Multiparticle azimuthal angular correlations in pA collisions

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In the Color Glass Condensate formalism, we evaluate the 3-dipole correlator up to the $\frac{1}{N_c}$ order with N_c being the number of colors, and compute the azimuthal cumulant c_{123} for 3-particle productions. In addition, we discuss the patterns appearing in the n -dipole formula in terms of $\frac{1}{N_c}$ expansions. This allows us to conjecture the N_c scaling of $c_n\{m\}$, which is crosschecked by our calculation of $c_2\{4\}$ in the dilute limit.

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I. INTRODUCTION

Experimental data in deep inelastic scatterings and proton-nucleus (pA) collisions indicates a sharp rise in the cross section at small Bjorken x , which is believed to be due to the fast growth of the gluon density in large nuclei. The gluon density increases steeply and tends to be saturated when going to a smaller x . The Color Glass Condensate (CGC) formalism [1–7] describes this gluon saturation regime by treating the fast-moving partons inside the dense nucleus as a classical color source of the soft gluons. In the McLerran-Venugopalan (MV) model [8], the small- x gluons are assumed as a classical Yang-Mills field. The soft gluons, whose color field A is associated with a random color source ρ , are radiated by the eikonal (high-energy and fixed at a transverse coordinate) partons [9].

With the fact that the gluon density inside the nucleus is large, one can obtain a weak coupling constant g . Nevertheless, due to the strong nuclear field $A \sim \frac{1}{g}$, the high-density regime is nonlinear and cannot be handled perturbatively. The Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) renormalization group equation [10,11] governs the nonlinear evolution of the gluon distribution function in the saturation regime. In the dilute regime, this equation reduces to the linear

BFKL equation, which describes the evolution of the unintegrated gluon density.

Within the framework introduced above, multiparticle productions in high-energy collisions are represented by dipoles, quadrupoles, and higher-point functions. These multipoint functions are written in forms of the Gaussian average of Wilson lines in the fundamental representation for quarks and the adjoint representation for gluons. However, Ref. [12] shows that, in the large- N_c limit, only dipoles and quadrupoles contribute to the particle production processes. The quadrupole corresponding to the Weizsäcker-Williams gluon distribution has been evaluated in Ref. [13] by the JIMWLK Hamiltonian method.

The CGC formalism generates multiparticle correlations in dilute-dense scatterings (pA collisions) from the initial state [14–37]. In order to study correlation observables, which include the n th moment of the m -particle cumulant $c_n\{m\}$ and the corresponding anisotropic flow $v_n\{m\}$, one has to derive multidipole correlators. Some recent work has been done in Refs. [14,31,32,38]. In this paper, we evaluate the 3-dipole correlator up to the $\frac{1}{N_c}$ order by the color transition matrices method developed in Refs. [39–42]. It helps to obtain an n -dipole formula in forms of a power series at large N_c . We calculate the 3-particle azimuthal cumulant c_{123} [43,44], within the model calculation approach proposed in Refs. [26,28,31,32]. Finally, we discuss the N_c scaling of $c_n\{m\}$, which is verified by the result of $c_2\{4\}$ in the dilute limit.

The rest of the paper is organized as follows. In Sec. II, we briefly review the MV model, and evaluate the 2-dipole correlator in forms of large N_c expansions. We calculate the 3-dipole correlator up to the $\frac{1}{N_c}$ order, then obtain the n -dipole expression. We subsequently proceed to compute the azimuthal cumulant c_{123} , and expand $c_2\{4\}$ at small saturation momentum scale Q_s in Sec. III. We discuss the

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N_c scaling of multiparticle correlations in Sec. IV. Finally, Sec. V is for the summary.

II. MULTIDIPOLE CORRELATORS IN THE MCLERRAN-VENUGOPALAN MODEL

We aim to compute the n -dipole correlator, namely,

$$\left\langle \prod_{i=1}^n D(\mathbf{x}_{2i-1\perp}, \mathbf{x}_{2i\perp}) \right\rangle \equiv \frac{1}{N_c^n} \left\langle \prod_{i=1}^n \text{Tr}[U(\mathbf{x}_{2i-1\perp}) U(\mathbf{x}_{2i\perp})^\dagger] \right\rangle, \quad (1)$$

where $U(\mathbf{x}_\perp)$ is a Wilson line in the fundamental representation given by

$$U(\mathbf{x}_\perp) = \mathcal{P} \exp \left[-ig \int_{-\infty}^{+\infty} dx^+ A_a^-(x^+, \mathbf{x}_\perp) t^a \right], \quad (2)$$

where $\mathcal{P}$ is x^+ ordering operator and t^a is a color matrix in the fundamental $\text{SU}(N_c)$ representation. In the MV model, A_a^- is the classical color field that obeys the classical Yang-Mills equation

$$-\nabla_\perp^2 A_a^-(x^+, \mathbf{x}_\perp) = g\rho_a(x^+, \mathbf{x}_\perp), \quad (3)$$

where ρ_a is the corresponding color charge density inside the nucleus. Deriving A_a^- from Eq. (3) gives

$$A_a^-(x^+, \mathbf{x}_\perp) = g \int d^2\mathbf{z}_\perp G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) \rho_a(x^+, \mathbf{z}_\perp),$$

$$G_0(\mathbf{x}_\perp) = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} \frac{e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}}{\mathbf{k}_\perp^2}, \quad (4)$$

where G_0 is the two-dimensional massless propagator.

