Scattering of light dark matter in atomic clocks

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We present a detailed analysis of the effect of light dark matter (DM) on atomic clocks, for the case where DM mass and density are such that occupation numbers are low and DM must be considered as particles scattering off the atoms, rather than a classical field. We show that the resulting atomic clock frequency shifts are first order in the scattering amplitudes and particularly suited to constrain DM models in the regime where the DM mass $m_\chi \ll \text{GeV}$. We provide some rough order of magnitude estimates of sensitivity that can be confronted to any DM model that allows for nonzero differential scattering amplitudes of the two atomic states involved in the clock.

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I. INTRODUCTION

In recent years, there has been increasing interest, both theoretical and experimental, in the possibility of ultralight (typically $<10^{-10}$ eV) dark matter (DM) detection using atomic clocks [1–8]. Most of that work is using a DM model where the DM is a massive scalar field that is nonminimally coupled to standard matter, with that field being described classically either as coherent oscillations or as some topological defects. The outstanding stability and accuracy of cold atom clocks has been used to set some of the most stringent limits on the coupling constants in this type of DM model [4–7].

In this work, and in more detail in the companion paper [9], we consider more general DM models (beyond only a classical scalar field) that may include nonscalar DM, which interacts with the spin of the atoms in atomic clocks. Additionally, for high masses, or if not all of the DM density in the galaxy is of this type, the mass and density may be such that occupation numbers are ≤ 1 , so that a classical field description is no longer appropriate. Then DM must be treated as particles that scatter the atoms in atomic clocks. This opens up a new range of laboratory DM searches in mass or density regions that cannot be probed by the usual weakly interacting massive particle or axion

experiments, while being different from the ultralow mass scalar fields described above.

More particularly, we elaborate the theory for the scattering of some particle with mass m_χ off the atoms with mass m_a for the particular case where $m_\chi \ll m_a$ (the mass range where m_χ is close to m_a is already well explored). We explicitly calculate the frequency and phase shift of atomic clocks as a function of the scattering amplitudes for different types of clocks (e.g., Ramsey and Rabi spectroscopy) and give expressions that allow confronting these results to any model for such particles and their interaction with the fundamental particles of the standard model. This analysis is quite general and applies to any light particle scattering off atoms, DM or not.

We then provide some very rough benchmark values for the sensitivities that could be achieved in terms of the corresponding scattering amplitudes in different scenarios that could apply to DM. In a companion article [9], we then use those results to estimate the sensitivities in terms of the parameters (masses, coupling constants) of some particular DM models. The present article is kept more general, the aim being to provide rough numbers that allow evaluating the potential sensitivity of atomic techniques to any DM model, provided it allows the derivation of scattering amplitudes for collisions of the light DM particles with atoms.

The scattering of particles off cold atoms in clocks has been discussed most recently in [10] in the context of frequency shifts in atomic clocks related to the presence of residual background gases. One of the results of that analysis is that in cold atom clocks only forward scattered atoms can reach the detection region, which leads to a loss of Ramsey fringe amplitude related to the interference between the scattered and unscattered wave in the forward

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direction. As we show below, this is, strictly speaking, only true for the case where the initial momentum of the scatterers $m_\chi v_{\chi i}$ is much larger than the maximum change of momentum $m_a \Delta v_a$ of the clock atoms that still allows them to reach the detection region. However, in this work we are interested in the opposite situation, where the scatterers are very light so that $m_\chi v_{\chi i} \ll m_a \Delta v_a$, and the conclusions in that case are phenomenologically different. For example, and as one would expect, there is now no loss of fringe amplitude as, to leading order in m_χ/m_a , the cold atoms do not change their trajectory in the scattering process, and thus all of them make it to the detection region. Furthermore, we extend our analysis to Rabi spectroscopy rather than only Ramsey spectroscopy as in [10], which turns out to be a promising method for actually observing such collisions.

Our article is organized as follows: We first recall some basics of quantum scattering theory in Sec. II A with particular emphasis on light scatterers in Sec. II B. We then apply it to atomic clocks in Ramsey (Sec. III A) or Rabi (Sec. III B) configuration. The final section discusses some strategies for DM detection using the previously obtained results, and provides some very rough sensitivity estimates based on present day technology.

II. QUANTUM SCATTERING OF TWO PARTICLES

The theory presented in this section is well known and can be found in many textbooks (e.g., [11]) or lecture courses (e.g., [12]). Its application to cold atom clocks has been discussed most recently in [10].

A. General theory

Consider a clock atom a interacting with some background particle χ . The two-particle wave function is $\Psi(t, \mathbf{x}_a, \mathbf{x}_\chi)$. The probability of detecting the atom in the detection region D at time t_D is

$$P_D = \int_D d^3x_a \int_{-\infty}^{\infty} d^3x_\chi |\Psi(t_D, \mathbf{x}_a, \mathbf{x}_\chi)|^2. \quad (1)$$

The evolution of $\Psi(t, \mathbf{x}_a, \mathbf{x}_\chi)$ is governed by the two-particle Schrödinger equation, where the Hamiltonian includes the interaction potential $V(\mathbf{x}_a - \mathbf{x}_\chi)$. With the change of variables to the center of mass and relative positions $\mathbf{x}_G = (m_a \mathbf{x}_a + m_\chi \mathbf{x}_\chi)/(m_a + m_\chi)$ and $\mathbf{x} = \mathbf{x}_a - \mathbf{x}_\chi$, the two-particle wave function can be factorized to $\Psi(t, \mathbf{x}_a, \mathbf{x}_\chi) = \Psi_G(t, \mathbf{x}_G) \Psi(t, \mathbf{x})$, where $\Psi_G(t, \mathbf{x}_G)$ is the center of mass wave function and $\Psi(t, \mathbf{x})$ is that of the “virtual particle” with mass $\mu = m_\chi m_a/(m_a + m_\chi)$ (see, e.g., [13]). The latter is obtained from the one-particle Schrödinger equation in relative coordinates $\mathbf{x} = \mathbf{x}_a - \mathbf{x}_\chi$, with the reduced mass μ and the interaction potential $V(\mathbf{x})$.

Let us first consider the case where $m_a \ll m_\chi$. Then, to zeroth order in m_a/m_χ , the center of mass position is that of

the scatterer $\mathbf{x}_G \simeq \mathbf{x}_\chi$ and the detection probability (1) becomes

$$\begin{aligned} P_D &\simeq \int_D d^3x |\Psi(t_D, \mathbf{x})|^2 \int_{\infty} d^3x_\chi |\Psi_G(t_D, \mathbf{x}_\chi)|^2 \\ &= \int_D d^3x |\Psi(t_D, \mathbf{x})|^2. \end{aligned} \quad (2)$$

This case is treated in [10]. It corresponds to considering only forward scattering of the virtual particle (essentially the atom in this case) as the integral is carried out only over the detection region, and cold atoms scattered away from that region are lost. Phenomenologically, one observes a frequency shift and a loss of fringe amplitude, the latter corresponding to cold atoms being scattered away from the detection region.

