

Novel black-hole solutions in Einstein-scalar-Gauss-Bonnet theories with a cosmological constant

A. Bakopoulos,^{1,*} G. Antoniou,^{2,†} and P. Kanti^{1,‡}

¹*Division of Theoretical Physics, Department of Physics, University of Ioannina, Ioannina GR-45110, Greece*

²*School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, USA*



(Received 21 December 2018; published 6 March 2019)

We consider the Einstein-scalar-Gauss-Bonnet theory in the presence of a cosmological constant Λ , either positive or negative, and look for novel, regular black-hole solutions with a nontrivial scalar hair. We first perform an analytic study in the near-horizon asymptotic regime and demonstrate that a regular black-hole horizon with a nontrivial hair may always be formed, for either sign of Λ and for arbitrary choices of the coupling function between the scalar field and the Gauss-Bonnet term. At the faraway regime, the sign of Λ determines the form of the asymptotic gravitational background leading to either a Schwarzschild-anti-de Sitter-type background ($\Lambda < 0$) or a regular cosmological horizon ($\Lambda > 0$), with a nontrivial scalar field in both cases. We demonstrate that families of novel black-hole solutions with scalar hair emerge for $\Lambda < 0$, for every choice of the coupling function between the scalar field and the Gauss-Bonnet term, whereas for $\Lambda > 0$, no such solutions may be found. In the former case, we perform a comprehensive study of the physical properties of the solutions found such as the temperature, entropy, horizon area, and asymptotic behavior of the scalar field.

DOI: [10.1103/PhysRevD.99.064003](https://doi.org/10.1103/PhysRevD.99.064003)

I. INTRODUCTION

As the ultimate theory of quantum gravity that would robustly describe gravitational interactions at high energies and facilitate their unification with the other forces is still eluding us, the interest in generalized gravitational theories remains unabated in the scientific literature. These theories include extra fields or higher-curvature terms in their action [1,2], and they provide the framework in the context of which several solutions of the traditional general relativity (GR) have been reexamined and, quite often, significantly enriched.

In this spirit, generalized gravitational theories containing scalar fields were among the first to be studied. However, the quest for novel black-hole solutions—beyond the three well-known families of GR—was abruptly stopped when the no-hair theorem was formulated [3], which forbade the existence of a static solution of this form

with a nontrivial scalar field associated with it. Nevertheless, counterexamples appeared in the years that followed and included black holes with Yang-Mills [4], Skyrme fields [5], or with a conformal coupling to gravity [6]. A novel formulation of the no-hair theorem was proposed in 1995 [7] but this, too, was evaded within a year with the discovery of the dilatonic black holes found in the context of the Einstein-dilaton-Gauss-Bonnet theory [8] (for some earlier studies that paved the way, see [9–13]). The colored black holes were found next in the context of the same theory completed by the presence of a Yang-Mills field [14,15], and higher-dimensional [16] or rotating versions [17–20] were also constructed (for a number of interesting reviews on the topic, see [21–24]).

This second wave of black-hole solutions were derived in the context of theories inspired by superstring theory [25]. During the past decade, though, the construction of generalized gravitational theories was significantly enlarged via the revival of the Horndeski [26] and Galileon [27] theories. Accordingly, novel formulations of the no-hair theorems were proposed that covered the case of standard scalar-tensor theories [28] and Galileon fields [29]. However, these recent forms were also evaded [30] and concrete black-hole solutions were constructed [31–33]. More recently, three independent groups [34–36] almost simultaneously demonstrated that a generalized gravitational theory that contains a scalar field and the quadratic Gauss-Bonnet (GB) term

*abakop@cc.uoi.gr

†anton296@umn.edu

‡pkanti@cc.uoi.gr

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

admits novel black-hole solutions with a nontrivial scalar hair. In a general theoretical argument, which we presented in [34], it was shown that the presence of the GB term was of paramount importance for the evasion of the novel no-hair theorem [7]. In addition, the exact form of the coupling function $f(\phi)$ between the scalar field and the GB term played no significant role for the emergence of the solutions: as long as the first derivative of the scalar field ϕ_h' at the horizon obeyed a specific constraint, an asymptotic solution describing a regular black-hole horizon with a nontrivial scalar field could always be constructed. Employing, then, several different forms of the coupling function $f(\phi)$, a large number of asymptotically flat black-hole solutions with scalar hair were determined [34]. Additional studies presenting novel black holes or compact objects in generalized gravitational theories have appeared [37–45] as well as further studies of the properties of these novel solutions [46–65].

In the present work, we will extend our previous analyses [34] that aimed at deriving asymptotically flat black-hole solutions by introducing in our theory a cosmological constant Λ , either positive or negative. In the context of this theory, we will investigate whether the previous, successful synergy between the Ricci scalar, the scalar field, and the Gauss-Bonnet term survives in the presence of Λ . The question of the existence of black-hole solutions in the context of a scalar-tensor theory, with scalar fields minimally coupled or conformally coupled to gravity, and a cosmological constant has been debated in the literature for decades [66–70]. In the case of a *positive* cosmological constant, the existing studies predominantly excluded the presence of a regular, black-hole solution with an asymptotic de Sitter behavior—a counterexample of a black hole in the context of a theory with a conformally coupled scalar field [71] was shown later to be unstable [72]. On the other hand, in the case of a *negative* cosmological constant, a substantial number of solutions with an asymptotically (anti-)de Sitter behavior have been found in the literature (for a nonexhaustive list, see [73–84]).

Here, we perform a comprehensive study of the existence of black-hole solutions with a nontrivial scalar hair and an asymptotically (anti-)de Sitter behavior in the context of a general class of theories containing the higher-derivative, quadratic GB term. To our knowledge, the only similar study is the one performed in the special case of the shift-symmetric Galileon theory [85], i.e., with a linear coupling function between the scalar field and the GB term. In this work, we consider the most general class of this theory by considering an arbitrary form of the coupling function $f(\phi)$, and we look for regular black-hole solutions with nontrivial scalar hair. Since the uniform distribution of energy associated with the cosmological constant permeates the whole spacetime, we expect Λ to have an effect on both the near-horizon and far-field asymptotic solutions. We will thus repeat our analytical calculations in both the

small and the large- r regimes to examine how the presence of Λ affects the asymptotic solutions both near and far away from the black-hole horizon. As we will see, our set of field equations admits regular solutions near the black-hole horizon with a nontrivial scalar hair for both signs of the cosmological constant. At the faraway regime, the analysis needs to be specialized since a positive or negative sign of Λ leads to either a cosmological horizon or an asymptotic Schwarzschild–anti-de Sitter–type gravitational background, respectively. Our results show that the emergence of a black-hole solution with a nontrivial scalar field strongly depends on the type of asymptotic background that is formed at large distances, and thus on the sign of Λ : whereas, for $\Lambda < 0$, solutions emerge with the same easiness as their asymptotically flat analogues, for $\Lambda > 0$, no such solutions were found.

In the former case, i.e., for $\Lambda < 0$, we present a large number of novel black-hole solutions with a regular black-hole horizon, a nontrivial scalar field, and a Schwarzschild–anti-de Sitter–type asymptotic behavior at large distances. These solutions correspond to a variety of forms of the coupling function $f(\phi)$: exponential, polynomial (even or odd), inverse polynomial (even or odd), and logarithmic. The motivation for an exponential coupling function is provided by the heterotic superstring effective theory [25] where the scalar field is the dilaton. On the other hand, the coupling function of the moduli fields to the GB term in the context of the same theory is given by the logarithm of the Dedekind function—this scalar-tensor theory was shown to lead to interesting singularity-free cosmological solutions [86]. In a subsequent work [87], it was shown that an arbitrary, even polynomial coupling function shares a number of characteristic features with the moduli coupling function and leads again to singularity-free cosmological solutions; this similarity has provided the motivation to consider also even polynomial coupling functions in our quest for novel black-hole solutions. The remaining choices for the coupling function (i.e., the inverse and odd polynomials) have admittedly no fundamental motivation, and their adoption serves to demonstrate that the synergy of a scalar field, the Ricci scalar, and the GB term is powerful enough to support regular black-hole solutions for every form of the coupling function. In this, we have drawn inspiration from the Horndeski [26] and Galileon [27] theories, which are the most general scalar-tensor theories with second-order derivatives of the field and metric while containing arbitrary coupling functions. After the derivation of our classes of black-hole solutions, we proceed to study their physical properties such as the temperature, entropy, and horizon area. We also investigate features of the asymptotic profile of the scalar field, namely its effective potential and rate of change at large distances since this greatly differs from the asymptotically flat case.

The outline of the present work is as follows: in Sec. II, we present our theoretical framework and perform our

analytic study of the near and faraway radial regimes as well as of their thermodynamical properties. In Sec. III, we present our numerical results for regular black-hole solutions in the case of $\Lambda < 0$ while our efforts in the case with $\Lambda > 0$ are presented in Sec. IV. We finish with our conclusions in Sec. V.

II. THE THEORETICAL FRAMEWORK

We consider a general class of higher-curvature gravitational theories described by the following action functional:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + f(\phi) R_{\text{GB}}^2 - 2\Lambda \right]. \quad (1)$$

In this, the quadratic GB term R_{GB}^2 , defined as

$$R_{\text{GB}}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \quad (2)$$

supplements the Einstein-Hilbert term, given by the Ricci scalar curvature R , and the kinetic term for a scalar field ϕ . A coupling term of the scalar field to the GB term, through a general coupling function $f(\phi)$, is necessary in order for the GB term—a total derivative in four dimensions—to contribute to the field equations. A cosmological constant Λ that may take either a positive or a negative value is also present in the theory.

By varying the action (1) with respect to the metric tensor $g_{\mu\nu}$ and the scalar field ϕ , we derive the gravitational field equations and the equation for the scalar field, respectively. These are found to have the form

$$G_{\mu\nu} = T_{\mu\nu}, \quad (3)$$

$$\nabla^2 \phi + \dot{f}(\phi) R_{\text{GB}}^2 = 0, \quad (4)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor, with the latter having the form

$$T_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (g_{\rho\mu} g_{\lambda\nu} + g_{\lambda\mu} g_{\rho\nu}) \times \eta^{\kappa\lambda\alpha\beta} \tilde{R}^{\rho\gamma}_{\alpha\beta} \nabla_\gamma \partial_\kappa f(\phi) - \Lambda g_{\mu\nu}. \quad (5)$$

In the above, the dot over the coupling function denotes its derivative with respect to the scalar field (i.e., $\dot{f} = df/d\phi$). We have also employed units in which $G = c = 1$ and used the definition

$$\tilde{R}^{\rho\gamma}_{\alpha\beta} = \eta^{\rho\gamma\sigma\tau} R_{\sigma\tau\alpha\beta} = \frac{\epsilon^{\rho\gamma\sigma\tau}}{\sqrt{-g}} R_{\sigma\tau\alpha\beta}. \quad (6)$$

Compared to the theory studied in [34], where Λ was zero, the changes in Eqs. (3) and (4) look minimal: the

scalar-field equation remains unaffected while the energy-momentum tensor T^μ_ν receives a constant contribution $-\Lambda \delta^\mu_\nu$. However, as we will see, the presence of the cosmological constant affects both of the asymptotic solutions, the properties of the derived black holes, and even their existence.

In the context of this work, we will investigate the emergence of regular, static, spherically symmetric but nonasymptotically flat black-hole solutions with a nontrivial scalar field. The line element of spacetime will accordingly take the form

$$ds^2 = -e^{A(r)} dt^2 + e^{B(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (7)$$

The scalar field will also be assumed to be static and spherically symmetric, $\phi = \phi(r)$. The coupling function $f(\phi)$ will retain a general form during the first part of our analysis and will be chosen to have a particular form only at the stage of the numerical derivation of specific solutions.

The nonvanishing components of the Einstein tensor G^μ_ν may easily be found by employing the line element (7), and they read

$$G^t_t = \frac{e^{-B}}{r^2} (1 - e^B - rB'), \quad (8)$$

$$G^r_r = \frac{e^{-B}}{r^2} (1 - e^B + rA'), \quad (9)$$

$$G^\theta_\theta = G^\phi_\phi = \frac{e^{-B}}{4r} [rA'^2 - 2B' + A'(2 - rB') + 2rA'']. \quad (10)$$

Throughout our analysis, the prime denotes differentiation with respect to the radial coordinate r . Using Eq. (5), the components of the energy-momentum tensor T^μ_ν take in turn the form

$$T^t_t = -\frac{e^{-2B}}{4r^2} [\phi'^2 (r^2 e^B + 16\dot{f}(e^B - 1)) - 8\dot{f}(B'\phi'(e^B - 3) - 2\phi''(e^B - 1))] - \Lambda, \quad (11)$$

$$T^r_r = \frac{e^{-B}\phi'}{4} \left[\phi' - \frac{8e^{-B}(e^B - 3)\dot{f}A'}{r^2} \right] - \Lambda, \quad (12)$$

$$T^\theta_\theta = T^\varphi_\varphi = -\frac{e^{-2B}}{4r} [\phi'^2 (re^B - 8\dot{f}A') - 4\dot{f}(A'^2\phi' + 2\phi'A'' + A'(2\phi'' - 3B'\phi'))] - \Lambda. \quad (13)$$

Matching the corresponding components of G^μ_ν and T^μ_ν , the explicit form of Einstein's field equations may easily be derived. These are supplemented by the scalar-field equation (4) whose explicit form reads

$$2r\phi'' + (4 + rA' - rB')\phi' + \frac{4\dot{f}e^{-B}}{r} \times [(e^B - 3)A'B' - (e^B - 1)(2A'' + A'^2)] = 0. \quad (14)$$

Although the system of equations involves three unknown functions, namely $A(r)$, $B(r)$, and $\phi(r)$, only two of them are independent. The metric function $B(r)$ may be easily shown to be a dependent variable: the (rr) component of field equations takes, in fact, the form of a second-order polynomial with respect to e^B , i.e., $\alpha e^{2B} + \beta e^B + \gamma = 0$, which easily leads to the following solution:

$$e^B = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \quad (15)$$

where

$$\begin{aligned} \alpha &= 1 - \Lambda r^2, & \beta &= \frac{r^2 \phi'^2}{4} - (2\dot{f}\phi' + r)A' - 1, \\ \gamma &= 6\dot{f}\phi'A'. \end{aligned} \quad (16)$$

Employing the above expression for e^B , the quantity B' may also be found to have the form

$$B' = -\frac{\gamma' + \beta'e^B + \alpha'e^{2B}}{2\alpha e^{2B} + \beta e^B}. \quad (17)$$

Therefore, by using Eqs. (15) and (17), the metric function $B(r)$ may be completely eliminated from the field equations. The remaining three equations then form a system of only two independent, ordinary differential equations of second order for the functions $A(r)$ and $\phi(r)$:

$$A'' = \frac{P}{S}, \quad (18)$$

$$\phi'' = \frac{Q}{S}. \quad (19)$$

The expressions for the quantities P , Q , and S , in terms of $(r, \phi', A', \dot{f}, \ddot{f})$, are given for the interested reader in Appendix A as they are quite complicated.

