Correction to the energy spectrum of 1S_0 heavy quarkonia due to two-gluon annihilation effect

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In this paper, the nonrelativistic asymptotic behavior of the transition $q\bar{q} \rightarrow 2g \rightarrow q\bar{q}$ in the 1S_0 channel is discussed. Different from the usual calculation which expands the physical amplitude around the threshold of a quark and antiquark pair, we take the quarks and antiquarks off shell and expand the expression on the three-momentum of the quarks and antiquarks. We calculate the results to order 6. The imaginary part of the results after applying the on-shell conditions can reproduce the nonrelativistic QCD (NRQCD) results in the leading order of α_s . The real part of the results can be used to estimate the mass shift of the 1S_0 heavy quark and antiquark system due to the $2g$ annihilation effect. The results can also be used to estimate the energy shifts of positronium due to two-photon annihilation.

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I. INTRODUCTION

The energy spectrum of quarkonia is a basic topic in the study of strong interaction. Many phenomenological methods have been applied to study this topic for a long time, such as the quark model [1], QCD sum rules [2], Dyson-Schwinger equation and Bethe-Salpeter equation [3], the unitary chiral model [4], etc. Due to the asymptotic freedom of QCD and the large masses of heavy quarks (c quark and b quark), heavy quarkonia provide a special window to study QCD since both the perturbative behavior and the nonperturbative behavior show their properties in such systems. For example, the spectrum of heavy quarkonia shows the nonperturbative confinement behavior, and the decay or the production of heavy quarkonia can be well described by the effective theory of nonrelativistic QCD (NRQCD) [5].

Experimentally, since 2003, the Belle [6], CDF [7], D0 [8], BABAR [9], Cleo-C [10], LHCb [11], BES [12], and CMS [13] collaborations have reported many new charmoniumlike states which cannot be well understood even at the phenomenological level. It is found that the experimental results and the theoretical calculations are compatible for the states whose masses are below the threshold of a D or D^* pair, while the situation is perplexing when the masses of the states are above the threshold of a D or D^* pair. This attracts a lot of interest from both theoretical and

experimental physicists and numerous studies have tried to understand these states. Details of these discussions can be found in recent reviews [14]. Physically, when the masses of the states lie above the threshold of a D or D^* pair, some new decay channels are opened. The interactions related with these decay channels not only result in the decay widths, but also shift the masses. A natural and important question is how large these effects are and how to estimate them. The effects on the energy spectrum of charmonium due to decay channels (annihilation effects) have been discussed in the original quark model [1] and the subsequent work [15] in a phenomenological way. The contributions are expected to be about 20 MeV in the former and are about 600 MeV in the later. In this paper, we plan to give a rigorous study on a similar effect due to two-gluon annihilation which is much clearer than the $D\bar{D}$, $D\bar{D}^*$, $D^*\bar{D}^*$ decay channels and can be well described in the pure perturbative frame.

In the quasipotential method, the effective potential of a nonrelativistic system is usually extracted from the matching conditions between the quantum mechanics and the full quantum field theory by expanding the physical amplitudes on the threshold order by order. Similar matching conditions are usually also applied between the effective theory and the full theory such as NRQCD and QCD. These matching conditions mean that the two theories are equivalent in the physical scattering region. To study the corrections to energy shifts of bound states due to two-gluon annihilation, in this paper we do not match the on-shell amplitudes but take the quark and antiquark pairs off shell and then expand the interaction kernel order by order. The gauge invariance of such expansion is also checked in a manifest way.

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FIG. 1. The Feynman diagrams for $q\bar{q} \rightarrow 2g \rightarrow q\bar{q}$. FIG. 2. The Feynman diagrams for $q\bar{q} \rightarrow 2g \rightarrow q\bar{q}$ in the $^{2s+1}L_J$ channel.

We organize the paper as follows. In Sec. II we give an introduction to the basic formula, in Sec. III we describe the method of our calculation and present the analytic result for the coefficients to order 6 after the nonrelativistic expansion, in Sec. IV we discuss the relation between our results and those given in NRQCD in the leading order of α_s (LO- α_s), and in Sec. V we estimate the corrections to the mass shifts numerically and give our conclusion.

II. BASIC FORMULA

In the perturbation theory, the Feynman diagrams for the transition of a heavy quark and antiquark pair to a heavy quark and antiquark pair via two-gluon annihilation $q\bar{q} \rightarrow 2g \rightarrow q\bar{q}$ are shown in Fig. 1.

When one takes the quark and antiquark pairs off shell, the corresponding Green function is part of the interaction kernel of the Bethe-Salpeter equation which plays a similar role with the potential in nonrelativistic quantum mechanics. The direct calculation of the corresponding Green function is a little tedious and there is no analytic expression in the full complex plane of momenta. Two methods are usually used to simplify the calculation. The first one is to project the quark and antiquark pairs to a special $^{2s+1}L_J$ state which can be described as Fig. 2. The second one is to study the asymptotic behavior of the Green function by expanding the expression on some small variables.

In the center of mass frame, the momenta can be chosen as follows:

$$\begin{aligned} p_1 &\triangleq \frac{1}{2}P + p_i, & p_2 &\triangleq \frac{1}{2}P - p_i, \\ p_3 &\triangleq \frac{1}{2}P + p_f, & p_4 &\triangleq \frac{1}{2}P - p_f, \end{aligned} \quad (1)$$

with $P \triangleq (\sqrt{s}, 0, 0, 0)$. Generally, there are six independent Lorentz invariant variables in the interaction kernel: P^2 , $P \cdot p_i$, $P \cdot p_f$, p_i^2 , p_f^2 , and $p_i \cdot p_f$. For simplification, we limit our discussion in the region with $P \cdot p_i = P \cdot p_f = 0$. The choice of this region corresponds to assuming the instantaneous approximation in the Bethe-Salpeter equation, which means that the contributions from the relative energy of the quark and antiquark pair in the bound states are neglected. The properties $P \cdot p_i = P \cdot p_f = 0$ naturally appear when the initial and final quark and antiquark pairs are taken on shell. This choice of the momenta gives us four independent Lorentz invariant variables and we can define $p_i \triangleq (0, \mathbf{p}_i)$, $p_f \triangleq (0, \mathbf{p}_f)$. This is different from the on-shell case, where there are only two independent Lorentz invariant variables $p_i \cdot p_f$ and P^2 . For the heavy quark and antiquark pair, we can take p_i^2/m^2 , p_f^2/m^2 , and $p_i \cdot p_f/m^2$ as small variables and leave $s \triangleq P^2$ as a free variable at first.

