Positive helicity Einstein-Yang-Mills amplitudes from the double copy method

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All positive helicity four-point gluon-graviton amplitudes in Einstein-Yang-Mills theory coupled to a dilaton and axion field are computed at the leading one-loop order using color-kinematics duality. In particular, all relevant contributions in the gravitational and gauge coupling are established. This extends a previous generalized unitarity based computation beyond the leading terms in the gravitational coupling κ . The resulting purely rational expressions take very compact forms. The previously seen vanishing of the single-graviton-three-gluon amplitude at leading order in κ is seen to be lifted at order κ^3 .

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I. INTRODUCTION

Scattering amplitudes involving only positive helicity gluon and graviton states take a very special role in minimally coupled gauge and gravitational theories: At tree-level they always vanish due to an effective (or hidden) supersymmetric Ward identity [1]

$$\mathcal{M}_{n,m}^{\text{tree}}(1_{a_1}^+, \dots, n_{a_n}^+; (n+1)^{++}, \dots, (n+m)^{++}) = 0. \quad (1)$$

This holds true for an amplitude in an arbitrary matter coupled Einstein-Yang-Mills (EYM) with n gluons and m gravitons and persists to all-loops in supersymmetric theories. In nonsupersymmetric theories, in particular in the “pure” Yang-Mills (YM) and Einstein gravity examples, the leading contribution to these amplitudes is at the one-loop order. The resulting one-loop pure gluon or pure graviton positive helicity amplitudes turn out to be remarkably simple rational functions of the kinematic invariants and spinor products. This is a mandatory property in order to have vanishing unitarity cuts in four dimensions for these amplitudes. In fact, these pure gluon or graviton one-loop amplitudes are identical to the ones one finds in self-dual YM or self-dual gravity, respectively. Here all-multiplicity expressions exist for the pure gluon and pure graviton amplitudes of uniform helicities [2–4].

Computing scattering amplitudes involving graviton via Feynman diagrams, as they follow from an expansion of the

metric tensor about a flat Minkowski background, is a daunting task, because of the sheer complexity and the infinite number of vertices involved. An important insight that arose from the study of these amplitudes, however, is that they are much simpler than expected and display an intimate connection to amplitudes in YM theory. This relation is known as the double copy, as gravitational amplitudes may be obtained by taking the product of two gauge theory quantities.

The earliest such connection is due to Kawai, Lewellen and Tye (KLT) who expressed gravitational tree-level amplitudes as sums over products of gauge theory amplitudes weighted by Mandelstam invariants, originating from a string theory analysis [5]. A decade ago Bern, Carrasco and Johansson (BCJ) [6] showed that these relations may be understood elegantly in terms of a specific diagrammatic expansion of the amplitudes: If one represents the gauge theory amplitudes in a trivalent fashion such that all kinematic numerators are arranged to obey a Jacobi-like relation mirroring the property of the color degrees of freedom, then the numerator of a gravitational amplitude follows by simply squaring the gauge theory kinematic numerator. This duality—known as the color kinematic duality (CKD)—was proven at tree-level [7] and conjectured to hold at loop-level as well [8]. Plenty of examples have been presented which confirm this conjecture and furnish the state of the art technique to compute highest loop orders in supergravities and beyond, see e.g., [9–26]. Generally it is difficult to find numerators that satisfy CKD at loop-level; therefore it has been shown in [27] that a modified double copy can be used to construct gravity integrands where contact terms are included due to the violation of kinematical Jacobi identities. Another one-loop generalization of the KLT-formula has been presented in [28,29].

In this paper we focus on the positive helicity sector of amplitudes for scattering processes involving gravitons

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interacting with gluons described by Einstein-Yang-Mills theory coupled to a dilaton and axion field (EYM),¹ as it arises as the low energy limit of bosonic string theory. The evaluation of tree-level S -matrix elements in pure EYM has made rapid progress in the recent years and may be considered as completely solved. The key insight was to relate color-ordered amplitudes of pure EYM to YM theory via an expansion of the form

$$\mathcal{M}_{n,m}^{\text{tree}}(1, \dots, n; h_1, \dots, h_m) = \sum_{\alpha \in \text{Perm}(2, \dots, n-1; h_1, \dots, h_m)} \mathcal{N}(1, \alpha, n) \times A_{\text{YM}}^{\text{tree}}(1, \alpha, n), \quad (2)$$

which has been proposed by [30] for one graviton and n gluons based on a string computation and proven by field theory techniques [31]. Higher graviton extensions of this result were presented in [31–33]. The generalization to the entire single-color-trace and multi-color-trace sector has been carried out in [34,35] by giving an algorithm to construct the coefficients $\mathcal{N}(1, \alpha, n)$. The color ordered amplitudes in YM theory on the other hand have been determined in [36], which implies the complete solution of the problem of EYM amplitudes at tree-level. An extension to arbitrary matter-single graviton amplitudes was recently established in [37].

The next step has been to tackle the loop amplitudes in (pure) EYM. In [38] all four point amplitudes at one-loop with one particle species circulating in the loop have been calculated for up to one negative helicity state. To make

the calculation tractable the authors used the two-particle cut method in $4 - 2\epsilon$ dimensions. Restricting to one particle species circulating the loop allowed them to use supersymmetric Ward–Takahashi identities to replace the virtual particles in the loop by complex scalars, since the all-plus and all-plus-but-one-minus helicity amplitudes vanish in supersymmetric and supergravitational theories [1]. However, these identities do not hold for a mixed propagation of gravitons and gluons in the loop, and therefore this technique is not applicable in the most general setting.

In this work we shall partly fill this gap by computing in four dimensions all positive helicity amplitudes of EYM at four points to all orders in κ at one-loop precision.² We now use the double copy method to generate the integrands of EYM by tensoring the two gauge theory integrands of YM and Yang-Mills coupled to a biadjoint scalar (YM + ϕ^3) as was established in a series of papers [18–20,33,39] starting from matter coupled $\mathcal{N} = 2$ supergravities. Focusing on the all-plus sector the calculation is greatly simplified since in YM theory in this case the four point amplitude only receives contributions from the box integrals, which implies that this side of the double copy trivially satisfies the CKD. Therefore in the YM + ϕ^3 sector also a non-duality respecting representation may be used to perform the double copy. The later we generate through usual Feynman diagrammatics.

We can report on rather compact formulas, the summary of our results reads³

$$\begin{aligned} \mathcal{M}^{1\text{-loop}}(1_a^+, 2_b^+, 3_c^+, 4_d^+) &= \frac{i}{(4\pi)^2} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \left(\left[-\frac{4}{3} g_{\text{YM}}^4 f^{a'ab'} f^{b'bc'} f^{c'cd'} f^{d'da'} \right. \right. \\ &\quad \left. \left. - \frac{\kappa^2 g_{\text{YM}}^2}{12} (4 f^{abe'} f^{e'cd} (U - T) + N S \delta^{ab} \delta^{cd}) \right. \right. \\ &\quad \left. \left. + \frac{\kappa^4}{960} \delta^{ab} \delta^{cd} (40 T U - (2 + N_g) S^2) \right] + \text{perm} \right), \\ \mathcal{M}^{1\text{-loop}}(1_a^+, 2_b^+, 3_c^+, 4^{++}) &= -\frac{\kappa^3 g_{\text{YM}} f^{abc}}{(8\pi)^2} \frac{[41][42][43][12]}{\sqrt{2} \langle 34 \rangle}, \\ \mathcal{M}^{1\text{-loop}}(1_a^+, 2_b^+, 3^{++}, 4^{++}) &= \frac{i}{(4\pi)^2} \delta^{ab} \frac{[12]^2 [34]^2}{\langle 34 \rangle^2} \left(-\frac{\kappa^2 g_{\text{YM}}^2}{24} N + \frac{\kappa^4}{1440} S(2 + N_g) \right), \\ \mathcal{M}^{1\text{-loop}}(1^{++}, 2^{++}, 3^{++}, 4^{++}) &= -\frac{i}{(4\pi)^2} \kappa^4 \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \frac{S^2 + T^2 + U^2}{1920} (2 + N_g). \end{aligned} \quad (3)$$

The outline of our paper is the following: In Sec. II we present the general strategy of our approach. In Sec. II A the BCJ representation in terms of cubic vertices of an amplitude is reviewed and we explain how gravity integrands can be obtained from gauge theory integrands by using the double copy technique. After we have discussed the construction of the one-loop

¹In this paper we call Einstein-Yang-Mills coupled to a dilaton and axion field EYM for short; it is defined in Eq. (11). The theory without these scalars we term “pure EYM”.