A. Asymptotic solution at black-hole horizon

As we are interested in deriving novel black-hole solutions, we will first investigate whether an asymptotic solution describing a regular black-hole horizon is admitted by the field equations. As a matter of fact, instead of assuming the usual power-series expression in terms of $(r - r_h)$, where r_h is the horizon radius, we will construct the solution as was done in [8,34]. To this end, we demand that, near the horizon, the metric function $e^{A(r)}$ should vanish (and $e^{B(r)}$ should diverge), whereas the scalar field

must remain finite. The first demand is reflected in the assumption that $A'(r)$ should diverge as $r \rightarrow r_h$ —this will be justified *a posteriori*—while $\phi'(r)$ and $\phi''(r)$ must be finite in the same limit.

Assuming the aforementioned behavior near the black-hole horizon, Eq. (15) may be expanded in terms of $A'(r)$ as follows¹:

$$e^B = \frac{(2\dot{f}\phi' + r)}{1 - \Lambda r^2} A' - \frac{2\dot{f}\phi'(r^2\phi'^2 - 12\Lambda r^2 + 8) + r(r^2\phi'^2 - 4)}{4(1 - \Lambda r^2)(2\dot{f}\phi' + r)} + \mathcal{O}\left(\frac{1}{A'}\right). \quad (20)$$

Then, substituting the above into the system (18) and (19), we obtain

$$A'' = \frac{W_1}{W_3} A'^2 + \mathcal{O}(A'), \quad (21)$$

$$\phi'' = \frac{W_2}{W_3} (2\dot{f}\phi' + r)A' + \mathcal{O}(1), \quad (22)$$

where

$$\begin{aligned} W_1 &= -(r^4 + 4r^3\dot{f}\phi' + 4r^2\dot{f}^2\phi'^2 - 24\dot{f}^2) + 24\Lambda^2 r^4 \dot{f}^2 \\ &\quad + \Lambda[4r^5\dot{f}\phi' + 4r^2\dot{f}^2(r^2\phi'^2 - 16) \\ &\quad - 64r\dot{f}^3\phi' - 64\dot{f}^4\phi'^2 + r^6], \end{aligned} \quad (23)$$

$$\begin{aligned} W_2 &= -r^3\phi'(1 - \Lambda r^2) - 32\Lambda\dot{f}^3\phi'^2 + 16\Lambda r\dot{f}^2\phi'(\Lambda r^2 - 3) \\ &\quad - 2\dot{f}[6 + r^2\phi'^2 + 2\Lambda^2 r^4 - \Lambda r^2(r^2\phi'^2 + 4)], \end{aligned} \quad (24)$$

and

$$W_3 = (1 - \Lambda r^2)[r^4 + 2r^3\dot{f}\phi' - 16\dot{f}^2(3 - 2\Lambda r^2) - 32\Lambda r\dot{f}^3\phi']. \quad (25)$$

From Eq. (20), we conclude that the combination $(2\dot{f}\phi' + r)$ near the horizon must be nonzero and positive for the metric function e^B to have the correct behavior, which is to diverge as $r \rightarrow r_h$ while being positive definite. Then, Eq. (22) dictates that, if we want ϕ'' to be finite, we must necessarily have

$$W_2|_{r=r_h} = 0. \quad (26)$$

The above constraint may be written as a second-order polynomial with respect to ϕ' , which can then be solved to yield

¹Note that only the (+) sign in the expression for e^B in Eq. (15) leads to the desired black-hole behavior.

$$\phi'_h = -\frac{r_h^3(1 - \Lambda r_h^2) + 16\Lambda r_h \dot{f}_h^2(3 - \Lambda r_h^2) \pm (1 - \Lambda r_h^2)\sqrt{C}}{4\dot{f}_h[r_h^2 - \Lambda(r_h^4 - 16\dot{f}_h^2)]}, \quad (27)$$

where all quantities have been evaluated at $r = r_h$. The quantity C under the square root stands for the following combination:

$$C = 256\Lambda \dot{f}_h^4(\Lambda r_h^2 - 6) + 32r_h^2 \dot{f}_h^2(2\Lambda r_h^2 - 3) + r_h^6 \geq 0, \quad (28)$$

and must always be non-negative for ϕ'_h to be real. This combination may be written as a second-order polynomial for \dot{f}_h^2 with roots

$$\dot{f}_h^2 = \frac{r_h^2[3 - 2\Lambda r_h^2 \pm \sqrt{3}\sqrt{3 - 2\Lambda r_h^2 + \Lambda^2 r_h^4}]}{16\Lambda(-6 + \Lambda r_h^2)}. \quad (29)$$

Then, the constraint on C becomes

$$C = (\dot{f}_h^2 - \dot{f}_-^2)(\dot{f}_h^2 - \dot{f}_+^2) \geq 0. \quad (30)$$

Therefore, the allowed regime for the existence of regular, black-hole solutions with scalar hair is given by $\dot{f}_h^2 \leq \dot{f}_-^2$ or $\dot{f}_h^2 \geq \dot{f}_+^2$, since $\dot{f}_+^2 > \dot{f}_-^2$. To obtain some physical insight on these inequalities, we take the limit of a small cosmological constant; then, the allowed ranges are

$$\dot{f}_h^2 \leq \frac{r_h^4}{96} \left(1 + \frac{\Lambda r_h^2}{6} + \dots\right) \quad \text{or} \quad \dot{f}_h^2 \geq \frac{r_h^4}{48} \left(1 - \frac{3}{\Lambda r_h^2} + \dots\right), \quad (31)$$

respectively. In the absence of Λ , Eq. (28) results in the simple constraint $\dot{f}_h^2 \leq r_h^4/96$ and defines a sole branch of solutions with a minimum allowed value for the horizon radius (and mass) of the black hole [34]. In the presence of a cosmological constant, this constraint is now replaced by $\dot{f}_h^2 \leq \dot{f}_-^2$ or by the first inequality presented in Eq. (31) in the small- Λ limit. This inequality leads again to a branch of solutions that—for chosen $f(\phi)$, ϕ_h , and Λ —terminates at a black-hole solution with a minimum horizon radius r_h^{\min} . We observe that, at least for small values of Λ , the presence of a positive cosmological constant relaxes the constraint, allowing for smaller black-hole solutions, while a negative cosmological constant pushes the minimum horizon radius toward larger values. The second inequality in Eq. (31) describes a new branch of black-hole solutions that does not exist when $\Lambda = 0$; this was also noted in [85] in the case of the linear coupling function. This branch of solutions describes a class of very small GB black holes and terminates instead at a black hole with a maximum horizon radius r_h^{\max} .

Returning now to Eq. (18) and employing the constraint (27), the former takes the form

$$A'' = -A'^2 + \mathcal{O}(A'). \quad (32)$$

Integrating the above, we find that $A'(r) \sim 1/(r - r_h)$, a result that justifies the diverging behavior of this quantity near the horizon that we assumed earlier. A second integration yields $A(r) \sim \ln(r - r_h)$, which then uniquely determines the expression of the metric function e^A in the near-horizon regime. Employing Eq. (20), the metric function B is also determined in the same regime. Therefore, the asymptotic solution of Eqs. (15), (18), and (19) that describes a regular, black-hole horizon in the limit $r \rightarrow r_h$ is given by the following expressions:

$$e^A = a_1(r - r_h) + \dots, \quad (33)$$

$$e^{-B} = b_1(r - r_h) + \dots, \quad (34)$$

$$\phi = \phi_h + \phi'_h(r - r_h) + \phi''_h(r - r_h)^2 + \dots, \quad (35)$$

where a_1 , b_1 , and ϕ_h are integration constants. We observe that the above asymptotic solution constructed for the case of a nonzero cosmological constant has exactly the same functional form as the one constructed in [34] for the case of vanishing Λ . The presence of the cosmological constant modifies though the exact expressions of the basic constraint (27) for ϕ'_h and of the quantity C given in (28), the validity of which ensures the existence of a regular black-hole horizon. As in [34], the exact form of the coupling function $f(\phi)$ does not affect the existence of the asymptotic solution; therefore regular black-hole solutions may emerge for a wide class of theories of the form (1).

The regularity of the asymptotic black-hole solution is also reflected in the nondiverging behavior of the components of the energy-momentum tensor and of the scale-invariant Gauss-Bonnet term. The components of the former quantity in this regime assume the form

$$T^t_t = \frac{2e^{-B}}{r^2} B' \phi' \dot{f} - \Lambda + \mathcal{O}(r - r_h), \quad (36)$$

$$T^r_r = -\frac{2e^{-B}}{r^2} A' \phi' \dot{f} - \Lambda + \mathcal{O}(r - r_h), \quad (37)$$

$$T^\theta_\theta = \frac{e^{-2B}}{r} (2A'' + A'^2 - 3A'B') \phi' \dot{f} - \Lambda + \mathcal{O}(r - r_h). \quad (38)$$

Employing the asymptotic expansions (33)–(35), one may see that all components remain indeed finite in the vicinity of the black-hole horizon. For future use, we note that the cosmological constant adds a positive contribution to all

components of the energy-momentum tensor T_ν^μ for $\Lambda < 0$, while it subtracts a positive contribution for $\Lambda > 0$. Also, all scalar curvature quantities, the explicit form of which may be found in Appendix B, independently exhibit a regular behavior near the black-hole horizon—when these are combined, the GB term, in the same regime, takes the form

$$R_{GB}^2 = +\frac{12e^{-2B}}{r^2}A'^2 + \mathcal{O}(r - r_h), \quad (39)$$

exhibiting, too, a regular behavior as expected.

B. Asymptotic solutions at large distances

The form of the asymptotic solution of the field equations at large distances from the black-hole horizon depends strongly on the sign of the cosmological constant. Therefore, in what follows, we study separately the cases of positive and negative Λ .

1. Positive cosmological constant

In the presence of a positive cosmological constant, a second horizon, the cosmological one, is expected to emerge at a radial distance $r = r_c > r_h$. We demand that this horizon is also regular, that is, that the scalar field ϕ and its derivatives remain finite in its vicinity. We may, in fact, follow a method identical to the one followed in Sec. II. 1 near the black-hole horizon: we again demand that, at the cosmological horizon, $g_{tt} \rightarrow 0$ while $g_{rr} \rightarrow \infty$; then, using that A' diverges there, the regularity of ϕ'' from Eq. (19) eventually leads to the constraint

$$\phi'_c = -\frac{r_c^3(1 - \Lambda r_c^2) + 16\Lambda r_c \dot{f}_c^2(3 - \Lambda r_c^2) \pm (1 - \Lambda r_c^2)\sqrt{\tilde{C}}}{4\dot{f}_c[r_c^2 - \Lambda(r_c^4 - 16\dot{f}_c^2)]}, \quad (40)$$

with \tilde{C} now being given by the non-negative expression

$$\tilde{C} = 256\Lambda \dot{f}_c^4(\Lambda r_c^2 - 6) + 32r_c^2 \dot{f}_c^2(2\Lambda r_c^2 - 3) + r_c^6 \geq 0. \quad (41)$$

Employing Eq. (40) in Eq. (18), the solution for the metric function A may again be constructed. Overall, the asymptotic solution of the field equations near a regular, cosmological horizon will have the form

$$e^A = a_2(r_c - r) + \dots, \quad (42)$$

$$e^{-B} = b_2(r_c - r) + \dots, \quad (43)$$

$$\phi = \phi_c + \phi'_c(r_c - r) + \phi''_c(r_c - r)^2 + \dots, \quad (44)$$

where care has been taken for the fact that $r \leq r_c$. One may see again that the above asymptotic expressions lead to

finite values for the components of the energy-momentum tensor and scalar invariant quantities. Once again, the explicit form of the coupling function $f(\phi)$ is of minor importance for the existence of a regular, cosmological horizon.