The angle brackets in Eq. (1) represent the color field average of a given physical quantity $f[A]$, where in the MV model, the average is a Gaussian weighted functional integral in form of

$$\langle f[A] \rangle = \int \mathcal{D}\rho \exp \left\{ - \int d\mathbf{x}^+ d^2\mathbf{x}_\perp \frac{\rho_a^2(x^+, \mathbf{x}_\perp)}{2\mu^2(x^+)} \right\} f[A], \quad (5)$$

where $\mu^2(x^+)$ is the variance of the Gaussian distribution of color field, which represents the squared color charge per unit transverse area at coordinate x^+ . The $\mu^2(x^+)$ integration over x^+ is proportional to the saturation momentum square Q_s^2 .

With the help of Wick's theorem, any correlator of ρ 's or A 's can be obtained by the most elementary correlators

$$\langle \rho_a(x_1^+, \mathbf{x}_{1\perp}) \rho_b(x_2^+, \mathbf{x}_{2\perp}) \rangle = \delta_{ab} \delta(x_1^+ - x_2^+) \delta(\mathbf{x}_{1\perp} - \mathbf{x}_{2\perp}) \mu^2(x_1^+), \quad (6)$$

$$g^2 \langle A_a^-(x_1^+, \mathbf{x}_{1\perp}) A_b^-(x_2^+, \mathbf{x}_{2\perp}) \rangle = \delta_{ab} \delta(x_1^+ - x_2^+) \mu^2(x_1^+) g^4 \times \int d^2\mathbf{z}_\perp G_0(\mathbf{x}_{1\perp} - \mathbf{z}_\perp) G_0(\mathbf{x}_{2\perp} - \mathbf{z}_\perp) = \delta_{ab} \delta(x_1^+ - x_2^+) \mu^2(x_1^+) L_{12}, \quad (7)$$

where

$$L_{ij} \equiv g^4 \int d^2\mathbf{z}_\perp G_0(\mathbf{x}_{i\perp} - \mathbf{z}_\perp) G_0(\mathbf{x}_{j\perp} - \mathbf{z}_\perp). \quad (8)$$

By using the color transition matrices method developed in Refs. [39–42], the 2-dipole correlator can be given by

$$\langle D(\mathbf{x}_1, \mathbf{x}_2) D(\mathbf{x}_3, \mathbf{x}_4) \rangle = \frac{T_{2\text{-dipole}}}{N_c^2} \begin{pmatrix} N_c^2 & N_c \end{pmatrix} \sum_{n=0}^{\infty} \frac{M_{2\text{-dipole}}^n}{n!} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9)$$

$$= \frac{T_{2\text{-dipole}}}{N_c^2} \begin{pmatrix} N_c^2 & N_c \end{pmatrix} e^{M_{2\text{-dipole}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (10)$$

where $T_{2\text{-dipole}} = e^{-\frac{c_F}{2}\mu^2 \sum_{i=1}^4 L_{ii}}$ is the so-called tadpole contribution corresponding to the configurations where each gluon link attaches to a single Wilson line. The $M_{2\text{-dipole}}$ is the color transition matrix of the 2-dipole configuration, which is written in terms of color transition factors F 's and L 's as

$$M_{2\text{-dipole}} = \mu^2 \begin{pmatrix} LF_{12,34} & \frac{1}{2}F_{1423} \\ \frac{1}{2}F_{1243} & LF_{14,32} \end{pmatrix} \equiv \begin{pmatrix} m_1 & m_4 \\ m_2 & m_3 \end{pmatrix}, \quad (11)$$

where $L_{ij,kl} \equiv L_{ij} + L_{kl}$, $LF_{ij,kl} \equiv C_F L_{ij,kl} + \frac{1}{2N_c} F_{ijkl}$, $\mu^2 \equiv \int d\mathbf{x}^+ \mu^2(x^+)$, and the color transition factor $F_{ijkl} \equiv L_{ik} + L_{jl} - L_{il} - L_{jk}$.

