Localization and reference frames in κ-Minkowski spacetime

Fedele Lizzi,[1,2,3,*] Mattia Manfredonia,[1,2,†] Flavio Mercati,[4,‡] and Timothé Poulain[5,§]

[1]*Dipartimento di Fisica "Ettore Pancini," Università di Napoli Federico II, 80126 Napoli, Italy*
[2]*INFN, 80126 Sezione di Napoli, Italy*
[3]*Departament de Física Quàntica i Astrofísica and Institut de Cíencies del Cosmos (ICCUB), Universitat de Barcelona, 08007 Barcelona, Spain*
[4]*Dipartimento di Fisica, Sapienza Università di Roma, 00185 Roma, Italy*
[5]*Laboratoire de Physique Théorique, Bâtiment 210, CNRS and Université de Paris-Sud 11, 91405 Orsay, France*

(Received 2 December 2018; published 8 April 2019)

We study the limits to the localizability of events and reference frames in the κ-Minkowski quantum spacetime. Our main tool will be a representation of the κ-Minkowski commutation relations between coordinates, and the operator and measurement theory borrowed from ordinary quantum mechanics. Spacetime coordinates are described by operators on a Hilbert space, and a complete set of commuting observables cannot contain the radial coordinate and time at the same time. The transformation between the complete sets turns out to be the Mellin transform, which allows us to discuss the localizability properties of states both in space and in time. We then discuss the transformation rules between inertial observers, which are described by the quantum κ-Poincaré group. These too are subject to limitations in the localizability of states, which impose further restrictions on the ability of an observer to localize events defined in a different observer's reference frame.

DOI: 10.1103/PhysRevD.99.085003

I. INTRODUCTION

The problem of quantum gravity suggests that the classical spacetimes at the basis of general relativity and quantum field theory may have to be replaced with quantum structures. A concrete realization of this idea is provided by noncommutative geometry. In this paper we consider the κ-Minkowski space [1–11], which is the homogeneous space of the κ-Poincaré Hopf algebra (quantum group) [12–16]. The commutation relations of the coordinate functions for κ-Minkowski are

$$[x^0, x^i] = \mathrm{i}\lambda x^i, \qquad [x^i, x^j] = 0, \qquad i, j = 1, 2, 3. \tag{1.1}$$

Often the deformation parameter λ is indicated by $\frac{1}{\kappa}$, hence the name. For us, as usual, $x^0 = ct$, where c is the speed of light, λ has the dimension of a length, and a natural scale for time is given by $\frac{\lambda}{c}$. The coordinate operators are assumed to be Hermitian $(x^\mu)^\dagger = x^\mu$. Our aim is to study the geometrical kinematics of spacetime, seen as a "quantum" object. The quantization parameter will be λ, a quantity presumably of the order of Planck length. Relations (1.1) suggest we use the theory of operators on a Hilbert space as the correct description. Since we are interested in the kinematics of spacetime alone, and will not discuss momentum for κ-Minkowski, the quantum of action $\hbar$ will not play a role, except when we reason in analogy with particle quantum mechanics.

The geometry described by Eq. (1.1) is a *noncommutative geometry*. One of the aims of this paper is to discuss what sorts of measurements of position and time are possible, and which are the states. Clearly, the presence of nontrivial commutation relations indicates that a version of Heisenberg's uncertainty relations is present:

$$\Delta x^0 \Delta x^i \geq \frac{\lambda}{2} |\langle x^i \rangle|, \tag{1.2}$$

and it will not be possible in general to localize states both in space and in time. In our treatment we will follow Dirac's correspondence principle; i.e., we will associate with the classical coordinates, and in general with the observables, operators on a Hilbert space, and consider their spectrum and eigenfunctions. We will also assume the eigenvalues to be the possible results of a measurement of the observables, and use the standard apparatus of quantum mechanics

*fedele.lizzi@na.infn.it
†mattia.manfredonia@na.infn.it
‡flavio.mercati@gmail.com
§timothe.poulain@th.u-psud.fr

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

(although, we repeat, we do not consider conjugate momenta and their commutations).

Let us make more precise what we mean by noncommutative geometry. An ordinary topological space is fully described by the algebra of continuous complex-valued functions (in the noncompact case, vanishing at infinity) on it. These form a commutative C^*-algebra, which can always be represented as operators on a Hilbert space. Further structures, such as smoothness, are encoded in other operators, such as the Dirac operator or its generalizations (for a review, see e.g., Ref. [17]). Usually, one introduces a deformation of this algebra by defining a noncommutative deformed $\star$-product so that the $\star$-commutator $[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu$ reproduces Eq. (1.1), usually based on the composition of plane waves [5,18]. There exist many versions of $\star$-products which reproduce the commutation relation (1.1); see, e.g., Refs. [10,19,20]. One of them has proved useful for the study of the quantum properties of various models of κ-Poincaré invariant scalar field theories [10,11]. Besides this, the geometric (spectral) properties, *à la* Connes, of the κ-Minkowski spacetime have been investigated in Refs. [21–24].

We are interested in the localizability of the states, i.e., the possibility to have a state of the system which describes a pointlike event, or a good approximation of it. In a noncommutative geometry, such as the quantum phase space of a particle, it may not be possible to localize points due to some version of the uncertainty principle (1.2). One might wonder whether the localizability properties of a state depend on the reference frame or not. This is not the case for the quantum phase space of one particle: the algebra of positions and momenta is invariant under classical translations and rotations. However, the algebra of Eq. (1.1) is clearly not invariant under the classical action of the Poincaré group (in particular, under translations and boosts). It is, however, invariant under a noncommutative generalization of the Poincaré group—as a matter of fact, it is *defined* as the homogeneous space of such generalization. This deformation of the Poincaré group makes the group manifold itself into a noncommutative space, and the transformation parameters relating different reference frames are subject to limitations to their localizability as well. As a consequence, different observers will not agree in general on the localizability properties of the same state.

Before we proceed with our treatment, we would like to remark that everything we do in the present paper pertains strictly to the *kinematics* of systems in κ-Minkowski spacetime: there is no dynamics. We are interested in the implications of the noncommutativity of this spacetime for the localizability of events, independently of the dynamical laws they are subject to, or which may be defining such events. In particular, seeing how, in the following, we represent certain noncommutative spacetime coordinates as differential operators, one might be tempted to interpret those as momenta conjugate to some of the coordinates, and wonder whether a phase space structure underlies our construction. We stress that the commutation relations (1.1) and the uncertainty relations (1.2) are not physically the Heisenberg algebra and Heisenberg's uncertainty principle between canonical coordinates and momenta. Indeed, the parameter λ setting the scale of noncommutativity has the dimension of a length, while the Planck constant $\hbar$ has the dimension of an action. The length λ parametrizes a noncommutative geometrical property of a noncommutative spacetime (essentially, it sets the limits to the localizability of spacetime regions), while $\hbar$ sets the limits to the localizability of *phase space* regions, a dynamical entity. The two constants may be connected—namely, λ may be Planck's length (the Compton length associated with the Planck mass)—but it may also have a different origin. If we want to find a connection between relations (1.1) and dynamics, it has to be looked for at a much deeper level, and its details (and consistency) are presently unknown. In fact, relations (1.1) are supposed to be an effective description of a quantum theory of gravity, whose ground state is not Minkowski spacetime, but rather a noncommutative deformation thereof. The limits to localizability then could be understood as the effective description of gravitational excitations that intervene when enough energy is concentrated into a small region in order to localize an event [25,26], and this would indeed be a dynamical effect. However, at the effective level, one would lose completely any trace of this dynamics, and would only be dealing with the effects that these limits to localizability have on the macroscopic dynamics. This connection with a fundamental quantum theory of gravity is, as we remarked, only crude. At the moment it can be made precise only in $2+1$ dimensions, where we have a better understanding of quantum gravity, and indeed, relations of the form (1.1) emerge when we integrate away the gravitational degrees of freedom in a model of quantum gravity coupled with matter [27,28].

If one were interested in going one step further, and discussing the dynamics of systems living in the noncommutative spacetime described by (1.1), then it would be necessary to introduce momenta and phase-space structures. It is then debatable whether it makes sense to introduce momenta that are conjugate to the coordinates of (1.1), because that would be subsuming the concept of a point particle, in a context where there are limits to the localizability of points. One way around this problem is to skip point particles altogether, and consider (quantum) field theories on noncommutative spacetimes, which then will have, in some limit, an approximate notion of particles as asymptotic solutions of the dynamics. One could then argue that it is only appropriate to talk about phase spaces at the level of field variables. These considerations, however, are beyond the scope of this work.

A. On localization and pure states

Consider first the phase space of *classical* mechanics, described through the commutative algebra of position and momentum operators q and p. Probability distributions $\rho(p, q)$ are only required to be integrable, so they belong to the function space $L^1(\mathbb{R}^{2d})$. We can represent the algebra as multiplication operators on $L^1(\mathbb{R}^{2d})$, and bounded operators will be continuous functions which vanish at infinity. However, a vector of $L^1(\mathbb{R}^{2d})$, being a function, is not a pure state, because it can always be written as the sum of two vectors obtained, e.g., by setting the original function to zero for $q^1 > 0$ or $q^1 < 0$, and adjusting the normalizations. Nevertheless, there are pure states, which can be obtained as limits: the Dirac δ's, also called evaluation maps in this case, so that if f is a function, the state is $\delta_{q_0,p_0}(f) = f(q_0, p_0)$. This is true for all commutative algebras. These states correspond also to irreducible representations and can be used to reconstruct the topology. The δ is not a vector of $L^1(\mathbb{R}^{2d})$, but is an acceptable distribution, and can be reached as a limit of normalized vectors.

When the algebra is noncommutative, this kind of pure states does not usually exist. Think, e.g., of the quantum-mechanical phase space algebra, i.e., the algebra of bounded operators of p and q, where $[p, q] = i\hbar$. In this case $\mathcal{H}$ is $L^2(\mathbb{R}^d)$, the space of wave functions, and pure states are any vector, while mixed states are mixed-density matrices. The noncommutativity of the algebra implies that there are no states which correspond to a single localized phase space point. Pure states in this case are normalized vectors of $L^2(\mathbb{R}^d)$, the "wave functions."[1] This is, of course, a manifestation of the Heisenberg uncertainty principle:

$$\Delta p \Delta q \geq \frac{\hbar}{2}, \tag{1.3}$$

which forbids the localization of phase space regions of area smaller than $\frac{\hbar}{2}$.

In what follows, we will be studying the states on the algebra (1.1), in a spirit similar to what is described here in the case of classical and quantum mechanics. In particular, we will focus on their localizability properties (i.e., to what extent one can be certain that an event took place within a certain region of an observer's coordinate system), and on the relationship between the states measured by different inertial observers. To achieve this, we will make use of specific representations of the commutation relations (1.1) as operators acting on some Hilbert space of functions.

[1]Treating the quantum phase space as a noncommutative geometry, in two dimensions, one gets the Moyal plane. It is easy to show that the $*$-product of a real function by itself is not definite positive, and therefore the evaluation maps cannot be states.

B. Outline of the paper

In Sec. II, we discuss the notions of states and events in a κ-Minkowski spacetime. To set the scene, we first present the well-known case study of ordinary quantum phase space in Sec. II A. Proceeding in analogy, we present the case of time and position in κ-Minkowski in Sec. II B, introducing the time operator and connecting its spectrum with Mellin transforms. We discuss the localization of states for an observer at the origin. In Sec. III, we briefly introduce the κ-Poincaré symmetries of our space, and in Sec. IV, we discuss the role of observer located away from the origin, using the deformed κ-Poincaré symmetry. This section is partly in $1 + 1$ dimension, where explicit representations are easier to control. A consistent part of the section is devoted to the physical interpretation of the results. A final section contains conclusions and outlook.

II. κ-MINKOWSKI SPACETIME: STATES AND EVENTS

In this section, we present a discussion on the states of the algebra of κ-Minkowski spacetime. To set the scene, however, we first present the well-known case of the single-particle quantum phase space of ordinary quantum mechanics.

A. A case study: The quantum phase space

Before we consider κ-Minkowski space, it is useful to consider the archetypical noncommutative geometry, that of the phase space of a single quantum particle. The content of this section is well known to every undergraduate student in physics, but we present it to set up a parallelism with what we will do in the next section.