2. Negative cosmological constant

For a negative cosmological constant, and at large distances from the black-hole horizon, we expect the spacetime to assume a form close to that of the Schwarzschild–anti-de Sitter (SAdS) solution. Thus, we assume the following approximate forms for the metric functions:

$$e^{A(r)} = \left(k - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2 + \frac{q_2}{r^2}\right) \left(1 + \frac{q_1}{r^2}\right)^2, \quad (45)$$

$$e^{-B(r)} = k - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2 + \frac{q_2}{r^2}, \quad (46)$$

where k , M , Λ_{eff} , and $q_{1,2}$ are, at the moment, arbitrary constants. Substituting the above expressions into the scalar field equation (14), we obtain at first order the constraint

$$\phi''(r) + \frac{4}{r}\phi'(r) - \frac{8\Lambda_{\text{eff}}\dot{f}}{r^2} = 0. \quad (47)$$

The gravitational equations, under the same assumptions, lead to two additional constraints, namely

$$\Lambda - \Lambda_{\text{eff}} + \frac{\Lambda_{\text{eff}}r^2\phi'}{12} \left(\phi' - \frac{16\Lambda_{\text{eff}}\dot{f}}{r}\right) = 0, \quad (48)$$

$$\Lambda - \Lambda_{\text{eff}} - \frac{4}{9}\dot{f}\Lambda_{\text{eff}}^2r^2 \left(\phi'' + \frac{3\phi'}{r}\right) - \frac{\Lambda_{\text{eff}}r^2}{12}\phi'^2 \left(1 + \frac{16\Lambda_{\text{eff}}\dot{f}}{3}\right) = 0. \quad (49)$$

Contrary to what happens close to the horizons (either black-hole or cosmological ones), the form of the coupling function $f(\phi)$ now affects the asymptotic form of the scalar field at large distances. The easiest case is that of a linear coupling function, $f(\phi) = \alpha\phi$ —that case was first studied in [85]; however, we review it again in the context of our analysis as it will prove to play a more general role. The scalar field, at large distances, may be shown to have the approximate form

$$\phi(r) = \phi_\infty + d_1 \ln r + \frac{d_2}{r^2} + \frac{d_3}{r^3} + \dots, \quad (50)$$

where again $(\phi_\infty, d_1, d_2, d_3)$ are arbitrary constant coefficients. The coefficients d_1 and Λ_{eff} may be determined through the first-order constraints (47) and (48), respectively, and are given by

$$d_1 = \frac{8}{3}\alpha\Lambda_{\text{eff}}, \quad \Lambda_{\text{eff}} \left(3 + \frac{80\alpha^2\Lambda_{\text{eff}}^2}{9} \right) = 3\Lambda. \quad (51)$$

The third first-order constraint, Eq. (49), is then trivially satisfied. In order to determine the values of the remaining coefficients, one needs to derive higher-order constraints. For example, the coefficients k , q_1 , and d_2 are found at third-order approximation to have the forms

$$\begin{aligned} k &= \frac{81 + 864\alpha^2\Lambda_{\text{eff}}^2 + 1024\alpha^4\Lambda_{\text{eff}}^4}{81 + 1008\alpha^2\Lambda_{\text{eff}}^2 + 2560\alpha^4\Lambda_{\text{eff}}^4}, \\ q_1 &= \frac{24\alpha^2\Lambda_{\text{eff}}(9 + 64\alpha^2\Lambda_{\text{eff}}^2)}{(9 + 32\alpha^2\Lambda_{\text{eff}}^2)(9 + 80\alpha^2\Lambda_{\text{eff}}^2)}, \\ d_2 &= -\frac{12\alpha(27 + 288\alpha^2\Lambda_{\text{eff}}^2 + 512\alpha^4\Lambda_{\text{eff}}^4)}{81 + 1008\alpha^2\Lambda_{\text{eff}}^2 + 2560\alpha^4\Lambda_{\text{eff}}^4}, \end{aligned} \quad (52)$$

while for q_2 or d_3 one needs to go even higher. In contrast, the coefficient M remains arbitrary and may be interpreted as the gravitational mass of the solution.

In the perturbative limit (i.e., for small values of the coupling constant α of the GB term), one may show that the above asymptotic solution is valid for all forms of the coupling function $f(\phi)$. Indeed, if we write

$$\phi(r) = \phi_0 + \sum_{n=1}^{\infty} \alpha^n \phi_n(r) \quad (53)$$

and define $f(\phi) = \alpha\tilde{f}(\phi)$, then, at first order, $\dot{f} \simeq \alpha\dot{\tilde{f}}(\phi_0)$. Therefore, independently of the form of $f(\phi)$, at first order in the perturbative limit, \dot{f} is a constant, as in the case of a linear coupling function. Then, a solution of the form of Eqs. (45), (46), and (50) is easily derived² with α in Eqs. (51) and (52) now being replaced by $\dot{f}(\phi_0)$.

For arbitrary values of the coupling constant α , though, or for a nonlinear coupling function $f(\phi)$, the approximate solution described by Eqs. (45), (46), and (50) will not, in principle, be valid any more. Unfortunately, no analytic form of the solution at large distances may be derived in these cases. However, as we will see in Sec. III, numerical solutions do emerge with a nontrivial scalar field and an asymptotic anti-de Sitter-type behavior at large distances. These solutions are also characterized by a finite GB term and finite, constant components of the energy-momentum tensor at the far asymptotic regime.

C. Thermodynamical analysis

In this subsection, we calculate the thermodynamical properties of the sought-for black-hole solutions, namely

²In the perturbative limit, at first order, one finds $d_1 = 8\Lambda\dot{f}(\phi_0)/3$, $\Lambda_{\text{eff}} = \Lambda$, $k = 1$, $q_1 = 0$, and $d_2 = -4\dot{f}(\phi_0)$. For more details on the perturbative analysis of the black-hole solutions that arise in the context of the general class of theories (1) and are either asymptotically flat or (anti-)de Sitter, see [88].

their temperature and entropy. The first quantity may easily be derived by using the following definition [89,90]:

$$T = \frac{k_h}{2\pi} = \frac{1}{4\pi} \left(\frac{1}{\sqrt{|g_{tt}g_{rr}|}} \left| \frac{dg_{tt}}{dr} \right| \right)_{r_h} = \frac{\sqrt{a_1 b_1}}{4\pi}, \quad (54)$$

which relates the black-hole temperature T to its surface gravity k_h . The above formula is valid for spherically symmetric black holes in theories that may contain also higher-derivative terms such as the GB term. The final expression of the temperature in Eq. (54) is derived by employing the near-horizon asymptotic forms (33) and (34) of the metric functions.

The entropy of the black hole may be calculated by using the Euclidean approach in which the entropy is given by the relation [91]

$$S_h = \beta \left[\frac{\partial(\beta F)}{\partial \beta} - F \right], \quad (55)$$

where $F = I_E/\beta$ is the Helmholtz free energy of the system given in terms of the Euclidean version of the action I_E , and $\beta = 1/(k_B T)$. The above formula has been used in the literature to determine the entropy of the asymptotically flat colored GB black holes [15] and of the family of novel black-hole solutions found in [34] for different forms of the GB coupling function. However, in the case of a nonasymptotically flat behavior, the above method needs to be modified: in the case of a de Sitter-type asymptotic solution, the Euclidean action needs to be integrated only over the causal spacetime $r_h \leq r \leq r_c$, whereas, for an anti-de Sitter-type asymptotic solution, the Euclidean action needs to be regularized [92,93], by subtracting the diverging, “pure” AdS-spacetime contribution.

Alternatively, one may employ the Noether current approach developed in [94] to calculate the entropy of a black hole. In this, the Noether current of the theory under diffeomorphisms is determined, with the Noether charge on the horizon being identified with the entropy of the black hole. In [95], the following formula was finally derived for the entropy:

$$S = -2\pi \oint d^2x \sqrt{h_{(2)}} \left(\frac{\partial \mathcal{L}}{\partial R_{abcd}} \right)_{\mathcal{H}} \hat{e}_{ab} \hat{e}_{cd}, \quad (56)$$

where \mathcal{L} is the Lagrangian of the theory, \hat{e}_{ab} the binormal to the horizon surface \mathcal{H} , and $h_{(2)}$ the two-dimensional projected metric on \mathcal{H} . The equivalence of the two approaches has been demonstrated in [93], in particular in the context of theories that contain higher-derivative terms such as the GB term. Here, we will use the Noether current approach to calculate the entropy of the black holes as it leads faster to the desired result.

To this end, we need to calculate the derivatives of the scalar gravitational quantities, appearing in the Lagrangian

of our theory (1), with respect to the Riemann tensor. In Appendix C, we present a simple way to derive those derivatives. Then, substituting in Eq. (56), we obtain

$$S = -\frac{1}{8} \oint d^2x \sqrt{h_{(2)}} \left\{ \frac{1}{2} (g^{ac} g^{bd} - g^{bc} g^{ad}) + f(\phi) [2R^{abcd} - 2(g^{ac} R^{bd} - g^{bc} R^{ad} - g^{ad} R^{bc} + g^{bd} R^{ac}) + R(g^{ac} g^{bd} - g^{bc} g^{ad})] \right\} \hat{e}_{ab} \hat{e}_{cd}. \quad (57)$$

The first term inside the curly brackets of the above expression comes from the variation of the Einstein-Hilbert term and leads to

$$S_1 = -\frac{1}{16} \oint d^2x \sqrt{h_{(2)}} (\hat{e}_{ab} \hat{e}^{ab} - \hat{e}_{ab} \hat{e}^{ba}). \quad (58)$$

We recall that \hat{e}_{ab} is antisymmetric and, in addition, satisfies $\hat{e}_{ab} \hat{e}^{ab} = -2$. Therefore, we easily obtain the result

$$S_1 = \frac{A_{\mathcal{H}}}{4}, \quad (59)$$

where $A_{\mathcal{H}} = 4\pi r_h^2$ is the horizon surface. The remaining terms in Eq. (57) are all proportional to the coupling function $f(\phi)$ and follow from the variation of the GB term. To facilitate the calculation, we notice that, on the horizon surface, the binormal vector is written as $\hat{e}_{ab} = \sqrt{-g_{00}g_{11}}|_{\mathcal{H}} (\delta_a^0 \delta_b^1 - \delta_a^1 \delta_b^0)$. This means that we may alternatively write

$$\left(\frac{\partial \mathcal{L}}{\partial R_{abcd}} \right)_{\mathcal{H}} \hat{e}_{ab} \hat{e}_{cd} = 4g_{00}g_{11}|_{\mathcal{H}} \left(\frac{\partial \mathcal{L}}{\partial R_{0101}} \right)_{\mathcal{H}}. \quad (60)$$

Therefore, the terms proportional to $f(\phi)$ may be written as

$$S_2 = -\frac{1}{2} f(\phi) g_{00}g_{11}|_{\mathcal{H}} \oint d^2x \sqrt{h_{(2)}} [2R^{0101} - 2(g^{00}R^{11} - g^{10}R^{01} - g^{01}R^{10} + g^{11}R^{00}) + g^{00}g^{11}R]_{\mathcal{H}}. \quad (61)$$

To evaluate the above integral, we will employ the near-horizon asymptotic solution (33)–(35) for the metric functions and scalar field. The asymptotic values of all quantities appearing inside the square brackets above are given in Appendix C. Substituting in Eq. (61), we straightforwardly find

$$S_2 = \frac{f(\phi_h) A_{\mathcal{H}}}{r_h^2} = 4\pi f(\phi_h). \quad (62)$$

Combining the expressions (59) and (62), we finally derive the result

$$S_h = \frac{A_h}{4} + 4\pi f(\phi_h). \quad (63)$$

The above describes the entropy of a GB black hole arising in the context of the theory (1), with a general coupling function $f(\phi)$ between the scalar field and the GB term, and a cosmological constant term. We observe that the above expression matches the one derived in [34] in the context of the theory (1) but in the absence of the cosmological constant. This was, in fact, expected on the basis of the more transparent Noether approach used here: the Λ term does not change the overall topology of the black-hole horizon, and it does not depend on the Riemann tensor; therefore, no modifications are introduced to the functional form of the entropy of the black hole due to the cosmological constant. However, the presence of Λ modifies in a quantitative way the properties of the black hole and therefore the value of the entropy, and temperature, of the found solutions.

III. ANTI-DE SITTER GAUSS-BONNET BLACK HOLES

In order to construct the complete black-hole solutions in the context of the theory (1), i.e., in the presence of both the GB and the cosmological constant terms, we need to numerically integrate the system of Eqs. (18) and (19). The integration starts at a distance very close to the horizon of the black hole, i.e., at $r \approx r_h + \mathcal{O}(10^{-5})$ (for simplicity, we set $r_h = 1$). The metric function A and scalar field ϕ in that regime are described by the asymptotic solutions (33) and (35). The input parameter ϕ'_h is uniquely determined through Eq. (27) once the coupling function $f(\phi) = \alpha \tilde{f}(\phi)$ is selected and the values of the remaining parameters of the model near the horizon are chosen. These parameters appear to be α , ϕ_h , and Λ . However, the first two are not independent: since it is their combination $\alpha \tilde{f}(\phi_h)$ that determines the strength of the coupling between the GB term and the scalar field, a change in the value of one of them may be absorbed in a corresponding change to the value of the other; as a result, we may fix α and vary only ϕ_h . The values of ϕ_h and Λ also cannot be totally uncorrelated as they both appear in the expression of C , Eq. (28), that must always be positive; therefore, once the value of the first is chosen, there is an allowed range of values for the second one for which black-hole solutions arise. This range of values are determined by the inequalities $\dot{f}_h^2 \leq \dot{f}_-^2$ and $\dot{f}_h^2 \geq \dot{f}_+^2$ according to Eq. (30), and lead in principle to two distinct branches of solutions. In fact, removing the square, four branches emerge depending on the sign of \dot{f}_h . However, in what follows we will assume that $\dot{f}_h > 0$, and thus study the two regimes $\dot{f}_h \leq \dot{f}_-$ and $\dot{f}_h \geq \dot{f}_+$; similar results emerge if one assumes instead that $\dot{f}_h < 0$.