The opposite case $m_a \gg m_\chi$ is the main subject of our article. We now have to zeroth order in m_χ/m_a : $\mathbf{x}_G \simeq \mathbf{x}_a$ and the detection probability (1) becomes

$$\begin{aligned} P_D &\simeq \int_{\infty} d^3x |\Psi(t_D, \mathbf{x})|^2 \int_D d^3x_a |\Psi_G(t_D, \mathbf{x}_a)|^2 \\ &= \int_{\infty} d^3x |\Psi(t_D, \mathbf{x})|^2, \end{aligned} \quad (3)$$

as $\Psi_G(t_D, \mathbf{x}_a) \sim \Psi_a(t_D, \mathbf{x}_a)$, which now propagates entirely to the detection region as the center of mass coincides with the atom and propagates freely along its initial direction. Phenomenologically, there is no loss of fringe amplitude now (all atoms make it to the detection region), but a frequency shift remains, as we will see below. Note that this applies entirely to cold atoms, provided that they are more massive than the scatterers, and in particular that $m_\chi v_{\chi i} \ll m_a \Delta v_a$ as mentioned in the Introduction. Equation (3) simply translates the fact that any, i.e., not only forward, scattering of the virtual particle (essentially the χ particle in this case) still allows the atom to be detected.

To make the criterion for using (2) or (3) more physical for atomic clocks, we still assume that $m_a > m_\chi$, but work to first order in m_χ/m_a . Classical energy-momentum conservation shows that the scattering angle θ of the virtual particle (i.e., the change in direction of $\mathbf{v} = \mathbf{v}_a - \mathbf{v}_\chi$) is given by

$$\tan \theta \simeq \mp \frac{m_a \Delta v_a}{m_\chi v_{\chi i}} \frac{\left(1 - \frac{m_a^2}{4m_\chi^2} \frac{\Delta v_a^2}{v_{\chi i}^2}\right)^{1/2}}{\left(1 - \frac{m_a^2}{2m_\chi^2} \frac{\Delta v_a^2}{v_{\chi i}^2}\right)}, \quad (4)$$

where $v_{\chi i}$ is the initial velocity of the light particle in the rest frame of the clock atom, and Δv_a is the maximum allowed change of velocity of the clock atom that still allows it to reach the detection region.

As an order of magnitude, consider cesium (Cs) clock atoms that are scattered off thermal H_2 molecules

($v_{\chi i} \approx 10^3$ m/s), as in [10]. For typical Cs fountain clock parameters, the clock atoms are still detected for $\Delta v_a \approx 10^{-3}$ m/s, corresponding to a 1 cm detection aperture 0.5 s after the scattering event. Then the terms in brackets in (4) are ≈ 1 and we have $\theta < 10^{-3}$ rad, which is quite compatible with considering only forward scattering [equivalent of (2)], as done in [10]. On the contrary, for DM particles with, e.g., $m_\chi \approx 1$ keV and $v_{\chi i} \approx 10^{-3}c$, the maximum possible value of Δv_a is $\approx 10^{-3}$ m/s and thus the atom is detected for any value of θ , which yields Eq. (3).

B. Particular case of light scatterers

We now further investigate the case where $m_a \gg m_\chi$. We want to evaluate the integral (3) with $\Psi(t_D, \mathbf{x})$, the wave function of the virtual particle solution of the single-particle Schrödinger equation, with mass μ and position $\mathbf{x} = \mathbf{x}_a - \mathbf{x}_\chi$. Assuming a short-range potential $V(\mathbf{x})$ that solution is standard in scattering theory [11,12] and takes the general form $\Psi_{\text{out}} = \Psi_{\text{inc}} + \Psi_{\text{sc}}$, where Ψ_{out} is the outgoing wave function after scattering,

$$\Psi_{\text{inc}} = N e^{i(kz - \omega t)} e^{-\frac{x^2 + y^2 + (z - vt)^2}{4d^2}}, \quad (5)$$

is the incident wave with $N = (\frac{1}{2\pi d^2})^{3/4}$, $\omega = \frac{\hbar k^2}{2\mu}$, $v = \hbar k/\mu$, and d the width of the Gaussian wave packet. The scattered wave,

$$\Psi_{\text{sc}} = N \frac{f(\theta)}{r} e^{i(kr - \omega t)} e^{-\frac{(r - vt)^2}{4d^2}}, \quad (6)$$

is a spherical wave moving outwards with Gaussian profile in r and scattering amplitude $f(\theta)$, where θ is the scattering angle.

The detection probability is then given by (3) in the form

$$\begin{aligned} P_D &= \int_{\infty} d^3x |\Psi_{\text{out}}|^2 \\ &= \int_{\infty} d^3x (|\Psi_{\text{inc}}|^2 + |\Psi_{\text{sc}}|^2 + |\Psi_{\text{int}}|^2), \end{aligned} \quad (7)$$

with the interference term $|\Psi_{\text{int}}|^2 \equiv \Psi_{\text{inc}}^* \Psi_{\text{sc}} + \Psi_{\text{inc}} \Psi_{\text{sc}}^*$.

The integral of the first term in (7) is equal to 1. The second term is simply given by probability conservation, which requires that $\int_{\infty} d^3x (|\Psi_{\text{sc}}|^2 + |\Psi_{\text{int}}|^2) = 0$.

The third term can be calculated explicitly [11,12], giving

$$\int_{\infty} d^3x |\Psi_{\text{int}}|^2 = -\frac{1}{2\pi d^2} \frac{4\pi}{k} \text{Im}[f(0)]. \quad (8)$$

This term represents the “fractional shadow” cast by the scatterer on the detection region, i.e., the part of the incident wave “lost” into the scattered one, and is known as the optical theorem. Note that only the forward scattering

amplitude ($\theta = 0$) is involved, as the interference term quickly averages to zero for $\theta \neq 0$.¹

Obviously, one then has a detection probability in (7) that is always $P_D = 1$ and unaffected by the scattering. However, as we will show below, for an initial superposition state the scattering may lead to a phase shift between the two parts of the superposition that can be measured.