²The all negative helicity amplitudes are trivially given by charge conjugation.

³We use spinor bracket notation in the conventions of [40] and the Mandelstam variables $S = \langle 12 \rangle [21]$, $T = \langle 23 \rangle [32]$ and $U = \langle 13 \rangle [31]$.

integrands of EYM, we explain in Sec. II B how to obtain the amplitude. The next part, Sec. III, is devoted to the evaluation of the amplitudes presented in Eq. (3). The paper is supplemented by three Appendixes. In Appendix A the four-dimensional-helicity regularization scheme we are using is presented, and Appendix B contains additional information about the double copy of EYM as well as the Feynman rules for computing the one-loop integrands of $YM + \phi^3$. The Appendix C contains all the integrands of $YM + \phi^3$ which shall be used to determine the integrands of EYM for one particular ordering of the external states.

II. PRELIMINARIES

A. Review on double copy construction

Any L -loop gauge theory amplitude with all particles in the adjoint representation of the gauge group $SU(N)$ can be written as

$$\mathcal{A}_m^{L\text{-loop}} = i^{L-1} g^{m-2+2L} \sum_{S_m} \sum_{j \in \Gamma} \int \frac{d^d L}{(2\pi)^d} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} D_{\alpha_j}}, \quad (4)$$

which separates the amplitude into three parts:

- (i) The color dependence is encoded in the color factors c_j which are a chain of the adjoint generators $i f^{abc}$.⁴ They obey the Jacobi identity which for the four particle case may be sketched as $c_s = c_t + c_u$.⁵ Furthermore, the adjoint generators $i f^{abc}$ are anti-symmetric in their indices which implies that the color factors are antisymmetric under permutation: $c_i = -c_j$.
- (ii) The set of all reduced Feynman propagators $1/(p^2 - m^2)$ associated to the j th graph are denoted by the inverse of the product $\prod_{\alpha_j} D_{\alpha_j}$.
- (iii) The numerators n_j account for the remaining kinematical dependence of the amplitude. Note that factors of $\pm i$ in the Feynman propagators are also absorbed in n_j .

The second sum runs over all distinct, nonisomorphic, trivalent graphs Γ and the first one over all $|S_m| = m!$ permutations of the external legs. Any overcounting of the j th diagram is removed by the symmetry factor S_j . Note that by using the identity $1 = D_{\alpha_j}/D_{\alpha_j}$ any graph in a diagrammatic expansion can be made trivalent formally.

Representing the amplitude in the form given in (4) reveals the parallel treatment of color degrees of freedom c_j and kinematical degrees of freedom n_j and is especially powerful if one arranges the kinematical numerators in

such a way that they obey the same algebraic relations as the corresponding color factors,

$$\left. \begin{aligned} c_s &= c_t + c_u \\ c_i &= -c_j \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} n_s &= n_t + n_u \\ n_i &= -n_j \end{aligned} \right. . \quad (5)$$

It has been conjectured by Bern, Carrasco and Johanson (BCJ) [6,8] and shown at tree-level ($L = 0$) in Refs. [7,41–45] that it is always possible to arrange all the numerators n_i of a diagram in such a way that they obey (5).

It is a striking feature of this color-kinematics duality (CKD) that integrands for gravity amplitudes can be easily constructed from integrands of gauge theories if at least one set of the gauge theory numerators n_j or $\tilde{n}_j$ satisfies (5),

$$\mathcal{M}_m^{L\text{-loop}} = i^{L-1} \left(\frac{\kappa}{4}\right)^{m-2+2L} \sum_{S_m} \sum_{j \in \Gamma_g} \int \frac{d^d L}{(2\pi)^d} \frac{1}{S_j} \frac{\tilde{n}_j n_j}{\prod_{\alpha_j} D_{\alpha_j}} . \quad (6)$$

The set Γ_g includes all pairs of trivalent graphs from both gauge theories with numerators n_j and $\tilde{n}_j$, respectively. This feature of obtaining gravity amplitude integrands has been proven at tree-level for pure gravity using BCFW recursion relations [46] in four dimensions [7]. Furthermore, gauge invariance of both gauge theory amplitudes implies invariance of the corresponding double copied amplitude under linearized diffeomorphisms; i.e., (6) is an amplitude of *some* gravity theory [33]. The unitarity method can extend this feature to loop-level by reducing amplitudes containing loops to tree-level amplitudes and demanding that CKD holds for all cuts. However, a general proof is still missing [47].⁶ This double copy (DC) construction can be viewed as a generalization of the Kawai-Lewellen-Tye relations [5] and a loop generalization thereof.

Since gauge theory integrands are much easier to calculate than gravity integrands, the DC prescription gives a powerful tool to build gravity integrands. In particular, it is enough to construct a certain number of master integrands, because all other integrands are determined by the relation (5). Diagrammatically this can be sketched as

⁴This coincides with the conventions given in [33] which differ from the conventions in [18].

⁵ $c_s := i f^{abe'} i f^{e'cd}$, $c_t := -i f^{ade'} i f^{e'bc}$ and $c_u := -i f^{ace'} i f^{e'db}$.

⁶We note that the conjectured formula (6) apparently fails at five loop for maximal supergravity if one requests the numerator factors to satisfy the Jacobi identity. Such a representation could not be found for the maximal super-Yang-Mills at five loops. Instead a generalized BCJ relation [25,48] is required.

$$\text{Triangle}(1,2,3,4) = \text{Box}(1,2,3,4) - \text{Box}(1,2,3,4) \quad (7)$$

Applying the same reasoning the bubble graphs may be reexpressed to triangle graphs. Therefore the master numerators for four points at one-loop are given by the box diagrams since all other diagrams can be obtained by applying the kinematical Jacobi identity (5).⁷

B. The double copy procedure for Einstein-Yang-Mills

In this section we outline the methods we use to obtain the amplitudes presented in (3). The strategy is to write the amplitude in the form (6), for which we have to determine the numerators of pure Yang-Mills (YM) and Yang-Mills coupled to a biadjoint scalar (YM + ϕ^3). Afterwards we perform the integrals by decomposing the tensor integral into scalar Feynman integrals and evaluating these in four dimensions.

We shall start our discussion by defining YM theory and YM + ϕ^3 theory through the Lagrangians,

$$\mathcal{L}^{\text{YM}} = -\frac{1}{4} F^{\mu\nu,a} F_{\mu\nu}^a, \quad (8)$$

$$\begin{aligned} \mathcal{L}^{\text{YM}+\phi^3} = & \mathcal{L}^{\text{YM}} + \frac{1}{2} (D_\mu \phi^A)^a (D^\mu \phi^A)^a \\ & - \frac{g^2}{4} f^{abe} f^{ecd} \phi^{Aa} \phi^{Bb} \phi^{Ac} \phi^{Bd} \\ & + \frac{1}{3!} \lambda g F^{ABC} f^{abc} \phi^{Aa} \phi^{Bb} \phi^{Cc} \end{aligned} \quad (9)$$

with the definitions

$$\begin{aligned} F_{\mu\nu}^c &= \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b, \\ (D_\mu \phi^A)^a &= \partial_\mu \phi^{Aa} + g f^{abc} A_\mu^b \phi^{Ac}. \end{aligned} \quad (10)$$

We have chosen the normalization $\text{Tr}(T^a T^b) = \delta^{ab}$ for the Hermitian (fundamental) generators T^a of the gauge group $\text{SU}(N)$, which implies that the structure constant is given by $f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}(T^a [T^b, T^c])$. For the global gauge group with structure constants F^{ABC} it is simply demanded that they shall obey the Jacobi identity and that they are antisymmetric in all of their indices $A, B, C, \dots$ ⁸ Dimensional analysis reveals that the coupling constant λ is of mass dimensions one.