The matrix exponential $e^{M_{2\text{-dipole}}}$ in Eq. (10) is the difficult part to calculate, which has been derived in Ref. [41] by using the eigenvalue method. In this paper we calculate the n th power of $M_{2\text{-dipole}}$ in Eq. (9), which gives us a platform to derive the 3-dipole correlator and conjecture a general n -dipole correlator in forms of $\frac{1}{N_c}$ power expansions.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Eq. (9), which represents the initial 2-dipole configuration, picks out the first column of the n th power of $M_{2\text{-dipole}}$

$$M_{2\text{-dipole}}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} m_1^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} m_2^j m_4^j \sum_{i=0}^{n-2j} \binom{j}{i+j} \binom{j-1}{n-i-j-1} m_1^i m_3^{n-2j-i} \\ \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} m_2^{j+1} m_4^j \sum_{i=0}^{n-2j-1} \binom{j}{i+j} \binom{j}{n-i-j-1} m_1^i m_3^{n-2j-1-i} \end{pmatrix}, \quad (12)$$

where $\lfloor \frac{n}{2} \rfloor \equiv \text{Floor}(\frac{n}{2})$, namely the rounding down function. Summing over n with the factor $\frac{1}{n!}$ gives the matrix exponential

$$e^{M_{2\text{-dipole}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{m_1 \sum_{j=0}^{\infty} m_2^j m_4^j \prod_{i=1}^{2j} \int_0^{\xi_{i-1}} d\xi_i e^{(-1)^{i+1} \xi_i (m_3 - m_1)}} \\ e^{m_1 \sum_{j=0}^{\infty} m_2^{j+1} m_4^j \prod_{i=1}^{2j+1} \int_0^{\xi_{i-1}} d\xi_i e^{(-1)^{i+1} \xi_i (m_3 - m_1)}} \end{pmatrix}. \quad (13)$$

Substituting Eq. (13) into Eq. (10) gives the 2-dipole correlator:

$$\langle D(\mathbf{x}_1, \mathbf{x}_2) D(\mathbf{x}_3, \mathbf{x}_4) \rangle = T_{2\text{-dipole}} e^{m_1} \sum_{j=0}^{\infty} \left[\underbrace{m_2^j m_4^j \prod_{i=1}^{2j} \int_0^{\xi_{i-1}} d\xi_i e^{(-1)^{i+1} \xi_i (m_3 - m_1)}}_{\frac{1}{N_c^2} \text{ order}} + \underbrace{\frac{m_2^{j+1} m_4^j}{N_c} \prod_{i=1}^{2j+1} \int_0^{\xi_{i-1}} d\xi_i e^{(-1)^{i+1} \xi_i (m_3 - m_1)}}_{\frac{1}{N_c^{2j+2}} \text{ order}} \right], \quad \xi_0 = 1, \quad (14)$$

where $m_{1,3}$ are of N_c^0 order, and $m_{2,4}$ are of $\frac{1}{N_c}$ order.

One can also write down the 3-dipole correlator by definition as

$$\langle D(\mathbf{x}_1, \mathbf{x}_2) D(\mathbf{x}_3, \mathbf{x}_4) D(\mathbf{x}_5, \mathbf{x}_6) \rangle = \frac{T_{3\text{-dipole}}}{N_c^3} \begin{pmatrix} N_c^3 & N_{c(1 \times 3)}^2 & N_{c(1 \times 2)} \end{pmatrix} \sum_{n=0}^{\infty} \frac{M_{3\text{-dipole}}^n}{n!} \begin{pmatrix} 1 \\ 0_{(5 \times 1)} \end{pmatrix}, \quad (15)$$

where $M_{3\text{-dipole}}$ is the corresponding color transition matrix given by

$$M_{3\text{-dipole}} = \mu^2 \begin{pmatrix} LF_{12,34,56} & \frac{1}{2} F_{1423} & \frac{1}{2} F_{1625} & \frac{1}{2} F_{3645} & 0 & 0 \\ \frac{1}{2} F_{1243} & LF_{14,32,56} & 0 & 0 & \frac{1}{2} F_{1645} & \frac{1}{2} F_{3625} \\ \frac{1}{2} F_{1265} & 0 & LF_{16,34,52} & 0 & \frac{1}{2} F_{3245} & \frac{1}{2} F_{1463} \\ \frac{1}{2} F_{3465} & 0 & 0 & LF_{12,36,54} & \frac{1}{2} F_{1623} & \frac{1}{2} F_{1425} \\ 0 & \frac{1}{2} F_{1465} & \frac{1}{2} F_{3425} & \frac{1}{2} F_{1263} & LF_{16,32,54} & 0 \\ 0 & \frac{1}{2} F_{3265} & \frac{1}{2} F_{1643} & \frac{1}{2} F_{1245} & 0 & LF_{14,36,52} \end{pmatrix} \\ \equiv \begin{pmatrix} M_{11(1 \times 1)} & M_{12(1 \times 3)} & 0 \\ M_{21(3 \times 1)} & M_{22(3 \times 3)} & M_{23(3 \times 2)} \\ 0 & M_{32(2 \times 3)} & M_{33(2 \times 2)} \end{pmatrix}, \quad (16)$$