III. APPLICATION TO ATOMIC CLOCKS

A. Ramsey spectroscopy

The detailed theory of atomic clocks and Ramsey spectroscopy can be found in, e.g., [14]. Here we consider a simplified situation with two ideal $\pi/2$ pulses separated by a free evolution time T and calculate the frequency shift from a single scattering event taking place during time T after the first and before the second pulse.

The ground and excited internal states of the atom are $|1\rangle$ and $|2\rangle$. The initial state before the first $\pi/2$ pulse is

$$\begin{aligned} \langle 2|\Psi \rangle &= 0, \\ \langle 1|\Psi \rangle &= \Psi_{\text{inc}}, \end{aligned} \quad (9)$$

with Ψ_{inc} given by (5). After the first pulse the state is

$$\begin{aligned} \langle 2|\Psi \rangle &= \frac{1}{\sqrt{2}} (-i\Psi_{\text{inc}}), \\ \langle 1|\Psi \rangle &= \frac{1}{\sqrt{2}} \Psi_{\text{inc}}, \end{aligned} \quad (10)$$

where we have set the phase of the light to zero. The scattering takes place between the two pulses, and at the second pulse the light has phase ϕ . The state after the second pulse is then

$$\begin{aligned} \langle 2|\Psi \rangle &= \frac{1}{2} (-i\Psi_{\text{out}}^{(2)} - i\Psi_{\text{out}}^{(1)} e^{-i\phi}), \\ \langle 1|\Psi \rangle &= \frac{1}{2} (\Psi_{\text{out}}^{(1)} - \Psi_{\text{out}}^{(2)} e^{i\phi}), \end{aligned} \quad (11)$$

where, as before, $\Psi_{\text{out}} = \Psi_{\text{inc}} + \Psi_{\text{sc}}$, and the superscript refers to scattering in state $|1\rangle$ or $|2\rangle$, i.e., we assume different scattering amplitudes $f_1(\theta)$ or $f_2(\theta)$ in (6). The probability of detecting, e.g., the internal state $|2\rangle$ is then

¹In the case where $m_\chi v_{\chi i} \gg m_a \Delta v_a$, (8) describes the loss of cold atoms, and thus fringe amplitude, due to scattering [10].

$$\begin{aligned}
P_2 &= \int_{\infty} d^3x |\langle 2|\Psi\rangle|^2 = \frac{1}{4} \int_{\infty} d^3x |\Psi_{\text{inc}}(1 + e^{-i\phi}) + \Psi_{\text{sc}}^{(2)} + \Psi_{\text{sc}}^{(1)} e^{-i\phi}|^2 \\
&= \frac{1}{2} (1 + \cos \phi) + \frac{1}{4} \int_{\infty} d^3x (|\Psi_{\text{sc}}^{(2)}|^2 + |\Psi_{\text{sc}}^{(1)}|^2 + |\Psi_{\text{int}}^{(2)}|^2 + |\Psi_{\text{int}}^{(1)}|^2 \\
&\quad + \Psi_{\text{sc}}^{(2)} \Psi_{\text{sc}}^{(1)*} e^{i\phi} + \Psi_{\text{sc}}^{(2)*} \Psi_{\text{sc}}^{(1)} e^{-i\phi} + \Psi_{\text{inc}}^* \Psi_{\text{sc}}^{(2)} e^{i\phi} + \Psi_{\text{inc}} \Psi_{\text{sc}}^{(2)*} e^{-i\phi} + \Psi_{\text{inc}}^* \Psi_{\text{sc}}^{(1)} e^{i\phi} + \Psi_{\text{inc}} \Psi_{\text{sc}}^{(1)*} e^{-i\phi}). \quad (12)
\end{aligned}$$

Probability conservation (see Sec. II B) requires that the first four terms in the integral of (12) cancel pairwise. To evaluate the remaining six terms, we first rewrite $\Psi_{\text{sc}}^{(\gamma)} \equiv f_{\gamma}(\theta) \Psi_0$, where $\gamma = 1, 2$ and the explicit form of Ψ_0 can be simply read off (6). With that notation, (12) becomes

$$P_2 = \frac{1}{2} (1 + \cos \phi) + \frac{1}{4} \int_{\infty} d^3x (f_1^* f_2 e^{i\phi} |\Psi_0|^2 + f_1 f_2^* e^{-i\phi} |\Psi_0|^2 + f_2 e^{i\phi} \Psi_{\text{inc}}^* \Psi_0 + f_2^* e^{-i\phi} \Psi_{\text{inc}} \Psi_0^* + f_1 e^{-i\phi} \Psi_{\text{inc}}^* \Psi_0 + f_1^* e^{i\phi} \Psi_{\text{inc}} \Psi_0^*). \quad (13)$$

For the first two terms in the integral, we evaluate the r integral with $|\Psi_0|^2$ from (6), and for the last four terms, we apply the optical theorem (8) to obtain

$$P_2 = \frac{1}{2} (1 + \cos \phi) + \frac{1}{4\pi d^2} (\text{Re}[e^{-i\phi} F_{12}]) - \frac{1}{2kd^2} (\text{Im}[e^{-i\phi} (f_1 - f_2^*)]), \quad (14)$$

where we have defined $F_{12} \equiv \int d\Omega f_1(\theta) f_2^*(\theta)$ and with the scattering amplitudes in the last line of (14) in the forward direction only, i.e., $f_{\gamma} = f_{\gamma}(0)$. More explicitly,

$$P_2 = \frac{1}{2} (1 + \cos \phi) + \frac{1}{d^2} \left(\frac{1}{4\pi} \text{Re}[F_{12}] - \frac{1}{2k} \text{Im}[f_1 + f_2] \right) \cos \phi + \frac{1}{d^2} \left(\frac{1}{4\pi} \text{Im}[F_{12}] + \frac{1}{2k} \text{Re}[f_1 - f_2] \right) \sin \phi. \quad (15)$$

For $f_1 = f_2$ we have $\text{Im}[F_{12}] = 0$. The optical theorem together with (6) gives directly $\text{Re}[F_{12}] = 4\pi \text{Im}[f]/k$. So Eq. (15) reduces to its standard form in the absence of scattering, as expected.

The quantity F_{12} , and more generally (15), can be further expanded in terms of partial wave phases (see Appendix A). However, for small cross sections we will be interested in the region where $k|f| \ll 1$, which implies that, of the scattering terms in (15), the last one will dominate by a factor $\simeq 1/(k|f|)$.² The result is thus simplified to

$$P_2 \simeq \frac{1}{2} (1 + \cos \phi) + \frac{1}{2kd^2} \text{Re}[f_1 - f_2] \sin \phi. \quad (16)$$

A similar calculation for state $|1\rangle$ yields

$$P_1 \simeq \frac{1}{2} (1 - \cos \phi) - \frac{1}{2kd^2} \text{Re}[f_1 - f_2] \sin \phi. \quad (17)$$

The sum satisfies $P_1 + P_2 = 1$ for any value of ϕ, f_1, f_2 , as expected when there is no loss of atoms due to scattering, i.e., in our case where $m_{\chi}/m_a \ll 1$.