To project the quark and antiquark pairs to the 1S_0 state, we use the same project matrix as in the on-shell case [16–18] and have the following:

$$\begin{aligned} \sum \bar{v}(p_2, s_2) T u(p_1, s_1) < \frac{1}{2} s_1; \frac{1}{2} s_2 | 00 > \triangleq \text{Tr}[T \cdot \Pi_{\text{ini}}], \\ \sum \bar{u}(p_3, s_3) T v(p_4, s_4) < \frac{1}{2} s_3; \frac{1}{2} s_4 | 00 > \triangleq \text{Tr}[T \cdot \Pi_{\text{fin}}], \end{aligned} \quad (2)$$

where the Clebsch-Gordan coefficients are the standard ones as in Ref. [17] and the Dirac spinors are normalized as $u^+ u = v^+ v = 1$, whose expressions are written as

$$\begin{aligned} u(p_1, s_1) &\triangleq \frac{\not{p}_1 + m}{\sqrt{E_1(E_1 + m)}} \begin{pmatrix} \xi^{s_1} \\ 0 \end{pmatrix}, \\ v(p_2, s_2) &\triangleq \frac{-\not{p}_2 + m}{\sqrt{E_2(E_2 + m)}} \begin{pmatrix} 0 \\ \eta^{s_2} \end{pmatrix}, \end{aligned} \quad (3)$$

with $E_{1,2} \triangleq \sqrt{\mathbf{p}_{1,2}^2 + m^2}$, $\xi^{1/2} = (1, 0)^T$, $\xi^{-1/2} = (0, 1)^T$, $\eta^{1/2} = (0, 1)^T$, and $\eta^{-1/2} = (-1, 0)^T$. The combination of the above expressions results in the following:

$$\begin{aligned} \Pi_{\text{ini}} &= \frac{1}{8\sqrt{2}E_i^2(E_i + m)} (\not{p}_1 + m)(2E_i + \not{p})\gamma^5(-\not{p}_2 + m), \\ \Pi_{\text{fin}} &= -\frac{1}{8\sqrt{2}E_f^2(E_f + m)} (-\not{p}_4 + m)\gamma^5(2E_f + \not{p})(\not{p}_3 + m), \end{aligned} \quad (4)$$

where $E_{i,f} \triangleq \sqrt{\mathbf{p}_{i,f}^2 + m^2}$ and one should note that there is a minus in the expression of Π_{fin} .

We want to point out that the form of such a project matrix is just for simplification in our calculation. In principle, the project matrix should be deduced from the Bethe-Salpeter equation and in the ultranonrelativistic limit the above expressions are expected to be true. In this paper, we do not discuss details of this project matrix but just take the same form as that in the references.

Using the above project method, the interaction kernel in the 1S_0 state can be expressed as follows:

$$\begin{aligned} G^{(a)}(^1S_0) &= -ic_f^{(2g)} \int \frac{d^D k}{(2\pi)^4} \text{Tr}[T_1 \cdot \Pi_{\text{ini}}] \\ &\quad \times \text{Tr}[T_2 \cdot \Pi_{\text{fin}}] D_{\mu\rho}(k) D_{\nu\lambda}(p_1 + p_2 - k), \\ G^{(b)}(^1S_0) &= -ic_f^{(2g)} \int \frac{d^D k}{(2\pi)^4} \text{Tr}[T_1 \cdot \Pi_{\text{ini}}] \\ &\quad \times \text{Tr}[T_3 \cdot \Pi_{\text{fin}}] D_{\mu\lambda}(k) D_{\nu\rho}(p_1 + p_2 - k), \end{aligned} \quad (5)$$

where the color factor $c_f^{(2g)}$ is

$$\begin{aligned} c_f^{(2g)} &= \left(\frac{\delta_{ij}}{\sqrt{N_c}} T_a^{jm} T_b^{mi} \right) \left(\frac{\delta_{i'j'}}{\sqrt{N_c}} T_c^{j'm'} T_d^{m'i'} \right) \delta_{ad} \delta_{bc} \\ &= \frac{C_A C_F}{2N_c} = \frac{N_c^2 - 1}{4N_c} = \frac{2}{3}, \end{aligned} \quad (6)$$

and the hard kernel T_i are

$$\begin{aligned} T_1 &= (-ig_s \gamma^\nu) \cdot S_F(p_1 - k) \cdot (-ig_s \gamma^\mu), \\ T_2 &= (-ig_s \gamma^\rho) \cdot S_F(p_3 - k) \cdot (-ig_s \gamma^\lambda), \\ T_3 &= (-ig_s \gamma^\rho) \cdot S_F(k - p_4) \cdot (-ig_s \gamma^\lambda), \end{aligned} \quad (7)$$

with

$$\begin{aligned} S_F(q) &= \frac{i(\not{q} + m)}{q^2 - m^2 + i\epsilon}, \\ D_{\mu\rho}(q) &= \frac{-i}{q^2 + i\epsilon} \left(g_{\mu\rho} - \xi \frac{q_\mu q_\rho}{q^2} \right). \end{aligned} \quad (8)$$

For the on-shell quark and antiquark pairs, one has $(P/2 \pm p_i)^2 = (P/2 \pm p_f)^2 = m^2$ and these conditions lead to the result that the denominators of the integrands in Eq. (5) include two terms $k^2 - (P - 2p_{i,f}) \cdot k + i\epsilon$. This situation leads to the expansion on $p_{i,f}$ and the integration of the loop momentum is noncommutative when one goes to estimate the asymptotic behavior of the expression. In this case, one can separate the loop integration into the hard part and the other parts using the region method [19]. For the off-shell quark and antiquark pairs, $(P^2/4 - m^2)$ is taken as a free finite quantity and one can commute the expansion and the integration of the loop momentum safely.