where $LF_{ab,cd,ef} \equiv C_F(L_{ab} + L_{cd} + L_{ef}) + \frac{1}{2N_c}(F_{abcd} + F_{abef} + F_{cdef})$. Substituting the block matrix in Eq. (16) into Eq. (15), one obtains the first column of the n th power of the $M_{3\text{-dipole}}$. In this computation, we only derive the result up to the $\frac{1}{N_c^4}$ order

$$M_{3\text{-dipole}}^n \begin{pmatrix} 1 \\ 0_{(5 \times 1)} \end{pmatrix} \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} \\ = \begin{pmatrix} M_{11}^n + \sum_{i=0}^{n-2} M_{12} M_{22}^{n-2-i} M_{21} M_{11}^i \begin{pmatrix} 1 \\ i+1 \end{pmatrix} + \sum_{j=0}^{n-4} \sum_{i=0}^{n-4-j} M_{12} M_{22}^{n-4-j-i} M_{21} M_{12} M_{22}^j M_{21} M_{11}^i \begin{pmatrix} 2 \\ i+2 \end{pmatrix} \\ + \sum_{k=0}^{n-4} \sum_{j=0}^{n-4-k} \sum_{i=0}^{n-4-k-j} M_{12} M_{22}^{n-4-k-j-i} M_{23} M_{33}^k M_{32} M_{22}^j M_{21} M_{11}^i \begin{pmatrix} 1 \\ i+1 \end{pmatrix} \\ \sum_{i=0}^{n-1} M_{22}^{n-1-i} M_{21} M_{11}^i + \sum_{j=0}^{n-3} \sum_{i=0}^{n-3-j} M_{22}^{n-3-j-i} M_{21} M_{12} M_{22}^j M_{21} M_{11}^i \begin{pmatrix} 1 \\ i+1 \end{pmatrix} \\ + \sum_{k=0}^{n-3} \sum_{j=0}^{n-3-k} \sum_{i=0}^{n-3-k-j} M_{22}^{n-4-k-j-i} M_{23} M_{33}^k M_{32} M_{22}^j M_{21} M_{11}^i \\ \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} M_{33}^{n-2-j-i} M_{32} M_{22}^j M_{21} M_{11}^i \end{pmatrix}. \quad (17)$$

Summing over n with a factor $\frac{1}{n!}$ in Eq. (15), one obtains the 3-dipole correlator up to the $\frac{1}{N_c^4}$ order

$$\begin{aligned} & \langle D(\mathbf{x}_1, \mathbf{x}_2) D(\mathbf{x}_3, \mathbf{x}_4) D(\mathbf{x}_5, \mathbf{x}_6) \rangle \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} \\ &= T_{\text{3-dipole}} \int_0^1 \prod_{i=0}^j d\xi_i \delta\left(\sum_{i=0}^j \xi_i - 1\right) \left[\delta_{j0} + \delta_{j2} e^{\xi_2 M_{11}} M_{12} e^{\xi_1 M_{22}} M_{21} + \delta_{j4} e^{\xi_4 M_{11}} M_{12} e^{\xi_3 M_{22}} M_{21} e^{\xi_2 M_{11}} M_{12} e^{\xi_1 M_{22}} M_{21} \right. \\ &+ \delta_{j4} e^{\xi_4 M_{11}} M_{12} e^{\xi_3 M_{22}} M_{23} e^{\xi_2 M_{33}} M_{32} e^{\xi_1 M_{22}} M_{21} + \frac{\delta_{j1}}{N_c} e^{\xi_1 M_{22}} M_{21} + \frac{\delta_{j3}}{N_c} e^{\xi_3 M_{22}} M_{21} e^{\xi_2 M_{11}} M_{12} e^{\xi_1 M_{22}} M_{21} \\ &\left. + \frac{\delta_{j3}}{N_c} e^{\xi_3 M_{22}} M_{23} e^{\xi_2 M_{33}} M_{32} e^{\xi_1 M_{22}} M_{21} + \frac{\delta_{j2}}{N_c} e^{\xi_2 M_{33}} M_{32} e^{\xi_1 M_{22}} M_{21} \right] e^{\xi_0 M_{11}}, \end{aligned} \quad (18)$$

where each term inside the square brackets has a clear physical pattern, which helps to conjecture the n -dipole correlator in the following sections. For illustration, we focus on the last term

$$\begin{aligned} & \frac{T_{\text{3-dipole}}}{N_c^2} \int_0^1 d\xi_0 d\xi_1 d\xi_2 \delta(\xi_0 + \xi_1 + \xi_2 - 1) \\ & \times e^{\xi_2 M_{33}} M_{32} e^{\xi_1 M_{22}} M_{21} e^{\xi_0 M_{11}}, \end{aligned} \quad (19)$$

which indicates that the original configuration (whose topology is represented by M_{11}) goes through two times of color transitions (represented by M_{21} and M_{32}), then becomes another two configurations (whose topologies are represented by M_{22} and M_{33}). This term can be graphically represented by the color transition approach below and in Fig. 1:

$$M_{11} \xrightarrow{M_{21}} M_{22} \xrightarrow{M_{32}} M_{33}, \quad (20)$$

Actually, the eight terms inside Eq. (18) correspond to eight kinds of color transition approaches, and all of them can be depicted as shown above.