To calculate the frequency shift, we set $\phi = \delta T$ in (16), where $\delta \equiv \omega - \omega_{12}$ is the detuning of the light from atomic

resonance. We then expand for $\delta T \ll 1$ and find the value δ_{max} for which P_2 is maximum, i.e., the position of the central Ramsey fringe maximum, by setting $\partial P_2 / \partial \delta = 0$. The result is

$$\delta_{\text{max}} \simeq \frac{1}{kd^2 T} \text{Re}[f_1(0) - f_2(0)], \quad (18)$$

where we recall that in (16) $f_{\gamma} = f_{\gamma}(0)$.

1. Discussion

The change of detection probability as given by Eq. (16) corresponds to the frequency in (18), but to no loss of fringe amplitude.

The physical interpretation is relatively straightforward: the scattering event corresponds to the atom experiencing the short-range interaction potential $V(\mathbf{x})$ for a short time. If that potential is different for the two atomic states, then the atomic resonance is shifted by that difference (the two states are “dressed” differently by the potential). Therefore, the scattering event leads to a transient shift of the atomic transition frequency ω_{12} , thus to a small change of the

²Unless $|f_1 - f_2| \ll |f|$, but we will not consider that case as it provides much lower sensitivity.

phase ϕ . During the Ramsey time T this phase shift leads to a shift of the resonance frequency of $\delta_{\max}$ as given in (18).

The process described above is independent of which direction the atoms or χ particles are scattered into, but only the atoms that arrive at the detection region contribute to the signal. In our case of $m_a \gg m_\chi$, all atoms make it to the detection region and there is no loss of signal amplitude. In the opposite case $m_a \ll m_\chi$ (or, more precisely, $m_\chi v_{\chi i} \gg m_a \Delta v_a$; see Sec. II A), only forward scattered atoms make it to the detection region and contribute to the frequency shift, so there is a shift *and* a signal loss [10].

In any case, the effect on the atomic transition frequency is also present at zero momentum transfer, which makes it ideal to look for very light scatterers or coherent effects as compared to other techniques based on atomic recoil.

B. Rabi spectroscopy

The physical interpretation from the previous section suggests that the effect of scattering on the clock frequency might be different for different types of interrogation sequences of the atom. Consider, for example, a single scattering event. In the case of a Ramsey sequence of duration T , the result on the detection probability is (16) irrespective of when the scattering takes place, as a transient change of ω_{12} has the same effect whether it happens toward the beginning, middle, or end of T . However, for a single Rabi pulse of duration T one would expect the effect to be maximum for the scattering occurring at $T/2$ and minimum for scattering near the beginning or the end of the pulse. This can be easily understood using the image of fictitious spin on a Bloch sphere or the sensitivity function formalism described, e.g., in [15]. An explicit derivation is provided below.

Consider a Rabi interrogation with a single pulse of duration T during which the scattering takes place. The initial state of the atom at $t = 0$ is

$$\begin{aligned}\langle 2|\Psi\rangle &= 0, \\ \langle 1|\Psi\rangle &= \Psi_{\text{inc}}.\end{aligned}\quad (19)$$

The collision takes place at time $t = t_c$. Just before the collision, the state is (see, e.g., [13,14,16])

$$\begin{aligned}\langle 2|\Psi\rangle &= -ie^{-i\delta t_c/2} \sin\theta \sin\left(\frac{\Omega_r t_c}{2}\right) \Psi_{\text{inc}}, \\ \langle 1|\Psi\rangle &= e^{i\delta t_c/2} \left[\cos\left(\frac{\Omega_r t_c}{2}\right) + i \cos\theta \sin\left(\frac{\Omega_r t_c}{2}\right) \right] \Psi_{\text{inc}},\end{aligned}\quad (20)$$

where δ is the detuning, Ω is the Rabi frequency, $\Omega_r = \sqrt{|\Omega|^2 + \delta^2}$ is the off-resonant Rabi frequency, $\sin\theta \equiv \Omega/\Omega_r$, and $\cos\theta \equiv -\delta/\Omega_r$, and where we have set the initial phase of the light to zero for simplicity.

The state just after the collision is then

$$\begin{aligned}\langle 2|\Psi\rangle &= -ie^{-i\delta t_c/2} \sin\theta \sin\left(\frac{\Omega_r t_c}{2}\right) \Psi_{\text{out}}^{(2)}, \\ \langle 1|\Psi\rangle &= e^{i\delta t_c/2} \left[\cos\left(\frac{\Omega_r t_c}{2}\right) + i \cos\theta \sin\left(\frac{\Omega_r t_c}{2}\right) \right] \Psi_{\text{out}}^{(1)},\end{aligned}\quad (21)$$

and at time T

$$\begin{aligned}\langle 2|\Psi\rangle &= e^{-i\delta T/2} \{ -i \sin\theta S_c [C_T - i \cos\theta S_T] \Psi_{\text{out}}^{(2)} \\ &\quad - i \sin\theta S_T [C_c + i \cos\theta S_c] \Psi_{\text{out}}^{(1)} \}, \\ \langle 1|\Psi\rangle &= e^{i\delta T/2} \{ -\sin^2\theta S_T S_c \Psi_{\text{out}}^{(2)} + [C_T + i \cos\theta S_T] \\ &\quad \times [C_c + i \cos\theta S_c] \Psi_{\text{out}}^{(1)} \},\end{aligned}\quad (22)$$

where we have used the shorthand notation $S_c \equiv \sin(\Omega_r t_c/2)$, $C_c \equiv \cos(\Omega_r t_c/2)$, $S_T \equiv \sin(\Omega_r(T - t_c)/2)$, and $C_T \equiv \cos(\Omega_r(T - t_c)/2)$. It is easy to check that, as required, (22) reduces to the standard expression [i.e., (20) with $t_c = T$] when $t_c = 0$ and setting $\Psi_{\text{out}}^{(1)} = \Psi_{\text{out}}^{(2)} = \Psi_{\text{inc}}$, i.e., in the absence of scattering.