⁷Here we do not consider the bubble-on-external-leg or tadpole diagrams that vanish in dimensional regularization after integration.

⁸After double copying we promote the global gauge group to the local gauge group $\text{SU}(N)$ of EYM.

The EYM Lagrangian emerging from the double copy of (8) and (9) reads explicitly [49]

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=0+\text{YM}} = & \frac{\sqrt{-G}}{\kappa^2} (-2R + \partial_\mu \varphi \partial^\mu \varphi + e^{2\varphi} \partial_\mu \chi \partial^\mu \chi) \\ & - \frac{\sqrt{-G}}{4} (e^{-\varphi} F_{\mu\nu}^a F^{\mu\nu,a} + i\chi F_{\mu\nu}^a \tilde{F}^{\mu\nu,a}), \end{aligned} \quad (11)$$

where G is the determinant of the metric $G_{\mu\nu}$, the scalar curvature is encoded in the Ricci tensor R and $\tilde{F}_{\mu\nu}^a = \frac{i}{2} \sqrt{-G} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma,a}$ represents the dual field strength tensor. The scalars φ and χ are the dilaton and axion, respectively.⁹ The authors of Ref. [49] confirmed at tree level that the DC procedure of the theories (8) and (9) give the same amplitudes as the one derived from (11). Clearly the dilaton as well as the axion interact with the gauge fields. To obtain the pure EYM result one has to remove the contributions originating from the two scalars. In [15] it has been shown that for $\mathcal{N} = 0$ supergravity the axion and dilaton can be subtracted by a DC of fields in the fundamental representations with opposite statistics.

After discussing the relevant Lagrangians of the theories, we start with the DC procedure. The first step is to determine both numerators n^{YM} and $n^{\text{YM}+\phi^3}$ which can be extracted from the corresponding one-loop integrands of pure YM and YM + ϕ^3 . It turns out that the four point YM numerator $n_{1+2+3+4+}^{\text{YM}}$ at one-loop for an all-plus helicity amplitude is very simple because it is exclusively given by graphs with box topology.¹⁰ Let us quickly review this property. The corresponding YM amplitude reads in the color trace basis

$$\begin{aligned} \mathcal{A}_4^{1\text{-loop}}(1_{a_1}^+, 2_{a_2}^+, 3_{a_3}^+, 4_{a_4}^+) &= \sum_{\sigma \in S_4/\mathbb{Z}_4} N \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(4)}}) A_{4;1}(\sigma(1^+), \dots, \sigma(4^+)) \\ &+ \sum_{\sigma \in S_4/S_{4;3}} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) \\ &\times A_{4;3}(\sigma(1^+), \dots, \sigma(4^+)), \end{aligned} \quad (12)$$

⁹In four dimensions the antisymmetric B-field, which naturally appears in $\mathcal{N} = 0$ supergravity, can be replaced by a scalar using a duality transformation. Details can be found e.g. in [50].

¹⁰This property follows from maximal cuts [51], because only box type diagrams of YM theory are consistent with its analytic structure and all other diagrams shall cancel.

where S_4 is the permutation group which is quotiented by the subgroups $\mathbb{Z}_4$ and $S_{4;3}$ that leave the single and double traces invariant, respectively. N is the degree of the Lie group $SU(N)$. In this case the primitive amplitude $A_{4;1}$ is related to the partial amplitude $A_{4;3}$ by $A_{4;3} = 6A_{4;1}$ [52]. In [53–56] $A_{4;1}$ has been calculated to all orders in ϵ . Its Veltman-Passarino [57] reduced form reads

$$A_{4;1}(1^+, 2^+, 3^+, 4^+) = \frac{2i}{(4\pi)^{2-\epsilon}} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} I_4[\mu^4; S, T],$$

where the scalar-box integral $I_4[\mu^4; S, T]$ is defined in the Appendix A and $[ij], \langle ij \rangle$ are the helicity spinor brackets of the external momenta p_i, p_j . The helicity spinor representation of these read $p^{\dot{\alpha}\alpha} = [i]^{\dot{\alpha}} \langle i |^{\alpha}$ (see [40,58,59] for reviews). μ is the fictitious mass of the propagating complex scalar field in the loop which needs to be integrated over [54] in order to emulate $4 - 2\epsilon$ dimensions.¹¹ Thus the corresponding numerator of the partial amplitude is

$$n_{1+2+3+4+}^{\text{YM}} = 2\mu^4 \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}. \quad (13)$$

Using the conventions for the color-ordered Feynman rules from [40,58,59] one can transform the fundamental generators T^a of the single and double trace color structure into adjoint generators f^{abc} . It turns out that the entire color structure of (12) is encoded precisely in the chain of structure constants

$$c^{a_1 a_2 a_3 a_4}, c^{a_1 a_2 a_4 a_3} \text{ and } c^{a_1 a_4 a_2 a_3}, \quad (14)$$

which are given by Feynman graphs with box topology, i.e., $c^{a_1 a_2 a_3 a_4} := f^{a' a_1 b'} f^{b' a_2 c'} f^{c' a_3 d'} f^{d' a_4 a'}$. This enables us to write the amplitude (12) in the form (4) with the YM numerators $n_{1+2+3+4+}^{\text{YM}}$. The exact relation between both numerators reads¹²

$$n_{1+2+3+4+}^{\text{YM}} = 4a_{1+2+3+4+}^{\text{YM}}. \quad (15)$$

Furthermore the numerators obey the following symmetry properties:

¹¹For more details on this regularization scheme see Appendix A.

¹²It is also possible to derive the numerator (15) directly in the structure constant color basis from the integrand of the maximally helicity violating (MHV) $\mathcal{N} = 4$ super YM amplitude. The authors of [2] conjectured a formula which relates at one-loop all-plus primitive amplitudes of YM to MHV primitive amplitudes of $\mathcal{N} = 4$ super YM. This formula also generalizes to higher multiplicity. We thank Radu Roiban for pointing this out to us.

$$\begin{aligned} n_{i^+ j^+ k^+ l^+}^{\text{YM}} &= n_{i^+ j^+ l^+ k^+}^{\text{YM}}, \\ n_{i^+ j^+ k^+ l^+}^{\text{YM}} &= n_{k^+ l^+ i^+ j^+}^{\text{YM}}, \\ n_{i^+ j^+ k^+ l^+}^{\text{YM}} &= n_{j^+ k^+ i^+ l^+}^{\text{YM}} = n_{k^+ l^+ i^+ j^+}^{\text{YM}}. \end{aligned} \quad (16)$$

These properties imply that for all 24 positions of the external legs the numerator reads as (15). Once the symmetries of the numerator are known it is trivial to prove that they obey color-kinematic duality (CKD). CKD demands the relation (5) to hold which in this case translates into

$$c^{a_1 a_2 a_3 a_4} - c^{a_1 a_2 a_4 a_3} = c^{\text{triangle}} := f^{a' a_1 b'} f^{b' a_2 c'} f^{c' a' d'} f^{d' a_4 a_3} \quad (17)$$

$$\Rightarrow n_{1+2+3+4+}^{\text{YM}} - n_{1+2+4+3+}^{\text{YM}} = n_{\text{triangle}}^{\text{YM}}. \quad (18)$$

It has been discussed that $n_{\text{triangle}}^{\text{YM}} = 0$. Besides, we have shown that all numerators with box topology are the same $n_{i^+ j^+ k^+ l^+}^{\text{YM}} = n_{1+2+3+4+}^{\text{YM}}$. These facts imply that (18) is trivially obeyed. An alternative way to obtain BCJ numerators using light cone Feynman rules for the one-loop rational pure YM amplitudes was introduced in [60].