Before starting our quest for black holes with an (anti-) de Sitter asymptotic behavior at large distances, we first

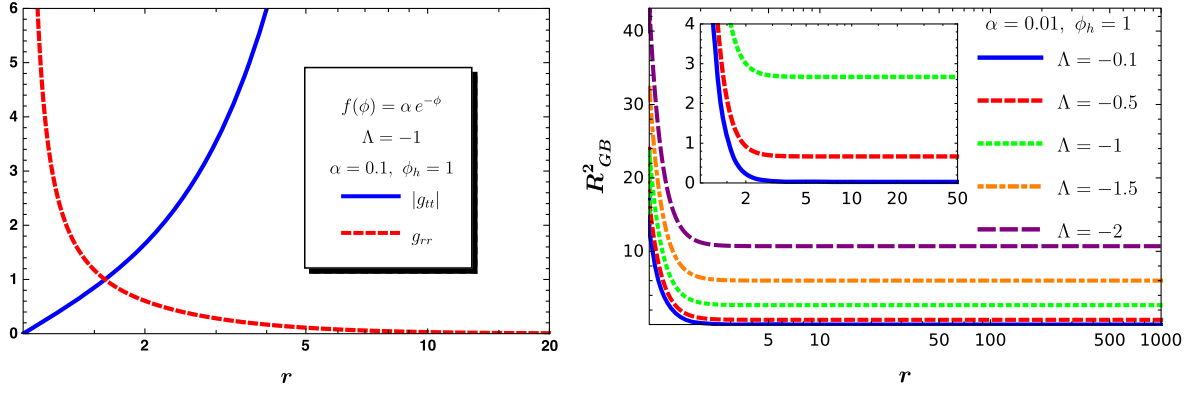


FIG. 1. The metric components $|g_{tt}|$ and g_{rr} (left plot), and the Gauss-Bonnet term R_{GB}^2 (right plot) in terms of the radial coordinate r , for $f(\phi) = \alpha e^{-\phi}$.

considered the case with $\Lambda = 0$ whereupon we successfully reproduced the families of asymptotically flat black holes derived in [34]. Then, we selected nonvanishing values of Λ and looked for novel black-hole solutions. For a negative cosmological constant ($\Lambda < 0$), the integration starts from the near-horizon regime with the asymptotic solutions (33) and (35), and proceeds toward large values of the radial coordinate until the form of the derived solution for the metric resembles the asymptotic solution (45) and (46) describing an anti-de Sitter-type background. The arbitrary coefficient a_1 that does not appear in the field equations may be fixed by demanding that, at very large distances, the metric functions satisfy the constraint $e^A \simeq e^{-B}$. We have considered a large number of forms for the coupling function $f(\phi)$, and, as we will now demonstrate, we have managed to produce a family of regular black-hole solutions with an anti-de Sitter asymptotic behavior, for every choice of $f(\phi)$.

A. Exponential coupling function

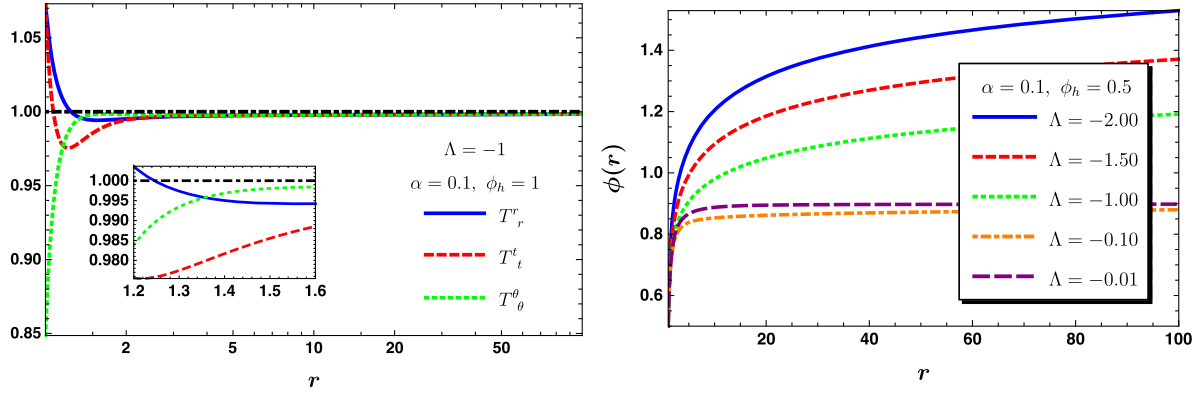
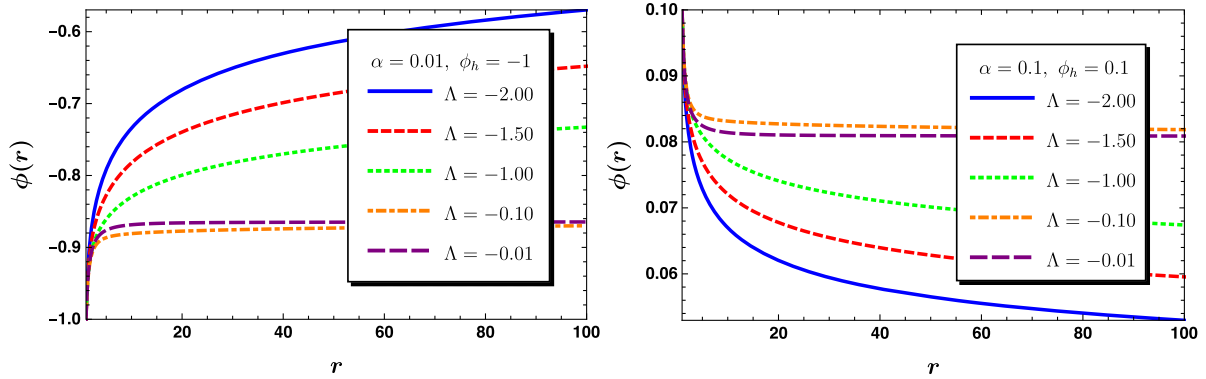
We will first discuss the case of an exponential coupling function, $f(\phi) = \alpha e^{-\phi}$. The solutions for the metric functions $e^{A(r)}$ and $e^{B(r)}$ are depicted in the left plot of Fig. 1. We may easily see that the near-horizon behavior, with $e^{A(r)}$ vanishing and $e^{B(r)}$ diverging, is eventually replaced by an anti-de Sitter regime with the exactly opposite behavior of the metric functions at large distances. The solution presented corresponds to the particular values $\Lambda = -1$ (in units of r_h^{-2}), $\alpha = 0.1$, and $\phi_h = 1$; however, we obtain the same qualitative behavior for every other set of parameters satisfying the constraint³ $\dot{f}_h \leq \dot{f}_-$ that follows from Eq. (28). The spacetime is regular in the whole radial regime, and this is reflected in the form of the scalar-invariant

Gauss-Bonnet term: this is presented in the right plot of Fig. 1, for $\alpha = 0.01$, $\phi_h = 1$ and for a variety of values of the cosmological constant. We observe that the GB term acquires its maximum value near the horizon regime, where the curvature of spacetime is larger, and reduces to a smaller, constant asymptotic value in the far-field regime. This asymptotic value is, as expected, proportional to the cosmological constant as this quantity determines the curvature of spacetime at large distances.

Although in Sec. II. B. 2, we could not find the analytic form of the scalar field at large distances from the black-hole horizon for different forms of the coupling function $f(\phi)$, our numerical results ensure that its behavior is such that the effect of the scalar field at the far-field regime is negligible, and it is only the cosmological term that determines the components of the energy-momentum tensor. In the left plot of Fig. 2, we display all three components of T^μ_ν over the whole radial regime, for the indicative solution $\Lambda = -1$, $\alpha = 0.1$, and $\phi_h = 1$. Far away from the black-hole horizon, all components reduce to $-\Lambda$, in accordance with Eqs. (11)–(13), with the effect of both the scalar field and the GB term there being negligible. Near the horizon, and according to the asymptotic behavior given by Eqs. (36)–(38), we always have $T^r_r \approx T^t_t$, since, at $r \simeq r_h$, $A' \simeq -B'$; also, the T^θ_θ component always has the opposite sign to that of T^r_r since $A'' \simeq -A'^2$. This qualitative behavior of T^μ_ν remains the same for all forms of the coupling function we have studied and for all solutions found; therefore we refrain from giving additional plots of this quantity for the other classes of solutions found.

From the results depicted in the left plot of Fig. 2, we see that, near the black-hole horizon, we always have $T^r_r \approx T^t_t > 0$. Comparing this behavior with the asymptotic forms (36)–(38), we deduce that, close to the black-hole horizon where $A' > 0$, we must have $(\phi' \dot{f})_h < 0$. In the case of a vanishing cosmological constant, the negative value of this quantity was of paramount importance for the evasion of the no-hair theorem [7] and the emergence of novel, asymptotically flat black-hole solutions [34].

³Note that no regular black-hole solutions were found that satisfy the alternative allowed choice $\dot{f}_h \geq \dot{f}_+$; for these choices of parameters, our numerical code does not lead to any solutions independently of the form of the coupling function. The same ill-defined behavior of this second potential branch of solutions with very small horizon radii was also found in [85].

FIG. 2. The energy-momentum tensor $T_{\mu\nu}$ (left plot) and scalar field ϕ (right plot) in terms of the radial coordinate r , for $f(\phi) = \alpha e^{-\phi}$.FIG. 3. The scalar field ϕ in terms of the radial coordinate r , for $f(\phi) = \alpha \phi^2$ (left plot) and $f(\phi) = \alpha \phi^3$ (right plot).

We observe that also in the context of the present analysis with $\Lambda \neq 0$, this quantity turns out to be again negative, and to lead once again to novel black-hole solutions. Coming back to our assumption of a decreasing exponential coupling function and upon choosing to consider $\alpha > 0$, the constraint $(\phi' \dot{f})_h < 0$ means that $\phi'_h > 0$ independently of the value of ϕ_h . In the right plot of Fig. 2, we display the solution for the scalar field in terms of the radial coordinate, for the indicative values of $\alpha = 0.1$, $\phi_h = 0.5$ and for different values of the cosmological constant. The scalar field satisfies indeed the constraint $\phi'_h > 0$ and increases away from the black-hole horizon.⁴ At large distances, we observe that, for small values of the cosmological constant, $\phi(r)$ assumes a constant value; this is the behavior found for asymptotically flat solutions [34] that the solutions with small Λ are bound to match. For increasingly larger values of Λ though, the profile of the scalar field deviates significantly from the series expansion in powers of $(1/r)$, thus allowing for a r -dependent ϕ even at infinity—in the perturbative limit, as we showed in the previous section, this dependence is given by the form $\phi(r) \simeq d_1 \ln r$.

⁴A complementary family of solutions arises if we choose $\alpha < 0$, with the scalar profile now satisfying the constraint $\phi'_h < 0$ and decreasing away from the black-hole horizon.

B. Polynomial coupling function

We will now consider the case of an even polynomial coupling function of the form $f(\phi) = \alpha \phi^{2n}$ with $n \geq 1$. The behavior of the solution for the metric functions matches the one depicted⁵ in the left plot of Fig. 1. The same is true for the behavior of the GB term and the energy-momentum tensor, whose profiles are similar to the ones displayed in Figs. 1 (right plot) and 2 (left plot), respectively. The positive-definite value of T^r_r near the black-hole horizon implies again that, there, we should have $(\dot{f} \phi')_h < 0$, or equivalently $\phi_h \phi'_h < 0$, for $\alpha > 0$. Indeed, two classes of solutions arise in this case: for positive values of ϕ_h , we obtain solutions for the scalar field that decrease away from the black-hole horizon, while for $\phi_h < 0$, solutions that increase with the radial coordinate are found. In Fig. 3

⁵Let us mention at this point that, for extremely large values of either the coupling constant α or the cosmological constant Λ , which are nevertheless allowed by the constraint (28), solutions that have their metric behavior deviating from the AdS-type form (45) and (46) were found; according to the obtained behavior, both metric functions seem to depend logarithmically on the radial coordinate instead of polynomially. As the physical interpretation of these solutions is not yet clear, we omit these solutions from the remainder of our analysis.

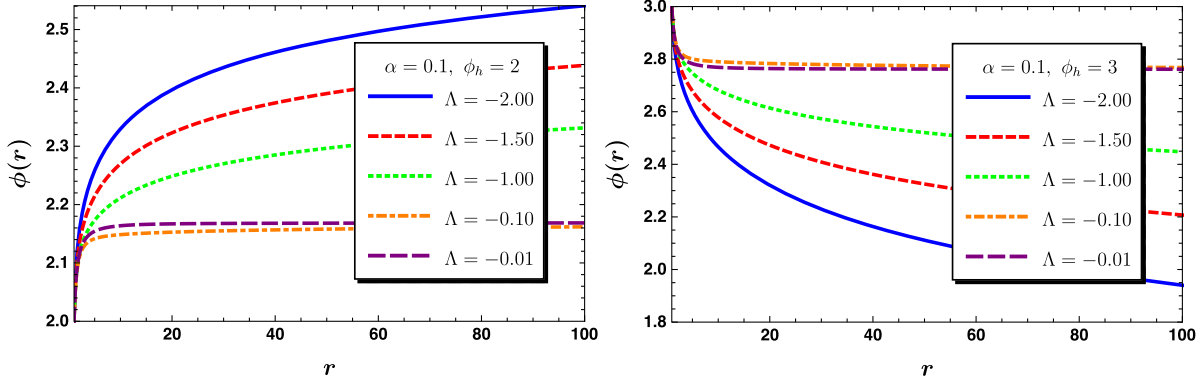


FIG. 4. The scalar field ϕ in terms of the radial coordinate r , for $f(\phi) = \alpha/\phi$ (left plot) and $f(\phi) = \alpha \ln \phi$ (right plot).

(left plot), we present a family of solutions for the case of the quadratic coupling function (i.e., $n = 1$), for $\phi_h = -1$ and $\alpha = 0.01$, arising for different values of Λ —since $\phi_h < 0$, the scalar field exhibits an increasing behavior as expected.

Let us examine next the case of an odd polynomial coupling function, $f(\phi) = \alpha\phi^{2n+1}$ with $n \geq 0$. The behavior of the metric functions, GB term, and energy-momentum tensor have the expected behavior for an asymptotically AdS background, as in the previous cases. The solutions for the scalar field near the black-hole horizon are found to satisfy the constraint $\alpha(\phi^{2n}\phi')_h < 0$ or simply $\phi'_h < 0$, when $\alpha > 0$. As this holds independently of the value of ϕ_h , all solutions for the scalar field are expected to decrease away from the black-hole horizon. Indeed, this is the profile depicted in the right plot of Fig. 3 where a family of solutions for the indicative case of a cubic coupling function (i.e., $n = 1$) is presented for $\alpha = 0.1$, $\phi_h = 0.1$, and various values of Λ .