The probability of detecting the atomic state 2 is obtained by integrating over all space. Thus,

$$\begin{aligned}P_2 &= \int_{\infty} d^3x |\langle 2|\Psi\rangle|^2 \\ &= \sin^2\theta \int_{\infty} d^3x [S_c^2 |X_T|^2 (|\Psi_{\text{inc}}|^2 + |\Psi_{\text{sc}}^{(2)}|^2 + |\Psi_{\text{int}}^{(2)}|^2) \\ &\quad + S_T^2 |X_c|^2 (|\Psi_{\text{inc}}|^2 + |\Psi_{\text{sc}}^{(1)}|^2 + |\Psi_{\text{int}}^{(1)}|^2) \\ &\quad + S_c S_T ((X_T^* X_c + X_T X_c^*) |\Psi_{\text{inc}}|^2 + X_T X_c^* \Psi_{\text{sc}}^{(2)} \Psi_{\text{sc}}^{(1)*} \\ &\quad + X_T^* X_c \Psi_{\text{sc}}^{(2)*} \Psi_{\text{sc}}^{(1)} + |\bar{\Psi}_{\text{int}}^{(2)}|^2 + |\bar{\Psi}_{\text{int}}^{(1)}|^2)],\end{aligned}\quad (23)$$

where $|\bar{\Psi}_{\text{int}}^{(2)}|^2 \equiv X_T X_c^* \Psi_{\text{inc}}^* \Psi_{\text{sc}}^{(2)} + X_T^* X_c \Psi_{\text{inc}} \Psi_{\text{sc}}^{(2)*}$, $|\bar{\Psi}_{\text{int}}^{(1)}|^2 \equiv X_T^* X_c \Psi_{\text{inc}}^* \Psi_{\text{sc}}^{(1)} + X_T X_c^* \Psi_{\text{inc}} \Psi_{\text{sc}}^{(1)*}$, and we have used other shorthand notations $X_T \equiv C_T - i \cos\theta S_T$ and $X_c \equiv C_c + i \cos\theta S_c$.

As in the Ramsey case, probability conservation requires that the last two terms in the first and second line of (23) cancel. The last four terms are identical to the last six terms of (12) with the substitutions $\phi \rightarrow \Delta\alpha$ and $1/4 \rightarrow S_c S_T A_c A_T$, where we have defined $X_T \equiv A_T e^{i\alpha_T}$, $X_c \equiv A_c e^{i\alpha_c}$, and $\Delta\alpha \equiv \alpha_T - \alpha_c$.

The result is then similar to (16),

$$\begin{aligned}P_2 &= \sin^2\theta \left\{ \sin^2\left(\frac{\Omega_r T}{2}\right) \right. \\ &\quad \left. + S_c A_T S_T A_c \frac{1}{2\pi d^2 k} \text{Re}[f_1 - f_2] \sin(\Delta\alpha) \right\},\end{aligned}\quad (24)$$

the second line of which is obviously equal to zero for $f_1 = f_2$ as required. Similarly, as required, one recovers the

standard result when $t_c = 0$ or $t_c = T$, as in that case $S_c = 0$ or $S_T = 0$, and the scattering term vanishes.

Expression (24) can be simplified when choosing T such that $\Omega_r T = \pi$, as in standard Rabi spectroscopy. We then expand for small δ , i.e., $\delta/\Omega \ll 1$. In the expansion, we keep terms up to $\mathcal{O}(\delta^2)$ in the first (classical) term of (24) and up to $\mathcal{O}(\delta)$ in the second (scattering) term, leading to

$$P_2 \simeq 1 - \frac{\delta^2}{\Omega^2} + \frac{2S_c^2 C_c^2}{kd^2} \text{Re}[f_1 - f_2] \left(\frac{2}{\sin(\Omega t_c)} \frac{\delta}{\Omega} \right), \quad (25)$$

where at the required order $\Omega \simeq \Omega_r$.

Setting the derivative $\partial P_2 / \partial \delta = 0$, we find the detuning for which P_2 is maximum, i.e., the frequency shift

$$\delta_{\max}(t_c) \simeq \sin(\Omega t_c) \frac{\pi}{2kd^2 T} \text{Re}[f_1(0) - f_2(0)], \quad (26)$$

where we recall that the result is valid for the particular case where $\Omega T \simeq \pi$, that $f_\gamma = f_\gamma(0)$ here, and that we work at first order in the interaction.

1. Discussion

Equation (26) is equivalent to the corresponding Eq. (18) for Ramsey spectroscopy, up to a factor $\frac{\pi}{2} \sin(\Omega t_c)$. This is consistent with the physical picture of the scattering provoking a transient shift of the atomic resonance frequency ω_{12} . In the Rabi case, the effect of that shift on the spectroscopy depends on the time t_c at which the collision occurs, contrary to the Ramsey sequence. As already mentioned, this can also be understood using the image of fictitious spin on a Bloch sphere or the sensitivity function formalism described, e.g., in [15]. Figure 1 shows the frequency shift for Rabi and Ramsey spectroscopy as a function of the time of the scattering t_c . For the Rabi case, the shift is zero at $t_c = 0$ and $t_c = T$ and maximum at $t_c = T/2$ as expected from the physical picture.

An interesting result is that when averaged over $T \simeq \pi/\Omega$ the Rabi frequency shift $\langle \delta_{\max}(t_c) \rangle_T$ is identical to the

FIG. 1. Frequency shift [in units of $\text{Re}[f_1 - f_2]/(kd^2 T)$] for Rabi (blue) and Ramsey (orange) spectroscopy as a function of the time t_c at which scattering takes place.

Ramsey one, i.e., $\langle \frac{\pi}{2} \sin(\Omega t_c) \rangle_T = 1$, as can be easily checked. In practice, that means that on *average* the scattering frequency shift for Ramsey or Rabi spectroscopy is the same. However, the fluctuations around that average will be larger for Rabi spectroscopy, which suggests one possible strategy for detecting scattering from light background particles, as discussed briefly in Sec. IV.

C. Summing over scatterers

In all of the above, we have considered a single scattering event during the Ramsey (Rabi) time T . The generalization to several scattering events is straightforward when assuming that the difference $\text{Re}[f_1 - f_2]$ is the same (or with non-vanishing average value) for the different scattering events. This point is explained in detail in [9] for different models. In the following, we simply assume that this quantity is the same for all the events. We further work in the limit $\Psi_{\text{sc}} \ll \Psi_{\text{inc}}$, which implies that for N_{sc} events the final wave function is $\Psi_{\text{out}} = \Psi_{\text{inc}} + N_{\text{sc}} \Psi_{\text{sc}}$, i.e., if we neglect the scattering of the scattered wave. Then all the scattering shifts simply add and the total shift is the individual one multiplied by N_{sc} . As shown in Appendix B, we have $N_{\text{sc}} \simeq 2n\pi d^2 v T$, where n is the scatterer number density.