On the other hand, one can check that each term of the multidipole correlators corresponds to an individual approach of color transitions between configurations, and vice versa. It enables us to conjecture an expression for the n -dipole correlator similar to the $2n$ -point correlator [45], which reads

FIG. 1. The original configuration with three fermion loops goes through color transitions two times, then it becomes another two configurations with one and two fermion loops, respectively.

$$\begin{aligned} & \left\langle \prod_{i=1}^n D(\mathbf{x}_{2i-1}, \mathbf{x}_{2i}) \right\rangle \\ &= T_{n\text{-dipole}} e^{M_1} \left[1 + \sum_{k=1}^{\infty} \sum_{I_k} N_c^{L(U_{k+1} + U_0) - n} \prod_{i=1}^k \frac{\mu^2}{2} F_{a_i b_i c_i d_i} \right. \\ & \times \left. \int_0^{\xi_{i-1}} d\xi_i e^{\xi_i \frac{\mu^2 N_c}{2} F_{a_i d_i c_i b_i}} \right], \end{aligned} \quad (21)$$

where

$$\begin{aligned} T_{n\text{-dipole}} &= e^{-\frac{\mu^2 C_F}{2} \sum_{i=1}^{2n} L_{ii}}, \\ M_1 &= \mu^2 \left(C_F \sum_{i=1}^n L_{2i-1, 2i} + \frac{1}{2N_c} \sum_{I_1} F_{a_1 b_1 c_1 d_1} \right), \quad \xi_0 = 1. \end{aligned} \quad (22)$$

I_k represents summing over all possible permutations $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, \dots, a_k, b_k, c_k, d_k$, which satisfy the following conditions

$$a_i < d_i; \quad (a_i, b_i), (d_i, c_i) \in U_i. \quad (23)$$

The recurrence relation between U_i and U_{i+1} is

$$U_{i+1} = U_i - \{(a_i, b_i), (d_i, c_i)\} + \{(a_i, c_i), (d_i, b_i)\}; \quad i \geq 1, \quad (24)$$

with its first two terms

$$\begin{aligned} U_0 &= \{(2, 1), (4, 3), \dots, (2n, 2n-1)\}, \\ U_1 &= \{(1, 2), (3, 4), \dots, (2n-1, 2n)\}. \end{aligned} \quad (25)$$

$U = \{(a, b), (b, c), \dots\}$, representing that the element a , is connected with b , and b is connected with $c, \dots$, is a set of two-dimensional row matrices. $L(U)$ in Eq. (21) is a function of U , which equals the number of elements' loops in U . For example, if $U = \{(1, 6), (2, 3), (3, 4), (4, 5), (5, 2), (6, 1)\}$, which consists of two loops $1 - 6 - 1$ and $2 - 3 - 4 - 5 - 2$, then $L(U) = 2$.

III. MULTI-PARTICLE AZIMUTHAL CUMULANTS

As the 3-dipole correlator involves such large numbers of color transitions, it is usually difficult to evaluate. As a first attempt, in this section, we calculate the 3-particle cumulant c_{123} by using the 3-dipole correlator in Eq. (18) to show how it works. Generally, an m -particle correlation takes the form

$$\langle e^{i(n_1\phi_1 + n_2\phi_2 + \dots + n_m\phi_m)} \rangle \equiv \frac{\kappa\{n_1, n_2, \dots, n_m\}}{\kappa\{0, 0, \dots, 0\}}, \quad (26)$$

where $n_1, n_2, \dots, n_m$ are integers which satisfy the azimuthal symmetry: $n_1 + n_2 + \dots + n_m = 0$. ϕ_i is the azimuthal angle of the i th outgoing particle's momentum $\mathbf{p}_i$. $\kappa\{n_1, n_2, \dots, n_m\}$ is the mixed harmonic of the m -particle distribution, which is defined by

$$\kappa\{n_1, n_2, \dots, n_m\} \equiv \int \prod_{i=1}^m d^2\mathbf{p}_i e^{in_i\phi_i} \frac{d^m N}{\prod_{i=1}^m d^2\mathbf{p}_i}, \quad (27)$$

where $\frac{d^m N}{\prod_{i=1}^m d^2\mathbf{p}_i}$ is the m -particle inclusive spectra [26,28,31,32] given by

$$\frac{d^m N}{\prod_{i=1}^m d^2\mathbf{p}_i} = \int \prod_{i=1}^m \frac{d^2\mathbf{b}_i d^2\mathbf{r}_i}{4\pi^3 B_p} e^{-\frac{b_i^2}{B_p} - \frac{r_i^2}{4B_p} + i\mathbf{p}_i \cdot \mathbf{r}_i} \times \left\langle \prod_{i=1}^m D\left(\mathbf{b}_i + \frac{\mathbf{r}_i}{2}, \mathbf{b}_i - \frac{\mathbf{r}_i}{2}\right) \right\rangle. \quad (28)$$