C. Inverse polynomial coupling function

The case of an inverse polynomial coupling function, $f(\phi) = \alpha\phi^{-k}$, with k either an even or an odd positive integer, was also considered. For odd k , i.e., $k = 2n + 1$, the positivity of T^r_r near the black-hole horizon demands again that $(f\phi')_h < 0$, or that $-\alpha/\phi^{2n+2}\phi' < 0$. For $\alpha > 0$, the solution for the scalar field should thus always satisfy $\phi'_h > 0$, regardless of our choices for ϕ_h or Λ . As an indicative example, in the left plot of Fig. 4, we present the case of $f(\phi) = \alpha/\phi$ with a family of solutions arising for $\alpha = 0.1$ and $\phi_h = 2$. The solutions for the scalar field clearly satisfy the expected behavior by increasing away from the black-hole horizon. On the other hand, for even k , i.e., $k = 2n$, the aforementioned constraint now demands that $\phi_h\phi'_h < 0$. As in the case of the odd polynomial coupling function, two subclasses of solutions arise: for $\phi_h > 0$, solutions emerge with $\phi'_h < 0$, whereas, for $\phi_h < 0$, we find solutions with $\phi'_h > 0$. The profiles of the solutions in this case are similar to the ones found

before, with ϕ approaching, at large distances, an almost constant value for small Λ but adopting a more dynamical behavior as the cosmological constant gradually takes on larger values.

D. Logarithmic coupling function

As a final example of another form of the coupling function between the scalar field and the GB term, let us consider the case of a logarithmic coupling function, $f(\phi) = \alpha \ln \phi$. Here, the condition near the horizon of the black hole gives $\alpha\phi'/\phi < 0$, and therefore, for $\alpha > 0$, we must have $\phi'_h\phi_h < 0$; for $\phi_h > 0$, this translates to a decreasing profile for the scalar field near the black-hole horizon. In the right plot of Fig. 4, we present a family of solutions arising for a logarithmic coupling function for fixed $\alpha = 0.01$ and $\phi_h = 1$, while varying the cosmological constant Λ . The profiles of the scalar field agree once again with the one dictated by the near-horizon constraint, and they all decrease in that regime. As in the previous cases, the metric functions approach asymptotically an anti-de Sitter background, the scalar-invariant GB term remains everywhere regular, and the same is true for all components of the energy-momentum tensor that asymptotically approach the value $-\Lambda$.

E. Physical properties of the solutions

It is of particular interest to study also the behavior of the effective potential of the scalar field, a role that in our theory is played by the GB term together with the coupling function, i.e., $V_\phi \equiv \dot{f}(\phi)R_{GB}^2$. In the left plot of Fig. 5, we present a combined graph that displays its profile in terms of the radial coordinate, for a variety of forms of the coupling function $f(\phi)$. As expected, the potential V_ϕ takes on its maximum value always near the horizon of the black hole, where the GB term is also maximized and thus sources the nontrivial form of the scalar field. On the other hand, as we move toward larger distances, V_ϕ reduces to an asymptotic constant value. Although this asymptotic value clearly depends on the

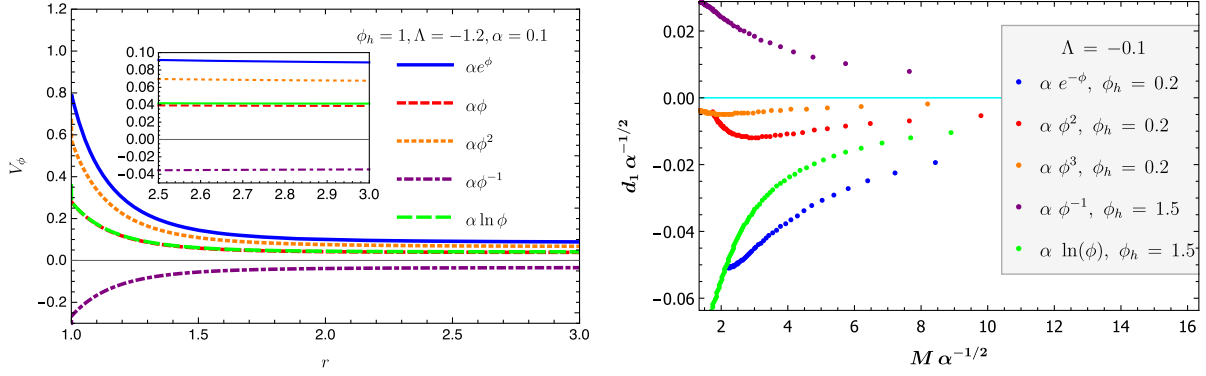


FIG. 5. The effective potential V_ϕ of the scalar field, in terms of the radial coordinate (left plot), and the coefficient d_1 (right plot) in terms of the mass M , for various forms of $f(\phi)$.

choice of the coupling function, its common behavior allows us to comment on the asymptotic behavior of the scalar field at large distances. Substituting a constant value V_∞ in the place of V_ϕ in the scalar-field equation (14), we arrive at the intermediate result

$$\partial_r [e^{(A-B)/2} r^2 \phi'] = -e^{(A+B)/2} r^2 V_\infty. \quad (64)$$

Then, employing the asymptotic forms of the metric functions at large distances (45) and (46), the above may easily be integrated with respect to the radial coordinate to yield a form for the scalar field identical to the one given in Eq. (50). We may thus conclude that the logarithmic form of the scalar field may adequately describe its far-field behavior even beyond the perturbative limit of very small α .

We now proceed to discuss the physical characteristics of the derived solutions. Because of the large number of solutions found, we will present, as for V_ϕ , combined graphs for different forms of the coupling function $f(\phi)$. Starting with the scalar field, we notice that no conserved quantity, such as a scalar charge, may be associated with the solution at large distances in the case of asymptotically anti-de Sitter black holes: the absence of an $\mathcal{O}(1/r)$ term in the far-field expression (50) of the scalar field, which would signify the existence of a long-range interaction term, excludes the emergence of such a quantity, even of secondary nature. One could attempt instead to plot the dependence of the coefficient d_1 , as a quantity that predominantly determines the rate of change of the scalar field at the far field, in terms of the mass of the black hole. This is displayed in the right plot of Fig. 5 for the indicative value $\Lambda = -0.1$ of the cosmological constant. We see that, for small values of the mass M , this coefficient takes in general a nonzero value, which amounts to having a nonconstant value of the scalar field at the far-field regime. As the mass of the black hole increases though, this coefficient asymptotically approaches a zero value. Therefore, the rate of change of the scalar field at infinity for massive GB black holes becomes negligible, and the

scalar field tends to a constant. This is the ‘‘Schwarzschild-AdS regime,’’ where the GB term decouples from the theory and the scalar-hair disappears—the same behavior was observed also in the case of asymptotically flat GB black holes [34] where, in the limit of large mass, all of our solutions merged with the Schwarzschild ones.

We present next the ratio of the horizon area of our solutions compared to the horizon area of the SAdS one with the same mass, for the indicative values of the negative cosmological constant $\Lambda = -0.001$ and $\Lambda = -0.1$ in the two plots of Fig. 6. These plots provide further evidence for the merging of our GB black-hole solutions with the SAdS solution in the limit of large mass. The left plot of Fig. 6 reveals that, for a small cosmological constant, all our GB solutions remain smaller than the scalar-hair-free SAdS solution independently of the choice for the coupling function $f(\phi)$ —this is in complete agreement with the profile found in the asymptotically flat case [34]. This behavior persists for even larger values of the negative cosmological constant for all classes of solutions apart from the one emerging for the logarithmic function whose horizon area is significantly increased in the small-mass regime, as may be seen from the right plot of Fig. 6. These plots also verify the termination of all branches of solutions at the point of a minimum horizon, or minimum mass, that all our GB solutions exhibit as a consequence of the inequality (28). We also observe that, as hinted by the small- Λ approximation given in Eq. (30), an increase in the value of the negative cosmological constant pushes upwards the lowest allowed value of the horizon radius of our solutions.

We now move to the thermodynamical quantities of our black-hole solutions. We start with their temperature T given by Eq. (54) in terms of the near-horizon coefficients (a_1, b_1) . In the left plot of Fig. 7, we display its dependence in terms of the cosmological constant Λ , for several forms of the coupling function. We observe that T increases, too, with $|\Lambda|$; we thus conclude that the more negatively curved the spacetime is, the hotter the black hole that is formed is. Note that the form of the coupling function plays almost

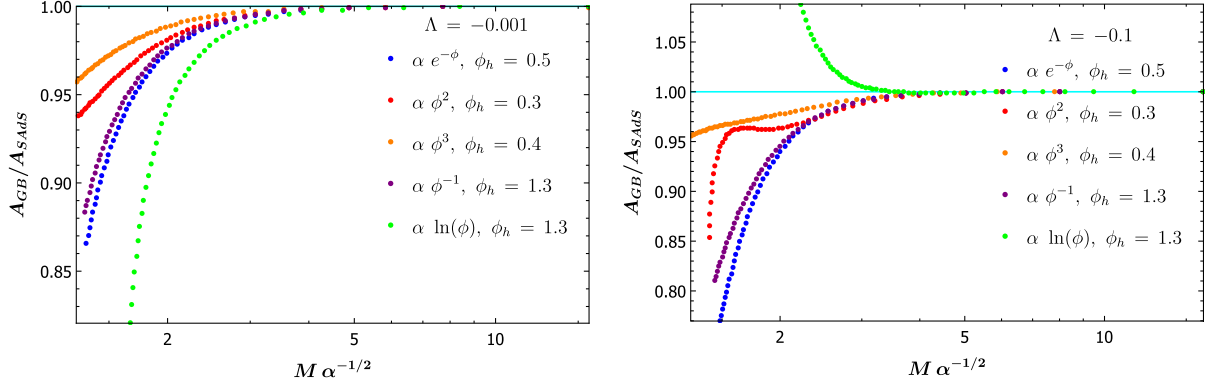


FIG. 6. The area ratio A_{GB}/A_{SAdS} of our solutions as a function of the mass M of the black hole, for various forms of $f(\phi)$, and for $\Lambda = -0.001$ (left plot) and $\Lambda = -0.1$ (right plot).

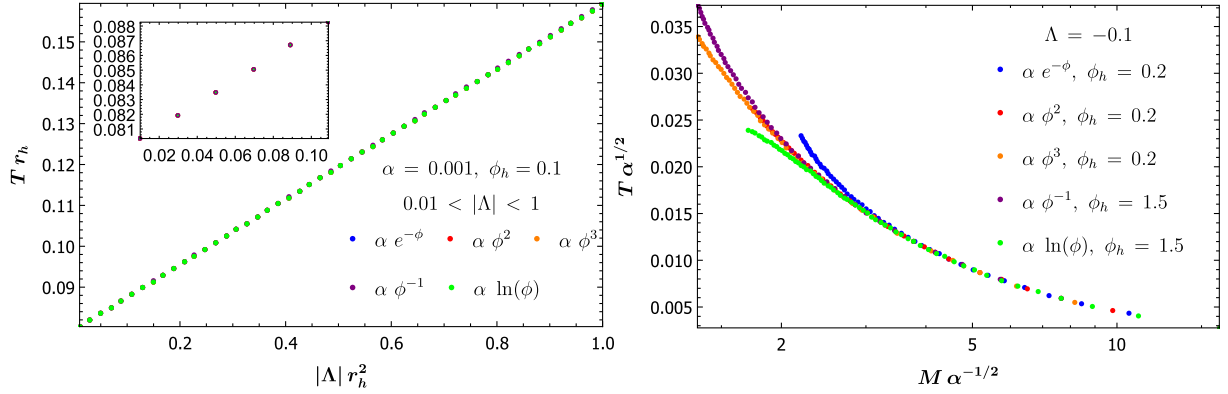


FIG. 7. The temperature T of the black hole as a function of the cosmological constant Λ (left plot) and the mass M of the black hole (right plot), for various forms of $f(\phi)$.

no role in this relation with the latter thus acquiring a universal character for all GB black-hole solutions. The dependence of the temperature of the black hole on its mass, as displayed in the right plot of Fig. 7, exhibits a decreasing profile, with the obtained solution being colder the larger its mass is. For small black-hole solutions, the exact dependence of T on M depends on the particular form of the coupling function but for solutions with a large mass

its role becomes unimportant as a common Schwarzschild-AdS regime is again approached.

Let us finally study the entropy of the derived black-hole solutions. In Fig. 8, we display the ratio of the entropy of our GB solutions over the entropy of the corresponding Schwarzschild-anti-de Sitter solution with the same mass, for the same indicative values of the negative cosmological constant as for the horizon area. i.e., for $\Lambda = -0.001$

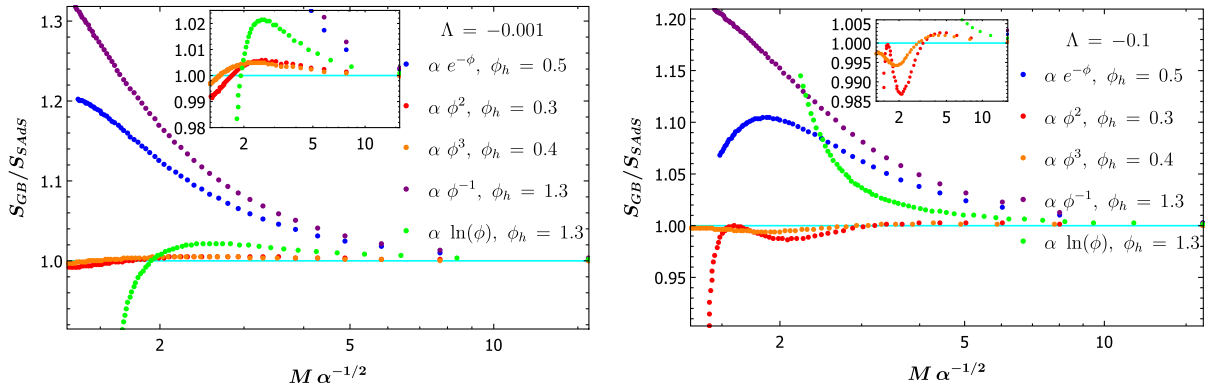


FIG. 8. The entropy ratio S_{GB}/S_{SAdS} of our solutions as a function of the mass M of the black hole, for various forms of $f(\phi)$, and for $\Lambda = -0.001$ (left plot) and $\Lambda = -0.1$ (right plot).