For Ramsey spectroscopy, the total frequency shift is then obtained by multiplying (18) by N_{sc} , which gives

$$\delta_{\max} \simeq \frac{2\pi n \hbar}{\mu} \text{Re}[f_1(0) - f_2(0)]. \quad (27)$$

The resulting frequency shift is independent of the Ramsey time T and the relative velocity v between the atom and the scatterers (provided f is independent of v). Both of these effects can be understood physically: For a single scatterer, the larger the velocity, the shorter the transient shift in ω_{12} relative to the Ramsey time T , i.e., the shift decreases with T and with v [hence, T and k in the denominator of (18)]. However, the larger T and v , the larger the number of scattering events N_{sc} , and thus T and v cancel in the final result.

For Rabi spectroscopy, the calculation is somewhat more involved and requires the distribution in time of the number of scattering events per unit time, i.e., $dN(t)/dt$ to be used in (26). The resulting frequency shift is

$$\delta_{\max} \simeq \frac{\pi^2 n \hbar}{\mu N_{\text{sc}}} \text{Re}[f_1 - f_2] \int_0^T dt_c \frac{dN(t_c)}{dt_c} \sin(\Omega t_c) \quad (28)$$

for the special case where $\Omega T \simeq \pi$ as in Eq. (26).

For a uniform distribution with $dN(t)/dt = \text{const}$, the resulting frequency shift is identical to (27), as already discussed at the end of Sec. III B.

IV. DARK MATTER DETECTION STRATEGIES

In this section, we propose some strategies for detecting possible collisions between the atoms in clocks and light dark matter particles with $m_\chi \ll m_a$. The methods described here are by no means exhaustive and are only

meant as a guide to the achievable uncertainties. They may be useful for any particular DM model. A class of such models is discussed in the companion article [9], but generally offers better sensitivity for atomic magnetometers rather than atomic clocks. At present, we are not aware of any model where atomic clocks yield the best prospects for detection, but the aim of our paper is to allow confronting any DM model, present or future, to clock-based detection. In general, any DM model that allows for a nonzero value of $\text{Re}[f_1(0) - f_2(0)]$ may be testable using one of the strategies below or some adapted version of them.

The different strategies are accompanied by very rough order of magnitude sensitivity estimates based on present day technology of Cs or Rb clocks (see, e.g., [17,18]). Depending on the particular DM model, other clocks may be more adapted, so the orders of magnitude given here provide only some benchmark numbers without exhausting the full potential of atomic clock detection techniques.

The aim here is to provide rough numbers to evaluate the potential sensitivity of atomic clocks to any DM model, provided it allows the derivation of scattering amplitudes for interactions with atoms. If the resulting sensitivity to the model parameters turns out to be interesting, a more detailed and specific analysis is required, which may then lead to a dedicated experiment.

A. Measuring an offset or slow modulations

The average frequency of the clock transition will be shifted by an amount given by Eq. (27). However, to measure the shift, one needs a reference frequency that is not affected by DM collisions.

A first strategy for strong enough interactions is shielding a reference clock that operates on the same atomic transition. Depending on the DM model under study, such shielding could be provided by, e.g., the Earth itself or the Earth's atmosphere. One possible configuration would consist of two identical clocks on opposite sides of the Earth compared via satellite links. As the Earth moves through Galactic DM at $10^{-3}c$, it acts as a shield with more DM present on the "forward" side. As a result, the measured frequency difference will oscillate at the Earth rotation frequency, a signal that could be searched for in the data. Possibly, the search could be further refined and confirmed by comparing clocks on the surface and in deep underground mines.

If a large fraction of DM is absorbed in the atmosphere, then one should be able to observe a constant frequency difference between ground and space clocks operating on the same transition, e.g., in the upcoming Atomic Clock Ensemble in Space (ACES) mission that will install a high-performance Cs clock on board the International Space Station in 2020, together with high-performance microwave and optical links that will allow frequency comparison to ground clocks [19].

If the scattering amplitudes depend on the relative velocity of DM with respect to the atoms, one would also

expect to observe modulations, as the amplitude and direction of that velocity vary due to the rotation of the Earth and its orbital motion. If the interaction is spin-dependent and DM is polarized, then one would expect modulations as the orientation of the atomic spins (when using $m_F \neq 0$ states) varies with the Earth's rotation.

Finally, one can also try to compare with clocks where $\text{Re}[f_2 - f_1] = 0$. This is the natural option in the case of DM interacting with the spin of the atomic state. In this particular situation, the comparison of clocks operating at a different value of m_F would generate a modulated signal [9] (we elaborate more on this possibility below).

The standard clock transition used in ^{133}Cs clocks is the $|F=3, m_F=0\rangle \leftrightarrow |F=4, m_F=0\rangle$ hyperfine transition of the $6S_{1/2}$ ground state at ≈ 9.2 GHz. In ^{87}Rb clocks it is the $|F=1, m_F=0\rangle \leftrightarrow |F=2, m_F=0\rangle$ hyperfine transition of the $5S_{1/2}$ ground state at ≈ 6.8 GHz. In both cases, and for any of the above scenarios, a rough order of magnitude for the uncertainty in the determination of the frequency shift is $\Delta\delta \approx 10^{-5}$ rad/s [17,18]. Inserting that into (27), one can expect a sensitivity of

$$\frac{\rho_\chi \hbar}{m_\chi^2} \text{Re}[f_1(0) - f_2(0)] \approx \frac{\Delta\delta}{2\pi}, \quad (29)$$

where ρ_χ is the local density of the χ particles ($0.4 \text{ GeV}/\text{cm}^3$ is expected if all of DM is made up of χ [20]) and $\mu \simeq m_\chi$ the DM mass. Equation (29) can be used to set constraints on the difference in scattering amplitudes for the two atomic states as a function of DM mass. Note that the constraint on the scattering amplitudes becomes more stringent for smaller DM masses, i.e., the sensitivity of clocks is best for small DM masses.