One of the nontrivial 3-particle cumulants is c_{123} , which is of the same definition with v_{123} in Refs. [43,44] and defined by

$$c_{123} \equiv \frac{\kappa\{1, 2, -3\}}{\kappa\{0, 0, 0\}}. \quad (29)$$

Within the framework above, the zeroth harmonic of the m -particle inclusive spectra in the MV model is derived by

$$\begin{aligned} \kappa\{0, 0, \dots, 0\} &= \int \prod_{i=1}^m d^2\mathbf{p}_i \frac{d^m N}{\prod_{i=1}^m d^2\mathbf{p}_i} \\ &= \int \prod_{i=1}^m \frac{d^2\mathbf{b}_i d^2\mathbf{r}_i d^2\mathbf{p}_i}{4\pi^3 B_p} e^{-\frac{b_i^2}{B_p} - \frac{r_i^2}{4B_p} + i\mathbf{p}_i \cdot \mathbf{r}_i} \\ &\quad \times \left\langle \prod_{i=1}^m D\left(\mathbf{b}_i + \frac{\mathbf{r}_i}{2}, \mathbf{b}_i - \frac{\mathbf{r}_i}{2}\right) \right\rangle \\ &= \int \prod_{i=1}^m \frac{d^2\mathbf{b}_i d^2\mathbf{r}_i}{\pi B_p} e^{-\frac{b_i^2}{B_p}} \delta^2(\mathbf{r}_i) \left\langle \prod_{i=1}^m D(\mathbf{b}_i, \mathbf{b}_i) \right\rangle \\ &= 1, \end{aligned} \quad (30)$$

where we used

$$\begin{aligned} \left\langle \prod_{i=1}^m D(\mathbf{b}_i, \mathbf{b}_i) \right\rangle &= \frac{1}{N_c^m} \left\langle \prod_{i=1}^m \text{Tr}[U(\mathbf{b}_i)U(\mathbf{b}_i)^\dagger] \right\rangle \\ &= \frac{1}{N_c^m} \left\langle \prod_{i=1}^m \text{Tr}[\mathbf{1}_{(N_c \times N_c)}] \right\rangle = 1. \end{aligned} \quad (31)$$

Substituting the factors L 's and F 's in Eq. (16) into Eq. (18), one obtains the explicit expression for the 3-dipole correlator up to the $\frac{1}{N_c^4}$ order

$$\begin{aligned} &\langle D(\mathbf{x}_1, \mathbf{x}_2) D(\mathbf{x}_3, \mathbf{x}_4) D(\mathbf{x}_5, \mathbf{x}_6) \rangle \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} \\ &= T_{3\text{-dipole}} e^{2C_F L_{12,34,56} + \frac{F_{1234,1256,3456}}{N_c}} \left\{ \frac{1}{3} + \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 F_{1243} F_{1423} e^{(\xi_1 - \xi_2) N_c F_{1342}} + \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \int_0^{\xi_3} d\xi_4 F_{1243} F_{1423} \right. \\ &\quad \times e^{(\xi_1 - \xi_2) N_c F_{1342}} [F_{1243} F_{1423} e^{(\xi_3 - \xi_4) N_c F_{1342}} + 2F_{1265} F_{1625} e^{(\xi_3 - \xi_4) N_c F_{1562}}] + 2 \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \int_0^{\xi_3} d\xi_4 F_{1243} F_{1465} \\ &\quad \times e^{N_c(\xi_1 F_{1342} + \xi_2 F_{1564})} [F_{1645} F_{1423} e^{N_c(\xi_3 F_{1546} + \xi_4 F_{1324})} + F_{3245} F_{1625} e^{N_c(\xi_3 F_{3542} + \xi_4 F_{1526})} + F_{1623} F_{3645} e^{N_c(\xi_3 F_{1326} + \xi_4 F_{3546})}] \\ &\quad + \int_0^1 d\xi_1 \frac{F_{1243}}{N_c} e^{\xi_1 N_c F_{1342}} + \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \left[\frac{F_{1243} F_{1423}}{N_c} e^{(\xi_1 - \xi_2) N_c F_{1342}} (F_{1243} e^{\xi_3 N_c F_{1342}} + F_{1265} e^{\xi_3 N_c F_{1562}}) \right. \\ &\quad \left. + \frac{F_{1265} F_{1625} F_{1243}}{N_c} e^{(\xi_1 - \xi_2) N_c F_{1562} + \xi_3 N_c F_{1342}} \right] + 2 \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \frac{F_{1243} F_{1465}}{N_c} e^{N_c(\xi_1 F_{1342} + \xi_2 F_{1564})} [F_{1645} e^{\xi_3 N_c F_{1546}} \\ &\quad \left. + F_{3245} e^{\xi_3 N_c F_{3542}} + F_{1623} e^{\xi_3 N_c F_{1326}}] + 2 \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \frac{F_{1243} F_{1465}}{N_c^2} e^{N_c(\xi_1 F_{1342} + \xi_2 F_{1564})} \right\} \\ &+ \text{The same expression with indices inside } L \text{ and } F \text{ factors replaced by: } \{12, 34, 56\} \rightarrow \{34, 56, 12\} \text{ and } \{56, 12, 34\}, \end{aligned} \quad (32)$$