A particle in three dimensions has a phase space which is a six-dimensional space spanned by the coordinates (q^i, p_i). What makes the particle quantum is promoting these coordinates to operators $(\hat{q}^i, \hat{p}_i)$ with nonvanishing commutation relations

$$[\hat{q}^i, \hat{p}_j] = \mathrm{i}\hbar\delta^i_j, \tag{2.1}$$

all other commutators being zero. The most common representations of position and momenta are as operators acting on the Hilbert space of square-integrable functions of position, $L^2(\mathbb{R}^3_q)$, as[2]

$$\hat{q}^i\psi(q) = q^i\psi(q); \qquad \hat{p}_i\psi(q) = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q). \tag{2.2}$$

We will indicate the operators with a hat ˆ. Both the $\hat{q}$'s and $\hat{p}$'s are unbounded self-adjoint operators with a dense

[2]To simplify the notation, we indicate by q and p the corresponding three-vectors, avoiding the use of a notation like $\vec{q}$.

domain. The spectrum is the real line (for each i). They have no eigenvectors but have improper eigenfunctions—namely, the eigenvalue problem is solved by a distribution. Since the $\hat{q}^i$'s commute among themselves, it is possible to have a simultaneous improper eigenvector of all of them; these are the Dirac distributions $\delta(q-\bar{q})$ for a particular position $\bar{q}$, which is a vector in $\mathbb{R}^3$. Similarly, for a particular momentum $\bar{p}$, the improper eigenfunctions of the $\hat{p}_i$ are the plane waves $e^{i\bar{p}_i q^i}$.

Formally, the eigenvalue equation

$$\partial_q \psi(q) = \alpha\psi(q), \qquad \alpha \in \mathbb{C}^3 \tag{2.3}$$

is solved by any function of the kind $e^{\alpha\cdot q}$. No function of this kind is square integrable, and therefore there are no eigenvalues or (proper) eigenfunctions. The operator $\hat{p}$ is self-adjoint on the domain of absolutely continuous functions, which is dense in $L^2(\mathbb{R}^3_q)$. One can see from Eq. (2.3) that α must be purely imaginary, $\alpha = ik$, $k \in \mathbb{R}^3$, for distributions to be well defined on the domain of self-adjointness of the operators. If α had a real part, $e^{\alpha\cdot q}$ would not be a solution of the eigenvalue problem even in the distributional sense. The improper eigenfunctions of momentum are physically interpreted as infinite plane waves of precise frequency. Since plane waves are not vectors of the Hilbert space, there is no quantum state which would give as its measure exactly the value $\hbar k$; nevertheless, we have all learned to live with this fact, and there is a well-defined sense in which we talk about "particles of momentum $\hbar k$."

The representation (2.2) is tantamount to the choice of $\hat{q}^i$ as a complete set of observables, and to the description of a quantum state as a function of positions. As usual, we interpret $|\psi(q)|^2$ for normalized functions as the probability density to find the particle at position q. The wave function, being a complex quantity, contains also the information about the density probability of the momentum operator. The connection is in the choice of the complete set of commuting observables and the Fourier transform. It is important that the Fourier transform be an isometry; i.e., it should map normalized functions of positions into normalized functions of momenta.

If we choose $\hat{p}_i$ as the complete set, then it is natural to express the state of the system as a function of the p's on which

$$\hat{q}^i\phi(p) = i\hbar\frac{\partial}{\partial p^i}\phi(p); \qquad \hat{p}_i\phi(p) = p_i\phi(p). \tag{2.4}$$

The functions $\psi(q)$ and $\phi(p)$ carry exactly the same information and are connected by a *Fourier* transform, which is but an expansion on the eigenfunction of $\hat{p}$:

$$\psi(q) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p\,\phi(p)e^{\frac{i}{\hbar}p\cdot q}. \tag{2.5}$$

The fact that $\hat{p}$ and $\hat{q}$ have been treated symmetrically (apart from signs) can be traced back to the symmetry of Eq. (2.1). If we choose a different set of commuting observables—for example, the number operator, the total angular momentum, and one of its components—the Hilbert space will look different (especially because these operators have discrete spectra).

All of this is, of course, well known. Let us now consider the case of κ-Minkowski in the same spirit.

B. Time and position of events in κ-Minkowski

In this section, we will use the techniques of the previous section. Let us begin by considering the $\hat{x}^i$'s as a complete set of observables on the Hilbert space $L^2(\mathbb{R}^3_x)$. We will represent the $\hat{x}^\mu$ as operators on this space.

1. The operator representation

The representations of the algebra generated by Eq. (1.1) are discussed in detail in Refs. [29,30]. In particular, the paper of Dabrowski and Piacitelli has been an important inspiration. In the following, we focus on the representation of time and position operators given by

$$\begin{aligned}&\hat{x}^i\psi(x) = x^i\psi(x),\\ &\hat{x}^0\psi(x) = i\lambda\left(\sum_i x^i\partial_{x^i} + \frac{3}{2}\right)\psi(x) = i\lambda\left(r\partial_r + \frac{3}{2}\right)\psi(x).\end{aligned} \tag{2.6}$$

The $\frac{3}{2}$ factor is necessary to have symmetric operators. In d dimensions, $\frac{1}{2}(r\partial_r + \partial_r r) = r\partial_r + \frac{d}{2}$. Here, $\hat{x}^0$ plays the role that $\hat{p}$ played in Sec. II A.

The representation (2.6) is far from being unique. In Ref. [31], Meljanac and Stojic have written (in the Euclidean context) the most general class of operator with the correct characteristic and have shown that they depend on two functions with some constraints. It would be interesting to consider these more general realizations, but we will not do it in this paper (see also Refs. [32,33]).

Let us discuss the relativistic invariance of the theory. The relation (2.6) appears to renounce the relativistic equivalence between space and time coordinates, because we are treating space and time differently. Indeed, in Eq. (2.6) we are representing the time coordinate as a dilation operator while the spatial coordinates are multiplicative operators. Clearly, there is a difference between $\hat{x}^0$ and $\hat{x}^i$. Already in the commutation relation (1.1), we can see that the commutator between space and time coordinates is proportional to the spatial coordinate. This clearly breaks *ordinary* Lorentz invariance, understood as a linear coordinate transformation $\hat{x}^\mu \to \Lambda^\mu{}_\nu\hat{x}^\nu$ in which $\Lambda^\mu{}_\nu$ are numerical entries of a Lorentz matrix. But the noncommutative spacetime [Eq. (2.6)] is a *quantum homogeneous space*, with a maximal degree of symmetry. Its symmetries, however, are nonordinary symmetries described by a Lie group: they are *quantum group symmetries*. As we will discuss in detail below in Sec. III, this type of symmetry requires Lorentz matrices $\Lambda^\mu{}_\nu$ and translations, which are noncommutative coordinates themselves—for example, in relations like Eq. (3.1) below. Our interpretation of the meaning of this is the following: one inertial observer, e.g., Alice, will be able to perform local experiments and establish that the limits to the localizability of events satisfy the uncertainty relations (1.2), where $\hat{x}^0$ is the direction of spacetime she calls "time." With repeated uncertainty measurements, she will be able to establish that her time direction and her spatial coordinates satisfy the commutation relations (1.1). A different observer, e.g., Bob, will perform similar measurements, perhaps on the same system studied by Alice (seen in his reference frame) and will conclude that *his own* time and space coordinates satisfy (1.1). But this is impossible if Bob's coordinates are related to Alice by linear relations like $\hat{x}'^\mu = \Lambda^\mu{}_\nu \hat{x}^\nu$. The solution to this apparent contradiction is that the transformation relating Bob's coordinates to Alice's is not a simple linear combination of Alice's coordinate operators, but it involves operator-valued Lorentz matrices (and translations), such that the transformed coordinates still satisfy the same commutation relations [Eq. (1.2)]. These commutation relations are then independent of the reference frame, and appear the same to all inertial observers, independently of their state of motion and translation. The secret to achieving this is the fact that the transformations act both on Alice's coordinates and on their *state*, which describes the event that Alice is studying, and they do so in such a way that the transformed event, when acted upon by Bob's coordinates, will show the same commutation relations, when referred to what Bob calls space and time coordinates.

The $\hat{x}^0$ operator is, up to constants, the dilation operator, and this suggests the use of a polar basis. The polar coordinates $\hat{\theta}$, $\hat{\varphi}$ do not correspond to well-defined self-adjoint operators, but we note that, defining $\hat{r}\cos\hat{\theta} = \hat{x}^3$ and $\hat{r}e^{i\hat{\varphi}} = (\hat{x}^1 + i\hat{x}^2)$, a simple calculation shows that

$$[\hat{x}^0, \cos\hat{\theta}] = [\hat{x}^0, e^{i\hat{\varphi}}] = 0, \qquad [\hat{x}^0, \hat{r}] = i\lambda\hat{r}. \tag{2.7}$$

In fact, $\hat{x}^0$ commutes with all spherical harmonics, or in general functions of $\hat{\theta}$ and $\hat{\varphi}$ independent on r. Hence, in the following we will consider the vectors of $L^2(\mathbb{R}^3_x)$ to be functions of the kind $\psi = \sum_{lm}\psi_{lm}(r)Y_{lm}(\theta,\varphi)$. Moreover, since the angular variables commute with everything, we will often concentrate on the radial parts, and consider functions of r alone. The uncertainty principle (1.2) has its polar version

$$\Delta\hat{x}^0\Delta\hat{r} \geq \frac{\lambda}{2}|\langle\hat{r}\rangle|. \tag{2.8}$$

The operator $\hat{x}^0$ is symmetric, but we should verify its self-adjointness domain. Since problems can only arise from the integration over r, we will assume that the angular degrees of freedom have been integrated out. Integrating by parts, one finds

$$\begin{aligned}\int \mathrm{d}r r^2\psi_1^* i\lambda\left(r\partial_r + \frac{3}{2}\right)\psi_2 \\ = i\lambda\int \mathrm{d}r r^2\psi_1^*\frac{3}{2}\psi_2 - \int \mathrm{d}r\, i\lambda\partial_r(r^3\psi_1^*)\psi_2 + \psi_1^* r^3\psi_2|_0^\infty.\end{aligned} \tag{2.9}$$

One can see that the boundary term vanishes if ψ_1 and ψ_2 vanish at infinity faster than $r^{-\frac{3}{2}}$, which is true for all square-integrable (according to the measure $\int \mathrm{d}r r^2$) functions. In the origin, the condition imposed is weaker than the one imposed by square integrability.

Let us now look for the spectrum and the (improper) eigenvectors. They will be the equivalent of the plane waves. Monomial in r are formal solutions of the eigenvalue problem:

$$i\lambda\left(r\partial_r + \frac{3}{2}\right)r^\alpha = i\lambda\left(\alpha + \frac{3}{2}\right)r^\alpha = \lambda_\alpha r^\alpha. \tag{2.10}$$

Therefore, eigenvalues are

$$\lambda_\alpha = i\lambda\left(\alpha + \frac{3}{2}\right). \tag{2.11}$$

These eigenvalues are real if and only if

$$\alpha = -\frac{3}{2} + i\tau, \tag{2.12}$$

with $-\infty < \tau < \infty$ a real number. In complete analogy with the momentum case previously discussed, unless the real part of α is $-3/2$, the improper eigenfunctions would not be acceptable distributions. The spectrum of the time operator is real and goes from minus infinity to plus infinity.

The distributions

$$T_\tau = \frac{r^{-\frac{3}{2}-i\tau}}{\lambda^{-i\tau}} = r^{-\frac{3}{2}}e^{-i\tau\log(\frac{r}{\lambda})} \tag{2.13}$$

are for time in classical κ-Minkowski space what plane waves are for momentum in quantum phase space. They are not physical states [vector of $L^2(\mathbb{R}^3_x)$], because their behavior at the origin and at infinity is bad, but "just about"—an epsilon slower at the origin and an epsilon faster at infinity would do, but then they would not be eigenfunctions of $\hat{x}^0$. They have a well-defined inner product with every vector in the domain of $\hat{x}^0$. The distribution has the correct dimension of a length to the power $3/2$; the factor of λ is there to avoid taking the logarithm of a dimensional quantity. Since λ is a natural scale for the model, this choice is natural, but not unique.

2. The spectrum of time and Mellin transforms

Since $\hat{x}^0$ is a self-adjoint operator, it will have a complete basis. As what matters to us is only the radial coordinates, we will leave θ and φ unchanged. We can therefore use in our set of complete observables either r or τ.

As noted earlier, the completeness of the observables implies that any function of r can be *isometrically* expanded in terms of the T_τ:

$$\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\tau r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau\log(\frac{r}{\lambda})}\tilde{\psi}(\tau,\theta,\varphi). \quad (2.14)$$

The integral above suggests $\psi(r,\theta,\varphi)$ to be some kind of integral transform of $\tilde{\psi}(\tau,\theta,\varphi)$, the analog in this context of the Fourier transform.