(left plot) and $\Lambda = -0.1$ (right plot). We observe that the profile of this quantity depends strongly on the choice of the coupling function $f(\phi)$ for solutions with small masses, whereas in the limit of large mass, where our solutions reduce to the SAdS ones, this ratio approaches unity as expected. For small values of Λ , the left plot of Fig. 8 depicts a behavior similar to the one found in the asymptotically flat case [34]: solutions emerging for the linear and the quadratic coupling functions exhibit smaller entropy compared to the SAdS one, while solutions for the exponential, logarithmic, and inverse-linear coupling functions lead to GB black holes with a larger entropy over the whole mass range or for particular mass regimes. As we increase the value of the cosmological constant (see right plot of Fig. 8), the entropy ratio is suppressed for all families of GB black holes apart from the one emerging for the logarithmic coupling function, which exhibits a substantial increase in this quantity over the whole mass regime. Together with the solutions for the exponential and inverse-linear coupling functions, they have an entropy ratio larger than unity while this ratio is now significantly lower than unity for all the other polynomial coupling functions. Although the question of the stability of the derived solutions is an important one and must be independently studied for each family of solutions found, the entropy profiles presented above may provide some hints regarding the thermodynamical stability of our solutions compared to the Schwarzschild–anti-de Sitter ones.

IV. DE SITTER GAUSS-BONNET BLACK HOLES

We now address the case of a positive cosmological constant, $\Lambda > 0$. We start our integration process at a distance close to the black-hole horizon, using the asymptotic solutions (33)–(35) and choosing ϕ_h to satisfy again the regularity constraint (27). The coupling function $f(\phi)$ is assumed to take on a variety of forms—namely exponential, even and odd polynomial, inverse even and odd polynomial, and logarithmic forms—as in the case of the negative cosmological constant. The numerical integration then proceeds outwards to meet the corresponding asymptotic solution (42)–(44) near the cosmological horizon.

Unfortunately, and despite our persistent efforts, no complete black-hole solution interpolating between the asymptotic solutions (33)–(35) and (42)–(44) was found. The same negative result concerning the existence of a black hole solution with an asymptotically de Sitter behavior was obtained in [85], where the case of a linear coupling function between the GB term and the scalar field was considered. It is, however, worth noting that the two asymptotic solutions near the black-hole and cosmological horizons do independently emerge—it is the effort to match them in a smooth way via an intermediate solution that fails.

To demonstrate this, in Fig. 9 we display the result of our numerical integration for the indicative case of $\alpha = 0.01$, $\phi_h = -1$, and $\Lambda = 0.01$. The coupling function has been chosen to be $f(\phi) = \alpha e^{-\phi}$; however, the same qualitative behavior was found for every choice of $f(\phi)$ we have considered. From the metric functions and the scalar-field profiles displayed in the two plots, we clearly see that an asymptotic solution describing a regular black-hole horizon is indeed formed. In this, the metric component $|g_{tt}|$ vanishes while the g_{rr} one diverges, as expected. The scalar field near the black-hole horizon assumes a finite, constant value while it decreases away from the horizon, in perfect agreement with the scalar-field profile found in the case of a negative cosmological constant. The integration proceeds uninhibited but stops abruptly close to the regime where the cosmological horizon should form. In fact, from the left plot of Fig. 9, we may see the expected behavior of the metric components near the cosmological horizon (i.e., the vanishing of $|g_{tt}|$ and divergence of g_{rr}) just to emerge.

The emergence of asymptotic solutions and the failure to smoothly match them strongly reminds us of the analysis involved in the no-hair theorems [3,7], where a similar situation holds. It is, however, difficult to generalize that analysis, or equivalently the argument for their evasion as developed in [34], in the present case of a nonvanishing cosmological constant.⁶ One could, nevertheless, gain some understanding of the situation by examining the form of the near-horizon value of the T^r_r component of the energy-momentum tensor given in Eq. (37)—the profile of this component is of paramount importance for the evasion of the novel no-hair theorem [7] and the emergence of novel solutions. For the evasion to be realized, this component must be positive and decreasing close to the black-hole horizon [8,34]. From Eq. (37), it becomes clear that the presence of a negative cosmological constant ($\Lambda < 0$) in the theory always gives a positive contribution to T^r_r and enhances the probability of obtaining regular black holes. This justifies the easiness in which novel black-hole solutions with an asymptotically anti-de Sitter behavior have emerged in the context of our analysis. On the other hand, the contribution of a positive cosmological constant ($\Lambda > 0$) to T^r_r is always negative, and this makes the evasion of the no-hair theorem less likely. It would indeed be interesting to readdress the arguments presented in [34] as well as the ones employed in the versions of the no-hair theorems for nonasymptotically flat black holes [66–70] to cover also the case where the GB term and the cosmological constant appear simultaneously in the theory.

Nevertheless, even if the evasion of the no-hair theorems may be realized for $\Lambda > 0$ in the presence of the GB

⁶A theoretical analysis is currently under way but has, so far, not given any conclusive results.

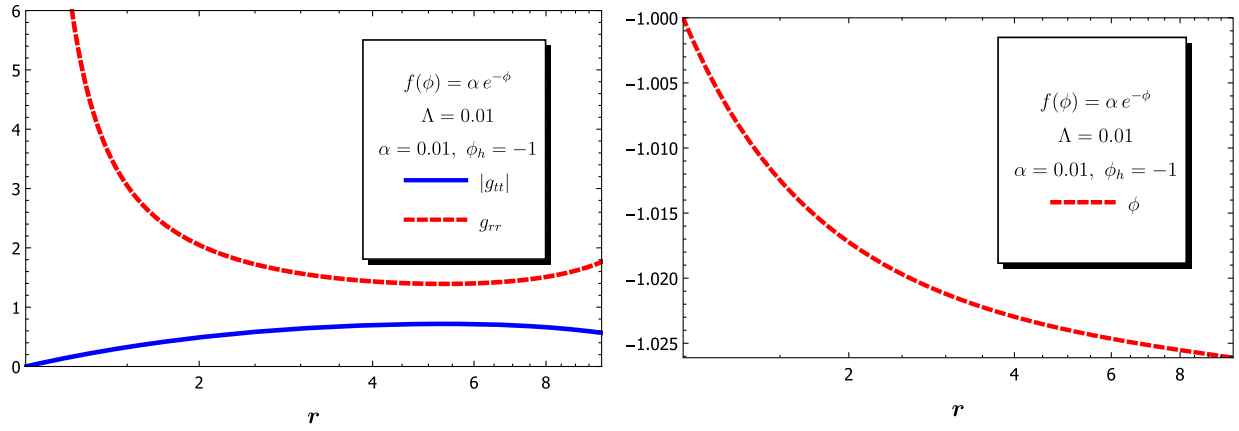


FIG. 9. The metric functions $|g_{tt}|$ and g_{rr} of the spacetime (left plot) and the scalar field ϕ (right plot) in terms of the radial coordinate r , for a positive cosmological constant and coupling function $f(\phi) = \alpha e^{-\phi}$.

term—for small values of Λ this seems quite likely—this merely opens the way to look for novel solutions, and it does not guarantee their existence. The emergence of a complete solution interpolating between the two horizons still demands the smooth matching of the two asymptotic solutions. It is quite likely that the system does not have enough freedom to simultaneously satisfy the requirements for the existence of a regular solution, namely Eqs. (27) and (28) and Eqs. (40) and (41)—this was also noted in [85]. Or, a very careful selection of parameters may be necessary for such a solution to emerge. In any case, further investigation is necessary, and we hope to return to this topic soon.

V. CONCLUSIONS

In this work, we have extended our previous analyses [34], on the emergence of novel, regular black-hole solutions in the context of the Einstein-scalar-GB theory, to include the presence of a positive or negative cosmological constant. Since the uniform distribution of energy associated with the cosmological constant permeates the whole spacetime, we expected Λ to have an effect on both the near-horizon and far-field asymptotic solutions. Indeed, our analytical calculations in the small- r regime revealed that the cosmological constant modifies the constraint that determines the value of ϕ'_h for which a regular, black-hole horizon forms. In addition, it was demonstrated that such a horizon is indeed formed, for either positive or negative Λ and for all choices of the coupling function $f(\phi)$.

In contrast, the behavior of the solution in the far-field regime depended strongly on the sign of the cosmological constant. For $\Lambda > 0$, a second horizon, the cosmological one, was expected to form at a distance $r_c > r_h$, whereas for $\Lambda < 0$, an anti-de Sitter-type of solution was sought for at asymptotic infinity. Both types of solutions were analytically shown to be admitted by the set of our field

equations at the limit of large distances, thus opening the way for the construction of complete black-hole solutions with an (anti-)de Sitter asymptotic behavior.

The complexity of the field equations prevented us from constructing such a solution analytically; therefore we turned to numerical analysis. Using our near-horizon analytic solution as a starting point, we integrated the set of field equations from the black-hole horizon and outwards. For a negative cosmological constant ($\Lambda < 0$), we demonstrated that regular black-hole solutions with an anti-de Sitter-type asymptotic behavior arise with the same easiness that their asymptotically flat counterparts emerge. We have produced solutions for an exponential, polynomial (even or odd), inverse polynomial (even or odd), and logarithmic coupling function between the scalar field and the GB term. In each and every case, once $f(\phi)$ was chosen, selecting the input parameter ϕ'_h to satisfy the regularity constraint (27) and the second input parameter ϕ_h to satisfy the inequality (28) a regular black hole solution always emerged. The metric components exhibited the expected behavior near the black-hole and asymptotic infinity with the scalar invariant GB term being everywhere regular. All solutions possessed non-trivial scalar hair, with the scalar field having a nontrivial profile both close to and far away from the black-hole horizon. For small negative values of Λ , we recovered the power-law falloff of the scalar field at infinity, found in the asymptotically flat case [34], whereas for large negative values of Λ the profile of ϕ was dominated by a logarithmic dependence on the radial coordinate. This behavior was analytically shown to emerge both in the linear coupling-function case and in the perturbative limit, in terms of the coupling parameter α , but it was numerically found to accurately describe all of our solutions at large distances.

The absence of a $(1/r)$ term in the expression of the scalar field at large distances excludes the presence of a scalar charge, even a secondary one. As a result, in the

presence of Λ , the black-hole scalar hair amounts to having a nontrivial, regular scalar field associated with the black hole—a feature also excluded by the traditional no-hair theorems—but not a conserved quantity. The coefficient d_1 in front of the logarithmic term in the expression of ϕ can give us information on how much the large-distance behavior of the scalar field deviates from the power-law one valid in the asymptotically flat case. We have found that this deviation is stronger for GB black holes with a small mass, whereas the more massive ones have a d_1 coefficient that tends to zero. The temperature of the black holes was found to increase with the cosmological constant independently of the form of the coupling function. The latter plays a more important role in the relation of T with the black-hole mass: while the temperature decreases with M for all classes of solutions found, the lighter ones exhibit a stronger dependence on $f(\phi)$. The same dependence on the form of the coupling function is observed in the entropy and horizon area of our solutions. For small masses, the entropy of each class of solutions has a different behavior, with the ones for the exponential, inverse-linear polynomial and logarithmic coupling functions exhibiting a ratio $S_{\text{GB}}/S_{\text{SAdS}}$ (over the entropy of the Schwarzschild–anti-de Sitter black hole with the same mass) larger than unity for the entire mass range, for large values of Λ . This feature hints toward the enhanced thermodynamical stability of our solutions compared to their GR analogues. In the limit of large mass, the entropy of all classes of our solutions tend to the one of the Schwarzschild–anti-de Sitter black hole with the same mass. The same holds for the horizon area: while for small masses, each class has its own pattern with M , with all solutions being smaller in size than the corresponding SAdS one apart from the logarithmic case, for large masses all black-hole solutions match the horizon area of the SAdS solution.

Based on the above, we conclude that our GB black-hole solutions with a negative cosmological constant smoothly merge with the SAdS ones, in the large mass limit. As in the asymptotically flat case, it is the small-mass range that provides the characteristic features for the GB solutions. These solutions have a modified dependence of both their temperature and horizon area on their mass compared to the SAdS solution. Another characteristic is also the minimum horizon, or minimum mass, that all our GB solutions possess due to the inequality (28).

Turning to GB solutions with a positive cosmological constant, our quest has failed to find any such solutions. Although the presence of a positive Λ does not obstruct the formation of a regular black-hole or cosmological horizon, our numerical integration did not manage to produce a complete solution that would interpolate between the two asymptotic regimes. This result holds independently of the choice of the GB coupling function $f(\phi)$ or the value of Λ .

In conclusion, we have demonstrated that the general classes of theories that contain the GB term and lead to novel black-hole solutions, continue to do so even in the presence of a negative cosmological constant in the theory. In contrast, the presence of a positive cosmological constant presents a severe obstacle for the formation of these solutions. A further investigation is clearly necessary in both cases: the relevance of the GB solutions with an anti-de Sitter-type asymptotic solution in the context of the AdS-CFT correspondence should be inquired, and the deeper reason for the absence of solutions with a positive cosmological constant should be investigated further. We hope to return soon with results on both issues.

ACKNOWLEDGMENTS

G. A. thanks Onassis Foundation for the financial support provided through its scholarship program. This research is co-financed by Greece and the European Union (European Social Fund-ESF) through the Operational Programme “Human Resources Development, Education and Lifelong Learning” in the context of the project “Strengthening Human Resources Research Potential via Doctorate Research” (MIS-5000432), implemented by the State Scholarships Foundation (IKY).