Many DM models involve spin dependence and may lead to a vanishing value of $\text{Re}[f_1(0) - f_2(0)]$ for the $m_F = 0$ states generally used in clocks and assumed in the estimate (29) above. They may, however, be sensitive to transitions involving $m_F \neq 0$ Zeeman states and would lead to additional modulation effects (see above). Unfortunately, the Zeeman states are first-order sensitive to magnetic fields, which strongly degrades the uncertainty in the frequency measurement. One way around that limitation is to use simultaneously two different atoms or isotopes in the same magnetic field and form a combination that is first-order insensitive to the Zeeman effect in spite of the $m_F \neq 0$ states involved. If that combination retains sensitivity to the DM model that is tested, a precise measurement can be performed. As a particular example consider ^{133}Cs and ^{87}Rb that are measured simultaneously in the same magnetic environment in the dual fountain clock FO2 at SYRTE [17,18]. Defining ν_i^{Cs} as the frequency of the $|F=3, m_F=1\rangle \leftrightarrow |F=4, m_F=i\rangle$ Zeeman transition of ^{133}Cs and similarly for ^{87}Rb , the combination $K(\nu_1^{\text{Cs}} - \nu_0^{\text{Cs}}) - (\nu_1^{\text{Rb}} - \nu_0^{\text{Rb}})$

is free of magnetic effects to leading order with $K \approx 2$. Then we directly obtain a sensitivity of

$$\frac{\rho_\chi \hbar}{m_\chi^2} (K \text{Re}[f_1 - f_2]^{\text{Cs}} - \text{Re}[f_1 - f_2]^{\text{Rb}}) \approx \frac{\Delta\delta}{2\pi}, \quad (30)$$

which allows a meaningful search provided that the DM model does not imply a vanishing left-hand side of (30). As shown in the companion paper [9], this is quite naturally the case in plausible DM models due to the different nuclear spin of the two atoms used. For the present order of magnitude estimates, we will assume that the measurement of the Zeeman-free combination $2\nu_1^{\text{Cs}} - \nu_1^{\text{Rb}}$ can be carried out with the same uncertainty as the $m_F = 0$ spectroscopy, i.e., $\Delta\delta \approx 10^{-5}$ rad/s in (30), particularly when searching for modulations at the rotation and orbital frequencies of the Earth as expected for spin-dependent interactions.

As shown in the companion paper ([9], Fig. 8), there are viable models of DM generating a frequency difference in (30) at levels measurable by current clocks and not yet excluded by experiment. Generically, these models may be tested to higher precision by atomic comagnetometers, although the data analysis for neither technique has been performed yet. Furthermore, since the final effect depends on the effective coupling of nuclei to the dark matter field and these experiments use different atomic samples, there are nongeneric situations where the signal of magnetometers may be reduced. Given this, and the fact that atomic clocks may be sensitive to other couplings where the phase differences (27) and (28) may be substantial (e.g., momentum-dependent couplings), we believe that the search for DM signals in atomic clocks and comagnetometers should be pursued in parallel.

B. Ramsey vs Rabi spectroscopy

As shown in Sec. III, the effect of scattering may lead to frequency fluctuations that are larger for Rabi spectroscopy than Ramsey spectroscopy (cf. Fig. 1) when using the same atomic transition and the same interrogation time T . Therefore, one could try and detect DM by comparing the clock stability of the same transition in the same clock but run on a Rabi or Ramsey sequence. For a single scattering event during the interrogation time T and for a single atom, the increase in the fluctuations when using

Rabi interrogation is given by the standard deviation of the frequency shift (28) as can be also seen in Fig. 1. It is

$$\begin{aligned} \sigma &= \frac{2\pi n \hbar}{\mu} \text{Re}[f_1 - f_2] \left(\frac{1}{T} \int_0^T dt_c \left(\frac{\pi}{2} \sin(\Omega t_c) - 1 \right)^2 \right)^{1/2} \\ &= \frac{2\pi n \hbar}{\mu} \text{Re}[f_1 - f_2] \left(\frac{\pi^2}{8} - 1 \right)^{1/2}, \end{aligned} \quad (31)$$

where $T \simeq \pi/\Omega$. When averaging over N_a atoms and N_{sc} scattering events, the value of σ is decreased by a factor $1/\sqrt{N_a N_{\text{sc}}}$.

The frequency fluctuations of the best Cs or Rb atomic clocks are limited by atomic shot noise at about $\sigma_a \approx 10^{-3}$ rad/s for a single interrogation cycle of the $m_F = 0$ clock transition with $N_a \approx 5 \times 10^6$ atoms [17]. It is proportional to $1/T$ and to $1/\sqrt{N_a}$. Although this is achieved in Ramsey spectroscopy only, we will somewhat optimistically assume that similar stability is possible in Rabi spectroscopy in a dedicated experiment. The resulting sensitivity is then

$$\frac{\rho_\chi \hbar}{m_\chi^2 \sqrt{N_a N_{\text{sc}}}} \text{Re}[f_1(0) - f_2(0)] \approx \frac{\sigma_a}{2\pi(\frac{\pi^2}{8} - 1)^{1/2}}. \quad (32)$$

A rough estimation of N_{sc} is provided in Appendix B. As N_{sc} is proportional to T , and σ_a is proportional to $1/T$, the sensitivity improves with $\sqrt{T}$. On the other hand, σ_a is proportional to $1/\sqrt{N_a}$, indicating that the sensitivity is independent of the total number of atoms N_a used.

For spin-dependent $m_F \neq 0$ transitions, the stability is, in general, limited by the magnetic field instability. For example, for the $|F=3, m_F=1\rangle \leftrightarrow |F=4, m_F=1\rangle$ Cs transition, the instability is typically about 3 orders of magnitude worse than for the standard $m_F=0$ clock transition. However, one could operate the clock with much less atoms so that the dominant noise is again atomic shot noise. The overall sensitivity then remains the same as in the $m_F=0$ case, as explained in the previous paragraph. Finally, one could of course use Zeeman-free combinations of different atoms as discussed in the previous section.

C. Summary

Table I gives some numerical sensitivity estimates for the scenarios discussed above with particular assumptions as specified. We stress again that these numbers are only

TABLE I. Numerical sensitivity estimates for the different detection strategies described above. We provide the minimum scattering amplitudes (or their combination) that could be detected, together with the assumptions made.

Sec.	Min. detectable f/m	Assumptions
IV A	$\text{Re}[f_1(0) - f_2(0)] \geq 10^{-22} \left(\frac{\rho_\chi}{0.4 \text{ GeV/cm}^3} \right) (m_\chi/\text{eV})^2$	$m_F = 0$, shielding or velocity dependence
IV A	$\text{Re}[f_1(0) - f_2(0)] \geq \times 10^{-22} \left(\frac{\rho_\chi}{0.4 \text{ GeV/cm}^3} \right) (m_\chi/\text{eV})^2$	$m_F \neq 0$, Zeeman-free Rb-Cs combination, Velocity dependence or polarized DM
IV B	$\text{Re}[f_1(0) - f_2(0)] \geq 10^{-14} \left(\frac{\rho_\chi}{0.4 \text{ GeV/cm}^3} \right) (m_\chi/\text{eV})^{3/2}$	Rabi vs Ramsey, $m_\chi > 10^4$ eV
IV B	$\text{Re}[f_1(0) - f_2(0)] \geq 10^{-10} \left(\frac{\rho_\chi}{0.4 \text{ GeV/cm}^3} \right) (m_\chi/\text{eV})^{1/2}$	Rabi vs Ramsey, $m_\chi < 10^4$ eV

rough benchmarks for potential sensitivity that should allow evaluating, for any particular DM model, whether a more detailed analysis, possibly followed by a dedicated experiment, is worth pursuing.