It is in fact a *Mellin* transform. Given a locally integrable function $f(x)$ with $x\in(0,\infty)$, the integral

$$\mathcal{M}[f,s] = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\mathrm{d}x\, x^{s-1}f(x) = \mathcal{F}(s) \quad (2.15)$$

defines the *Mellin transform* of f, when Eq. (2.15) converges. The integral in Eq. (2.15) converges for $\mathrm{Re}(s)\in(A,B)$, where A and B are real numbers such that

$$f(x) = \begin{cases} O(x^{-A-\epsilon}) & \text{as} \quad \chi\to 0_+ \\ O(e^{-B+\epsilon}) & \text{as} \quad \chi\to +\infty \end{cases}, \qquad \forall\ \epsilon>0, A<B. \quad (2.16)$$

The interval (A,B) is the so-called *strip of analyticity* of $\mathcal{M}[f,s]$. The inverse of the Mellin transform is[3]

$$\mathcal{M}^{-1}[\mathcal{F}(s),x] = \frac{1}{\mathrm{i}\sqrt{2\pi}}\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty}\mathrm{d}s\, x^{-s}\mathcal{F}(s), \quad A<C<B. \quad (2.17)$$

We require a transform which is an isometry between square-integrable functions of r with measure $\mathrm{d}rr^2$ and functions of τ. Therefore, we define

$$\begin{aligned}\psi(r,\theta,\varphi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}\tau\, r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log(\frac{r}{\lambda})}\tilde{\psi}(\tau,\theta,\varphi) \\ &= \mathcal{M}^{-1}[\tilde{\psi}(\tau,\theta,\varphi),r], \end{aligned} \quad (2.18)$$

$$\begin{aligned}\tilde{\psi}(\tau,\theta,\varphi) &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\mathrm{d}r\, r^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\tau\log(\frac{r}{\lambda})}\psi(r,\theta,\varphi) \\ &= \mathcal{M}\left[\psi(r,\theta,\varphi), -\frac{3}{2}+\mathrm{i}\tau\right]. \end{aligned} \quad (2.19)$$

Thus, $\tilde{\psi}$ is the Mellin transform of ψ, with $s=3/2+\mathrm{i}\tau$. Hereafter, we will often omit the explicit dependence on θ and φ when there is no confusion. The above-defined transformations conserve the norms:

$$\int_0^{\infty}\mathrm{d}rr^2|\psi(r)|^2 = \int_{-\infty}^{\infty}\mathrm{d}\tau|\tilde{\psi}(\tau)|^2. \quad (2.20)$$

Likewise, there is a Parseval identity:

$$\begin{aligned}\langle\psi_1|\psi_2\rangle &= \int_0^{\infty}\mathrm{d}r\, r^2\bar{\psi}_1(r)\psi_2(r) \\ &= \int_{-\infty}^{\infty}\mathrm{d}\tau\overline{\tilde{\psi}_2}(\tau)\tilde{\psi}_1(\tau) = \langle\tilde{\psi}_1|\tilde{\psi}_2\rangle. \end{aligned} \quad (2.21)$$

Assuming the usual measurement theory, we have that the average time measured by a particle in the state described by ψ with spherical symmetry is given by

$$\langle\hat{x}^0\rangle_\psi = 4\pi\int r^2\mathrm{d}r\bar{\psi}(r)\mathrm{i}\lambda\left(r\partial_r+\frac{3}{2}\right)\psi(r). \quad (2.22)$$

If ψ is real, it results in $\langle\hat{x}^0\rangle_\psi = 0$. In fact,

$$\begin{gathered}\int r^3\mathrm{d}r\bar{\psi}(r)\partial_r\psi(r) = r^3|\psi|^2|_0^{\infty} - \int r^3\mathrm{d}r\psi(r)\partial_r\bar{\psi}(r) - 3\int r^2\mathrm{d}r|\psi(r)|^2 \\ \Downarrow \\ \psi=\bar{\psi} \Rightarrow \int r^3\mathrm{d}r\bar{\psi}(r)\partial_r\psi(r) = -\frac{3}{2}\int r^2\mathrm{d}r|\psi(r)|^2, \end{gathered} \quad (2.23)$$

which implies that the two terms in Eq. (2.22) cancel each other. Hence, only complex-valued functions will have a nonzero mean value for a measurement of time. One may note the analogy with quantum phase space, where real functions have a vanishing mean value of the momentum. The probability of measuring a given value of τ is given by $|\tilde{\psi}(\tau)|^2$ for normalized functions.

To get familiar with this representation, let us give a few examples. Consider the following state, localized on a shell of radius r_0: $\psi(r) = \delta(r-r_0)/r_0^2$. Then

[3] A more detailed discussion can be found in Ref. [34].

$$\tilde{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} r_0^{-\frac{3}{2}} \left(\frac{r_0}{\lambda}\right)^{\mathrm{i}\tau} = \frac{1}{\sqrt{2\pi}} r_0^{-\frac{3}{2}} \mathrm{e}^{\mathrm{i}\tau\log(\frac{r_0}{\lambda})}, \qquad (2.24)$$

and the probability $|\psi(\tau)|^2$ does not depend on τ, which means that all values of time are equally probable, just like in quantum mechanics, where a localized particle has all values of momentum equally probable. Not surprisingly, the function $\tilde{\psi}(\tau)$ in Eq. (2.24) is not normalizable. We can regularize the delta function by approximating it with a constant function with support on a "thick spherical shell":

$$\psi(r) = \begin{cases} 0 & r < R_1 \\ \sqrt{\frac{3}{4\pi(R_2^3 - R_1^3)}} & R_1 \le r \le R_2 \\ 0 & R_2 < r \end{cases}. \qquad (2.25)$$

Its Mellin transform is

$$\tilde{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{4\pi(R_2^3 - R_1^3)}} \left(\frac{R_2^{\frac{3}{2}+\mathrm{i}\tau} - R_1^{\frac{3}{2}+\mathrm{i}\tau}}{\lambda^{\mathrm{i}\tau}}\right) \frac{2}{3+2\mathrm{i}\tau} \qquad (2.26)$$

with the probability density

$$|\tilde{\psi}(\tau)|^2 = \frac{3}{8\pi^2(R_2^3 - R_1^3)} \times \left[R_2^3 + R_1^3 - 2R_1^{\frac{3}{2}} R_2^{\frac{3}{2}} \cos\left(\tau \log\frac{R_2}{R_1}\right)\right] \frac{4}{9+4\tau^2}, \qquad (2.27)$$

which is an even function, which explains why the average value of $\hat{x}^0$ vanishes. The probability density (2.27) now is not constant: it is now peaked around $\tau = 0$, and it decreases like τ^{-2} away from the origin. In the limit $R_1 \to R_2$, the Mellin transform (2.26) tends to (be proportional to) the Mellin transform of the delta function (2.24).

It is useful to have an idea of the dimensional quantities involved. If we call t the eigenvalue of the time operator $\frac{x^0}{c}$, then $\tau = t\frac{c}{\lambda}$. Note that $\frac{c}{\lambda}$ is a dimensional quantity. If we choose for λ the Planck length, then $\frac{c}{\lambda} \sim 2 \times 10^{43}$ Hz. In other words, if $t = 1$ s, then $\tau = 2 \times 10^{43}$, an extremely large number. If t is of the order of Planck time, then $\tau \sim 1$.

C. Localized states

The aim of this section is to show that the localization properties (in space and time) of a particle at the origin are different from those away from it. To this extent, we will consider the Hilbert-space vectors for particles in the two cases and compare them and their limit to a distribution. The relation (2.8) implies a generalized uncertainty principle which will limit the simultaneous localizabilty of a particle in space and time; we wish to see its explicit consequences for localized states. We have chosen to present the results of this section using concrete examples for clarity; we do not, however, have at present a general theory encompassing all possible states. This will have to wait for further work.

1. Point localized at a finite distance from the origin

Consider a wave function localized in space in a small region of size a around a point at distance z_0 along the z axis. The wave function can have constant value inside that region, and the normalization condition fixes that value. In spherical coordinates we can write

$$\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\lambda}{2a\pi((a+z_0)^3 - z_0^3)}}, & z_0 \le r \le (z_0+a) \quad \text{and} \quad \cos\theta > 1 - \frac{a}{\lambda} \\ 0, & \text{otherwise} \end{cases}. \qquad (2.28)$$

The shape of the region we are considering is shown in Fig. 1. For any nonzero (positive) a, the wave function is normalized and is a well-defined state of the Hilbert space $L^2(\mathbb{R}^3_x)$. In the limit $a \to 0$, $\psi_{z_0,a}$ goes to a δ function localized at a distance z_0 from the origin along the positive z axis. It is possible to calculate its Mellin transform

$$\tilde{\psi}_{z_0}(\tau,\theta,\varphi) = \frac{\sqrt{3\lambda}}{\pi} \frac{(z_0+a)^{\frac{3}{2}+\mathrm{i}\tau} - z_0^{\frac{3}{2}+\mathrm{i}\tau}}{\lambda^{\mathrm{i}\tau}(3+\mathrm{i}2\tau)\sqrt{a((a+z_0)^3 - z_0^3)}} \Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right) \qquad (2.29)$$

and the associated probability density

$$|\tilde{\psi}_{z_0,a}|^2 = \frac{3\lambda}{\pi^2} \frac{z_0^3 + (z_0+a_0)^3 - 2(z_0(a+z_0))^{3/2}\cos(\tau\log(\frac{z_0}{z_0+a}))}{(4\tau^2+9)a((a+z_0)^3 - z_0^3)} \Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right)$$
$$= \left[\frac{\lambda}{4\pi^2 z_0} - \frac{\lambda a}{8(\pi^2 z_0^2)} + \mathcal{O}(a^2)\right] \Theta\left(\cos\theta - 1 + \frac{a}{\lambda}\right). \qquad (2.30)$$

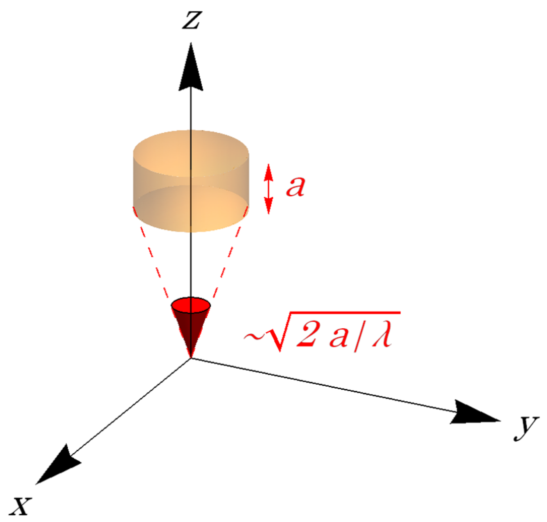

FIG. 1. The support of the wave function (2.28).

We can integrate the above function in θ, which gives a factor a/λ:

$$\int |\tilde{\psi}_{z_0,a}|^2 \sin\theta\, \mathrm{d}\theta = \frac{a}{4\pi^2 z_0} - \frac{a^2}{8\lambda(\pi^2 z_0^2)} + \mathcal{O}(a^3). \tag{2.31}$$

The Mellin transformed function has been plotted in Fig. 2. In the limit $a \to 0$, the Mellin-transformed wave function tends to a constant $\frac{\lambda}{4\pi^2 z_0}$ localized in θ in a cone of angle $\arccos(1-\frac{a}{\lambda}) - \pi/2 \sim \sqrt{\frac{2a}{\lambda}}$. The angular average tends to a constant which vanishes as $a \to 0$ (because of the normalization). This implies that in the limit, the state is not an L^2 function anymore, and is instead a function with zero scalar product with all L^2 functions.

Note also that (not surprisingly) the series expansion for a around 0, and that for z_0 around ∞ are the same:

$$\begin{aligned} |\tilde{\psi}_{z_0}|^2 &= \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}(a^3) \\ &= \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}(z_0^{-4}). \end{aligned}$$

This means that a sharp localization of a particle far away from the origin implies that the particle cannot be localized in time. And this is in accordance with the generalized uncertainty principle (2.8).