III. THE ANALYTIC RESULTS FOR THE ASYMPTOTIC BEHAVIOR

In our calculation, we at first use the package FeynCalc [20] to do the trace of Dirac matrices in D -dimensions as in Ref. [21]; then we expand the expression of the variables p_i^2 , p_f^2 and $p_i \cdot p_f$ to a special order. This expansion is equivalent to expanding the expression of the momenta p_i and p_f directly. After the expansion, we use the following tensor decomposition to reexpress the loop integrations:

$$\begin{aligned}
\int_k \frac{(p_i \cdot k)^6}{k^2(P-k)^2((P/2-k)^2-m^2)^n} &= \int_k \frac{-15[k^2s - (k \cdot P)^2]^3 |\mathbf{p}_i|^6}{(D+3)(D^2-1)s^3k^2(P-k)^2((P/2-k)^2-m^2)^n}, \\
\int_k \frac{(p_i \cdot k)^4}{k^2(P-k)^2((P/2-k)^2-m^2)^n} &= \int_k \frac{3[(k \cdot P)^2 - k^2s]^2 |\mathbf{p}_i|^4}{(D^2-1)s^2k^2(P-k)^2((P/2-k)^2-m^2)^n}, \\
\int_k \frac{(p_i \cdot k)^2}{k^2(P-k)^2((P/2-k)^2-m^2)^n} &= \int_k \frac{[sk^2 - (k \cdot P)^2] |\mathbf{p}_i|^2}{(1-D)sk^2(P-k)^2((P/2-k)^2-m^2)^n}, \\
\int_k \frac{(p_i \cdot k)^4 (p_f \cdot k)^2}{k^2(P-k)^2((P/2-k)^2-m^2)^n} &= \int_k \frac{-3(4\beta^2+1)[k^2s - (k \cdot P)^2]^3 |\mathbf{p}_i|^4 |\mathbf{p}_f|^2}{(D+3)(D^2-1)s^3k^2(P-k)^2((P/2-k)^2-m^2)^n}, \\
\int_k \frac{(p_i \cdot k)^2 (p_f \cdot k)^2}{k^2(P-k)^2((P/2-k)^2-m^2)^n} &= \int_k \frac{(2\beta^2+1)[(k \cdot P)^2 - k^2s]^2 |\mathbf{p}_i|^2 |\mathbf{p}_f|^2}{(D^2-1)s^2k^2(P-k)^2((P/2-k)^2-m^2)^n}, \tag{9}
\end{aligned}$$

where $\beta \triangleq \mathbf{p}_i \cdot \mathbf{p}_f / |\mathbf{p}_i| |\mathbf{p}_f|$ and $\int_k \triangleq \int d^D k$. After the expansion and the tensor decomposition, for simplification we directly use the package FIESTA [22] to do the sector decomposition and output the database. Finally we use Mathematica to do the analytic integration.

In the practical calculation, we expand the expression on $|\mathbf{p}_{i,f}|$ to order 6. The gauge parameter ξ is also kept in the practical calculation and we find that the result is not dependent on ξ which means the result is gauge independent although the momenta $P/2 \pm p_{i,f}$ are not on shell. The direct numerical calculation is also used to check the analytic result which shows that the $i\epsilon$ in the propagators should be carefully dealt with.

After the loop integration, we apply the following property to reduce the variables since we only care about the matrix element of the interaction kernel between the 1S_0 states:

$$\begin{aligned}
\int d\Omega_{\mathbf{p}_i} d\Omega_{\mathbf{p}_f} \mathbf{p}_i \cdot \mathbf{p}_f &= 0, \\
\int d\Omega_{\mathbf{p}_i} d\Omega_{\mathbf{p}_f} (\mathbf{p}_i \cdot \mathbf{p}_f)^2 &= \frac{1}{3} |\mathbf{p}_i|^2 |\mathbf{p}_f|^2 \int d\Omega_{\mathbf{p}_i} d\Omega_{\mathbf{p}_f}. \tag{10}
\end{aligned}$$

The high terms like $(\mathbf{p}_i \cdot \mathbf{p}_f)^4$ do not appear in the expression and we need not care about them.

Equation (10) means that we can replace the terms $(\mathbf{p}_i \cdot \mathbf{p}_f)^2$ and $(\mathbf{p}_i \cdot \mathbf{p}_f)$ with $\frac{1}{3} |\mathbf{p}_i|^2 |\mathbf{p}_f|^2$ and 0 in our discussion. After this replacement, the final result can be expressed as

$$\begin{aligned}
\bar{G}^{(a+b)}(^1S_0)|_{|\mathbf{p}_{i,f}|} &= c_f^{(2g)} \alpha_s^2 \left[\frac{1}{m^2} c_0^{\text{full}} + \frac{1}{m^4} c_2^{\text{full}} + \frac{1}{m^6} c_4^{\text{full}} \right. \\
&\quad \left. + \frac{1}{m^8} c_6^{\text{full}} + \text{higher order} \right], \tag{11}
\end{aligned}$$

where $\bar{G}$ refers to the result of G after the replacement, the subindexes $|\mathbf{p}_{i,f}|$ mean expanding the expression on $|\mathbf{p}_{i,f}|$, and c_i^{full} are some functions on $r_s \triangleq s/4m^2$ with

corresponding orders on $|\mathbf{p}_{i,f}|$. The manifest expressions of c_i^{full} are a little complex and we list them in the Appendix.

An interesting property of the coefficients c_i^{full} is that the expressions for their imaginary parts are discontinuous at $r_s = 0$ and 1 due to the terms $\log(1-r_s)$ and $\log r_s$ in the expressions of $\text{Re}[c_i^{\text{full}}]$. When r_s goes from the region $0 < r_s < 1$ to the region $r_s > 1$, the term $\log(1-r_s)$ gives additional contributions to the imaginary parts of the coefficients. In the practical calculation, this analytic continuation is also checked by the direct independent calculations in the two regions. Physically, this property is natural since in the purely perturbative theory when $0 < r_s < 1$ there is only a contribution from the $2g$ intermediate states to the imaginary part; when $r_s > 1$ there is also a contribution from the $q\bar{q} + 2g$ intermediate states (after the NR expansion and using the off-shell conditions for the momenta). To check this property, we also carry a calculation directly on the decay width $\int d\Phi_{\text{phasespace}} |\mathcal{M}(^1S_0 \rightarrow 2g)|^2$ and get the consistent results with the expressions $\text{Im}[c_i^{\text{full}}]$ in the region $0 < r_s < 1$. Since the physical quarks are confined, we prefer to apply the expressions of $\text{Im}[c_i^{\text{full}}]$ in the region $0 < r_s < 1$ to discuss their properties in the region $r_s > 1$ but not to use the expressions calculated by the purely perturbative theory.