where in order to make the expression shorter, each L and F factor absorbs a $\frac{\mu^2}{2}$, and the integral variables ξ in Eq. (18) are replaced by

$$\begin{aligned} & \int_0^1 d\xi_0 d\xi_1 d\xi_2 d\xi_3 \delta(\xi_0 + \xi_1 + \xi_2 + \xi_3 - 1) f(\xi_0, \xi_1, \xi_2, \xi_3) \\ &= \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 f(1 - \xi_1, \xi_1 - \xi_2, \xi_2 - \xi_3, \xi_3). \end{aligned} \quad (33)$$

Using the functional forms [41] of the color transition factors F 's and L 's as

$$\begin{aligned} \mu^2(L_{ii} + L_{jj} - 2L_{ij}) &= \frac{Q_s^2}{2C_F} (\mathbf{x}_i - \mathbf{x}_j)^2, \\ \mu^2 F_{1234} &= \frac{Q_s^2}{2C_F} (\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{x}_3 - \mathbf{x}_4), \end{aligned} \quad (34)$$

and replacing the coordinates by

$$\mathbf{x}_{1,2} = \mathbf{b}_1 \pm \frac{\mathbf{r}_1}{2}, \quad \mathbf{x}_{3,4} = \mathbf{b}_2 \pm \frac{\mathbf{r}_2}{2}, \quad \mathbf{x}_{5,6} = \mathbf{b}_3 \pm \frac{\mathbf{r}_3}{2}, \quad (35)$$

one reaches the final expression for $\langle \prod_{i=1}^3 D(\mathbf{b}_i + \frac{\mathbf{r}_i}{2}, \mathbf{b}_i - \frac{\mathbf{r}_i}{2}) \rangle$ to the $\frac{1}{N_c^4}$ order. We substitute this correlator into the definition of c_{12} and c_{123} in Eq. (29), and integrate over the variables $\mathbf{p}, \mathbf{b}, \mathbf{r}$ by the following five steps:

(a) Integrate over ϕ (the azimuthal angular of $\mathbf{p}$) by

$$\int_0^{2\pi} d\phi e^{i(\mathbf{p} \cdot \mathbf{r} + n\phi)} = 2\pi i^n J_n(pr) e^{in\theta}; \quad (36)$$

(b) Integrate the Gaussian type functions of $\mathbf{b}$;

(c) We truncate the expansion series of $e^{\frac{Q_s^2}{2} \mathbf{r}_i \cdot \mathbf{r}_j}$ at order N as $\sum_{n=0}^N \frac{(\frac{Q_s^2}{2} \mathbf{r}_i \cdot \mathbf{r}_j)^n}{n!}$, and integrate over r, p by

$$\int_0^\infty r dr p dp e^{-ar^2} J_n(pr) r^m = \begin{cases} \frac{\Gamma(\frac{m}{2})^n}{a^{\frac{n}{2}}}, & n > 0, \\ \frac{\Gamma(\frac{m}{2})^n}{a^{\frac{n}{2}} (-1)^n}, & n < 0; \end{cases} \quad (37)$$

(d) Integrate over θ_i (the azimuthal angular of $\mathbf{r}_i$) analytically, and integrate over ξ numerically.

Then we reach the values of the 3-particle cumulant c_{123} (with negative values) up to the $\frac{1}{N_c^4}$ order in the MV model, which is plotted as a function of $Q_s^2 B_p$ in Fig. 2. Some 3-particle charge-dependent azimuthal correlations have been measured at the Relativistic Heavy Ion Collider and the LHC [46,47] to search for the chiral magnetic effect. We see that our $-c_{123}$ (with a falloff at large Q_s^2) has a similar magnitude and trend to the three-particle correlations $\langle \cos(\phi_\alpha + \phi_\beta - 2\phi_c) \rangle$ (whose magnitude gradually decreases as the event multiplicity increases) observed in pPb collisions at the CMS [47], where Q_s is related to the multiplicity (centrality) by growing (falling) with the multiplicity (centrality). In order to obtain a number of order unity, we scale the mixed cumulant c_{123} by the corresponding 2-particle anisotropic flows $v_n\{2\} \equiv \sqrt{c_n\{2\}}$ [48], and plot the ratios as in Fig. 2, which also shows a similar magnitude and trend to the result in Ref. [44].