APPENDIX A: SET OF DIFFERENTIAL EQUATIONS

Here, we display the explicit expressions of the coefficients S , P , and Q that appear in the system of differential equations (18) and (19) whose solution determines the metric function A and the scalar field ϕ . Note that in these expressions we have eliminated, via Eq. (17), B' , which involves A'' and ϕ'' , but retained e^B for notational simplicity. They are

$$\begin{aligned}
 S = & 2304A'\dot{f}^3\phi'^2 + 8e^B[-128rA'\dot{f}^2\phi' - 448A'\dot{f}^3\phi'^2 + 32r^2\dot{f}^2\phi'^3 - 80\dot{f}^2\phi'] \\
 & + 8e^{2B}[16r^2A'f' + 160rA'\dot{f}^2\phi' + 160A'\dot{f}^3\phi'^2 - 12r^3\dot{f}\phi'^2 - 16r^2\dot{f}^2\phi'^3 \\
 & - 64\Lambda r^2\dot{f}^2\phi' + 16r\dot{f} + 160\dot{f}^2\phi'] + 8e^{3B}[-16r^2A'\dot{f} - 32rA'\dot{f}^2\phi' + 4r^3\dot{f}\phi'^2 \\
 & + 16\Lambda r^3\dot{f} + 64\Lambda r^2\dot{f}^2\phi' - 32r\dot{f} - 80\dot{f}^2\phi' + r^4\phi'] + 8e^{4B}[16r\dot{f} - 16\Lambda r^3\dot{f}],
 \end{aligned} \tag{A1}$$

$$\begin{aligned}
P = & -128e^{4B}\Lambda^2r^3\dot{f}(rA' + 2e^B - 2) + 16A'^3\dot{f}[-2e^B(-14e^B + 3e^{2B} + 19)r\dot{f}\phi' \\
& + 8(-8e^B + 3e^{2B} + 9)\dot{f}^2\phi'^2 - e^{2B}(3e^B - 5)r^2] + 4e^BA'^2\{e^Br\dot{f}[(5e^B - 19)r^2\phi'^2 \\
& + 12(e^B - 1)^2] - 4\dot{f}^2\phi'[(9e^B - 17)r^2\phi'^2 + 8(e^B - 1)^2] + e^{2B}r^4\phi'\} \\
& + 4e^{2B}2\Lambda\{-e^{2B}r^3(-2 + rA')\phi' - 16A'\dot{f}^2\phi'[6(3 - 4e^B + e^{2B}) + (-5 + e^B)rA'] \\
& + 4e^B\dot{f}[-3r^2A'^2(1 + e^B) + 4(4(-1 + e^B)^2 - r^2\phi'^2) + 2rA'(3 - 3e^B + r^2\phi'^2)]\} \\
& - 2e^{2B}r\phi'\{-8\dot{f}\phi'[4e^B(-1 + e^B) + r^2\phi'^2(-2 + e^B)] - 4re^B(-1 + e^B) \\
& - r\phi'^2[r^2e^B - 16\dot{f}(-1 + e^B)]\} - A'e^B\{32r\dot{f}^2\phi'^2\phi'(9 - 4e^B + 3e^{2B}) \\
& - r^3\phi'e^B[4e^B(1 + e^B) - \phi'^2(r^2e^B + 16\dot{f}(1 + e^B))]\} \\
& + 8e^B\dot{f}[4(-1 + e^B)^2 + r^2\phi'^2(-7 + 3e^B) - 2\phi'^4(r^4 + 8r^2\dot{f})]\}, \tag{A2}
\end{aligned}$$

and

$$\begin{aligned}
Q = & 2304A'\dot{f}^2\dot{f}\phi'^4 - 1152A'^2\dot{f}^3\phi'^3 + e^B[-144r^2A'\dot{f}^2\phi'^4 + 672rA'^2\dot{f}^2\phi'^2 \\
& + 768A'^2\dot{f}^3\phi'^3 - 384A'\dot{f}^2\phi'^2 - 1024rA'\dot{f}\dot{f}\phi'^3 - 3584A'\dot{f}^2\dot{f}\phi'^4 \\
& + 480r\dot{f}^2\phi'^4 + 64r^2\dot{f}\dot{f}\phi'^5 - 640\dot{f}\dot{f}\phi'^3] + e^{2B}[128r^2A'\dot{f}\phi'^2 + 52r^3A'\dot{f}\phi'^3 \\
& + 80r^2A'\dot{f}^2\phi'^4 - 128r^2A'^2\dot{f}\phi' - 576\Lambda r^2A'\dot{f}^2\phi'^2 - 320rA'^2\dot{f}^2\phi'^2 \\
& + 176rA'\dot{f}\phi' - 128A'^2\dot{f}^3\phi'^3 + 640A'\dot{f}^2\phi'^2 + 1280rA'\dot{f}\dot{f}\phi'^3 + 1280A'\dot{f}^2\dot{f}\phi'^4 \\
& - 16r^3\dot{f}\phi'^4\phi'^4 + 128r\dot{f}\phi'^2 - 4r^4\dot{f}\phi'^5 - 152r^2\dot{f}\phi'^3 - 256r\dot{f}^2\phi'^4 + 384\Lambda r\dot{f}^2\phi'^2 \\
& + 160\dot{f}\phi' - 64r^2\dot{f}\dot{f}\phi'^5 - 512\Lambda r^2\dot{f}\dot{f}\phi'^3 + 1280\dot{f}\dot{f}\phi'^3] + e^{3B}[-128r^2A'\dot{f}\phi'^2 \\
& + 208\Lambda r^3A'\dot{f}\phi' + 32r^2A'^2\dot{f}\phi' + 320\Lambda r^2A'\dot{f}^2\phi'^2 + 32rA'^2\dot{f}^2\phi'^2 - 224rA'\dot{f}\phi' \\
& - 256A'\dot{f}^2\phi'^2 - 256rA'\dot{f}\dot{f}\phi'^3 - 6r^4A'\phi'^2 + 8r^3A'^2 - 24r^2A' + 16r^3\dot{f}\phi'^4 \\
& + 128\Lambda r^3\dot{f}\phi'^2 - 256r\dot{f}\phi'^2 + 16\Lambda r^4\dot{f}\phi'^3 + 24r^2\dot{f}\phi'^3 - 12r^3A'\dot{f}\phi'^3 + 32r\dot{f}^2\phi'^4 \\
& + 224\Lambda r^2\dot{f}\phi' - 512\Lambda r\dot{f}^2\phi'^2 - 320\dot{f}\phi' + 512\Lambda r^2\dot{f}\dot{f}\phi'^3 - 640\dot{f}\dot{f}\phi'^3 + r^5\phi'^4 \\
& + 12r^3\phi'^2 - 32r] + e^{4B}[-48\Lambda r^3A'\dot{f}\phi' + 48rA'\dot{f}\phi' - 24\Lambda r^4A' + 24r^2A' \\
& - 128\Lambda r^3\dot{f}\phi'^2 + 128r\dot{f}\phi'^2 + 128\Lambda^2r^4\dot{f}\phi' - 224\Lambda r^2\dot{f}\phi' + 128\Lambda r\dot{f}^2\phi'^2 + 160\dot{f}\phi' \\
& - 4\Lambda r^5\phi'^2 + 4r^3\phi'^2 - 64\Lambda r^3 + 64r] + e^{5B}[-32\Lambda^2r^5 + 64\Lambda r^3 - 32r]. \tag{A3}
\end{aligned}$$

APPENDIX B: SCALAR QUANTITIES

By employing the metric components of the line element (7), one may compute the following scalar-invariant gravitational quantities:

$$R = +\frac{e^{-B}}{2r^2}(4e^B - 4 - r^2A'^2 + 4rB' - 4rA' + r^2A'B' - 2r^2A''), \tag{B1}$$

$$\begin{aligned}
R_{\mu\nu}R^{\mu\nu} = & +\frac{e^{-2B}}{16r^4}[8(2 - 2e^B + rA' - rB')^2 + r^2(rA'^2 - 4B' - rA'B' + 2rA'')^2 \\
& + r^2(rA'^2 + A'(4 - rB') + 2rA'')^2], \tag{B2}
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = & +\frac{e^{-2B}}{4r^4}[r^4A'^4 - 2r^4A'^3B' - 4r^4A'B'A'' + r^2A'^2(8 + r^2B'^2 + 4r^2A'') \\
& + 16(e^B - 1)^2 + 8r^2B'^2 + 4r^4A''^2], \tag{B3}
\end{aligned}$$

$$R_{\text{GB}}^2 = +\frac{2e^{-2B}}{r^2}[(e^B - 3)A'B' - (e^B - 1)A'^2 - 2(e^B - 1)A'']. \tag{B4}$$

APPENDIX C: VARIATION WITH RESPECT TO THE RIEMANN TENSOR

Here, we derive the derivatives of the Lagrangian of the theory (1) with respect to the Riemann tensor. A simple way to do this is to take the derivatives ignoring the symmetries, which the final expression should possess, and restore them afterwards. For example, if A_{abcd} is a four-rank tensor and A the corresponding scalar quantity, we may write

$$\frac{\partial A}{\partial A_{abcd}} = \frac{\partial}{\partial A_{abcd}} (g^{\mu\rho} g^{\nu\sigma} A_{\mu\nu\rho\sigma}) = g^{\mu\rho} g^{\nu\sigma} \delta_\mu^a \delta_\nu^b \delta_\rho^c \delta_\sigma^d = g^{ac} g^{bd}. \quad (C1)$$

Now, if $A_{abcd} = R_{abcd}$, it should satisfy the following relations:

$$A_{abcd} = A_{cdab} = -A_{abdc} \quad \text{and} \quad A_{abcd} + A_{acdb} + A_{adbc} = 0. \quad (C2)$$

Restoring the symmetries, we arrive at

$$\frac{\partial R}{\partial R_{abcd}} = \frac{1}{2} (g^{ac} g^{bd} - g^{bc} g^{ad}). \quad (C3)$$

Alternatively, we could have explicitly written

$$\begin{aligned} \frac{\partial R}{\partial R_{abcd}} &= \frac{\partial}{\partial R_{abcd}} (g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}) \\ &= \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \frac{\partial}{\partial R_{abcd}} (R_{\mu\nu\rho\sigma} - R_{\nu\mu\rho\sigma}) \\ &= \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} (\delta_\mu^a \delta_\nu^b \delta_\rho^c \delta_\sigma^d - \delta_\nu^a \delta_\mu^b \delta_\rho^c \delta_\sigma^d) \\ &= \frac{1}{2} (g^{ac} g^{bd} - g^{bc} g^{ad}), \end{aligned} \quad (C4)$$

which clearly furnishes the same result.

We now proceed to the higher derivative terms. Let us start with the Kretschmann scalar for which we find

$$\frac{\partial R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}}{\partial R_{abcd}} = 2 R^{\mu\nu\rho\sigma} \frac{\partial R_{\mu\nu\rho\sigma}}{\partial R_{abcd}} = 2 R^{abcd}. \quad (C5)$$

The above result does not need any correction as it is already proportional to R_{abcd} and satisfies all the desired identities. We now move to the $R_{\mu\nu} R^{\mu\nu}$ term and employ again the simple method used above. Then

$$\begin{aligned} \frac{\partial A_{\mu\nu} R^{\mu\nu}}{\partial A_{abcd}} &= 2 A^{\mu\nu} \frac{\partial A_{\mu\nu}}{\partial A_{abcd}} = 2 A^{\mu\nu} g^{\kappa\lambda} \frac{\partial A_{\kappa\lambda\mu\nu}}{\partial A_{abcd}} \\ &= g^{ac} A^{bd} - g^{bc} A^{ad}. \end{aligned} \quad (C6)$$

If $A_{abcd} = R_{abcd}$ and $A_{\mu\nu} = R_{\mu\nu}$, the above result will have all the right properties if it is rewritten as

$$\frac{\partial R_{\mu\nu} R^{\mu\nu}}{\partial R_{abcd}} = \frac{1}{2} (g^{ac} R^{bd} - g^{bc} R^{ad} - g^{ad} R^{bc} + g^{bd} R^{ac}), \quad (C7)$$

which is indeed the correct result. Finally, we easily derive that

$$\frac{\partial R^2}{\partial R_{abcd}} = R (g^{ac} g^{bd} - g^{bc} g^{ad}). \quad (C8)$$

In order to compute the integral appearing in Eq. (61), we use the near-horizon solution (33)–(35) for the metric functions and scalar field. Then recalling that, near the horizon, the relations $A'' \approx -A'^2$ and $B' \approx -A'$ also hold, we find the results

$$\begin{aligned} R^{0101}|_{\mathcal{H}} &= -\frac{1}{4} e^{-A-2B} (-2A'' + A'B' - A'^2)|_{\mathcal{H}} \rightarrow 0, \\ &- 2(g^{00} R^{11} - g^{10} R^{01} - g^{01} R^{10} + g^{11} R^{00})|_{\mathcal{H}} \\ &\rightarrow \frac{4}{r_h} e^{-A-2B} A'|_{\mathcal{H}} \approx -\frac{4b_1^2}{a_1 r_h}, \\ g^{00} g^{11} R|_{\mathcal{H}} &\rightarrow \frac{e^{-A-2B}}{r^2} (4rA' - 2e^B)|_{\mathcal{H}} \approx \frac{4b_1^2}{a_1 r_h} - \frac{2b_1}{a_1 r_h^2}, \\ g_{00} g_{11}|_{\mathcal{H}} &= e^{A+B}|_{\mathcal{H}} \rightarrow a_1/b_1. \end{aligned}$$

Substituting these into Eq. (61), we readily obtain the result (62).

-
- [1] K. S. Stelle, *Phys. Rev. D* **16**, 953 (1977).
 [2] T. P. Sotiriou, *Lect. Notes Phys.* **892**, 3 (2015); E. Berti *et al.*, *Classical Quantum Gravity* **32**, 243001 (2015).
 [3] J. D. Bekenstein, *Phys. Rev. Lett.* **28**, 452 (1972); C. Teitelboim, *Lett. Nuovo Cimento* **3**, 397 (1972).