To compare the above results with those given in NRQCD, we further expand the above results on p^2 with $p^2 \triangleq s/4 - m^2$, rearrange the results according to their order, and get the following expression:

$$\begin{aligned}
\bar{G}^{(a+b)}(^1S_0)|_{|\mathbf{p}_i|, |\mathbf{p}_f|, p^2} &= c_f^{(2g)} \alpha_s^2 \left[\frac{1}{m^2} c_0 + \frac{1}{m^4} c_2 + \frac{1}{m^6} c_4 \right. \\
&\quad \left. + \frac{1}{m^8} c_6 + \text{higher order} \right], \tag{12}
\end{aligned}$$

where c_i are the combinations of terms with corresponding orders on $\mathbf{p}_{i,f}^2$ and p^2 . In the region $0 < r_s < 1$ or $-m^2 < p^2 < 0$, c_i are expressed as follows:

$$\begin{aligned}
\text{Im}[c_0^{(p^2 < 0)}] &= 2\pi, \\
\text{Im}[c_2^{(p^2 < 0)}] &= -\frac{\pi}{6}[17(|\mathbf{p}_f|^2 + |\mathbf{p}_i|^2) - 18p^2], \\
\text{Im}[c_4^{(p^2 < 0)}] &= \frac{\pi}{360}[1119(|\mathbf{p}_i|^4 + |\mathbf{p}_f|^4) + 1445|\mathbf{p}_i|^2|\mathbf{p}_f|^2 - 1410(|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2)p^2 + 225p^4], \\
\text{Im}[c_6^{(p^2 < 0)}] &= -\frac{\pi}{10080}[32652(|\mathbf{p}_i|^6 + |\mathbf{p}_f|^6) + 44387(|\mathbf{p}_f|^4|\mathbf{p}_f|^2 + |\mathbf{p}_f|^2|\mathbf{p}_f|^4) \\
&\quad - 42525(|\mathbf{p}_f|^4 + |\mathbf{p}_f|^4)p^2 - 51170|\mathbf{p}_i|^2|\mathbf{p}_f|^2p^2 + 6825(|\mathbf{p}_f|^2 + |\mathbf{p}_f|^2)p^4 + 1260p^6)], \quad (13)
\end{aligned}$$

and

$$\begin{aligned}
\text{Re}[c_0^{(p^2 < 0)}] &= 4(1 - \log 2), \\
\text{Re}[c_2^{(p^2 < 0)}] &= \frac{1}{3}[(17\log 2 - 10)(|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2) - 18\log 2]p^2, \\
\text{Re}[c_4^{(p^2 < 0)}] &= \frac{1}{180}\left\{[180\log(-r_{p^2}) + 936 - 1119\log 2](|\mathbf{p}_i|^4 + |\mathbf{p}_f|^4) + [180\log(-r_{p^2}) + 665 - 1445\log 2]|\mathbf{p}_i|^2|\mathbf{p}_f|^2 \right. \\
&\quad - [540\log(-r_{p^2}) + 210 - 1410\log 2](|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2)p^2 + [540\log(-r_{p^2}) - 315 - 225\log 2]p^4 \\
&\quad \left. - \frac{45(|\mathbf{p}_f|^6 + |\mathbf{p}_f|^4|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2|\mathbf{p}_i|^4 + |\mathbf{p}_i|^6)}{p^2}\right\}, \\
\text{Re}[c_6^{(p^2 < 0)}] &= \frac{1}{5040}\{[-420\log(-r_{p^2}) - 16581 + 32652\log 2](|\mathbf{p}_i|^6 + |\mathbf{p}_f|^6) \\
&\quad - [9240\log(-r_{p^2}) + 25795 - 44387\log 2](|\mathbf{p}_i|^4|\mathbf{p}_f|^2 + |\mathbf{p}_i|^4|\mathbf{p}_f|^2) \\
&\quad + [420\log(-r_{p^2}) - 10269 - 42525\log 2](|\mathbf{p}_i|^4 + |\mathbf{p}_f|^4)p^2 + [25200\log(-r_{p^2}) + 4760 - 51170\log 2]|\mathbf{p}_i|^2|\mathbf{p}_f|^2p^2 \\
&\quad + [2940\log(-r_{p^2}) + 18900 + 6825\log 2](|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2)p^4 - [12600\log(-r_{p^2}) - 1050 - 1260\log 2]p^6\}, \quad (14)
\end{aligned}$$

where $r_{p^2} \triangleq p^2/m^2 = r_s - 1$. Similarly with the expressions in the Appendix, in the region $r_{p^2} > 0$ the term $\log(-r_{p^2})$ should be taken as $\log(-r_{p^2} - i\epsilon)$ with $\epsilon = 0^+$, which gives the same real parts and an additional contribution to the imaginary parts of the coefficients. For convenience we also list the expressions in the region $p^2 > 0$:

$$\begin{aligned}
\text{Im}[c_0^{(p^2 > 0)}] &= \text{Im}[c_0^{(p^2 < 0)}], \\
\text{Im}[c_2^{(p^2 > 0)}] &= \text{Im}[c_2^{(p^2 < 0)}], \\
\text{Im}[c_4^{(p^2 > 0)}] &= \frac{\pi}{360}[759(|\mathbf{p}_i|^4 + |\mathbf{p}_f|^4) + 1085|\mathbf{p}_i|^2|\mathbf{p}_f|^2 - 330(|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2)p^2 - 855p^4], \\
\text{Im}[c_6^{(p^2 > 0)}] &= -\frac{\pi}{10080}[31822(|\mathbf{p}_i|^6 + |\mathbf{p}_f|^6) + 25907(|\mathbf{p}_f|^4|\mathbf{p}_f|^2 + |\mathbf{p}_f|^2|\mathbf{p}_f|^4) - 41685(|\mathbf{p}_f|^4 + |\mathbf{p}_f|^4)p^2 - 770|\mathbf{p}_i|^2|\mathbf{p}_f|^2p^2 \\
&\quad + 12705(|\mathbf{p}_f|^2 + |\mathbf{p}_f|^2)p^4 - 23940p^6)]. \quad (15)
\end{aligned}$$

The real parts of the expressions in the region of $p^2 > 0$ are the same as those in the expressions in the region of $p^2 < 0$ if we change $-r_{p^2}$ to r_{p^2} .