Thus, in restricting ourselves to the all-plus sector, only the color structures (14) appear for the YM amplitudes and their numerators satisfy CKD automatically. Therefore to obtain integrands of EYM we have to collect all diagrams of $\text{YM} + \phi^3$ theory which have the same color structures as (14) and multiply the two numerators divided by the stripped propagators. More details of the DC construction are given in Appendix B.

After we have analyzed the first gauge theory let us turn to the second one, i.e., $\text{YM} + \phi^3$. The corresponding amplitude of $\text{YM} + \phi^3$ at one-loop reads

$$\mathcal{A}_4^{1\text{-loop}}|_{g^4 \lambda^n} = g^4 \lambda^n \sum_{S_m} \sum_{j \in \Gamma} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{S_j} \frac{c_j n_j^{\text{YM}+\phi^3}}{\prod_{\alpha_j} D_{\alpha_j}}. \quad (19)$$

It follows from the Lagrangian (9) that the ϕ^3 interaction in $\text{YM} + \phi^3$ is proportional to the coupling constants $g\lambda$. Thus the exponent $n \leq 4$ indicates how often the ϕ^3 -interaction appears in the Feynman diagrammatic decomposition of the amplitude. $n_j^{\text{YM}+\phi^3}$ shall be computed by Feynman diagrams generated from the Feynman rules of the theory given in Appendix B.

Once we know both gauge theory numerators we can determine the gravity amplitude via

$$\begin{aligned} \mathcal{M}_4^{1\text{-loop}}|_{\kappa^4 - n g_{\text{YM}}^n} &= \left(\frac{\kappa}{4}\right)^{4-n} g_{\text{YM}}^n \sum_{S_m} \int \frac{d^4 l}{(2\pi)^4} n^{\text{YM}} \sum_{j \in \Gamma_g} \frac{1}{S_j} \frac{n_j^{\text{YM}+\phi^3}}{\prod_{\alpha_j} D_{\alpha_j}}. \end{aligned} \quad (20)$$

In [18,33] it has been deduced how the global color structure constant F^{ABC} of $\text{YM} + \phi^3$ maps into the local one f^{abc} of EYM. Furthermore, it follows from (19) and (20) that the mapping of the coupling constants from the gauge theories to EYM is given by

$$F^{ABC} \rightarrow f^{abc}, \quad (g^2, \lambda) \rightarrow \left(\frac{\kappa}{4}, 4 \frac{g_{\text{YM}}}{\kappa} \right). \quad (21)$$

Thus the first step to obtain the full result in four dimensions for the amplitudes presented in Sec. III is to evaluate the contributing numerators $n_j^{\text{YM}+\phi^3}$ by Feynman diagrams. After the gravity numerators of the amplitude are determined we shall decompose the tensor integral into scalar integrals by using the technique of Veltman-Passarino reduction [57]. This feature is nicely implemented in the MATHEMATICA package FeynCalc [61,62], which is used in the following calculations. The final scalar Veltman-Passarino functions still depend on the fictitious mass squared μ^2 . The next step is then to evaluate the remaining integral over μ^2 . However, as the authors of [54] have shown, we can express the integrals containing μ^2 as higher dimensional loop integrals, which gives an easy analytic way to determine the four-dimensional amplitude. The relevant formulas to this reduction are given in Appendix A.

III. AMPLITUDES

In this section, all positive helicity amplitudes in EYM shall be calculated using the DC construction discussed in Sec. II. New results are obtained for $\langle 1^+ 2^+ 3^+ 4^+ \rangle$ and $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle$ at order κ^3 and κ^4 , respectively. Furthermore, we also present the κ -corrections at order κ^2 and κ^4 for the four-gluon amplitude $\langle 1^+ 2^+ 3^+ 4^+ \rangle$. Note that a DC expression of these integrands for arbitrary helicity configurations has been given in [18] by using the results

FIG. 1. Graphs of these topologies are the only ones that have to be considered in $\text{YM} + \phi^3$ at order $\lambda^2 g^4$. The other graphs do not contain the color structures (14). Curly lines represent propagating gluons and dashed lines represent scalar fields. The internal momenta Q_i are d dimensional.

of [12]; however, the integrated amplitude has not been published. In the last part of this section we present the result for $\langle 1^{++} 2^{++} 3^{++} 4^{++} \rangle$. The relevant integrands of $\text{YM} + \phi^3$ which shall be used in the DC prescription are collected in Appendix C.

A. Amplitudes: $\langle 1^+ 2^+ 3^+ 4^+ \rangle$

We shall start the evaluation by calculating gravitational corrections to the four-gluon amplitude. At first we compute the κ^2 correction to $\langle 1^+ 2^+ 3^+ 4^+ \rangle|_{\kappa^2}$ with the DC method introduced in the previous section. Therefore we have to calculate in $\text{YM} + \phi^3$ all Feynman graphs which are proportional to λ^2 according to the DC dictionary presented in (21). In Sec. II B we also pointed out that only the box diagrams are nonvanishing for the all-plus YM amplitude. Hence, only the diagrams which carry the same color structure as the box diagrams have to be determined in $\text{YM} + \phi^3$. A careful analysis shows that only the graph topologies shown in Fig. 1 contribute. The integrands in Fig. 1 have a very simple form and can be obtained by the Feynman rules given in Appendix B. For example the integrand of the first graph in Fig. 1 reads

$$\begin{aligned} & \text{Diagram 1} = \frac{g^4 \lambda^2 c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABE'} F^{E'CD} [(p_3 + q_3) \cdot (q_1 - p_2) - \mu^2]. \end{aligned}$$

The factors in the denominator $D_i = Q_i^2 + i\epsilon = q_i^2 - \mu^2 + i\epsilon$ are the denominators of the Feynman propagators. These are defined in Appendix A Eq. (A2). The color structure reads $c^{abcd} = f^{a'b'} f^{b'bc'} f^{c'cd'} f^{d'da'}$ and the global $\text{SU}(N)$ group information is encoded in $F^{ABE'} F^{E'CD}$, which will be mapped into the adjoint gauge group generators of EYM in the DC.

The amplitude representation (19) only contains cubic graphs; however, we see that in Fig. 1 Feynman graphs with quartic vertices also appear; e.g., the first graph in the second line of Fig. 1 reads

$$\begin{aligned} & \text{Diagram 2} = \frac{g^4 \lambda^2}{D_0 D_1 D_2} \left[c^{abcd} \left(F^{ACE'} F^{E'DB} - \delta^{CD} F^{A'AB'} F^{B'BA'} \right) \right. \\ & \quad \left. + c^{abdc} \left(F^{ADE'} F^{E'CB} - \delta^{CD} F^{A'AB'} F^{B'BA'} \right) + c^{\text{triangle}} \dots \right]. \end{aligned}$$

We use the Jacobi-identity $c^{abdc} \equiv c^{abcd} + c^{\text{triangle}}$ of (17) in our computations which implies that we have to add the numerators of the first two terms. However, we also know that all numerators in YM theory associated to color structures different from (14) vanish. This implies that the parts of the integrand which effectively contribute to EYM are

$$\begin{aligned} \text{eff} &= \frac{g^4 \lambda^2}{D_0 D_1 D_2} c^{abcd} \left(F^{ACE'} F^{E'DB} + F^{ADE'} F^{E'CB} \right. \\ &\quad \left. - 2\delta^{CD} F^{A'AB'} F^{B'BA'} \right). \end{aligned}$$

The next step is to insert $\frac{D_3}{D_3}$ which makes the graph trivalent formally such that the gravity integrand can be obtained by (20).

