y_j^{1/4}}{X_{j+1}^{1/2} Y_{j+1}^{1/4}} = \frac{\cos(\frac{\sqrt{3}}{2} kja)}{\cos(\frac{\sqrt{3}}{2} k(j+1)a)}, \\ a &= \frac{\pi R}{N}. \end{aligned} \quad (6.16)$$

With conformal coupling to gravity, $\xi = 3/8$, all CW massive modes are shifted by a positive constant $4\xi k^2 = \frac{3}{2}k^2$, and similarly, the massless CW Goldstone boson gets a mass $m_{a_0}^2 = \frac{3}{2}k^2$ and the final $U(1)$ symmetry is broken. Given that the conformal coupling only introduces a fixed mass term for all sites, $V \supset 4\xi k^2 \phi_j^2 = 3/2 k^2 \phi_j^2$, it only shifts the masses as written above, and it does not change the $\mathcal{O}_{ji}$, where $i, j \in [0, N]$. This is because given a system of eigenvalues and eigenvectors, adding a matrix proportional to the identity, only shifts the eigenvalues by the constant of proportionality, and keeps the same eigenvectors. The outcome is a CW model where the breaking of the $U(1)_{\text{CW}}$ is due to the conformal coupling to gravity, masses $\tilde{m}_{a_k}^2 = \frac{3}{2}k^2 + m_{a_k}^2$ where $m_{a_k}^2$ are the CW massive modes in the absence of conformal coupling, and with site-dependent charges q_j as described in the Appendix. The massless mode is:

$$\mathcal{O}_{j0} = \mathcal{N}_0 \left\{ 1, \frac{1}{q_1}, \frac{1}{q_1 q_2}, \frac{1}{q_1 q_2 q_3}, \dots, \frac{1}{\prod_i q_i} \right\}. \quad (6.17)$$

Substituting the q_j s, specifically for the massless mode, one gets a simple expression of

¹³Using the z coordinate X is some hypergeometric function.

$$\begin{aligned} \mathcal{O}_{j0} = & \tilde{\mathcal{N}} \cos\left(\frac{\sqrt{3}}{2}ka\right) \left\{ \cos\left(\frac{\sqrt{3}}{2}ka\right), \right. \\ & \cos\left(\frac{\sqrt{3}}{2}2ka\right), \cos\left(\frac{\sqrt{3}}{2}3ka\right), \dots, \\ & \left. \cos\left(\frac{\sqrt{3}}{2}k(N+1)a\right) \right\}. \end{aligned} \quad (6.18)$$

So while the mass gap is diminished as all modes have masses $m^2 \sim k^2$, the special functional dependence of the original massless mode $\mathcal{O}_{j0} \sim \frac{1}{\Pi_i q_i}$ remains. However, due to the “telescopic” nature of each q_j we are left with a massless mode that depends rather democratically on all sites.

Before we conclude, in the above analysis, we have not taken into account effects caused by putting branes in the extra dimension when the continuum limit is discussed, as in [3,8,12,17]. Such construction can have interesting consequences, and we defer such investigation to future work.

VII. CONCLUSIONS

In this paper, we have attempted to generalize the clockwork idea in several directions, while keeping the original notion of natural generation of hierarchy in a theory whose fundamental parameters are of similar size. Or, in mundane terms, getting an exponential hierarchy, at the price of considering $N+1$ fields. From a lattice point of view, we have demonstrated that coupling beyond nearest neighbors leads to enhanced symmetry, depending on the number of neighbors each site couples to. This is interpreted as higher derivative terms from the extra-dimensional point of view. If we allow all possible number of neighbors interactions, the massive eigenstates are shifted considerably, but a massless mode and a residual $U(1)$ remain in tact. The masses of the axions in these constructions are modified and could come close to the mass of the radial modes. We have further generalized the clockwork to global $O(N)$ models.

Generalization to SUGRA is a delicate issue. If we use spurions or stabilizers, the SUSY breaking either has to be as large as the clockwork term for viable phenomenology, or more ingredients have to be added, such as further uplifting of N flat directions, or having the same energy scale for the clockwork and the CC. Simple successful CW SUGRA proceeds either by tuning w_0 in the superpotential to cancel the spurions/stabilizers contributions, or by discarding such fields and building a manifestly shift symmetric superpotential. We have also demonstrated how to break SUSY spontaneously and simultaneously conserve or break the CW symmetry for model building purposes. A particularly interesting result, is the fact that the CW symmetry actually dictates the SUSY breaking direction, if one does not want to change the field content of the theory. This direction is exactly the CW massless mode, that gets a finite mass, and neither SUSY nor CW are left

intact. Finally, in the mass basis, a generic SUSY breaking direction which is not the massless mode is available by trivial solution of the extremum equations.