Using the above expressions and the quasipotential method, one can directly get the corresponding nonrelativistic potential in LO- α_s . In the momentum space it is expressed as

$$V_{\text{eff}}(^1S_0) = -G^{(a+b)}(^1S_0), \quad (16)$$

since we have normalized the Dirac spinors as $u^+u = v^+v = 1$.

Comparing the above effective potential with the non-perturbative potential in the quark model or the non-perturbative potential V_{LPG+L} from the lattice calculation [23], one can take the above effective potential as a perturbative interaction. The corresponding energy shift in leading order can be obtained directly as

$$\begin{aligned}
\Delta E(s) &= \langle V_{\text{eff}}^{^1S_0} \rangle \\
&= \int d^3\mathbf{p}_i d^3\mathbf{p}_f \Phi^*(\mathbf{p}_f) V_{\text{eff}}(^1S_0, p_i^2, p_f^2, p_i \cdot p_f, s) \Phi(\mathbf{p}_i), \quad (17)
\end{aligned}$$

where $\Phi(\mathbf{p}_{i,f}) \triangleq \phi(|\mathbf{p}_{i,f}|)Y_{00}$ are the wave functions of 1S_0 states in the momentum space normalized as $\int d^3\mathbf{p} \Phi^*(\mathbf{p})\Phi(\mathbf{p}) = 1$.

Correspondingly, the result after expanding and integrating is expressed as

$$\begin{aligned} \Delta E(s)|_{|\mathbf{p}_i|,|\mathbf{p}_f|} = & -2\pi^3 c_f^{(2g)} \alpha_s^2 \left[\frac{c_{01}^{\text{full}}}{m^2} \psi(0)^2 - \frac{c_{21}^{\text{full}}}{m^4} 2\psi^{(2)}(0)\psi(0) \right. \\ & + \frac{c_{41}^{\text{full}}}{m^6} 2\psi^{(4)}(0)\psi(0) + \frac{c_{42}^{\text{full}}}{m^6} \psi^{(2)}(0)^2 \\ & \left. - \frac{c_{61}^{\text{full}}}{m^8} 2\psi^{(6)}(0)\psi(0) - \frac{c_{62}^{\text{full}}}{m^8} 2\psi^{(2)}(0)\psi^{(4)}(0) \right], \end{aligned} \quad (18)$$

where the coefficients can be found in the Appendix and $\psi^{(n)}(0)$ is defined as

$$\psi^{(n)}(0) \triangleq \frac{(-i)^n}{(2\pi)^{3/2}} \int |\mathbf{p}|^n \Phi(\mathbf{p}) d^3\mathbf{p}, \quad (19)$$

with n even. $\psi^{(n)}(0)$ corresponds to the values of the n th derivative of the wave functions in the coordinate space at $r = 0$ since

$$\Phi(\mathbf{p}) = \int \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{r}} \psi(\mathbf{r}) d^3\mathbf{r}. \quad (20)$$

One should note that the angle part Y_{00} is included in the wave function $\psi(\mathbf{r})$.

Furthermore, one can expand the result on p^2 , which gives

$$\Delta E(s)|_{|\mathbf{p}_i|,|\mathbf{p}_f|,p^2} = -2\pi^3 c_f^{(2g)} \alpha_s^2 \left[\frac{c_0}{m^2} \psi(0)^2 + \text{higher terms} \right]. \quad (21)$$

Physically, the imaginary part of $\Delta E(s)$ corresponds to the decay width of a state by the relation $\Gamma = -2\text{Im}[\Delta E(s)]$ with $\sqrt{s} = M_{^1S_0}$ and the real part of $\Delta E(s)$ corresponds to the mass shift.

Equation (18) can also be directly used to estimate the decay widths of $\eta_{c,b} \rightarrow 2\gamma$ and the mass shifts and decay widths of positronium in 1S_0 states due to the annihilation effect $e^+e^- \rightarrow 2\gamma$. For $\eta_{c,b} \rightarrow 2\gamma$, we should replace the factor $c_f^{(2g)} \alpha_s^2$ with $c_f^{(2\gamma)} Q_{c,b}^4 \alpha_{\text{QED}}^2$ where $c_f^{(2\gamma)} = 3$ and $Q_{b,c}$ are the electric charges of b, c quarks. For positronium, we should replace the factor $c_f^{(2g)} \alpha_s^2$ with α_{QED}^2 .

IV. COMPARISON WITH THE RESULTS OF NRQCD IN LO- α_s

To compare the above results with the corresponding results in NRQCD in LO- α_s , we can apply the on-shell

conditions to the coefficients c_i . The on-shell conditions mean $|\mathbf{p}_i|^2 = |\mathbf{p}_f|^2 = -p_i^2 = -p_f^2 = p^2$, which lead to the following results:

$$\begin{aligned} \text{Im}[c_0^{\text{on}}] &= 2\pi, \\ \text{Im}[c_2^{\text{on}}] &= -\frac{8\pi}{3} |\mathbf{p}_i|^2, \\ \text{Im}[c_4^{\text{on}}] &= \frac{136\pi}{45} |\mathbf{p}_i|^4, \\ \text{Im}[c_6^{\text{on}}] &= -\frac{1024\pi}{315} |\mathbf{p}_i|^6, \end{aligned} \quad (22)$$

where the index on refers to the results after applying the on-shell conditions to $\text{Im}[c_i^{(p^2 < 0)}]$. We want to point out that the expressions $\text{Im}[c_i^{(p^2 > 0)}]$ and $\text{Im}[c_i^{(p^2 < 0)}]$ give the same results to order 6 after using the on-shell conditions.