After collecting all the nonvanishing contributions and building up the complete integrand, we Veltman-Passarino reduce the amplitude with the MATHEMATICA package FeynCalc [61,62]. Using the mapping conventions given in (21) the full amplitude at order κ^2 reads

$$\begin{aligned} \mathcal{M}(1^+ 2^+ 3^+ 4^+) |_{\kappa^2 g_{\text{YM}}^2} &= -\frac{i}{(4\pi)^2} \left(\frac{\kappa g_{\text{YM}}}{4} \right)^2 \frac{4}{3} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} (4f^{abe'} f^{e'cd} (U - T) + 4f^{ade'} f^{e'bc} (S - U) \\ &\quad + 4f^{ace'} f^{e'db} (T - S) + NS\delta^{ab}\delta^{cd} + NT\delta^{ad}\delta^{cb} + NU\delta^{ac}\delta^{bd}) \\ &= -i \frac{\kappa^2 g_{\text{YM}}^2}{192\pi^2} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} (4f^{abe'} f^{e'cd} (U - T) + NS\delta^{ab}\delta^{cd} + \text{perm}), \end{aligned} \quad (22)$$

where perm indicates the permutations of the legs 2 and 3 as well as 2 and 4. The kinematical dependence is encoded in the spinor brackets and the Mandelstam variables $S = \langle 12 \rangle [21]$, $T = \langle 23 \rangle [32]$ and $U = \langle 13 \rangle [31]$.

The next correction term $\langle 1^+ 2^+ 3^+ 4^+ \rangle |_{\kappa^4}$ can be obtained by the same technique as for $\langle 1^+ 2^+ 3^+ 4^+ \rangle |_{\kappa^2 g_{\text{YM}}^2}$. All the graphs which contribute are depicted in Fig. 2. The numerators are fairly simple and are listed in the Appendix C 1. A straight forward calculation gives the following integrated amplitude:

$$\begin{aligned} \mathcal{M}(1_a^+, 2_b^+, 3_c^+, 4_d^+) |_{\kappa^4} &= \frac{i}{(4\pi)^2} \frac{\kappa^4}{4^4} \frac{4}{15} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} (\delta^{ab}\delta^{cd}(40TU - (2 + N_g)S^2) \\ &\quad + \delta^{ac}\delta^{bd}(40ST - (2 + N_g)U^2) + \delta^{ad}\delta^{bc}(40SU - (2 + N_g)T^2)) \\ &= \frac{i}{(16\pi)^2} \frac{\kappa^4}{60} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} [\delta^{ab}\delta^{cd}(40TU - (2 + N_g)S^2) + \text{perm}], \end{aligned} \quad (23)$$

FIG. 2. At order g^4 the box, triangle and bubble topologies are nonvanishing after double copying.

FIG. 3. To obtain $\langle 1^+ 2^+ 3^+ 4^{++} \rangle|_{\kappa^3 g_{\text{YM}}}$ we have to analyze this type of graphs in $\text{YM} + \phi^3$.

where again perm indicates the permutations of legs 2 and 3 as well as 2 and 4 and $N_g = \delta^{a'a} = N^2 - 1$ is the number of adjoint generators of the Lie algebra.

B. Amplitudes: $\langle 1^+ 2^+ 3^+ 4^{++} \rangle$

It has been explicitly shown that $\langle 1^+ 2^+ 3^+ 4^{++} \rangle|_{\kappa^3 g_{\text{YM}}}$ vanishes in four dimensions [38] using both generalized unitarity and the DC. Therefore the DC calculation is not reproduced in this paper.

We therefore move on to the order κ^3 contribution to this amplitude. It can be proven that for the last graph in Fig. 3 all box color structures vanish after applying Jacobi identities.

A typical integrand of Fig. 3 is of the form

$$\begin{aligned}
 & \text{Diagram: Box with legs } 1_a^A, 2_b^B, 3_c^C, 4_d^+ \text{ and internal lines } Q_0, Q_1, Q_2, Q_3. \\
 &= \frac{-ig^4 \lambda}{D_0 D_1 D_2 D_3} c^{abcd} F^{ABC} [(p_2 + q_2) \cdot (q_0 - p_1) - \mu^2] \frac{\langle r_4 | q_0 + q_3 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle},
 \end{aligned}$$

where four-dimensional spinor helicity variables are used to represent the polarization vector. Compared to the previous two amplitudes we have now a gauge choice that is encoded in the reference vector r_4 which can be chosen arbitrarily but not such that it is proportional to p_4 . We have done the calculation with the choices $r_4 \in \{p_1, p_2, p_3\}$. Since the amplitude has to be invariant under different gauges, this represents a powerful cross-check for the final result.

After all integrands of $\text{YM} + \phi^3$ have been determined one can construct the gravity integrand using (20). Evaluating and reducing this expression yields the simple result

$$\mathcal{M}(1_a^+, 2_b^+, 3_c^+, 4^{++})|_{g_{\text{YM}} \kappa^3} = -\frac{\kappa^3 g_{\text{YM}} f^{abc} [41][42][43][12]}{(8\pi)^2 \sqrt{2} \langle 34 \rangle}. \quad (24)$$

FIG. 4. This pair of graph topologies represent graphs which appear at leading order in $\lambda^2 g^2$ for $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle$.

C. Amplitudes: $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle$

The leading order in κ^2 for the amplitude $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle|_{\kappa^2 g_{\text{YM}}^2}$ has been determined in [38] using the unitarity based two cut method. We shall see that with the DC prescription (20) this result is much easier obtained. Only the graph topologies drawn in Fig. 4 have to be evaluated on the $\text{YM} + \phi^3$ side. Besides, the calculation can even further be simplified by choosing the reference momenta r_i for the gluon polarization vectors at the legs three and four to be the same such that the integrands containing the quartic vertex give zero identically.¹³

Thus only the first type of graph from Fig. 4 has to be determined. The resulting amplitude is given by

$$\begin{aligned}
 & \mathcal{M}(1_a^+, 2_b^+, 3^{++}, 4^{++})|_{g_{\text{YM}} \kappa^3} \\
 &= \frac{i}{(4\pi)^2} \left(\frac{\kappa g_{\text{YM}}}{2} \right)^2 f^{a'ab'} f^{b'ba'} \frac{S [12][34]^2}{6 \langle 12 \rangle \langle 34 \rangle^2}. \quad (25)
 \end{aligned}$$

This result agrees with the expression given in [38] if one inserts the pair of coupling constants $(g_{\text{YM}} \kappa/2)^2$ and the color structure $N \text{Tr}(T^a T^b) = N \delta^{ab} = f^{a'ab'} f^{b'ba'}$ following from the decomposition of a one-loop amplitude into partial amplitudes. Since the result stated in [38] is the only partial amplitude which contributes, both expression coincide.

¹³This follows from $\epsilon_3^+ \cdot \epsilon_4^+ \sim \langle r_3 r_4 \rangle$.

FIG. 5. At g^4 these graph topologies shall contribute to the gravity amplitude $\langle 1^{++}2^{++}3^{++}4^{++} \rangle|_{\kappa^4}$.

The κ^4 contribution can be determined by the graphs given in Fig. 5. Applying the same steps as before we arrive at the result,

$$\mathcal{M}(1_a^+, 2_b^+, 3^{++}, 4^{++})|_{\kappa^4} = i \frac{\kappa^4}{(16\pi)^2} \frac{[21]^2 [43]^3}{\langle 34 \rangle} \frac{2 + N_g}{90} \delta^{ab}. \quad (26)$$

Here N_g represents again the dimension of the adjoint representation of the gauge group. This result has been calculated for the following choices of reference momenta $r_3 \in \{p_1, p_2, p_4\}$, $r_4 \in \{p_1, p_2, p_3\}$ yielding identical results.