Conformal coupling of the CW metric in 5D, makes the massless mode tachyonic upon discretization. This is a generic property of negatively curved 5D manifolds once we canonically normalize the kinetic terms of the scalar field and discretize. A 5D mass term, generically uplifts the massless mode, though for certain value of parameters the masslessness can be restored. This conclusion is valid also in the presence of conformal coupling to gravity in negatively curved 5D manifolds. Finally, with a positively curved 5D manifold, one can generate mass for the CW massless mode, but then the charges of the CW scalars become site-dependent. All in all, it seems we have only started to unravel the various possibilities of the clockwork mechanism.

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APPENDIX: CLOCKWORK WITH VARIOUS CHARGES

Let us consider only nearest neighbors interaction, but with a different charge, q_i at each site. In such a case, if $q_i > 3$ we get a nonrenormalizable theory, and we have to divide by some mass scale Λ^{q_i-3} . Thus the general potential looks like:

$$\begin{aligned} V(\phi_j) = & \sum_{j=0}^N \left(-\tilde{m}^2 \phi_j^\dagger \phi_j + \frac{\lambda}{4} |\phi_j^\dagger \phi_j|^2 \right) \\ & + \sum_{j=0}^{N-1} \left(\epsilon \frac{\phi_j^\dagger \phi_{j+1}^{q_j}}{\Lambda^{q_j-3}} + \text{H.c.} \right). \end{aligned} \quad (A1)$$

We implicitly assume that Λ is *larger than any other energy scale of the problem*. Let us check when can we continue with the separation between the radial and axial modes. We would like to give each radial field an approximate vev as before of $\langle |\phi_j|^2 \rangle = f^2 \equiv 2\tilde{m}^2/\lambda$, $\forall j$, as well as $f < \Lambda$. For this to happen, we need:

$$\epsilon \frac{f^{1+q_j}}{\Lambda^{q_j-3}} \ll \frac{\lambda}{f^4} \Rightarrow \epsilon \ll \lambda \left(\frac{\Lambda}{f} \right)^{q_j-3}. \quad (A2)$$

Thus, for $q > 3$, the desired hierarchy is easier to fulfill than the original CW. This will also be true when we couple beyond nearest neighbors. The masses of the radial modes are negligibly shifted to $m_{r_k}^2 \sim f^2(1 + \epsilon(f/\Lambda)^{q_k-3}) \sim f^2$.

Below the breaking scale $\sqrt{\lambda}f$, we have a theory of $N+1$ Goldstone bosons with $U_j = e^{i\pi_j(x)/f}$ and $j = 0, \dots, N$:

$$\mathcal{L} = - \sum_{j=0}^N f^2 \partial U_j^\dagger \partial U_j + \frac{m^2}{2} \frac{f^{q_j-1}}{\Lambda^{q_j-3}} \sum_{j=0}^{N-1} (U_j^\dagger U_{j+1}^{q_j} + \text{H.c.}) \quad (\text{A3})$$

with $m^2 = 2\epsilon f^2$. The CW massless mode will now have the following eigenvector:

$$\mathcal{O}_{j0} = \mathcal{N}_0 \left\{ 1, \frac{1}{q_1}, \frac{1}{q_1 q_2}, \frac{1}{q_1 q_2 q_3} \cdots, \frac{1}{\prod_i q_i} \right\}. \quad (\text{A4})$$

The masses of the axial modes behave as $m_{a_k}^2 \sim \epsilon f^2 (f/\Lambda)^{q_k-3}$. The exact diagonalization is straightforward.