Comparing these results to the decay width of $\Gamma(\eta_c \rightarrow 2\gamma)$ in NRQCD [17], which is expressed as

$$\begin{aligned} \Gamma(H(^1S_0) \rightarrow \gamma\gamma) = & \frac{2\alpha_{\text{QED}}^2 Q^4 \pi}{m^2} \langle H(^1S_0) | \mathcal{O}_{\text{em}}(^1S_0) | H(^1S_0) \rangle \\ & - \frac{8\alpha_{\text{QED}}^2 Q^4 \pi}{3m^4} \langle H(^1S_0) | \mathcal{P}_{\text{em}}(^1S_0) | H(^1S_0) \rangle \\ & + \frac{20\alpha_{\text{QED}}^2 Q^4 \pi}{9m^6} \langle H(^1S_0) | \mathcal{Q}'_{\text{em}}(^1S_0) | H(^1S_0) \rangle \\ & + \frac{4\alpha_{\text{QED}}^2 Q^4 \pi}{5m^6} \langle H(^1S_0) | \mathcal{Q}''_{\text{em}}(^1S_0) | H(^1S_0) \rangle, \end{aligned} \quad (23)$$

with

$$\begin{aligned} \mathcal{O}_{\text{em}}(^1S_0) &= \psi^\dagger \chi |0\rangle \langle 0| \chi^\dagger \psi, \\ \mathcal{P}_{\text{em}}(^1S_0) &= \frac{1}{2} \psi^\dagger \left(-\frac{i \leftrightarrow D}{2} \right)^2 \chi |0\rangle \langle 0| \chi^\dagger \psi + \text{H.c.}, \\ \mathcal{Q}'_{\text{em}}(^1S_0) &= \psi^\dagger \left(-\frac{i \leftrightarrow D}{2} \right)^2 \chi |0\rangle \langle 0| \chi^\dagger \left(-\frac{i \leftrightarrow D}{2} \right)^2 \psi, \\ \mathcal{Q}''_{\text{em}}(^1S_0) &= \frac{1}{2} \psi^\dagger \left(-\frac{i \leftrightarrow D}{2} \right)^4 \chi |0\rangle \langle 0| \chi^\dagger \psi + \text{H.c.}, \end{aligned} \quad (24)$$

one can find that the $\mathbf{p}_{i,f}$ in Eq. (22) corresponds to the operator $-\frac{i \leftrightarrow D}{2}$ in Eq. (24); $\text{Im}[c_{0,2,4}^{\text{on}}]$ are the same as the coefficients of the first and second terms and the sum of the third and the fourth terms of Eq. (23), except for a global factor $\alpha_{\text{QED}}^2 Q^4$ and a normalized factor $1/m^n$. This is natural at first glance since we take s as a free variable at first and then apply the on-shell conditions after the loop integration, and NRQCD takes the on-shell conditions at first. The two results should be the same if the two steps are commutative. For the imaginary parts in LO- α_s , the two steps are really commutative since there is no divergence.

FIG. 3. The numerical results for the corrections to the decay width and the mass shift for the 1S_0 state in the leading order of α_s and v , where $r_s \triangleq s/4m^2$.

The coefficient $\text{Im}[c_0^{\text{on}}]$ corresponds to the sum of the coefficients in NRQCD in order v^6 [21]. In the above calculation we take the quantities p^2 , $|\mathbf{p}_i|$, $|\mathbf{p}_f|$, and $\mathbf{p}_i \cdot \mathbf{p}_f$ as independent variables at first and then, using the on-shell conditions $p^2 = |\mathbf{p}_i|^2 = |\mathbf{p}_f|^2$, this means that we cannot distinguish $|\mathbf{p}_i|$ and $|\mathbf{p}_f|$ after applying the on-shell conditions and we can only give the sum of the corresponding NRQCD coefficients. The real part of c_i^{on} can also be obtained easily from Eq. (14) as follows:

$$\begin{aligned} \text{Re}[c_0^{\text{on}}] &= 4(1 - \log 2), \\ \text{Re}[c_2^{\text{on}}] &= \frac{4}{3} |\mathbf{p}_i|^2 (4 \log 2 - 5), \\ \text{Re}[c_4^{\text{on}}] &= \frac{1}{90} |\mathbf{p}_i|^4 (811 - 544 \log 2), \\ \text{Re}[c_6^{\text{on}}] &= \frac{1}{315} |\mathbf{p}_i|^6 (2048 \log 2 - 3855), \end{aligned} \quad (25)$$

where $\text{Re}[c_0^{\text{on}}]$ is the same as the result given in Ref. [24]. We want to point out that for the real physical systems, we do not suggest using Eqs. (22) and (25) but instead suggest using the corresponding full expressions without the on-shell conditions.

V. THE NUMERICAL RESULT AND CONCLUSION

For the visualization, we list some numerical results in this section. In NRQCD, the contribution to the decay width in LO- v and LO- α_s is determined by the coefficient c_0^{on} , the wave function at zero point $\psi(0)$, and the strong coupling α_s . In the decay of charmonium and bottomonium, the corresponding values of α_s are about 0.3 and 0.2. In our calculation, we expand the expression on $|\mathbf{p}_i|$, $|\mathbf{p}_f|$, and $\mathbf{p}_i \cdot \mathbf{p}_f$ and do not take the on-shell conditions to fix s . This results in an interesting property that the decay width

is also dependent on s . The ratio $\text{Im}[c_0^{\text{full}}]/\text{Im}[c_0^{\text{on}}]$ reflects the corresponding correction to the decay width in NRQCD in the leading order due to the off-shell effect. Such a correction is not dependent on the nonperturbative parameter $\psi(0)$ and the strong coupling α_s but only on the ratio r_s . The corresponding numerical results are presented in the left panel of Fig. 3, where the expression $\text{Im}[c_0^{\text{full}}]$ in the region $0 < r_s < 1$ is used to estimate its values in the region $r_s > 1$ due to our argument in the above.

In the right panel of Fig. 3, we also present a similar correction to the real part of the coefficient. The ratio $\text{Re}[c_0^{\text{full}}]/\text{Re}[c_0^{\text{on}}]$ reflects the correction to the mass shift due to the off-shell effect. For positronium, the bound energies are very small compared with the mass of the electron. This means the ratios $\sqrt{r_s} = (2m - \Delta E)/2m$ are very close to 1 and the corrections are very small. For

FIG. 4. The numerical results for the ratio of the mass shift and the decay width for the 1S_0 state in the leading order of α_s and v .

charmonium, the bound energies are not small compared with the quark mass and $\sqrt{r_s}$ values for the real physical states are also not small. For example, if we take the masses of the b , c quarks as constants, then the relative ratios $\sqrt{r_s(\eta_c(2S))}/\sqrt{r_s(\eta_c(1S))}=M_{\eta_c(2S)}/M_{\eta_c(1S)}=1.22$ and $\sqrt{r_s(\eta_b(2S))}/\sqrt{r_s(\eta_b(1S))}=M_{\eta_b(2S)}/M_{\eta_b(1S)}=1.06$, which are not small values. This suggests the corrections are not small.