D. Amplitude: $\langle 1^{++}2^{++}3^{++}4^{++} \rangle$

The remaining all-plus amplitude of EYM only contains gravitons as asymptotic states. In principle all the diagrams given in Fig. 6 contribute to the integrand. It can be seen

that the graphs in the first row arise from pure YM theory. It has been discussed in Sec. II B that the calculation of these diagrams simplifies by choosing an appropriate gauge such that the graphs with box topology are the only nonvanishing ones; i.e., the only integrand which survives is exactly given by (15). For example this has been demonstrated in [56] using unitarity cuts. To simplify their analysis, the propagating gluon has been replaced by a complex scalar which is possible due to the supersymmetric Ward-Takahashi identities. Since a complex scalar has two degrees of freedom we can represent this relation diagrammatically by

This immediately shows that the graphs in the first and second row are intimately related. Hence we can write the

FIG. 6. These topologies in $\text{YM} + \phi^3$ have to be evaluated to obtain $\langle 1^{++}2^{++}3^{++}4^{++} \rangle|_{\kappa^4}$.

amplitude as the sum of three scalar boxes in $4 - 2\epsilon$ dimensions, which are defined in Eq. (A1) of the Appendix A,

$$\begin{aligned} \mathcal{M}(1^{++}, 2^{++}, 3^{++}, 4^{++}) \\ = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\kappa^4}{4} \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \\ \times (I_4[\mu^8; S, T] + I_4[\mu^8; T, U] + I_4[\mu^8; U, S]) \left(1 + \frac{N_g}{2} \right). \end{aligned}$$

In four dimensions the result simply reduces to

$$\begin{aligned} \mathcal{M}(1^{++}, 2^{++}, 3^{++}, 4^{++}) \\ = -\frac{i}{(4\pi)^2} \kappa^4 \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \frac{S^2 + T^2 + U^2}{1920} (2 + N_g). \quad (27) \end{aligned}$$

According to [15] we can remove the dilaton and axion by subtracting twice the contribution generated by the adjoint scalar circulating in the loop. For the all-plus amplitude the scalar part is given by

$$\begin{aligned} \mathcal{M}^{\text{scalar}}(1^{++}, 2^{++}, 3^{++}, 4^{++}) \\ = -\frac{i}{(4\pi)^2} \kappa^4 \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \frac{S^2 + T^2 + U^2}{3840}, \end{aligned}$$

which implies that the pure EYM result reads

$$\begin{aligned} \mathcal{M}^{\text{EYM}}(1^{++}, 2^{++}, 3^{++}, 4^{++}) \\ = -\frac{i}{(4\pi)^2} \kappa^4 \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \frac{S^2 + T^2 + U^2}{1920} (1 + N_g). \quad (28) \end{aligned}$$

We note that the pure gravity part of (28) also agrees with [4, 63].¹⁴

IV. CONCLUSIONS

In this paper we have calculated the one-loop corrections to EYM for all four point amplitudes with positive helicity configuration. Using the DC of YM and YM + ϕ^3 as well as the fact that diagrams with box topology are the only ones on the YM side that have a nonvanishing contribution, we have evaluated all the diagrams of YM + ϕ^3 which contain the color structure of graphs with box topology. Furthermore, we have checked that our results obtained for $\langle 1^+, 2^+, 3^{++}, 4^{++} \rangle|_{\kappa^2 g_{\text{YM}}^2}$ and the pure gravitational part of $\langle 1^{++}, 2^{++}, 3^{++}, 4^{++} \rangle|_{\kappa^4}$ coincide with known results in the literature.

Generally, it would be very interesting to be able to obtain the pure EYM results from the amplitudes calculated

in this paper. An analysis of possible interaction terms generated by the Lagrangian (11) shows that the axion and dilaton cannot contribute at one loop for a four point amplitude if the number of asymptotic gluon states matches the power of the coupling constant g . The reason is that in this case only the gauge fields can propagate in the loop, because all fields couple at least quadratically to it. Similarly, we were able to separate the axion and dilaton contribution to the four graviton amplitude from the gluon and graviton contributions. However, all these cases have in common that only one particle type is circulating in the loop. All the remaining amplitudes contain a mixed type of particles circulating in the loop, which makes a subtraction of the axion and dilaton rather difficult.

Therefore it would be very interesting to remove the axion and dilaton contribution from all the amplitudes to obtain pure EYM amplitudes. It might be possible to subtract these fields by a similar method used in [15].

Moreover, one can compute the remaining amplitudes with arbitrary helicity configurations at four points. These can be obtained by using the DC method in the same way as it has been done here because in [12] the color-kinematics representation for the YM amplitudes at one-loop has been presented for arbitrary helicity configurations. However, the calculation is much more tedious since the numerators given in [12] are nonzero for all possible graph topologies. However, this is left for future work.

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APPENDIX A: REGULARIZATION

To regularize the Feynman integrals we shall use four-dimensional helicity (FDH) regularization. This is a dimensional regularization scheme in which fermions and gauge bosons of observed and unobserved particles have two helicity states and the momenta of the observed particles are also kept in four dimensions. Only the momenta of unobserved particles are continued to $4 - 2\epsilon$ dimensions with $\epsilon < 0$ [53, 54, 64]. These properties make this scheme in particular useful for expressing the amplitudes in spinor helicity formalism.

Thus we separate the d -dimensional vector $L := (l_4, l_{-2\epsilon}) := (l, \mu)$, where the two vector spaces are

¹⁴Note that $\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} = -\frac{ST}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$.

orthogonal $l \cdot \mu = 0$. Therefore we have $L^2 = l^2 - \mu^2$ using the four-dimensional mostly minus convention $(+, -, -, -)$. So we can view a higher dimensional vector as a lower dimensional vector whose mass-squared is shifted by μ^2 .¹⁵

In particular a scalar product with a four-dimensional vector projects always on the four-dimensional vector space e.g., $\epsilon_4 \cdot L = \epsilon_4 \cdot l$. This implies one can treat the loop integration with “massless” loop momentum L in d dimensions as a massive loop momentum l in four dimensions. Since we are dealing with four external particles and the dimension of integration is $d > 4$, one can think of all external particles lying in a four-dimensional subspace. Thus the external particles can always be represented by spinor helicity variables.

Following the conventions of [38] the d -dimensional scalar Feynman integrals are defined by the expression

$$\frac{i}{(4\pi)^{2-\epsilon}} I_n[\mu^{2r}] := \int \frac{d^4 l}{(2\pi)^4} \int \frac{d^{-2\epsilon} \mu}{(2\pi)^{-2\epsilon}} \frac{\mu^{2r}}{D_0 \cdots D_{n-1}}, \quad (\text{A1})$$

where

$$D_i = Q_i^2 + i\epsilon = q_i^2 - \mu^2 + i\epsilon, \quad q_j = l + \sum_{i=1}^j p_i. \quad (\text{A2})$$

Using the formula

$$I_n^{d=4-2\epsilon}[\mu^{2r}] = -\epsilon(1-\epsilon)(2-\epsilon) \cdots (r-1-\epsilon) I_n^{d=4+2r-2\epsilon}[1], \quad (\text{A3})$$

which can be derived by transforming the $d^{-2\epsilon} \mu$ -integration into spherical polar coordinates [54]. Hence formula (A3) removes the dependence on the fictitious mass μ^{2r}

parameter by shifting the dimension of the integration variable to $4 + 2r - 2\epsilon$.

From the $4 - 2\epsilon$ -dimensional expression for the bubble, the one-mass triangle,

$$\begin{aligned} I_2[1; S] &= r_\Gamma \frac{(-S)^{-\epsilon}}{\epsilon(1-2\epsilon)}, \\ I_3[1; S] &= -\frac{r_\Gamma}{\epsilon^2} (-S)^{-1-\epsilon} \\ \text{with } r_\Gamma &:= \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \end{aligned}$$

and the zero-mass box,

$$\begin{aligned} I_4[1; S, T] &= r_\Gamma \frac{2}{ST} \left[\frac{(-S)^{-\epsilon}}{\epsilon^2} {}_2F_1\left(1, -\epsilon, 1-\epsilon; 1 + \frac{S}{T}\right) \right. \\ &\quad \left. + \frac{(-T)^{-\epsilon}}{\epsilon^2} {}_2F_1\left(1, -\epsilon, 1-\epsilon; 1 + \frac{T}{S}\right) \right], \end{aligned}$$

one can derive exact expressions in arbitrary dimensions. The reason is that these expression are exact to all orders in ϵ such that one obtains higher dimensional integrals by simply shifting the value of ϵ ,

$$d = 6 - 2\epsilon \text{ is obtained by } \epsilon \rightarrow \epsilon - 1,$$

$$d = 8 - 2\epsilon \text{ is obtained by } \epsilon \rightarrow \epsilon - 2,$$

etc.,

which follows from the expression (A1). Together with the formula (A3), all scalar Veltman-Passarino functions can be expressed in terms of Mandelstam variables $S = \langle 12 \rangle [21]$, $T = \langle 14 \rangle [41]$, and $U = \langle 13 \rangle [31]$.