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- [1] D.E. Kaplan and R. Rattazzi, Large field excursions and approximate discrete symmetries from a clockwork axion, *Phys. Rev. D* **93**, 085007 (2016).
- [2] K. Choi and S. H. Im, Realizing the relaxation from multiple axions and its UV completion with high scale supersymmetry, *J. High Energy Phys.* **01** (2016) 149.
- [3] G.F. Giudice and M. McCullough, A clockwork theory, *J. High Energy Phys.* **02** (2017) 036.
- [4] K. Choi, H. Kim, and S. Yun, Natural inflation with multiple sub-Planckian axions, *Phys. Rev. D* **90**, 023545 (2014).
- [5] J. E. Kim, H. P. Nilles, and M. Peloso, Completing natural inflation, *J. Cosmol. Astropart. Phys.* **01** (2005) 005.
- [6] I. Ben-Dayan, F. G. Pedro, and A. Westphal, Hierarchical Axion Inflation, *Phys. Rev. Lett.* **113**, 261301 (2014).
- [7] S. H. H. Tye and S. S. C. Wong, Helical inflation and cosmic strings, *arXiv:1404.6988*.
- [8] T. Hambye, D. Teresi, and M. H. G. Tytgat, A clockwork WIMP, *J. High Energy Phys.* **07** (2017) 047.
- [9] A. Kehagias and A. Riotto, Clockwork inflation, *Phys. Lett. B* **767**, 73 (2017).
- [10] A. Ahmed and B. M. Dillon, Clockwork composite Higgses, *Phys. Rev. D* **96**, 115031 (2017).
- [11] M. Farina, D. Pappadopulo, F. Rompineve, and A. Tesi, The photo-philic QCD axion, *J. High Energy Phys.* **01** (2017) 095; T. Higaki, K. S. Jeong, N. Kitajima, and F. Takahashi, The QCD axion from aligned axions and diphoton excess, *Phys. Lett. B* **755**, 13 (2016); Quality of the Peccei-Quinn symmetry in the aligned QCD axion and cosmological implications, *J. High Energy Phys.* **06** (2016) 150; T. Higaki, K. S. Jeong, N. Kitajima, T. Sekiguchi, and F. Takahashi, Topological defects and nano-Hz gravitational waves in aligned axion models, *J. High Energy Phys.* **08** (2016) 044.
- [12] N. Craig, I. Garcia Garcia, and D. Sutherland, Disassembling the clockwork mechanism, *J. High Energy Phys.* **10** (2017) 018.
- [13] G. F. Giudice and M. McCullough, Comment on “Disassembling the clockwork mechanism”, *arXiv:1705.10162*.
- [14] R. Coy, M. Frigerio, and M. Ibe, Dynamical clockwork axions, *J. High Energy Phys.* **10** (2017) 002.
- [15] I. Antoniadis, A. Arvanitaki, S. Dimopoulos, and A. Giveon, Phenomenology of TeV Little String Theory from Holography, *Phys. Rev. Lett.* **108**, 081602 (2012).
- [16] M. Baryakhtar, Graviton phenomenology of linear dilaton geometries, *Phys. Rev. D* **85**, 125019 (2012).
- [17] P. Cox and T. Gherghetta, Radion dynamics and phenomenology in the linear dilaton model, *J. High Energy Phys.* **05** (2012) 149.
- [18] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, De Sitter vacua in string theory, *Phys. Rev. D* **68**, 046005 (2003).
- [19] J. J. Blanco-Pillado, C. P. Burgess, J. M. Cline, C. Escoda, M. Gomez-Reino, R. Kallosh, A. D. Linde, and F. Quevedo, Racetrack inflation, *J. High Energy Phys.* **11** (2004) 063.
- [20] T. W. Grimm, Axion inflation in type II string theory, *Phys. Rev. D* **77**, 126007 (2008).
- [21] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, Four-dimensional string compactifications with D-branes, orientifolds and fluxes, *Phys. Rep.* **445**, 1 (2007).
- [22] I. Ben-Dayan, S. Jing, A. Westphal, and C. Wieck, Accidental inflation from Kähler uplifting, *J. Cosmol. Astropart. Phys.* **03** (2014) 054.
- [23] I. Ben-Dayan, F. G. Pedro, and A. Westphal, Towards natural inflation in string theory, *Phys. Rev. D* **92**, 023515 (2015).
- [24] R. Brustein and S. P. de Alwis, Moduli potentials in string compactifications with fluxes: Mapping the discretuum, *Phys. Rev. D* **69**, 126006 (2004).
- [25] I. Ben-Dayan, R. Brustein, and S. P. de Alwis, Models of modular inflation and their phenomenological consequences, *J. Cosmol. Astropart. Phys.* **07** (2008) 011.
- [26] M. Kawasaki, M. Yamaguchi, and T. Yanagida, Natural Chaotic Inflation in Supergravity, *Phys. Rev. Lett.* **85**, 3572 (2000).
- [27] R. Kallosh, A. Linde, and T. Rube, General inflaton potentials in supergravity, *Phys. Rev. D* **83**, 043507 (2011).
- [28] I. Ben-Dayan and M. B. Einhorn, Supergravity Higgs inflation and shift symmetry in electroweak theory, *J. Cosmol. Astropart. Phys.* **12** (2010) 002.
- [29] R. Hofmann and M. Pospelov, Conformal matter in warped backgrounds, *Phys. Lett. B* **556**, 203 (2003).