D. Points localized at the origin of space and limit to eigenstates of the origin

We now present a one-parameter family of L^2 functions which tends to a state completely localized at the spatial origin (while in time it might be completely localized around any value of τ, or it may be nonlocal). This is all allowed by the κ-Minkowski uncertainty relations (1.2), in which the presence of $\langle \hat{x}^i \rangle$ on the right-hand side suggests that, although general localized states are impossible to achieve, in the special case of states localized at the spatial origin, perfect localization should be possible. Just like delta functions and plane waves in ordinary quantum mechanics (as described in Sec. I A), it should be possible to obtain the mentioned states localized at the spatial origin as limits of normalized vectors of our Hilbert space (see Fig. 4). The key is to find functions that saturate the uncertainty bounds. In the case of the quantum phase space algebra, these are Gaussians (coherent states), as is well known. The κ-Minkowski algebra, however, is not canonical, and Gaussians are not minimal uncertainty states for this algebra. This role is played by *log-Gaussian* normalized wave functions, as plotted in Fig. 3:

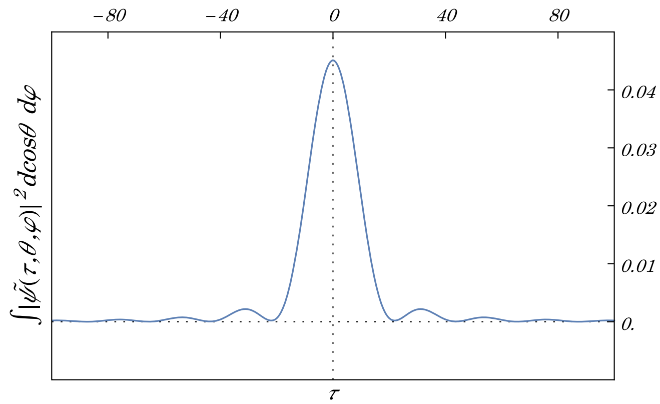

FIG. 2. The τ dependence of the Mellin transform of the wave function (2.28).

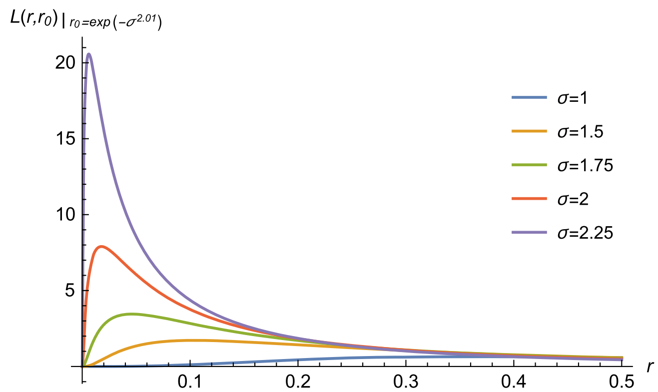

FIG. 3. The $\sigma \to \infty$ limit of $L(r, r_0)$ when $\xi = e^{-\sigma^{(2+\epsilon)}}$, for $\epsilon = 0.01$.

$$L(r, r_0) = N e^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = \frac{e^{-\left(\frac{\log(\frac{r}{r_0})}{\sigma}\right)^2} e^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}}. \tag{2.32}$$

They have a maximum in $r = r_0$, and they localize at $r = r_0$ as $\sigma \to 0$, and at $r = 0$ as $r_0 \to 0$, for any value of $\sigma \geq 0$.

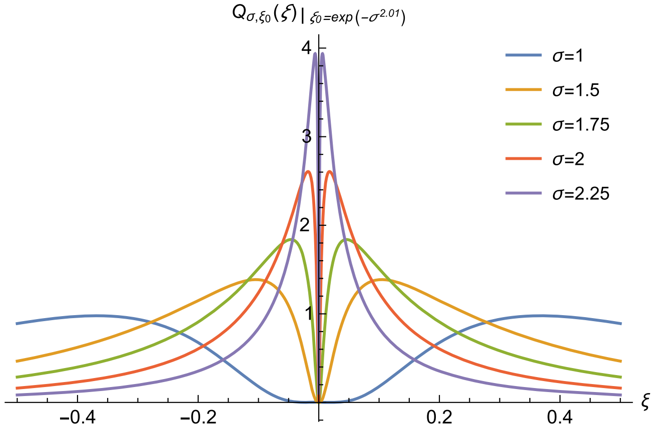

FIG. 4. The $\sigma \to \infty$ limit, or $Q_{\sigma,\xi_0}(\xi)$ when $\xi = e^{-\sigma^{(2+\epsilon)}}$, for $\epsilon = 0.01$.

The calculation of the average values of $\hat{r}^n$ is straightforward,

$$\langle \hat{r}^n \rangle_L = \mathrm{e}^{\frac{\sigma^2}{8} n(n+6)} r_0^n, \tag{2.33}$$

and shows that they all vanish for $r_0 \to 0$. In order to calculate the quantity $\langle r^n \rangle_L$, it is best to Mellin-transform; the function in τ space is remarkably simple:

$$\tilde{L}(\tau, r_0) = \frac{\sigma^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{4}\sigma^2 \tau(\tau - 3\mathrm{i})}}{2\sqrt[4]{2}\pi^{3/4}} \left(\frac{r_0}{\lambda}\right)^{\mathrm{i}\tau}, \tag{2.34}$$

The interesting fact is that

$$|\tilde{L}(\tau, r_0)|^2 = \frac{\sigma \mathrm{e}^{-\frac{\sigma^2 \tau^2}{2}}}{4\sqrt{2}\pi^{3/2}}. \tag{2.35}$$

Namely, in τ space, the probability density is a Gaussian independent on r_0. It is now trivial to see that

$$\langle (\hat{x}^0)^n \rangle_L = \frac{1}{4\pi} \left(\frac{\lambda}{\sigma}\right)^n \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases}. \tag{2.36}$$

We can see that there is a double limit,[4] $r_0 \to 0$ and $\sigma \to \infty$, which gives a state which is localized both in space (at $r = 0$) and in time. In the example above, the time localization is at $\tau = 0$, but it is possible to shift the state by multiplying the function by $r^{\mathrm{i}\tau_0}$. Moreover, one can attribute any wave function to time while still having the spatial coordinates localized at the origin, just by convoluting this with a function of τ. We have then introduced a state—which we can call the "eigenstate of the origin," and refer to as[5] $|o\rangle$—that is completely localized at the origin of spacetime and can be obtained as a limit of normalized elements of $L^2(\mathbb{R}^3_x)$. Moreover, we have a one-parameter family of states, which we indicate from now on with $|o_\tau\rangle$, which are localized at the origin of space, at a nonzero time. These states, too, can be obtained as limits of normalized elements of $L^2(\mathbb{R}^3_x)$.

III. κ-POINCARÉ SYMMETRY

In this section, we briefly introduce the deformed symmetry of our space. We still concentrate on the group rather than the algebra. We will opt for an intuitive presentation, rather than a mathematically rigorous one. In the following, to lighten the notation, we will suppress the hat symbol we have used so far to distinguish quantum operators.

A. The κ-Poincaré quantum group

The algebra (1.1) emerges as the quantum homogeneous space of a Hopf-algebra deformation of the Poincaré group, known as *κ-Poincaré* [1,12–14]. This object has historical precedence over κ-Minkowski, which was introduced by Majid and Ruegg after recognizing the "bi-cross-product" structure of the κ-Poincaré group [1]. The κ-Poincaré group is part of a very small family of possible Hopf-algebra deformations of the Poincaré group with a deformation parameter with the dimensions of (the inverse of) energy [16,35]. Moreover, under the requirement of undeformed spatial isotropy, the version of κ-Poincaré corresponding to Eq. (1.1) is singled out uniquely [16].

We introduce κ-Poincaré as the noncommutative algebra of functions $\mathcal{P}_\kappa$, generated by $\Lambda^\mu{}_\nu$ and a^μ, that leave the commutation relations (1.1) invariant under the transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1. \tag{3.1}$$

We ask that the above map, from the κ-Minkowski algebra $\mathcal{M}_\kappa$ to the tensor product algebra $\mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$, be a *left-coaction*. This entails that the map is a homomorphism with respect to the noncommutative product of $\mathcal{M}_\kappa$, hence the covariance of the commutation relations (1.1). In other words, we require that

$$[x'^\mu, x'^\nu] = \mathrm{i}\lambda(\delta^\mu{}_0 x'^\nu - \delta^\nu{}_0 x'^\mu). \tag{3.2}$$

This fixes some commutation relations between the κ-Poincaré group coordinates[6]:

[4]For example, it is sufficient to take $r_0 = e^{-\sigma^{2+\epsilon}}$ for any $\epsilon > 0$, such that all $\langle \hat{r}^n \rangle_L$ in Eq. (2.33) and all $\langle (\hat{x}^0)^n \rangle_L$ in Eq. (2.36) go to zero as $\sigma \to \infty$.

[5]While we have seen that there is a state corresponding to $|o\rangle$, there is not a normalized vector corresponding to it. Here (and in the following), we are performing the usual abuse of notation made when one uses the ket notation $|x\rangle$ in ordinary quantum mechanics.

[6]The metric used here is $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

$$[a^\mu, a^\nu] = \mathrm{i}\lambda(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu), \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0,$$
$$[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]. \tag{3.3}$$

Also, the group laws (the group product or composition law, the inverse, and the identity), here encoded with a *coproduct* $\Delta: \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa$,

$$\Delta(a^\mu) = a^\nu \otimes \Lambda^\mu{}_\nu + 1 \otimes a^\mu, \qquad \Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu, \tag{3.4}$$

an *antipode* $S: \mathcal{P}_\kappa \to \mathcal{P}_\kappa$,

$$S(a^\mu) = -a^\nu (\Lambda^{-1})^\mu{}_\nu, \qquad S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu, \tag{3.5}$$

and a *counit* $\varepsilon: \mathcal{P}_\kappa \to \mathbb{C}$,

$$\varepsilon(a^\mu) = 0, \qquad \varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu, \tag{3.6}$$

have to be homomorphisms with respect to the commutation relations (3.3). In this way we make sure that our noncommutative algebra of functions on the Poincaré group is compatible with the group structure. Finally, in order to have a proper Hopf algebra, the group maps, together with the noncommutative product, have to satisfy two identities. One is the *coassociativity* of the coproduct:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{3.7}$$

which ensures that we can combine two coproducts in either order, and the result is the same. As we said, the coproduct encodes the group combination law. Combining two coproducts means that we are making three subsequent transformations in the two possible orders, and the combined transformation is the same. This is just one of the axioms of ordinary Lie groups: the associativity of the group product. The other axioms are the existence of the identity (ensured by the existence of the counit map), and the relation between the group inverse and the identity. This is now encoded in the *Hopf identity:*

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon, \tag{3.8}$$

which ensures that the antipode provides a left- and right-inverse for the coproduct [$\mu: \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa \to \mathcal{P}_\kappa$ stands for the (noncommutative) multiplication map].

B. A representation of the κ-Poincaré quantum group

The operators $\Lambda^\mu{}_\nu$ in Eq. (3.3) should not be understood as 16 independent operators, but rather as 16 redundant functions satisfying the relations $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, which reduce the independent components to six. Since all components of $\Lambda^\mu{}_\nu$ commute with each other, the standard representation theory of the Lorentz group applies, and we can write

$$\Lambda^\mu{}_\nu = (\exp\omega)^\mu{}_\nu, \qquad \omega^\mu{}_\rho \eta^{\rho\nu} = -\omega^\nu{}_\rho \eta^{\rho\mu}, \tag{3.9}$$

where the (Lorentzian) antisymmetry relation above reduces the independent components of $\omega^\mu{}_\nu$ to six. These components commute with each other,

$$[\omega^\mu{}_\nu, \omega^\rho{}_\sigma] = 0, \tag{3.10}$$

but they do *not* commute with a^μ. The structure of the commutation relations (3.3) suggests the representation of the a^μ's as vector fields,

$$a^\rho = -\mathrm{i}\lambda[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]\frac{\partial}{\partial \Lambda^\mu{}_\nu}, \tag{3.11}$$

and the exponential relation between $\omega^\mu{}_\nu$ and $\Lambda^\mu{}_\nu$ implies $\frac{\partial}{\partial \Lambda^\mu{}_\nu} = \Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha}$, which allows us to write the above representation as vector fields acting on the space of $\omega^\mu{}_\nu$ coordinates:

$$a^\rho = -i\lambda[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha}. \tag{3.12}$$

Interestingly, the above vector fields already "know" about the commutation relations between the translation operators. In fact, the commutator of two of these vector fields acts on wave functions of $\omega^\mu{}_\nu$ as the Lie bracket between the vector fields, and computing this Lie bracket yields $[a^\mu, a^\nu] = \mathrm{i}\lambda(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu)$.

We found a representation of the κ-Poincaré algebra, in which $\Lambda^\mu{}_\nu$ is represented as multiplication operators on wave functions of $\omega^\mu{}_\nu$,

$$\Lambda^\mu{}_\nu \phi(\omega) = (\exp\omega)^\mu{}_\nu \phi(\omega), \tag{3.13}$$

while the translation operators act as vector fields,

$$a^\rho \phi(\omega) = -i\lambda[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha \frac{\partial \phi(\omega)}{\partial \omega^\mu{}_\alpha}. \tag{3.14}$$

The wave functions can be taken as belonging to $L^2(SO(3,1))$, with the scalar product constructed, e.g., with the Haar measure on the Lorentz group.