V. CONCLUSION

We have presented a theoretical analysis of the sensitivity of atomic clocks to interaction with dark matter beyond the more usual massive scalar field models widely used in recent years [1–8]. More particularly, we focus on DM whose mass is much less than the mass of the atoms in the clocks and density such that the occupation numbers are <1 and hence cannot be treated as classical fields. We have analyzed the scattering effects on the clock frequency, showing that such effects are first order in the differential forward scattering amplitudes $f_1(0) - f_2(0)$ of the involved atomic states. The effect has a simple physical interpretation as a transient differential shift of the atomic state energies by the DM-atom interaction potential as the DM particle flies past the atom.

As a consequence, different detection strategies can be imagined, which we describe in Sec. IV. We also give some very rough order of magnitude sensitivities to the differential scattering amplitudes as a function of DM mass and density in Table I, which can be confronted to any DM model that can be expressed in terms of such differential scattering amplitudes. A number of such models are treated in much more detail in the companion paper [9]. As discussed in more detail in that paper, for certain models, atomic comagnetometers (e.g., [21,22]) can be treated in a very similar way as the clocks in this work and may provide competitive bounds as well.

We hope that our work will inspire confronting different DM models to atomic clock (or comagnetometer) data, with possibly dedicated or opportunistic experiments in the near future. Our results are also relevant for understanding the effects of other astrophysical backgrounds in atomic clocks, such as astrophysical neutrinos or gravitational waves. We hope to explore these ideas further in the future.

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APPENDIX A: PARTIAL WAVE EXPANSION

We provide a partial wave expansion of the full detection probability (15). To do so, we use

$$f_\gamma(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) [e^{2i\delta_{\gamma\ell}} - 1] P_\ell(\cos\theta), \quad (\text{A1})$$

where $\delta_{\gamma\ell}$ are the partial wave phases and $P_\ell(x)$ are the Legendre polynomials. Then the integral in F_{12} can be

evaluated over all solid angles by making use of the orthogonality $\int_{-1}^1 dx P_n(x) P_m(x) = 2\delta_{nm}/(2n+1)$, giving

$$F_{12} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) [e^{2i\delta_{1\ell}} - 1] [e^{2i\delta_{2\ell}} - 1], \quad (\text{A2})$$

which when separated into real and imaginary parts is

$$\begin{aligned} \text{Re}[F_{12}] &= \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) (\sin^2\delta_{1\ell} \sin^2\delta_{2\ell} \\ &\quad + \sin(2\delta_{1\ell}) \sin(2\delta_{2\ell})/4), \\ \text{Im}[F_{12}] &= \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) (\sin\delta_{1\ell} \sin\delta_{2\ell} \sin(\delta_{1\ell} - \delta_{2\ell})). \end{aligned} \quad (\text{A3})$$

Substituting (A3) into (15) and replacing the $f_\gamma(0)$ by (A1), one obtains after some simplification

$$\begin{aligned} P_2 &= \frac{1}{2} (1 + \cos\phi) \\ &\quad - \frac{1}{2k^2 d^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2(\delta_{1\ell} - \delta_{2\ell}) \cos\phi \\ &\quad + \frac{1}{4k^2 d^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin(2\delta_{1\ell} - 2\delta_{2\ell}) \sin\phi. \end{aligned} \quad (\text{A4})$$

When summing over all scatterers, i.e., multiplying (A4) by $N_{\text{sc}} \simeq 2n\pi d^2 vT$ (cf. Sec. III C), one obtains an expression that is equivalent to Eq. (2) of [10].

APPENDIX B: ROUGH ESTIMATE OF THE NUMBER OF SCATTERING EVENTS

The number of scatterers in time T crossing a circular area with radius r_A is $N_{r_A} = nvT\pi r_A^2$. The wave function $\Psi(t, \mathbf{x})$ describes the relative position of the two particles (see Sec. II A). We assume it is Gaussian with the corresponding probability distribution having a $1/e^2$ radius equal to d [cf. (5)]. The number of scattering events in time T is then the probability of the relative position being $\leq r_A$ multiplied by N_{r_A}

$$N_{\text{sc}} = nvT\pi r_A^2 (1 - e^{-\frac{r_A^2}{2d^2}})^{-1}, \quad (\text{B1})$$

where r_A is some characteristic length, typically the range of the interaction potential.

The relative wave function $|\Psi(t, \mathbf{x})|^2$ gives the probability of finding the two particles at a relative position $\mathbf{x}$ at time t . Therefore, its radius d is roughly equal to the radius of the larger one of the two single-particle wave functions in the lab frame.

The minimum size of the atomic wave function is given by the uncertainty principle and the atomic temperature (μK , cm/s velocity spread) at roughly $d_a \approx 10^{-8}$ m.

The minimum size of the DM wave function is also given by the uncertainty principle and the virial velocity spread of DM in the Galaxy ($\Delta v \approx 10^{-3}c$) at roughly $d_\chi \approx 10^{-4}/(m_\chi/\text{eV})$ m. So for DM masses $< 10^4$ eV it is d_χ that needs to be used in the expression for N_{sc} . The DM number density is given by $n \approx 4 \times 10^{14}/(m_\chi/\text{eV}) \text{ m}^{-3}$ for the density of $0.4 \text{ GeV}/\text{cm}^3$.

Finally, the relevant radius r_A is typically the range of the interaction potential. If this is less than the value of d , i.e., $r_A \ll 10^{-8}$ m, then Eq. (B1) reduces to

$$N_{\text{sc}} \simeq 2\pi n d_a^2 v T \approx 10^5 \left(\frac{\text{eV}}{m_\chi} \right) \left(\frac{\rho_\chi}{0.4 \text{ GeV}/\text{cm}^3} \right) \quad (\text{B2})$$

for $m_\chi > 10^4$ eV, and

$$N_{\text{sc}} \simeq 2\pi n d_\chi^2 v T \approx 10^{13} \left(\frac{\text{eV}}{m_\chi} \right)^3 \left(\frac{\rho_\chi}{0.4 \text{ GeV}/\text{cm}^3} \right) \quad (\text{B3})$$

for $m_\chi \leq 10^4$ eV.

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