- [4] M. S. Volkov and D. V. Galtsov, *JETP Lett.* **50**, 346 (1989); P. Bizon, *Phys. Rev. Lett.* **64**, 2844 (1990); B. R. Greene, S. D. Mathur, and C. M. O'Neill, *Phys. Rev. D* **47**, 2242 (1993); K. i. Maeda, T. Tachizawa, T. Torii, and T. Maki, *Phys. Rev. Lett.* **72**, 450 (1994).

- [5] H. Luckock and I. Moss, *Phys. Lett. B* **176**, 341 (1986); S. Droz, M. Heusler, and N. Straumann, *Phys. Lett. B* **268**, 371 (1991).
- [6] J. D. Bekenstein, *Ann. Phys. (N.Y.)* **82**, 535 (1974); **91**, 75 (1975).
- [7] J. D. Bekenstein, *Phys. Rev. D* **51**, R6608 (1995).
- [8] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, *Phys. Rev. D* **54**, 5049 (1996); **57**, 6255 (1998).
- [9] G. W. Gibbons and K. i. Maeda, *Nucl. Phys. B* **298**, 741 (1988).
- [10] C. G. Callan, Jr., R. C. Myers, and M. J. Perry, *Nucl. Phys. B* **331**, 673 (1989).
- [11] B. A. Campbell, M. J. Duncan, N. Kaloper, and K. A. Olive, *Phys. Lett. B* **251**, 34 (1990); B. A. Campbell, N. Kaloper, and K. A. Olive, *Phys. Lett. B* **263**, 364 (1991).
- [12] S. Mignemi and N. R. Stewart, *Phys. Rev. D* **47**, 5259 (1993).
- [13] P. Kanti and K. Tamvakis, *Phys. Rev. D* **52**, 3506 (1995).
- [14] T. Torii, H. Yajima, and K. i. Maeda, *Phys. Rev. D* **55**, 739 (1997).
- [15] P. Kanti and K. Tamvakis, *Phys. Lett. B* **392**, 30 (1997); P. Kanti and E. Winstanley, *Phys. Rev. D* **61**, 084032 (2000).
- [16] Z. K. Guo, N. Ohta, and T. Torii, *Prog. Theor. Phys.* **120**, 581 (2008); K. i. Maeda, N. Ohta, and Y. Sasagawa, *Phys. Rev. D* **80**, 104032 (2009); N. Ohta and T. Torii, *Prog. Theor. Phys.* **124**, 207 (2010).
- [17] B. Kleihaus, J. Kunz, and E. Radu, *Phys. Rev. Lett.* **106**, 151104 (2011); B. Kleihaus, J. Kunz, S. Mojica, and E. Radu, *Phys. Rev. D* **93**, 044047 (2016).
- [18] P. Pani, C. F. B. Macedo, L. C. B. Crispino, and V. Cardoso, *Phys. Rev. D* **84**, 087501 (2011); P. Pani, E. Berti, V. Cardoso, and J. Read, *Phys. Rev. D* **84**, 104035 (2011).
- [19] C. A. R. Herdeiro and E. Radu, *Phys. Rev. Lett.* **112**, 221101 (2014).
- [20] D. Ayzenberg and N. Yunes, *Phys. Rev. D* **90**, 044066 (2014); **91**, 069905(E) (2015).
- [21] E. Winstanley, *Lect. Notes Phys.* **769**, 49 (2009).
- [22] C. Charmousis, *Lect. Notes Phys.* **769**, 299 (2009).
- [23] C. A. R. Herdeiro and E. Radu, *Int. J. Mod. Phys. D* **24**, 1542014 (2015).
- [24] J. L. Blazquez-Salcedo *et al.*, *IAU Symp.* **12**, 265 (2016).
- [25] R. R. Metsaev and A. A. Tseytlin, *Nucl. Phys. B* **293**, 385 (1987).
- [26] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [27] A. Nicolis, R. Rattazzi, and E. Trincherini, *Phys. Rev. D* **79**, 064036 (2009).
- [28] T. P. Sotiriou and V. Faraoni, *Phys. Rev. Lett.* **108**, 081103 (2012).
- [29] L. Hui and A. Nicolis, *Phys. Rev. Lett.* **110**, 241104 (2013).
- [30] T. P. Sotiriou and S. Y. Zhou, *Phys. Rev. Lett.* **112**, 251102 (2014).
- [31] E. Babichev and C. Charmousis, *J. High Energy Phys.* **08** (2014) 106.
- [32] T. P. Sotiriou and S. Y. Zhou, *Phys. Rev. D* **90**, 124063 (2014); R. Benkel, T. P. Sotiriou, and H. Witek, *Classical Quantum Gravity* **34**, 064001 (2017); *Phys. Rev. D* **94**, 121503 (2016).
- [33] N. Yunes and L. C. Stein, *Phys. Rev. D* **83**, 104002 (2011).
- [34] G. Antoniou, A. Bakopoulos, and P. Kanti, *Phys. Rev. Lett.* **120**, 131102 (2018); *Phys. Rev. D* **97**, 084037 (2018).
- [35] D. D. Doneva and S. S. Yazadjiev, *Phys. Rev. Lett.* **120**, 131103 (2018).
- [36] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, *Phys. Rev. Lett.* **120**, 131104 (2018).
- [37] C. Charmousis, T. Kolyvaris, E. Papantonopoulos, and M. Tsoukalas, *J. High Energy Phys.* **07** (2014) 085.
- [38] F. Correa, M. Hassaine, and J. Oliva, *Phys. Rev. D* **89**, 124005 (2014).
- [39] D. D. Doneva and S. S. Yazadjiev, *J. Cosmol. Astropart. Phys.* **04** (2018) 011.
- [40] H. Motohashi and M. Minamitsuji, *Phys. Lett. B* **781**, 728 (2018); *Phys. Rev. D* **98**, 084027 (2018).
- [41] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, *Phys. Rev. Lett.* **121**, 101102 (2018); T. Delsate, C. Herdeiro, and E. Radu, *Phys. Lett. B* **787**, 8 (2018); Y. Brihaye, C. Herdeiro, and E. Radu, *Phys. Lett. B* **788**, 295 (2019).
- [42] D. D. Doneva, S. Kiorpelidi, P. G. Nedkova, E. Papantonopoulos, and S. S. Yazadjiev, *Phys. Rev. D* **98**, 104056 (2018).
- [43] M. Butler, A. M. Ghezelbash, E. Massaelli, and M. Motaharfar, *arXiv:1808.03217*.
- [44] B. Danila, T. Harko, F. S. N. Lobo, and M. K. Mak, *arXiv:1811.02742*.
- [45] M. M. Stetsko, *arXiv:1811.05030*.
- [46] K. Yagi, L. C. Stein, N. Yunes, and T. Tanaka, *Phys. Rev. D* **85**, 064022 (2012); **93**, 029902(E) (2016); D. Ayzenberg, K. Yagi, and N. Yunes, *Phys. Rev. D* **89**, 044023 (2014).
- [47] S. R. Dolan, S. Ponglertsakul, and E. Winstanley, *Phys. Rev. D* **92**, 124047 (2015).
- [48] J. L. Blazquez-Salcedo, C. F. B. Macedo, V. Cardoso, V. Ferrari, L. Gualtieri, F. S. Khoo, J. Kunz, and P. Pani, *Phys. Rev. D* **94**, 104024 (2016).
- [49] S. Bhattacharya and S. Chakraborty, *Phys. Rev. D* **95**, 044037 (2017); I. Banerjee, S. Chakraborty, and S. SenGupta, *Phys. Rev. D* **96**, 084035 (2017).
- [50] O. J. Tattersall, P. G. Ferreira, and M. Lagos, *Phys. Rev. D* **97**, 084005 (2018).
- [51] S. Mukherjee and S. Chakraborty, *Phys. Rev. D* **97**, 124007 (2018).
- [52] S. Chakraborty, *Eur. Phys. J. C* **78**, 296 (2018).
- [53] E. Berti, K. Yagi, and N. Yunes, *Gen. Relativ. Gravit.* **50**, 46 (2018).
- [54] Y. Brihaye and B. Hartmann, *Classical Quantum Gravity* **35**, 175008 (2018).
- [55] K. Prabhu and L. C. Stein, *Phys. Rev. D* **98**, 021503 (2018).
- [56] Y. S. Myung and D. C. Zou, *Phys. Rev. D* **98**, 024030 (2018).
- [57] J. L. Blazquez-Salcedo, D. D. Doneva, J. Kunz, and S. S. Yazadjiev, *Phys. Rev. D* **98**, 084011 (2018); J. L. Blazquez-Salcedo, Z. Altaha Motahar, D. D. Doneva, F. S. Khoo, J. Kunz, S. Mojica, K. V. Staykov, and S. S. Yazadjiev, *arXiv:1810.09432*.
- [58] R. Benkel, N. Franchini, M. Saravani, and T. P. Sotiriou, *Phys. Rev. D* **98**, 064006 (2018).
- [59] L. Iorio and M. L. Ruggiero, *J. Cosmol. Astropart. Phys.* **10** (2018) 021.
- [60] J. Ovalle, R. Casadio, R. da Rocha, A. Sotomayor, and Z. Stuchlik, *Europhys. Lett.* **124**, 20004 (2018); J. Ovalle, *Phys. Lett. B* **788**, 213 (2019).

- [61] L. Barack *et al.*, [arXiv:1806.05195](#).
- [62] Y. X. Gao, Y. Huang, and D. J. Liu, *Phys. Rev. D* **99**, 044020 (2019).
- [63] B. H. Lee, W. Lee, and D. Ro, *Phys. Rev. D* **99**, 024002 (2019).
- [64] H. Witek, L. Gualtieri, P. Pani, and T. P. Sotiriou, [arXiv:1810.05177](#).
- [65] H. Motohashi and S. Mukohyama, [arXiv:1810.12691](#).
- [66] T. Torii, K. Maeda, and M. Narita, *Phys. Rev. D* **59**, 064027 (1999).
- [67] E. Winstanley, *Found. Phys.* **33**, 111 (2003); *Classical Quantum Gravity* **22**, 2233 (2005).
- [68] S. Bhattacharya and A. Lahiri, *Phys. Rev. Lett.* **99**, 201101 (2007).
- [69] Y. Bardoux, M. M. Caldarelli, and C. Charmousis, *J. High Energy Phys.* **05** (2012) 054.
- [70] J. Sultana and D. Kazanas, *Gen. Relativ. Gravit.* **50**, 137 (2018).
- [71] C. Martinez, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **67**, 024008 (2003).
- [72] T. J. T. Harper, P. A. Thomas, E. Winstanley, and P. M. Young, *Phys. Rev. D* **70**, 064023 (2004).
- [73] M. Henneaux, C. Martinez, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **70**, 044034 (2004); C. Martinez, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **70**, 084035 (2004); C. Erices and C. Martinez, *Phys. Rev. D* **97**, 024034 (2018).
- [74] E. Radu and E. Winstanley, *Phys. Rev. D* **72**, 024017 (2005).
- [75] A. Anabalon and H. Maeda, *Phys. Rev. D* **81**, 041501 (2010).
- [76] D. Hosler and E. Winstanley, *Phys. Rev. D* **80**, 104010 (2009).
- [77] C. Charmousis, T. Kolyvaris, and E. Papantonopoulos, *Classical Quantum Gravity* **26**, 175012 (2009); T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, *Gen. Relativ. Gravit.* **43**, 163 (2011).
- [78] K. i. Maeda, N. Ohta, and Y. Sasagawa, *Phys. Rev. D* **83**, 044051 (2011); Z. K. Guo, N. Ohta, and T. Torii, *Prog. Theor. Phys.* **121**, 253 (2009); N. Ohta and T. Torii, *Prog. Theor. Phys.* **121**, 959 (2009); N. Ohta and T. Torii, *Prog. Theor. Phys.* **122**, 1477 (2009).
- [79] S. G. Saenz and C. Martinez, *Phys. Rev. D* **85**, 104047 (2012).
- [80] M. M. Caldarelli, C. Charmousis, and M. Hassaine, *J. High Energy Phys.* **10** (2013) 015.
- [81] P. A. Gonzalez, E. Papantonopoulos, J. Saavedra, and Y. Vasquez, *J. High Energy Phys.* **12** (2013) 021.
- [82] M. Bravo Gaete and M. Hassaine, *Phys. Rev. D* **88**, 104011 (2013); M. Bravo Gaete and M. Hassaine, *J. High Energy Phys.* **11** (2013) 177.
- [83] G. Giribet, M. Leoni, J. Oliva, and S. Ray, *Phys. Rev. D* **89**, 085040 (2014).
- [84] J. Ben Achour and H. Liu, [arXiv:1811.05369](#).
- [85] Y. Brihaye, B. Hartmann, and J. Urrestilla, *J. High Energy Phys.* **06** (2018) 074; Y. Brihaye and B. Hartmann, [arXiv:1810.05108](#).
- [86] I. Antoniadis, J. Rizos, and K. Tamvakis, *Nucl. Phys.* **B415**, 497 (1994).
- [87] P. Kanti, J. Rizos, and K. Tamvakis, *Phys. Rev. D* **59**, 083512 (1999).
- [88] A. Bakopoulos, G. Antoniou and P. Kanti (to be published).
- [89] J. W. York, Jr., *Phys. Rev. D* **31**, 775 (1985).
- [90] G. W. Gibbons and R. E. Kallosh, *Phys. Rev. D* **51**, 2839 (1995).
- [91] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2752 (1977).
- [92] S. W. Hawking and D. N. Page, *Commun. Math. Phys.* **87**, 577 (1983).
- [93] S. Dutta and R. Gopakumar, *Phys. Rev. D* **74**, 044007 (2006).
- [94] R. M. Wald, *Phys. Rev. D* **48**, R3427 (1993).
- [95] V. Iyer and R. M. Wald, *Phys. Rev. D* **50**, 846 (1994).