Unfortunately, the representation we just considered is not good enough: it is not *faithful*. In fact, we can write combinations of the $\Lambda^\mu{}_\nu$ and a^ρ operators that are represented as the null operator:

$$
\begin{aligned}
&\eta_{\rho\mu}(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)a^\rho\triangleright\phi(\omega)\\
&\quad=[\eta_{\rho\beta}\Lambda^\rho{}_\nu(\delta^\kappa{}_0\Lambda^\beta{}_\kappa-\delta^\beta{}_0)(\delta^\sigma{}_0\Lambda^\mu{}_\sigma-\delta^\mu{}_0)+(\delta^\sigma{}_0\Lambda^\mu{}_\sigma-\delta^\mu{}_0)(\delta^0{}_\sigma\Lambda^\sigma{}_\nu-\delta^0{}_\nu)]\Lambda^\nu{}_\alpha\frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha}\\
&\quad=(\delta^\sigma{}_0\Lambda^\mu{}_\sigma-\delta^\mu{}_0)[\eta_{\rho\beta}\Lambda^\rho{}_\nu(\delta^\kappa{}_0\Lambda^\beta{}_\kappa-\delta^\beta{}_0)+(\delta^0{}_\sigma\Lambda^\sigma{}_\nu-\delta^0{}_\nu)]\Lambda^\nu{}_\alpha\frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha}\\
&\quad=(\delta^\sigma{}_0\Lambda^\mu{}_\sigma-\delta^\mu{}_0)[(\eta_{00}-1)\delta^0{}_\nu+(1-\eta_{00})\Lambda^0{}_\nu]\Lambda^\nu{}_\alpha\frac{\partial\phi(\omega)}{\partial\omega^\mu{}_\alpha}=0,
\end{aligned}\tag{3.15}
$$

where the last line is zero because $\eta_{00}=+1$ in our convention. The operator

$$\eta_{\rho\mu}(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)a^\rho \tag{3.16}$$

is nontrivial and, at least in order to admit a good classical limit, some of its expectation values should not be vanishing. We conclude that the representation (3.14) is not faithful, and it needs to be enlarged. The simplest way to do this is to write a direct sum of representations: the above one and the (at this point familiar) representation (2.6) of κ-Minkowski coordinates, which reproduces the commutation rules between translation operators, but commutes with Lorentz transformations. The Hilbert space now has to be enlarged with three additional coordinates, $q^i\in\mathbb{R}$, $i=1,2,3$, so it is $L^2(SO(3,1)\times\mathbb{R}^3)$; the Lorentz matrices still represent as multiplicative operators (3.13); and the translation operators are represented as follows:

$$a^\rho=-\mathrm{i}\frac{\lambda}{2}[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)\Lambda^\rho{}_\nu+(\Lambda^\sigma{}_\nu\delta^0{}_\sigma-\delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha}+\mathrm{i}\frac{\lambda}{2}\left(\delta^\rho{}_0q^i\frac{\partial}{\partial q^i}+\delta^\rho{}_iq^i\right)+\frac{1}{2}\mathrm{H.c.},\tag{3.17}$$

where by "H.c." we mean the Hermitian conjugate of the previous expression. This ensures that the operator is self-adjoint on some domain. The final form of our representation is

$$
\begin{aligned}
a^\rho\phi(q,\omega)&=\mathrm{i}\lambda\delta^\rho{}_0\left(\frac{3}{2}\phi(q,\omega)+q^i\frac{\partial\phi(q,\omega)}{\partial q^i}\right)+\delta^\rho{}_iq^i\phi(q,\omega)\\
&\quad-\mathrm{i}\lambda:[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)\Lambda^\rho{}_\nu+(\Lambda^\sigma{}_\nu\delta^0{}_\sigma-\delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha}:\phi(q,\omega),\\
\Lambda^\mu{}_\nu\phi(q,\omega)&=\Lambda^\mu{}_\nu(\omega)\phi(\omega)=(\exp\omega)^\mu{}_\nu\phi(q,\omega);
\end{aligned}\tag{3.18}
$$

that is,

$$
\begin{aligned}
a^\rho\phi(q,\omega)&=\mathrm{i}\lambda\delta^\rho{}_0\left(\frac{3}{2}\phi(q,\omega)+q^i\frac{\partial\phi(q,\omega)}{\partial q^i}\right)+\delta^\mu{}_iq^i\phi(q,\omega)\\
&\quad-\frac{\mathrm{i}\lambda}{2}[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)\Lambda^\rho{}_\nu+(\Lambda^\sigma{}_\nu\delta^0{}_\sigma-\delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha\frac{\partial\phi(q,\omega)}{\partial\omega^\mu{}_\alpha}\\
&\quad-\frac{\mathrm{i}\lambda}{2}\phi(q,\omega)\frac{\partial}{\partial\Lambda^\mu{}_\nu}[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0-\delta^\mu{}_0)\Lambda^\rho{}_\nu+(\Lambda^\sigma{}_\nu\delta^0{}_\sigma-\delta^0{}_\nu)\eta^{\mu\rho}],\\
\Lambda^\mu{}_\nu\phi(q,\omega)&=\Lambda^\mu{}_\nu(\omega)\phi(\omega)=(\exp\omega)^\mu{}_\nu\phi(q,\omega).
\end{aligned}\tag{3.19}
$$

It is trivial to check that, since the derivatives with respect to $\omega^\mu{}_\nu$ commute with the functions of q^i, and the derivatives with respect to q^i commute with the functions of $\omega^\mu{}_\nu$, the representation splits into a direct sum of representations, and the commutation relations between a^μ's are satisfied.

The representation (3.19) is complicated, and its explicit functional form depends on the coordinate system on the Lorentz group we choose. In two spacetime dimensions, the situation is greatly simplified by the fact that the Lorentz group is one dimensional, and everything can be made very explicit. In the next section, we will repeat the steps that led us to introduce the representation (3.19) in the $(1+1)$-dimensional case, a useful exercise both for pedagogical reasons, and in order to have an example that can be worked out explicitly. This will be useful later.

C. The representation of κ-Poincaré in $1+1$ dimensions

The great advantage of working in $1+1$ dimensions is that we have an explicit (and simple) coordinatization of the Lorentz group:

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi, \qquad \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi \tag{3.20}$$

in this parametrization. The commutation relations of κ-Poincaré (3.3) take the form

$$\begin{aligned} &[a^0, a^1] = \mathrm{i}\lambda a^1, \qquad [\cosh\xi, a^0] = -\mathrm{i}\lambda \sinh^2\xi, \\ &[\cosh\xi, a^1] = -\mathrm{i}\lambda(\cosh\xi - 1)\sinh\xi, \\ &[\sinh\xi, a^0] = -\mathrm{i}\lambda\sinh\xi\cosh\xi, \\ &[\sinh\xi, a^1] = -\mathrm{i}\lambda(\cosh\xi - 1)\cosh\xi, \end{aligned} \tag{3.21}$$

which can be simplified to

$$[a^0,a^1] = \mathrm{i}\lambda a^1, \quad [\xi,a^0] = -\mathrm{i}\lambda\sinh\xi, \quad [\xi,a^1] = \mathrm{i}\lambda(1-\cosh\xi). \tag{3.22}$$

It is evident that a^0 and a^1 act on ξ like vector fields:

$$a^0 = \mathrm{i}\lambda\sinh\xi\frac{\partial}{\partial\xi}, \qquad a^1 = \mathrm{i}\lambda(\cosh\xi - 1)\frac{\partial}{\partial\xi}. \tag{3.23}$$

The above representation would be acceptable, as it reproduces the $[a^0, a^1]$ commutation relations. In this case, we can easily show this explicitly:

$$\begin{aligned} [a^0,a^1] &= -\lambda^2\left[\sinh\xi\frac{\partial}{\partial\xi}(\cosh\xi-1) - (\cosh\xi-1)\frac{\partial}{\partial\xi}\sinh\xi\right]\frac{\partial}{\partial\xi} \\ &= -\lambda^2[\sinh^2\xi - (\cosh\xi-1)\cosh\xi]\frac{\partial}{\partial\xi} \\ &= -\lambda^2(\cosh\xi-1)\frac{\partial}{\partial\xi} = \mathrm{i}\lambda a^1. \end{aligned} \tag{3.24}$$

As before, this representation cannot be faithful, because the operator

$$\begin{aligned} &(\cosh\xi - 1)a^0 - \sinh\xi a^1 \\ &\quad = -\mathrm{i}\lambda(\cosh\xi - 1) \\ &\qquad \times \sinh\xi\frac{\partial}{\partial\xi} + \mathrm{i}\lambda\sinh\xi(\cosh\xi - 1)\frac{\partial}{\partial\xi} = 0, \end{aligned} \tag{3.25}$$

which is the $(1+1)$-dimensional version of Eq. (3.16), is represented as the null operator. Again, it is sufficient to add to the above representation the familiar representation of the κ-Minkowski algebra in $1+1$ dimensions:

$$a^0 = \mathrm{i}\lambda q\frac{\partial}{\partial q} + \mathrm{i}\lambda\sinh\xi\frac{\partial}{\partial\xi}, \qquad a^1 = q + \mathrm{i}\lambda(\cosh\xi - 1)\frac{\partial}{\partial\xi}. \tag{3.26}$$

The two parts commute with each other and separately satisfy the commutation relations and the Jacobi identity, and therefore they provide a good representation of our algebra on the Hilbert space $L^2(SO(1,1)\times\mathbb{R}) \sim L^2(\mathbb{R}^2)$ of square-integrable functions of ξ and q. This representation is not self-adjoint, but it can be made so by Weyl-ordering it:

$$\begin{aligned} a^0 &= \frac{\mathrm{i}\lambda}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right) + \frac{\mathrm{i}\lambda}{2}\left(\sinh\xi\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi}\sinh\xi\right), \\ a^1 &= q + \frac{\mathrm{i}\lambda}{2}\left((\cosh\xi - 1)\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi}(\cosh\xi - 1)\right), \end{aligned} \tag{3.27}$$

which can be written

$$\begin{aligned} a^0 &= \mathrm{i}\lambda\left(\frac{1}{2} + q\frac{\partial}{\partial q}\right) + \mathrm{i}\lambda\left(\frac{1}{2}\cosh\xi + \sinh\xi\frac{\partial}{\partial\xi}\right), \\ a^1 &= q + \mathrm{i}\lambda\left(\frac{1}{2}\sinh\xi + (\cosh\xi - 1)\frac{\partial}{\partial\xi}\right). \end{aligned} \tag{3.28}$$

It is easy to check that the above reproduces the commutation relations (3.3).

D. From κ-Poincaré to κ-Minkowski

We can now make it precise, within the framework of the representations we introduced for κ-Minkowski and κ-Poincaré, in which sense the κ-Minkowski noncommutative spacetime is the quantum homogeneous space obtained by quotienting the κ-Poincaré quantum group by the Lorentz group. The idea is that there are enough states in the representation of κ-Poincaré that we can reproduce any vector in the Hilbert space of the representation of κ-Minkowski [i.e., $L^2(\mathbb{R})$] as an appropriate limit of vectors belonging to the representation of κ-Poincaré [$L^2(SO(3,1)\times\mathbb{R})$], in which the wave function on the Lorentz group becomes localized at the identity (in the limit).

We illustrate this explicitly in the $(1+1)$-dimensional case. Consider the representation (3.28): if it is restricted to act on functions which are localized around $\xi \sim 0$, we can expand all the functions of ξ on the right-hand side around $\xi = 0$, and at first order in ξ, the representation looks like

$$\begin{aligned} a^0 &= \mathrm{i}\lambda\left(\frac{1}{2} + q\frac{\partial}{\partial q}\right) + \mathrm{i}\lambda\left(\frac{1}{2} + \xi\frac{\partial}{\partial\xi}\right) + \mathcal{O}(\xi^2), \\ a^1 &= q + \frac{\mathrm{i}\lambda}{2}\xi + \mathcal{O}(\xi^2). \end{aligned} \tag{3.29}$$

This reveals the underlying structure: on wave functions sufficiently localized around $\xi = 0$, the representation looks like two copies of the κ-Poincaré representation (2.6), one acting on q and one on ξ (the only difference being that the ξ part of a^1 is multiplied by $\mathrm{i}\lambda/2$, which is irrelevant in our discussion). We are interested in defining a sequence of wave functions that localize at $\xi = 0$, maintaining the freedom in the choice of the q dependence. The form (3.29) suggests taking nonentangled states:

$$\psi_{\sigma,\xi_0}(q,\xi) = f(q)Q_{\sigma,\xi_0}(\xi), \tag{3.30}$$

where Q_{σ,ξ_0} is a log-Gaussian similar to (2.32):

$$Q_{\sigma,\xi_0}(\xi) = \frac{e^{-\frac{\sigma^2}{16}}}{\sqrt{\sqrt{2\pi}\xi_0\sigma}} e^{-\left(\frac{\log(\xi^2)-\log(\xi_0^2)}{2\sigma}\right)^2}, \tag{3.31}$$

which is a function which attributes to ξ^n a zero expectation value for n positive and odd, and $e^{\frac{1}{8}n(n+2)\sigma^2}$ for n positive and even.