In Fig. 4, the ratios $\text{Re}[c_0^{\text{full}}]/-2\text{Im}[c_0^{\text{full}}]$ and $\text{Re}[c_0^{\text{on}}]/-2\text{Im}[c_0^{\text{on}}]$ are presented. These ratios reflect the relation between the mass shifts and the decay widths in LO. If one assumes that the LO gives the most of the contribution, one can use the experimental decay width to estimate the mass shift. For heavy quarkonia, if we take the approximation $\sqrt{r_s(\eta_c)} \approx 1$ and $\sqrt{r_s(\eta_b)} \approx 1$, then the corrections to the mass shifts are about $-(1-\log 2)/\pi\Gamma(\eta_c) \approx -3.1 \text{ MeV}$ for η_c , $0.078\Gamma(\eta_c(2S))=0.88 \text{ MeV}$ for $\eta_c(2S)$, $-(1-\log 2)/\pi\Gamma(\eta_b) \approx -6.1 \text{ MeV}$ for η_b , and $-0.04\Gamma(\eta_b(2S))$ for $\eta_b(2S)$. The interesting property is that the mass shifts are strongly dependent on the masses or the binding energies of the 1S_0 states. This means that the corrections to different 1S_0 states are also very different and cannot be subtracted or hidden in a unified way.

In summary, the nonrelativistic asymptotic behavior of the transition $q\bar{q} \rightarrow 2g \rightarrow q\bar{q}$ in the 1S_0 channel is discussed. In our discussion, the momenta of the quarks and antiquarks are not limited on the mass shell after projecting the quark and antiquark pairs to the 1S_0 state. We calculate the results by expanding the expression on the three-momentum of quarks and antiquarks to order 6. The imaginary part of the first three terms of our results after applying the on-shell conditions can reproduce the NRQCD results in the leading order of α_s . The real part

of our results can be used to estimate the mass shift of 1S_0 heavy quarkonia due to the $2g$ annihilation effect. The results can also be used to estimate the energy shifts of the 1S_0 states of positronium.

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APPENDIX: EXPRESSIONS FOR THE COEFFICIENTS c_i^{full}

In this appendix, the expressions for c_i^{full} are listed. According on their dependence on $|\mathbf{p}_{i,f}|$ we separate c_i^{full} into a few parts as follows:

$$\begin{aligned} c_0^{\text{full}} &= c_{01}^{\text{full}}, \\ c_2^{\text{full}} &= (|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2) c_{21}^{\text{full}}, \\ c_4^{\text{full}} &= (|\mathbf{p}_i|^4 + |\mathbf{p}_f|^4) c_{41}^{\text{full}} + |\mathbf{p}_i|^2 |\mathbf{p}_f|^2 c_{42}^{\text{full}}, \\ c_6^{\text{full}} &= (|\mathbf{p}_i|^6 + |\mathbf{p}_f|^6) c_{61}^{\text{full}} + (|\mathbf{p}_i|^4 |\mathbf{p}_f|^2 + |\mathbf{p}_i|^2 |\mathbf{p}_f|^4) c_{62}^{\text{full}}. \end{aligned} \quad (\text{A1})$$

In the region $0 < r_s < 1$ or $p^2 > 0$ with $r_s \triangleq s/4m^2$ and $p^2 = s/4 - m^2$, we have the following expressions for the imaginary and real parts of c_{ij}^{full} :

$$\text{Im}[c_{01}^{\text{full},(p^2 < 0)}] = \frac{\pi r_s^2}{2(r_s + 1)^2} (r_s + 3)^2, \quad (\text{A2})$$

$$\text{Im}[c_{21}^{\text{full},(p^2 < 0)}] = -\frac{\pi r_s^2}{24(r_s + 1)^4} (15r_s^4 + 124r_s^3 + 342r_s^2 + 388r_s + 219), \quad (\text{A3})$$

$$\text{Im}[c_{41}^{\text{full},(p^2 < 0)}] = \frac{\pi r_s^2}{240(r_s + 1)^6} (165r_s^6 + 1670r_s^5 + 6547r_s^4 + 13148r_s^3 + 14643r_s^2 + 8286r_s + 3285), \quad (\text{A4})$$

$$\text{Im}[c_{42}^{\text{full},(p^2 < 0)}] = \frac{\pi r_s^2}{288(r_s + 1)^6} (15r_s^3 + 79r_s^2 + 105r_s + 73)^2, \quad (\text{A5})$$

$$\begin{aligned} \text{Im}[c_{61}^{\text{full},(p^2 < 0)}] &= -\frac{\pi r_s^2}{13440(r_s + 1)^8} (9765r_s^8 + 117320r_s^7 + 586600r_s^6 + 1623520r_s^5 + 2755906r_s^4 + 2993320r_s^3 + 2098672r_s^2 \\ &\quad + 715568r_s + 244545), \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \text{Im}[c_{62}^{\text{full},(p^2 < 0)}] &= -\frac{\pi r_s^2}{2880(r_s + 1)^8} (2475r_s^8 + 30660r_s^7 + 155480r_s^6 + 435388r_s^5 + 761518r_s^4 + 855004r_s^3 + 636176r_s^2 \\ &\quad + 289956r_s + 79935), \end{aligned} \quad (\text{A7})$$

$$\text{Re}[c_{01}^{\text{full},(p^2 < 0)}] = -\frac{(3 + r_s)^2}{4r_s(1 + r_s)^2} [-r_s(1 + r_s)^2 - (-1 + r_s)^2(1 + 2r_s) \log(1 - r_s) + 2r_s^3 \log(4r_s)], \quad (\text{A8})$$