We shall use the following explicit expressions in four dimensions:

$$\begin{aligned} I_2[\mu^2; S] &= -\frac{S}{6} + \mathcal{O}(\epsilon), & I_2[\mu^4; S] &= -\frac{S^2}{60} + \mathcal{O}(\epsilon), & I_2[\mu^6; S] &= -\frac{S^3}{420} + \mathcal{O}(\epsilon), \\ I_3[\mu^2; S] &= \frac{1}{2} + \mathcal{O}(\epsilon), & I_3[\mu^4; S] &= \frac{S}{24} + \mathcal{O}(\epsilon), & I_3[\mu^6; S] &= \frac{S^2}{180} + \mathcal{O}(\epsilon), \\ I_4[\mu^2; S, T] &= \mathcal{O}(\epsilon), & I_4[\mu^4; S, T] &= -\frac{1}{6} + \mathcal{O}(\epsilon), & I_4[\mu^6; S, T] &= -\frac{S+T}{60} + \mathcal{O}(\epsilon), \\ I_4[\mu^8; S, T] &= -\frac{1}{840} (2S^2 + ST + 2T^2) + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A4})$$

¹⁵We have $L^2 = l^2 - \mu^2$ because we have a mostly minus signature, and therefore the -2ϵ should have only a minus signature. If $L^2 = M^2$ then $l^2 = L^2 + \mu^2 = M^2 + \mu^2$.

APPENDIX B: FEYNMAN RULES

We shall give a short review of how we can determine the actual gravity amplitude from the DC of the two theories (8). It turns out that in this case the gravity theory is uniquely determined by its spectrum and its cubic coupling, i.e., its three point amplitude. Scrutinizing the spectrum of the gravity theory ($\text{YM} + \phi^3$) $\otimes_{\text{DC}}$ YM one can deduce that it contains a graviton, an antisymmetric two-form and the dilaton. However, the latter two can be removed by

introducing ghost fields in the double copy of the loop amplitudes. Thus the next step is to normalize the three point vertex in $\mathcal{L}^{\text{YM}+\phi^3}$ appropriately to obtain EYM. This has been done in [18,33] by explicitly calculating the three point amplitude of $\mathcal{N} = 2$ Einstein-Yang-Mills and then truncating this to the bosonic sector. The final integrand of EYM amplitudes is obtained by evaluating the integrands of the two gauge theories $\text{YM} + \phi^3$ and YM which is done by using the Feynman rules listed in Table I.

TABLE I. Feynman rules for $\text{YM} + \phi^3$ derived from the Lagrangian (9). In our conventions all momenta are outgoing. For the gluon propagator the Feynman gauge is used.

	$-gf^{abc}[(r_\mu - q_\mu)\eta_{\nu\rho} + (p_\nu - r_\nu)\eta_{\rho\mu} + (q_\rho - p_\rho)\eta_{\mu\nu}]$
	$-ig^2[f^{abe'}f^{e'cd}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) + f^{ace'}f^{e'db}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) + f^{ade'}f^{e'bc}(\eta_{\mu\nu}\eta_{\sigma\rho} - \eta_{\mu\rho}\eta_{\sigma\nu})]$
	$i\lambda gf^{abc}F^{ABC}$
	$-ig^2[f^{abe'}f^{e'cd}(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}) + f^{ace'}f^{e'db}(\delta^{AD}\delta^{BC} - \delta^{AB}\delta^{CD}) + f^{ade'}f^{e'bc}(\delta^{AB}\delta^{CD} - \delta^{AC}\delta^{BD})]$
	$gf^{abc}\delta^{AC}(p_\mu - q_\mu)$
	$ig^2(f^{abe'}f^{e'cd} + f^{cbe'}f^{e'ad})\delta^{BD}\eta_{\mu\nu}$
	$\frac{-i\eta^{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$
	$\frac{i\delta^{AB}\delta^{ab}}{p^2 + i\epsilon}$

APPENDIX C: INTEGRANDS

All integrands are presented in four dimensions. We shall use the notation of the color factors from (14). The inverse of the stripped propagators is given by $D_j = Q_j^2 + i\epsilon = q_j^2 + i\epsilon = (\sum_{k=0}^j p_k + l)^2 - \mu^2 + i\epsilon$.

Integrands which are proportional to the color ordering c^{abdc} shall be written with permuted external momenta p_i , i.e., $\tilde{Q}_0 = l$, $\tilde{Q}_1 = p_2$, $\tilde{Q}_2 = l + p_2 + p_3$, $\tilde{Q}_3 = Q_3$. The corresponding stripped propagators are given by $(\tilde{D}_j)^{-1} = (\tilde{Q}_j^2 + i\epsilon)^{-1}$.

We only write the parts of the integrand in $\text{YM} + \phi^3$ which contribute to the gravity amplitude in EYM; hence all terms that are not proportional to the color structure (14) are not displayed.

To simplify the expression for the amplitudes we shall make use of the explicit value of the quadratic Casimir of the adjoint representation c_A which for $\text{SU}(N)$ is given by

$$f^{a'ab'} f^{b'ba'} = c_A \delta^{ab} = N \delta^{ab}.$$

1. Integrands for $\langle 1_a^A, 2_b^B, 3_c^C, 4_d^D \rangle$

These are the two types of integrands that appear in the computation of $\langle 1_a^A, 2_b^B, 3_c^C, 4_d^D \rangle|_{\lambda^2 g^4}$. The remaining integrands can be obtained by permuting the external legs. In total twelve nonequal box integrands and six nonequal triangle integrands contribute.

$$= \frac{\lambda^2 g^4 c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABE'} F^{E'CD} [(p_3 + q_3) \cdot (q_1 - p_2) - \mu^2]$$

$$\stackrel{\text{eff}}{=} -\frac{g^4 \lambda^2 c^{abdc}}{D_0 D_1 D_2} \left(F^{ADE'} F^{E'BC} + F^{ACE'} F^{E'BD} + 2\delta^{CD} F^{A'AB'} F^{B'BA'} \right)$$

For the computation of $\langle 1_a^A, 2_b^B, 3_c^C, 4_d^D \rangle|_{g^4}$ the following diagrams have to be evaluated. $N_g = N^2 - 1$ is the number of adjoint generators of $\text{SU}(N)$. By permuting the external legs the remaining nonequal integrands can be obtained (six box graphs, twelve triangles and six bubbles).

$$= \frac{g^4 c^{abcd}}{D_0 D_1 D_2 D_3} \delta^{AB} \delta^{CD} [(q_1 - p_2) \cdot (p_3 + q_3) - \mu^2] [(p_1 + q_1) \cdot (q_3 - p_4) - \mu^2]$$

$$\stackrel{\text{eff}}{=} \frac{g^4 c^{abdc}}{D_0 D_1 D_2} (\delta^{AC} \delta^{BD} + \delta^{AD} \delta^{BC} - 2\delta^{AB} \delta^{CD}) [(q_0 - p_1) \cdot (q_2 + p_2) - \mu^2]$$

$$\stackrel{\text{eff}}{=} -2 \frac{g^4 c^{abdc}}{D_0 D_1 D_2} \delta^{AB} \delta^{CD} [(q_1 - p_2) \cdot (p_1 + q_1) - \mu^2]$$

$$\stackrel{\text{eff}}{=} 2 \frac{g^4}{D_0 D_2} c^{abdc} [2(N_g - 2) \delta^{AB} \delta^{CD} + \delta^{AD} \delta^{BC} + \delta^{AC} \delta^{BD}]$$

$$\stackrel{\text{eff}}{=} 16 \frac{g^4}{D_0 D_2} c^{abdc} \delta^{AB} \delta^{CD}$$