All the expectation values of $(a^\mu)^n$ tend to

$$\langle\psi_{\sigma,\xi_0}|(a^\mu)^n|\psi_{\sigma,\xi_0}\rangle \xrightarrow[\xi_0\to 0,\sigma\to\infty]{} \langle f|(x^\mu)^n|f\rangle$$
$$= \int \mathrm{d}q\bar{f}(q)(x^\mu)^n f(q), \tag{3.32}$$

where $x^1 = q$ and $x^0 = \mathrm{i}\lambda(\frac{1}{2} + q\frac{\partial}{\partial q})$ is the familiar κ-Poincaré representation, and the limits $\xi_0 \to 0$, $\sigma \to \infty$ are taken in such a way[7] that $e^{c\sigma^2}\xi_0 \to 0$ for all $c > 0$.

This is the fundamental content of the statement that κ-Minkowski is the homogeneous space of κ-Poincaré: we can reproduce any vector f in $L^2(\mathbb{R}_x)$ by taking the limit of the product of f with the log-Gaussian (3.31), and all expectation values of powers of translation operators will coincide with the expectation values of the corresponding powers of x^μ operators on the vector f. We reproduce all we know of κ-Minkowski by taking particular states on κ-Poincaré and "silencing" the boost part localizing around $\xi = 0$.

IV. OBSERVERS AND REFERENCE FRAMES

We are representing the algebra (1.1) as generators of operators on the Hilbert space of functions of position. This algebra and its states represent the position in κ-Minkowski. We have to specify, however, the *observer* making the observations, and we have been implicitly considering an observer located at the origin. In order to change the observer, usually a Poincaré transformation is performed. But in our case, the symmetry is the quantum κ-Poincaré. Accordingly, it will be impossible to locate the position of the transformed observer, since translations do not commute. In the spirit of this paper, we will consider the algebra generated by the a's and Λ's, and associate with a translated and Lorentz-transformed observer a state of this algebra. We first consider the observer located at the origin, which is reached via the identity transformation.

A. The identity transformation state

Looking at the commutation relations (3.3), it is possible to define a state $|o\rangle_{\mathcal{P}}$ of $\mathcal{P}_\kappa$ with the property

$${}_{\mathcal{P}}\langle o|f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f), \tag{4.1}$$

where $f(a,\Lambda)$ is a generic element of the κ-Poincaré algebra (i.e., a generic noncommutative function of translations and Lorentz-transformation matrices), and ε is the counit of the κ-Poincaré algebra defined in Eq. (3.8). In other words, the state returns the value of the function on the identity transformation.

We interpret this state in the enlarged algebra as describing the Poincaré transformation between two coincident observers—i.e., between an observer and a second one located at the origin of the coordinate system of the first observer. It is not difficult to see, looking at Eq. (3.3), that the state is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in κ-Minkowski spacetime.

Note also that all the Λ's commute among themselves, and will therefore have common eigenvectors. It is clear from this that the localizability uncertainties have to do with translations, not Lorentz transformations.

This state can easily be obtained as a limit of vectors in the Hilbert space. It suffices to take a succession of functions which converge to a δ as far as a^μ and the diagonal elements of $\Lambda^\mu{}_\nu$ are concerned, and to zero for the off-diagonal elements of the Λ's.

B. Physical interpretation

We propose an interpretation for the operators x^μ we have been using all along, and the operators x'^μ that appear in Eq. (3.1): they are the coordinate systems associated with two inertial observers—say, Alice and Bob—which are translated and in relative motion with respect to each other. A spacetime event (i.e., the clicking of a particle detector) seen by Alice will be described by the expectation value of its coordinates $\langle x^\mu\rangle$; their variance $\langle(x^\mu - \langle x^\mu\rangle)^2\rangle$, which measures how localized it is; the skewness $\langle(x^\mu - \langle x^\mu\rangle)^3\rangle$ measuring how asymmetric it is around the expectation value; and all higher moments $\langle(x^\mu - \langle x^\mu\rangle)^n\rangle$, which describe in increasingly finer details the distribution of probability where the event can be localized. The same event, seen by Bob, will be described by a tower of moments of the transformed coordinate operators: $\langle(x'^\mu - \langle x'^\mu\rangle)^n\rangle$, which are in general different from

[7] As before, we could take $\xi_0 = e^{-\sigma^{(2+\epsilon)}}$ and get everything we want from the $\sigma \to \infty$ limit.

Alice's, unless the transformation that connects Alice and Bob is the identity described in Sec. IV A.

What does it mean to take expectation values of the operators x'^μ and their powers? x'^μ belongs to the tensor-product algebra $\mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$. We can obtain a representation for this algebra by taking the direct sum of the representation (3.19) of $\mathcal{P}_\kappa$ with the representation (2.6) of $\mathcal{M}_\kappa$. Clearly the x^μ algebra (Alice's coordinates) is lifted to elements of the kind $\mathbb{1} \otimes \mathcal{M}_\kappa$, where the identity of $\mathcal{P}_\kappa$ is given by $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, $a^\mu = 0$. The representation of $\mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$ will act on the Hilbert space $\mathcal{H}_\mathcal{P}\times L^2(\mathbb{R}^3_x) \sim L^2(SO(3,1)\times \mathbb{R}^3_q \times \mathbb{R}^3_x)$, in the following way:

$$\begin{aligned} x'^\mu f(\omega,q,x) &= \mathrm{i}\lambda \Lambda^\mu{}_\nu(\omega)\left[\delta^\nu{}_0\left(\frac{3}{2}f(\omega,q,x) + x^i\frac{\partial f(\omega,q,x)}{\partial x^i}\right) + \delta^\nu{}_i x^i f(\omega,q,x)\right] \\ &\quad + \mathrm{i}\lambda\delta^\mu{}_0\left(\frac{3}{2}f(\omega,q,x) + q^i\frac{\partial f(\omega,q,x)}{\partial q^i}\right) + \delta^\mu{}_i q^i f(\omega,q,x) \\ &\quad - \frac{\mathrm{i}\lambda}{2}[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]\Lambda^\nu{}_\alpha\frac{\partial f(\omega,q,x)}{\partial\omega^\mu{}_\alpha} \\ &\quad - \frac{\mathrm{i}\lambda}{2}f(\omega,q,x)\frac{\partial}{\partial\Lambda^\mu{}_\nu}[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0)\Lambda^\rho{}_\nu + (\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu)\eta^{\mu\rho}]. \end{aligned}$$

In the $(1+1)$-dimensional case, we have a more intelligible expression for our representation:

$$\begin{aligned} x'^0 f(\xi,q^1,x^1) &= \mathrm{i}\lambda\cosh\xi\left(\frac{1}{2}f + x^1\frac{\partial f}{\partial x^1}\right) + \sinh\xi x^1 f + \mathrm{i}\lambda\left(\frac{1}{2}f + q^1\frac{\partial f}{\partial q^1}\right) + \mathrm{i}\lambda\left(\frac{1}{2}\cosh\xi f + \sinh\xi\frac{\partial f}{\partial\xi}\right), \\ x'^1 f(\xi,q^1,x^1) &= \mathrm{i}\lambda\sinh\xi\left(\frac{1}{2}f + x^1\frac{\partial f}{\partial x^1}\right) + \cosh\xi x^1 f + q^1 f + \mathrm{i}\lambda\left(\frac{1}{2}\sinh\xi f + (\cosh\xi - 1)\frac{\partial f}{\partial\xi}\right). \end{aligned} \tag{4.2}$$

Our Hilbert space will admit nonentangled states, i.e., objects of the kind

$$|g,\psi\rangle = |g\rangle\otimes|\psi\rangle, \tag{4.3}$$

with $|g\rangle \in \mathcal{H}_\mathcal{P} = L^2[SO(3,1)]\times\mathbb{R}^3_q$ and $|\psi\rangle \in L^2(\mathbb{R}^3)$. It represents the state of the coordinates x'^μ of a Poincaré-transformed observer. If we want to calculate the expectation values of the coordinates of the transformed observer, we have to do the following:

$$\langle x'^\mu\rangle = \langle g|\otimes\langle\psi|(\Lambda^\mu{}_\nu\otimes x^\nu + a^\mu\otimes 1)|g\rangle\otimes|\psi\rangle = \langle g|\Lambda^\mu{}_\nu|g\rangle\langle\psi|x^\nu|\psi\rangle + \langle g|a^\mu|g\rangle, \tag{4.4}$$

where we use the normalization condition $\langle\psi|\psi\rangle = 1$. Similarly, one can calculate all the higher momenta of the coordinates as

$$\langle x'^{\mu_1}\ldots x'^{\mu_n}\rangle = \langle g|\otimes\langle\psi|(x'^{\mu_1}\ldots x'^{\mu_n})|g\rangle\otimes|\psi\rangle. \tag{4.5}$$

Examining again the relations (4.2), notice how the coordinates x'^μ of a Poincaré-transformed observer (e.g., Bob) act on states describing an event in this observer's reference frame with *two copies* of the now-familiar representation (2.6). One acts on the state of the original observer (Alice), which, if the state is a product state as in Eq. (4.3), is written as a function of $x^i \in \mathbb{R}^3$. The other acts on the state of the Poincaré group coordinates, which, in the product state case, is written as a function of $q^i \in \mathbb{R}^3$ and $\Lambda^\mu{}_\nu \in SO(3,1)$.

C. Transforming the states

We will now derive some general results regarding the properties of these transformed states, which do not depend on a representation except for assuming the existence of the identity state.

1. Poincaré-transforming the origin state

Consider the following state, which Poincaré-tranforms the origin:

$$|g,0\rangle = |g\rangle\otimes|o\rangle. \tag{4.6}$$

If we want to know what the *Poincaré-transformed observer* measures with the coordinates centered on her reference frame, we have to use the operators $x'^\mu = \Lambda^\mu{}_\nu\otimes x^\nu + a^\mu\otimes 1$, which act on $L^2(\mathbb{R}^3_x)\times\mathcal{H}_\mathcal{P}$. Their expectation values on our transformed state are

$$\langle x'^\mu \rangle = \langle g| \otimes \langle o|x'^\mu|g\rangle \otimes |o\rangle = \langle g|\Lambda^\mu{}_\nu|g\rangle\langle o|x^\nu|o\rangle + \langle g|a^\mu|g\rangle\langle o|o\rangle, \quad (4.7)$$

the state $|o\rangle$ is normalized so that $\langle o|o\rangle = 1$, and moreover, the expectation value of x^μ on $|o\rangle$ is, as we have shown before, zero. We get

$$\langle x'^\mu \rangle = \langle g|a^\mu|g\rangle, \quad (4.8)$$

and the expectation value of the transformed coordinates is completely determined by the expectation value of the translation operators on the chosen κ-Poincaré state. This is natural: the different observers are comparing positions, not directions. Now consider, more in general, an arbitrary monomial in the transformed coordinates: $x'^{\mu_1}x'^{\mu_2}\ldots x'^{\mu_n}$. Its expectation value on $|g\rangle \otimes |o\rangle$ is

$$\begin{aligned}\langle x'^{\mu_1}\ldots x'^{\mu_n}\rangle &= \langle g| \otimes \langle o|(a^{\mu_1} \otimes 1 + \Lambda^{\mu_1}{}_{\nu_1} \otimes x^{\nu_1})\cdots(a^{\mu_n} \otimes 1 + \Lambda^{\mu_n}{}_{\nu_n} \otimes x^{\mu_n})|g\rangle \otimes |o\rangle \\ &= \langle g|a^{\mu_1}\cdots a^{\mu_n}|g\rangle\langle o|o\rangle + \langle g|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)|g\rangle\langle o|x^\nu|o\rangle + \cdots \\ &\quad + \langle g|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\nu_2}(a,\Lambda)|g\rangle\langle o|x^{\nu_1}x^{\nu_2}|o\rangle + \langle g|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(a,\Lambda)|g\rangle\langle o|x^{\nu_1}\ldots x^{\nu_n}|o\rangle, \end{aligned} \quad (4.9)$$

and since we showed that $|o\rangle$ is such that $\langle o|x^{\nu_1}\ldots x^{\nu_n}|o\rangle = 0\ \forall\ n$,

$$\langle x'^{\mu_1}\ldots x'^{\mu_n}\rangle = \langle g|a^{\mu_1}\ldots a^{\mu_n}|g\rangle\langle o|o\rangle = \langle g|a^{\mu_1}\ldots a^{\mu_n}|g\rangle. \quad (4.10)$$

Therefore, Poincaré-transforming the origin state $|o\rangle$ by a state with wave function $|g\rangle$ in the representation of the κ-Poincaré algebra a^μ, $\Lambda^\mu{}_\nu$, the resulting state will assign, to all polynomials in the transformed coordinates $x'^\mu = a^\mu \otimes 1 + \Lambda^\mu{}_\nu \otimes x^\nu$, the same expectation value as that assigned by $|g\rangle$ to the corresponding polynomials in a^μ. In other words, the state of x'^μ is identical to the state of a^μ. So, e.g., all uncertainty in the transformed coordinates $\Delta x'^\mu$ is introduced by the uncertainty in the state of the translation operator, Δa^μ. Let us stress again the fact that, although the new observer is measuring these expectation values, since the a^μ closes a noncommutative algebra, we cannot know, with absolute precision in time and direction, *where* the new observer is, unless she has just time-translated the origin, i.e., $|g\rangle = |o_{a^0}\rangle_{\mathcal{P}}$.