$$\begin{aligned} \text{Re}[c_{21}^{\text{full},(p^2<0)}] = & \frac{(3+r_s)}{48r_s^2(1+r_s)^4} \{-r_s(1+r_s)^2[-24-r_s+97r_s^2+73r_s^3+15r_s^4] \\ & + [24+37r_s-19r_s^2+64r_s^3+54r_s^4-7r_s^5-123r_s^6-30r_s^7] \log(1-r_s) \\ & + 2r_s^4[73+105r_s+79r_s^2+15r_s^3] \log(4r_s)\}, \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} \text{Re}[c_{41}^{\text{full},(p^2<0)}] = & -\frac{(3+r_s)}{480r_s^3(1+r_s)^6} \{-r_s(1+r_s)^2[216+612r_s+768r_s^2+1667r_s^3+2702r_s^4+2740r_s^5+1114r_s^6+165r_s^7] \\ & - 2[108+468r_s+834r_s^2+854r_s^3+45r_s^4-522r_s^5-1643r_s^6+898r_s^7+1643r_s^8+990r_s^9+165r_s^{10}] \log(1-r_s) \\ & + 2r_s^5[1095+2397r_s+4082r_s^2+3022r_s^3+1175r_s^4+165r_s^5] \log(4r_s)\}, \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \text{Re}[c_{42}^{\text{full},(p^2<0)}] = & -\frac{1}{576r_s^3(1+r_s)^6} \{-r_s(1+r_s)^2[432-168r_s-3323r_s^2-138r_s^3+10855r_s^4+15340r_s^5+8579r_s^6 \\ & + 2246r_s^7+225r_s^8] + [-432-480r_s+3611r_s^2+9638r_s^3+11964r_s^4+2352r_s^5-7190r_s^6-17300r_s^7 \\ & - 19972r_s^8-14240r_s^9-4365r_s^{10}-450r_s^{11}] \log(1-r_s) + 2r_s^5[73+105r_s+79r_s^2+15r_s^3]^2 \log(4r_s)\}, \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} \text{Re}[c_{61}^{\text{full},(p^2<0)}] = & \frac{(3+r_s)}{26880(-1+r_s)r_s^4(1+r_s)^8} \{-r_s(1+r_s)^2[11520+46800r_s+64976r_s^2+11453r_s^3-126012r_s^4 \\ & - 212164r_s^5-276660r_s^6+46006r_s^7+243804r_s^8+213340r_s^9+74692r_s^{10}+9765r_s^{11}] \\ & + [-11520-64080r_s-134216r_s^2-96857r_s^3+94860r_s^4+184247r_s^5+90006r_s^6-530402r_s^7 \\ & + 389392r_s^8+318086r_s^9+372748r_s^{10}-167589r_s^{11}-290220r_s^{12}-134925r_s^{13}-19530r_s^{14}] \log(1-r_s) \\ & + 2r_s^6[-81515-129836r_s-417756r_s^2-158964r_s^3+132126r_s^4+333420r_s^5+234500r_s^6 \\ & + 78260r_s^7+9765r_s^8] \log(4r_s)\}, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} \text{Re}[c_{62}^{\text{full},(p^2<0)}] = & \frac{1}{5760(-1+r_s)r_s^4(1+r_s)^8} \{-r_s(1+r_s)^2[3456+4752r_s-14280r_s^2-25338r_s^3-41079r_s^4 \\ & - 117330r_s^5-209254r_s^6-62302r_s^7+176892r_s^8+231498r_s^9+115582r_s^{10}+27088r_s^{11}+2475r_s^{12}] \\ & - 2[1728+4968r_s-3720r_s^2-24801r_s^3-16111r_s^4+48974r_s^5+22471r_s^6-16675r_s^7-164646r_s^8-186212r_s^9 \\ & - 87722r_s^{10}+120473r_s^{11}+172613r_s^{12}+99950r_s^{13}+26235r_s^{14}+2475r_s^{15}] \log(1-r_s) \\ & + 2r_s^6[-79935-210021r_s-346220r_s^2-218828r_s^3+93486r_s^4 \\ & + 326130r_s^5+279908r_s^6+124820r_s^7+28185r_s^8+2475r_s^9] \log(4r_s)\}. \end{aligned} \quad (\text{A13})$$

For the expressions in the region $r_s > 1$ or $p^2 > 0$, one should combine the imaginary and real parts of the above expressions and take $\log(1-r_s)$ as $\log(1-r_s-i\epsilon)$ to do analytic continuation. This analytic continuation is also independently checked by the manifest calculation. The results of this analytic continuation are equivalent to replacing the term $\log(1-r_s)$ with $\log(r_s-1)$ for the real parts. The terms $\log(1-r_s-i\epsilon)$ give some additional contributions to the imaginary parts and the final expressions for $\text{Im}[c_{ij}^{\text{full}}]$ in the region $r_s > 1$ or $p^2 > 0$ are expressed as follows:

$$\text{Im}[c_{01}^{\text{full},(p^2>0)}] = \frac{\pi(r_s+3)^2}{4r_s(r_s+1)^2} (3r_s^2-1), \quad (\text{A14})$$

$$\text{Im}[c_{21}^{\text{full},(p^2>0)}] = -\frac{\pi}{48r_s^2(r_s+1)^4} (35r_s^7+308r_s^6+809r_s^5+664r_s^4+173r_s^3-20r_s^2+135r_s+72), \quad (\text{A15})$$

$$\begin{aligned} \text{Im}[c_{41}^{\text{full},(p^2>0)}] = & \frac{\pi}{240r_s^3(r_s+1)^6} (185r_s^{10}+1934r_s^9+7321r_s^8+13592r_s^7+13737r_s^6+4806r_s^5-989r_s^4 \\ & - 3396r_s^3-2970r_s^2-1512r_s-324), \end{aligned} \quad (\text{A16})$$

$$\text{Im}[c_{42}^{\text{full},(p^2>0)}] = \frac{\pi}{576r_s^3(r_s+1)^6} (375r_s^{10} + 4542r_s^9 + 17588r_s^8 + 27818r_s^7 + 23470r_s^6 + 13010r_s^5 + 11964r_s^4 + 9638r_s^3 + 3611r_s^2 - 480r_s - 432), \quad (\text{A17})$$

$$\text{Im}[c_{61}^{\text{full},(p^2>0)}] = -\frac{\pi}{26880r_s^4(r_s+1)^8} (21595r_s^{13} + 265160r_s^{12} + 1300751r_s^{11} + 3435504r_s^{10} + 5346662r_s^9 + 4901016r_s^8 + 2772582r_s^7 + 329336r_s^6 + 294511r_s^5 + 942112r_s^4 + 1129835r_s^3 + 705048r_s^2 + 238320r_s + 34560), \quad (\text{A18})$$

$$\text{Im}[c_{62}^{\text{full},(p^2>0)}] = -\frac{\pi}{2880r_s^4(r_s+1)^8} (1950r_s^{13} + 26820r_s^{12} + 134115r_s^{11} + 339772r_s^{10} + 520980r_s^9 + 488364r_s^8 + 306790r_s^7 + 113444r_s^6 + 11038r_s^5 - 37936r_s^4 - 21825r_s^3 + 2976r_s^2 + 6696r_s + 1728). \quad (\text{A19})$$

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