2. Integrand for $\langle 1_a^A, 2_b^B, 3_c^C, 4_d^+ \rangle$

These four types of integrands can appear in general for the computation of $\langle 1_a^A, 2_b^B, 3_c^C, 4_d^+ \rangle|_{g^2\lambda}$. However, once the double copy is performed the last type of graphs are vanishing. For the first type of box graphs and the triangle graphs we have two additional permutations whereas for the other box graphs we have two.

$$\begin{aligned}
 & \stackrel{\text{eff}}{=} i \frac{g^4 \lambda c^{abdc}}{D_0 D_1 D_2} F^{ABC} \frac{\langle r_4 | q_1 - p_2 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} \\
 & \stackrel{\text{eff}}{=} 0 \\
 & = -i \frac{g^4 \lambda c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABC} \left[(p_2 + q_2) \cdot (q_0 - p_1) - \mu^2 \right] \frac{\langle r_4 | q_0 + q_3 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} \\
 & = i \frac{g^4 \lambda c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABC} \left[\left[(p_1 + q_1) \cdot (q_3 - p_4) - \mu^2 \right] \frac{\langle r_4 | q_2 - p_3 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} \right. \\
 & \quad \left. + \left[(q_2 - p_3) \cdot (p_4 + q_0) - \mu^2 \right] \frac{\langle r_4 | p_1 + q_1 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} - \left[(q_2 - p_3) \cdot (p_1 + q_1) - \mu^2 \right] \frac{\langle r_4 | q_0 + q_3 | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} \right]
 \end{aligned}$$

3. Integrands for $\langle 1_a^A, 2_b^B, 3_c^+, 4_d^+ \rangle$

The gauge choice $r_3 = r_4$ reduces the amount of diagrams to compute. This gauge choice implies that only the three box graphs contribute to $\langle 1_a^A, 2_b^B, 3_c^+, 4_d^+ \rangle|_{g^2\lambda^2}$.

$$= - \frac{g^4 \lambda^2 c^{abcd}}{D_0 D_1 D_2 D_3} F^{A'BB'} F^{B'AA'} \frac{\langle r_4 | q_2 + q_3 | 3 \rangle \langle r_4 | q_0 + q_3 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle}$$

For $\langle 1_a^A, 2_b^B, 3_c^+, 4_d^+ \rangle|_{g^4}$ this gauge choice sets all bubble graphs to zero. The remaining graphs give a contribution to $\langle 1_a^A, 2_b^B, 3_c^+, 4_d^+ \rangle|_{g^4}$. To all graphs with box topology we obtain also one other distinguished integrand. The next three triangles are the only diagrams which can be drawn with this topology. The last two graphs can be drawn in four inequivalent ways.

$$\begin{aligned}
&= -\frac{g^4 c^{abcd} \delta^{AB}}{D_0 D_1 D_2 D_3} (p_2 + Q_2) \cdot (Q_0 - p_1) \frac{\langle r_4 | q_2 + q_3 | 3 \rangle \langle r_4 | q_0 + q_3 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} \\
&= \frac{g^4 c^{adbc} \delta^{AB}}{\tilde{D}_0 \tilde{D}_1 \tilde{D}_2 \tilde{D}_3} \frac{\langle r_4 | \tilde{q}_0 + \tilde{q}_3 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} \left[\langle r_4 | p_1 + \tilde{q}_3 | 3 \rangle (p_3 + \tilde{Q}_2) \cdot (\tilde{Q}_0 - p_2) \right. \\
&\quad \left. + \langle r_4 | \tilde{q}_0 - p_2 | 3 \rangle (\tilde{Q}_1 - p_3) \cdot (\tilde{Q}_3 + p_1) - \langle r_4 | \tilde{q}_1 + \tilde{q}_2 | 3 \rangle (p_1 + \tilde{Q}_3) \cdot (\tilde{Q}_0 - p_2) \right] \\
&= -\frac{g^4 c^{abcd} \delta^{AB}}{2 D_0 D_1 D_2 D_3} \left[\frac{\langle r_4 | q_1 - p_2 | 3 \rangle \langle r_4 | q_2 - p_3 | 4 \rangle - \langle r_4 | q_3 + q_2 | 3 \rangle \langle r_4 | q_1 - p_2 | 4 \rangle}{\langle r_4 3 \rangle \langle r_4 4 \rangle} \right. \\
&\quad \times (Q_3 - p_4) \cdot (p_1 + Q_1) + \frac{\langle r_4 | p_1 + q_1 | 4 \rangle}{\langle r_4 3 \rangle \langle r_4 4 \rangle} \left[\langle r_4 | p_4 + q_0 | 3 \rangle (Q_1 - p_2) \cdot (Q_3 + p_3) \right. \\
&\quad \left. + \langle r_4 | q_1 - p_2 | 3 \rangle (Q_2 - p_3) \cdot (Q_0 + p_4) - \langle r_4 | q_2 + q_3 | 3 \rangle (Q_1 - p_2) \cdot (Q_0 + p_4) \right] \\
&\quad \left. - \frac{\langle r_4 | q_0 + q_3 | 4 \rangle}{\langle r_4 3 \rangle \langle r_4 4 \rangle} \left[\langle r_4 | q_1 - p_2 | 3 \rangle (Q_2 - p_3) \cdot (Q_1 + p_1) \right. \right. \\
&\quad \left. \left. + \langle r_4 | p_1 + q_1 | 3 \rangle (Q_1 - p_2) \cdot (Q_3 + p_3) - \langle r_4 | q_2 + q_3 | 3 \rangle (Q_1 - p_2) \cdot (Q_1 + p_1) \right] \right] \\
&\stackrel{\text{eff}}{=} -\frac{g^4 c^{abcd}}{D_0 D_2 D_3} \delta^{AB} \frac{\langle r_4 | q_2 + q_3 | 3 \rangle \langle r_4 | q_0 + q_3 | 4 \rangle}{\langle r_4 3 \rangle \langle r_4 4 \rangle} (1 - N_g) \\
&\stackrel{\text{eff}}{=} \frac{g^4 c^{abcd}}{D_0 D_2 D_3} \frac{\delta^{AB}}{\langle r_4 3 \rangle \langle r_4 4 \rangle} \left[2 \langle r_4 | q_2 + q_3 | 3 \rangle \langle r_4 | q_0 + q_3 | 4 \rangle \right. \\
&\quad \left. + \langle r_4 | q_0 + p_4 | 3 \rangle \langle r_4 | p_3 + q_3 | 4 \rangle + \langle r_4 | q_3 - p_4 | 3 \rangle \langle r_4 | q_2 - p_3 | 4 \rangle \right] \\
&\stackrel{\text{eff}}{=} -\frac{g^4 c^{abcd}}{D_0 D_1 D_2} \delta^{AB} \left[\frac{\langle r_4 | q_1 - p_2 | 3 \rangle \langle r_4 | p_1 + q_1 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} + \frac{\langle r_4 | p_1 + q_1 | 3 \rangle \langle r_4 | q_1 - p_2 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} \right] \\
&\stackrel{\text{eff}}{=} -\frac{g^4 c^{adbc}}{\tilde{D}_1 \tilde{D}_2 \tilde{D}_3} \delta^{AB} \frac{\langle r_4 | \tilde{q}_1 + \tilde{q}_2 | 3 \rangle \langle r_4 | p_1 - \tilde{q}_2 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} \\
&\stackrel{\text{eff}}{=} -\frac{g^4 c^{adbc}}{\tilde{D}_1 \tilde{D}_2 \tilde{D}_3} \delta^{AB} \left[\frac{\langle r_4 | \tilde{q}_3 + p_1 | 3 \rangle \langle r_4 | p_3 + \tilde{q}_2 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} - \frac{\langle r_4 | \tilde{q}_1 + \tilde{q}_2 | 3 \rangle \langle r_4 | p_1 + \tilde{q}_3 | 4 \rangle}{2 \langle r_4 3 \rangle \langle r_4 4 \rangle} \right]
\end{aligned}$$

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