2. Poincaré-transforming an arbitrary state with the identity transformation

A second useful result we present now is the effect of the identity transformation on an arbitrary state of the κ-Minkowski coordinates. We start from an arbitrary element of the Hilbert space of our representation of the κ-Minkowski algebra, $|\psi\rangle \in L^2(\mathbb{R}^3_x)$. We transform the state as in Eq. (4.3) but use the identity state $|o\rangle_{\mathcal{P}}$ in place of the generic $|g\rangle$. In the transformed state $|o\rangle_{\mathcal{P}} \otimes |\psi\rangle$, all of the expectation values of the polynomials in the transformed coordinates x'^μ take the form

$$\begin{aligned}\langle x'^{\mu_1}\ldots x'^{\mu_n}\rangle &= {}_{\mathcal{P}}\langle o| \otimes \langle\psi|(a^{\mu_1} \otimes 1 + \Lambda^{\mu_1}{}_{\nu_1} \otimes x^{\nu_1})\cdots(a^{\mu_n} \otimes 1 + \Lambda^{\mu_n}{}_{\nu_n} \otimes x^{\mu_n})|o\rangle_{\mathcal{P}} \otimes |\psi\rangle \\ &= {}_{\mathcal{P}}\langle o|a^{\mu_1}\ldots a^{\mu_n}|o\rangle_{\mathcal{P}}\langle\psi|\psi\rangle + {}_{\mathcal{P}}\langle o|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)|o\rangle_{\mathcal{P}}\langle\psi|x^\nu|\psi\rangle + {}_{\mathcal{P}}\langle o|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\nu_2}(a,\Lambda)|o\rangle_{\mathcal{P}}\langle\psi|x^{\nu_1}x^{\nu_2}|\psi\rangle \\ &\quad + \cdots + {}_{\mathcal{P}}\langle o|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(a,\Lambda)|o\rangle_{\mathcal{P}}\langle\psi|x^{\nu_1}\ldots x^{\nu_n}|\psi\rangle \\ &= \epsilon(a^{\mu_1}\ldots a^{\mu_n})\langle\psi|\psi\rangle + \epsilon[O^{\mu_1\cdots\mu_n}_{\nu}(a,\Lambda)]\langle\psi|x^\nu|\psi\rangle + \epsilon[\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\nu_2}]\langle\psi\rangle x^{\nu_1}x^{\nu_2}|\psi\rangle \\ &\quad + \cdots + \epsilon[\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(a,\Lambda)]\langle\psi|x^{\nu_1}\ldots x^{\nu_n}|\psi\rangle. \end{aligned} \quad (4.11)$$

Now, the algebra elements $O^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_m}(a,\Lambda)$ are monomials in a^μ, $\Lambda^\mu{}_\nu$, without a particular ordering. However, we know that the mth element contains m Lorentz matrix generators and $n-m$ translation generators. Using the homomorphism property of the counit map ϵ, and the fact that $\epsilon(a^\mu) = 0$, $\epsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu$, we can prove that

$$\epsilon[O^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_m}(a,\Lambda)] = 0 \quad \text{unless } m = n \quad (4.12)$$

and

$$\epsilon[O^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(a,\Lambda)] = \delta^{\mu_1}{}_{\nu_1}\ldots\delta^{\mu_n}_{\nu_n}. \quad (4.13)$$

We conclude that

$${}_{\mathcal{P}}\langle o| \otimes \langle\psi|x'^{\mu_1}\ldots x'^{\mu_n}|o\rangle_{\mathcal{P}} \otimes |\psi\rangle = \langle\psi|x^{\mu_1}\ldots x^{\mu_n}|\psi\rangle, \quad (4.14)$$

i.e., the identity transformation does not change any expectation value—the original observer [who uses the coordinate operators x^μ and the Hilbert space $L^2(\mathbb{R}^3_x)$], and the transformed one [using the coordinates operators x'^μ

and the Hilbert space $\mathbb{H}_{\mathcal{P}} \otimes L^2(\mathbb{R}^3_x)$] agree on all measurements if the state of $\mathcal{H}_{\mathcal{P}}$ that defines the transformation is $|o\rangle_{\mathcal{P}}$.

3. κ-Poincaré and coordinate uncertainty

Consider a generic transformation of a generic state: $|\psi\rangle \to |g\rangle \otimes |\psi\rangle$. We want to study the relationship between the uncertainty in the transformed coordinates $\Delta x'^{\mu}$ and that in the original ones Δx^{μ}.

First, the simplest example: a pure translation, $x'^{\mu} = 1 \otimes x^{\mu} + a^{\mu} \otimes 1$. Calculating the variance of x^{μ},

$$\begin{aligned}\Delta(x'^{\mu})^2 &= \langle (x'^{\mu})^2\rangle - \langle x'^{\mu}\rangle^2 = \langle (x^{\mu})^2 + (a^{\mu})^2 + x^{\mu}a^{\mu} + a^{\mu}x^{\mu}\rangle \\ &\quad - \langle x^{\mu}\rangle^2 - \langle a^{\mu}\rangle^2 - 2\langle x^{\mu}\rangle\langle a^{\mu}\rangle \\ &= \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2 + 2\mathrm{cov}(x^{\mu}, a^{\mu}). \end{aligned} \tag{4.15}$$

The covariance between a^{μ} and x^{μ} is zero, because they belong to different sides of the tensor product:

$$\begin{aligned} 2\mathrm{cov}(x^{\mu}, a^{\mu}) &= \langle g| \otimes \langle\psi|(x^{\mu}a^{\mu} + a^{\mu}x^{\mu})|g\rangle \otimes |\psi\rangle \\ &\quad - 2\langle\psi|x^{\mu}|\psi\rangle\langle g|a^{\mu}|g\rangle \\ &= \langle\psi|x^{\mu}|\psi\rangle\langle g|a^{\mu}|g\rangle + \langle g|a^{\mu}|g\rangle\langle\psi|x^{\mu}|\psi\rangle \\ &\quad - 2\langle\psi|x^{\mu}|\psi\rangle\langle g|a^{\mu}|g\rangle = 0. \end{aligned} \tag{4.16}$$

We conclude that

$$\Delta(x'^{\mu})^2 = \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2 \geq \Delta(x^{\mu})^2, \tag{4.17}$$

i.e., a translation can only increase the uncertainty of the coordinates. One is simply adding uncorrelated variables, and their uncertainties get square-summed.[8]

Performing a translation results in an increase of the uncertainty in the coordinates, unless the translation parameter has zero uncertainty. This happens only in the cases of the identity transformation or of a *purely temporal translation*, which can have zero uncertainty in all of the a^{μ}'s, in analogy with the discussion in the Introduction. We have the nice result that the uncertainty does not depend on time translations.

Consider a state which looks uncertain to the observer Alice located at the origin. One could think that there would be another observer, Bob, translated with respect to Alice, such that this same state is perfectly localized for him. One could naively think to start (in 1+1D) from the state $\psi(x^1)$ for x^1, and then make a translation with the wave function $\psi(-q^1)$ where ψ is the same function. One would think that the translated state is localized at the origin. Relation (4.17) shows that this is impossible. Calculating the expectation value of $(x'^1)^n = (x^1 + a^1)^n$, a Newton binomial sum of this kind is obtained:

$$\begin{aligned}\langle (x^1 + a^1)^n\rangle &= \sum_{m=0}^{n} \binom{n}{m} \langle\psi(x^1)|(x^1)^{n-m}|\psi(x^1)\rangle\langle\psi(-q)|(a^1)^m|\psi(-q)\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} \langle\psi|(x^1)^{n-m}|\psi\rangle\langle\psi|(-x^1)^m|\psi\rangle. \end{aligned} \tag{4.18}$$

The above expression can never be zero. For example, for $n = 2$,

$$\langle (x^1 + a^1)^2\rangle = \langle (x^1)^2\rangle + 2\langle x^1 a^1\rangle + \langle (a^1)^2\rangle = 2\langle (x^1)^2\rangle - 2\langle x^1\rangle^2 = 2\Delta(x^1)^2. \tag{4.19}$$

The variance doubles; it does not go to zero.

The process of translating a state and then "undoing" it with a change of observer does not lead to an identification of states. Of course, the symmetry between Alice and Bob is preserved—each has a set of states which is isomorphic, but the quantum nature of the transformation implies that this set of states are not transformed into each other by a translation.

Now, let us consider general κ-Poincaré transformations—for example, the transformation of the spatial coordinate in $1 + 1$ dimensions,

$$x'^1 = \cosh\xi \otimes x^1 + \sinh\xi \otimes x^0 + a^1 \otimes 1, \tag{4.20}$$

calculating the difference between its variance and the variance of x^1:

[8] Notice that this conclusion is a consequence of the fact that we assumed that transformed states are product states $|g\rangle \otimes |\psi\rangle$. If we allowed for entanglement between the transformation part $|g\rangle$ and the state $|\psi\rangle$ describing the event in the initial reference frame, we would have opened the possibility of reducing the uncertainty of x^{μ} with a translation. This, however, conflicts with the basic physical intuition that the relationship between inertial observers should be independent of the state of the system that the observers are studying.

$$\begin{aligned}\Delta(x'^1)^2 = {} & \Delta(x^1)^2 + \Delta(a^1)^2 + \langle x^1\rangle^2\Delta(\cosh\xi)^2 + \langle x^0\rangle^2\Delta(\sinh\xi)^2 \\ & + \langle\sinh\xi\rangle^2\Delta(x^0)^2 + \Delta(\sinh\xi)^2\Delta(x^0)^2 + \langle\cosh\xi\rangle^2\Delta(x^1)^2 + \Delta(\cosh\xi)^2\Delta(x^1)^2 \\ & + 2\mathrm{cov}(x^1,x^0)\langle\cosh\xi\rangle\langle\sinh\xi\rangle + 2\mathrm{cov}(a^1,\sinh\xi)\langle x^0\rangle + 2\mathrm{cov}(a^1,\cosh\xi)\langle x^1\rangle \\ & + 2\mathrm{cov}(\cosh\xi,\sinh\xi)(\mathrm{cov}(x^0,x^1) + \langle x^0\rangle\langle x^1\rangle) - \Delta(x^1)^2. \end{aligned} \tag{4.21}$$

The above expression can be rewritten as

$$\begin{aligned}\Delta(x'^1)^2 = {} & \Delta(x^1)^2 + \langle\sinh^2\xi\rangle(\Delta(x^0)^2 + \Delta(x^1)^2) \\ & + \Delta[\cosh\xi]^2\langle x^1\rangle^2 + \Delta[\sinh\xi]^2\langle x^0\rangle^2 + 2\mathrm{cov}(\cosh\xi,\sinh\xi)\langle x^0\rangle\langle x^1\rangle \\ & + \Delta[a^1]^2 + 2\mathrm{cov}(\cosh\xi,a^1)\langle x^1\rangle + 2\mathrm{cov}(\sinh\xi,a^1)\langle x^0\rangle \\ & + 2\langle\cosh\xi\sinh\xi\rangle\mathrm{cov}(x^0,x^1); \end{aligned} \tag{4.22}$$

the second and third lines above can be rewritten as the squared uncertainty of the operator $a^1 + \sinh\xi\langle x^0\rangle + \cosh\xi\langle x^1\rangle$, which is positive, and we get

$$\begin{aligned}\Delta(x'^1)^2 - \Delta(x^1)^2 = {} & \Delta[a^1 + \sinh\xi\langle x^0\rangle + \cosh\xi\langle x^1\rangle]^2 \\ & + \langle\sinh^2\xi\rangle(\Delta(x^0)^2 + \Delta(x^1)^2) + 2\langle\cosh\xi\sinh\xi\rangle\mathrm{cov}(x^0,x^1). \end{aligned} \tag{4.23}$$

Now, we assume that $\langle x^0\rangle = \langle x^1\rangle$ so that the first term reduces to the uncertainty of a^1. Moreover, we rewrite the covariance of x^0 and x^1 as $2\mathrm{cov}(x^0,x^1) = \Delta(x^0+x^1)^2 - \Delta(x^0)^2 - \Delta(x^1)^2$:

$$\Delta(x'^1)^2 - \Delta(x^1)^2 = \Delta(a^1)^2 + (\langle\sinh^2\xi\rangle - \langle\cosh\xi\sinh\rangle)(\Delta(x^0)^2 + \Delta(x^1)^2) + \langle\cosh\xi\sinh\xi\rangle\Delta(x^0+x^1)^2. \tag{4.24}$$

It is easy to prove that

$$\langle\sinh^2\xi\rangle + \langle\cosh\xi\sinh\xi\rangle = \frac{1}{2}(\langle e^{2\xi}\rangle - 1), \tag{4.25}$$

so that

$$\Delta(x'^1)^2 - \Delta(x^1)^2 = \Delta(a^1)^2 + \frac{1}{2}(\langle e^{2\xi}\rangle - 1)(\Delta(x^0)^2 + \Delta(x^1)^2) + \langle\cosh\xi\sinh\xi\rangle\Delta(x^0+x^1)^2. \tag{4.26}$$

One linear combination of x^0 and x^1 can always be made arbitrarily localized, so we can make $\Delta(x^0+x^1)^2$ arbitrarily small. The same, of course, holds for $\Delta(a^1)^2$, without putting any constraint on the other quantities except the uncertainty of ξ, which, however, does not limit much our ability to manipulate the state in order to adjust the values of $\langle e^{2\xi}\rangle$ and $\langle\cosh\xi\sinh\xi\rangle$. It does not take long to convince oneself that we can concoct a state such that $\langle e^{2\xi}\rangle < 1$ (e.g., it is sufficient that the wave function over ξ be supported on the $\xi < 0$ region) and $\langle\cosh\xi\sinh\xi\rangle$ is $\mathcal{O}(1)$. Then the expression above will be dominated by $\frac{1}{2}(\langle e^{2\xi}\rangle - 1)(\Delta(x^0)^2 + \Delta(x^1)^2)$, which is negative.

We proved that the variances of x^μ can only increase after a pure translation, but under particular circumstances, they can decrease after a Poincaré transformation. In particular, states with a zero expectation value of x^μ, such that the uncertainty of (x^0+x^1) is sufficiently small, can reduce their uncertainty if we perform a κ-Poincaré transformation with sufficiently localized translation and a Lorentz transformation such that $\langle e^{2\xi}\rangle < 1$ and $\langle\cosh\xi\sinh\xi\rangle = \mathcal{O}(1)$. We postpone to further work the study of the physical consequences of this observation.

V. CONCLUSIONS AND OUTLOOK

In this paper, we discussed a way to look at the κ-Minkowski quantum space with the tools of the algebra of operators and the theory of measurement initially developed for ordinary quantum mechanics. This enables a coherent way to look at states, localization, and transformations. The picture of quantum κ-Minkowski spacetime which emerges is, in our opinion, quite fascinating. There are no absolutely localized points, but it is nevertheless possible to find states which approximately localize. The role of Fourier transformation from position to momentum is here played by Mellin transforms, which connect time with (radial) position. We also laid out the foundations of a discussion of the deformed transformations of this space. This is an aspect which will deserve further scrutiny for a complete understanding of transformation theory. In this paper, we presented a series of basic results valid in $3+1$ dimensions, and we discussed in quantitative details the $(1+1)$-dimensional case. Generalizing all of our results to the $(3+1)$-dimensional case seems technically more complicated, but there do not seem to be any conceptual obstacles. A possible future development could be addressing the fact that we used a particular representation of the operators, while others are possible. It should be investigated if the alternatives are, at least qualitatively, similar.

Finally, the next challenge: we considered a regime which is not very natural in physics—namely, we considered the effects of a quantum spacetime for which the noncommutativity parameter of space λ is nonzero, while $\hbar$ can be ignored. Bringing $\hbar$ back into the picture would require us to consider momenta (either in the form of wave modes in a field-theoretical setting, or as a quantity of motion of particles). The space of momenta in κ-Minkowski is curved [36–39], and this has led us to introduce the principle of relative locality [38,40,41]. The relationship between the relaxations of locality that we found in the present paper and those introduced by relative locality is an interesting open issue, worth exploring.

ACKNOWLEDGMENTS

The authors acknowledge the COST action QSPACE, and in particular the Short Term Scientific Missions which enabled two visits of T. P. in Napoli. F. L. and M. M. acknowledge the support of the INFN Iniziativa Specifica GeoSymQFT; F. L. acknowledges the Spanish MINECO under Project No. MDM-2014-0369 of ICCUB (Unidad de Excelencia "Maria de Maeztu"). F. M. has received funding from the European Union's research and innovation program under a Marie Skłodowska-Curie grant through the INdAM-COFUND-2012 program of the Italian Institue of High Mathematics (INdAM). We thank Florio M. Ciaglia, Max Kurkov, Marco Laudato, Patrizia Vitale, and Jean-Christophe Wallet for discussions.

[1] S. Majid and H. Ruegg, Bicrossproduct structure of κ-Poincare group and non-commutative geometry, Phys. Lett. B **334**, 348 (1994).

[2] S. Majid, Algebraic approach to quantum gravity II: Noncommutative spacetime, in *Approaches to Quantum Gravity, Toward a New Understanding of Space, Time and Matter*, edited by D. Oriti (Cambridge University Press, Cambridge, England, 2009), pp. 466–492.

[3] P. Kosinski, P. Maslanka, J. Lukierski, and A. Sitarz, Generalized κ deformations and deformed relativistic scalar fields on noncommutative Minkowski space, in *Mexico City 2002, Topics in Mathematical Physics, General Relativity and Cosmology* (Cinvestav, Mexico City, 2003), pp. 255–277.

[4] P. Kosiński, J. Lukierski, and P. Maślanka, Local $D=4$ field theory on κ-deformed Minkowski space, Phys. Rev. D **62**, 025004 (2000).

[5] A. Agostini, F. Lizzi, and A. Zampini, Generalized Weyl systems and κ-Minkowski space, Mod. Phys. Lett. A **17**, 2105 (2002).

[6] M. Dimitrijevic, L. Jonke, L. Moller, E. Tsouchnika, J. Wess, and M. Wohlgenannt, Deformed field theory on κ space-time, Eur. Phys. J. C **31**, 129 (2003).

[7] M. Arzano, J. Kowalski-Glikman, and A. Walkus, Lorentz invariant field theory on κ-Minkowski space, Classical Quantum Gravity **27**, 025012 (2010).

[8] F. Mercati, Quantum κ-deformed differential geometry and field theory, Int. J. Mod. Phys. D **25**, 1650053 (2016).

[9] F. Mercati and M. Sergola, Light cone in a quantum spacetime, Phys. Lett. B **787**, 105 (2018).

[10] T. Poulain and J.-C. Wallet, κ-Poincaré invariant quantum field theories with Kubo-Martin-Schwinger weight, Phys. Rev. D **98**, 025002 (2018).

[11] T. Poulain and J.-C. Wallet, κ-Poincaré invariant orientable field theories at 1-loop: Scale-invariant couplings, J. High Energy Phys. 01 (2019) 064.

[12] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoy, q-deformation of Poincaré algebra, Phys. Lett. B **264**, 331 (1991).

[13] J. Lukierski, A. Nowicki, and H. Ruegg, New quantum Poincaré algebra and κ-deformed field theory, Phys. Lett. B **293**, 344 (1992).

[14] S. Zakrzewski, Quantum Poincaré Group related to the κ-Poincare algebra, J. Phys. A **27**, 2075 (1994).

[15] J. Lukierski and H. Ruegg, Quantum κ-Poincaré in any dimension, Phys. Lett. B **329**, 189 (1994).

[16] F. Mercati and M. Sergola, Physical constraints on quantum deformations of spacetime symmetries, Nucl. Phys. **B933**, 320 (2018).

[17] P. Aschieri, M. Dimitrijevi, P. Kulish, F. Lizzi, and J. Wess, Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory, Lect. Notes Phys. **774**, 89 (2009).

[18] T. Jurić, T. Poulain, and J.-C. Wallet, Involutive representations of coordinate algebras and quantum spaces, J. High Energy Phys. 07 (2017) 116.

[19] B. Durhuus and A. Sitarz, Star product realizations of κ-Minkowski space, J. Noncommut. Geom. **7**, 605 (2013).

[20] S. Meljanac and S. Kresic-Juric, Generalized κ-deformed spaces, $\star$-products, and their realizations, J. Phys. A **41**, 235203 (2008).

[21] F. D'Andrea, Spectral geometry of κ-Minkowski space, J. Math. Phys. (N.Y.) **47**, 062105 (2006).

[22] B. Iochum, T. Masson, T. Schucker, and A. Sitarz, Compact κ-deformation and spectral triples, Rep. Math. Phys. **68**, 37 (2011).

[23] B. Iochum, T. Masson, and A. Sitarz, κ-deformation, affine group and spectral triples, Banach Cent. Pub. **98**, 261 (2012).

[24] M. Matassa, A modular spectral triple for κ-Minkowski space, J. Geom. Phys. **76**, 136 (2014).

[25] S. Doplicher, K. Fredenhagen, and J. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, Commun. Math. Phys. **172**, 187 (1995).

[26] C. Alden Mead, Possible connection between gravitation and fundamental length, Phys. Rev. **135**, B849 (1964).

[27] L. Freidel and E. R. Livine, 3D Quantum Gravity and Effective Non-Commutative Quantum Field Theory, Phys. Rev. Lett. **96**, 221301 (2006).

[28] H.-J. Matschull and M. Welling, Quantum mechanics of a point particle in $2+1$ dimensional gravity, Classical Quantum Gravity **15**, 2981 (1998).

[29] A. Agostini, κ-Minkowski representations on Hilbert spaces, J. Math. Phys. (N.Y.) **48**, 052305 (2007).

[30] L. Dabrowski and G. Piacitelli, Canonical κ-Minkowski spacetime, arXiv:1004.5091.

[31] S. Meljanac and M. Stojic, New realizations of Lie algebra κ-deformed Euclidean space, Eur. Phys. J. C **47**, 531 (2006).

[32] S. Meljanac, D. Meljanac, F. Mercati, and D. Pikutić, Noncommutative spaces and Poincaré symmetry, Phys. Lett. B **766**, 181 (2017).

[33] N. Loret, S. Meljanac, F. Mercati, and D. Pikutić, Vectorlike deformations of relativistic quantum phase-space and relativistic kinematics, Int. J. Mod. Phys. D **26**, 1750123 (2017).

[34] R. Paris and D. Kaminski, Asymptotics and Mellin-Barnes integrals, *Encyclopedia of Mathematics and its Applications, No. 85* (Cambridge University Press, Cambridge, England, 2001).

[35] S. Zakrzewski, Quantum Poincaré Group related to the κ-Poincaré algebra, Commun. Math. Phys. **185**, 285 (1997).

[36] J. Kowalski-Glikman and S. Nowak, Doubly special relativity and de Sitter space, Classical Quantum Gravity **20**, 4799 (2003).

[37] M. Arzano, Anatomy of a deformed symmetry: Field quantization on curved momentum space, Phys. Rev. D **83**, 025025 (2011).

[38] G. Gubitosi and F. Mercati, Relative locality in κ-Poincaré, Classical Quantum Gravity **30**, 145002 (2013).

[39] F. Mercati and M. Sergola, Pauli-Jordan function and scalar field quantization in κ-Minkowski noncommutative spacetime, Phys. Rev. D **98**, 045017 (2018).

[40] G. Amelino-Camelia, M. Matassa, F. Mercati, and G. Rosati, Taming Nonlocality in Theories with Planck-Scale Deformed Lorentz Symmetry, Phys. Rev. Lett. **106**, 071301 (2011).

[41] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, The principle of relative locality, Phys. Rev. D **84**, 084010 (2011).