IV. THE N_c SCALING OF MULTIPARTICLE CORRELATIONS

For the $2m$ -particle harmonic, one usually defines

$$\begin{aligned} \kappa_n\{2m\} &\equiv \int \prod_{i=1}^m d^2\mathbf{p}_i e^{in(\phi_1 + \dots + \phi_m - \phi_{m+1} - \dots - \phi_{2m})} \frac{d^m N}{\prod_{i=1}^m d^2\mathbf{p}_i}, \\ c_n\{2m\} &\equiv \frac{\kappa_n\{2m\}}{\kappa_0\{2m\}}, \end{aligned} \quad (38)$$

for convenience.

Since the m -particle cumulant $c_n\{m\}$ demonstrates the correlations of these m particles, as a prior guest, its N_c

FIG. 2. Left: $-c_{123}$ depending on $Q_s^2 B_p$, where we truncate the expansion series at orders seven and eight. We see in the Taylor expansion approach, $-c_{123}$ converges quickly in small $Q_s^2 B_p$ region, and gradually converges in large $Q_s^2 B_p$ region. Right: The scaled cumulant $-c_{123}$ as a function of $Q_s^2 B_p$, where the factors in the denominator are written by $v_n \equiv v_n\{2\}$ for short.

FIG. 3. The left topology represents that all the four dipoles connect together. The right topology represents that dipoles 3,4 change from connected to disconnected, and finally dipoles 1,2,3 connect together. Both of them are of $\frac{1}{N_c^6}$ order.

scaling should depend on the configurations where all dipoles (particles) connect or ever connected together as in Fig. 3.

As each gluon exchanging between two unconnected dipoles brings a $\frac{1}{N_c^2}$, the cumulants should scale as [14]

$$c_n\{m\} \sim \frac{1}{N_c^{2(m-1)}}, \quad (39)$$

which is well understood graphically, and this can also be proved by the matrix method. We consider the $\frac{1}{N_c^4}$ order of $c_n\{4\} \equiv \frac{\kappa_n\{4\}}{\kappa_0\{4\}} - 2\left(\frac{\kappa_n\{2\}}{\kappa_0\{2\}}\right)^2 = \langle e^{in(\phi_1-\phi_2+\phi_3-\phi_4)} \rangle - 2\langle e^{in(\phi_1-\phi_2)} \rangle^2$, which means only two gluons exchanging between dipoles (1, 2), (3, 4) or (1, 4), (3, 2) contribute, and thus we only use these two 2-gluon exchanging matrices instead of the full 4-dipole color transition matrix as

$$\begin{aligned} M_{(1,2),(3,4)} &= \mu^2 \begin{pmatrix} LF_{12,34;56,78} & \frac{1}{2}F_{5867} & \frac{1}{2}F_{1423} & 0 \\ \frac{1}{2}F_{5687} & LF_{12,34;58,76} & 0 & \frac{1}{2}F_{1423} \\ \frac{1}{2}F_{1243} & 0 & LF_{14,32;56,78} & \frac{1}{2}F_{5867} \\ 0 & \frac{1}{2}F_{1243} & \frac{1}{2}F_{5687} & LF_{14,32;58,76} \end{pmatrix} \\ &= \mu^2 \begin{pmatrix} LF_{12,34} & \frac{1}{2}F_{1423} \\ \frac{1}{2}F_{1243} & LF_{14,32} \end{pmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes \mu^2 \begin{pmatrix} LF_{56,78} & \frac{1}{2}F_{5687} \\ \frac{1}{2}F_{5867} & LF_{58,76} \end{pmatrix} \\ &= M_{(1,2)} \otimes \mathbf{1} + \mathbf{1} \otimes M_{(3,4)} \\ &\equiv M_{(1,2)} \oplus M_{(3,4)}, \end{aligned} \quad (40)$$

and $M_{(1,4),(3,2)} = M_{(1,4)} \oplus M_{(3,2)}$ in the same way, where $LF_{ab,cd;ef,gh} \equiv LF_{ab,cd} + LF_{ef,gh}$. Thus we see that $M_{(1,2),(3,4)}$ can be decomposed into the Kronecker sum of two individual 2-dipole color transition matrices $M_{(1,2)}$ and $M_{(3,4)}$, and so can $M_{(1,4),(3,2)}$. Then the first term of $c_n\{4\}$ gives

$$\begin{aligned} \langle e^{in(\phi_1-\phi_2+\phi_3-\phi_4)} \rangle \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} &= \int \prod_{i=1}^4 \frac{d^2b_i d^2r_i d^2p_i}{4\pi^3 B_p} e^{\frac{b_i^2}{B_p} - \frac{r_i^2}{4B_p} + ip_i \cdot r_i} e^{in(\phi_1-\phi_2+\phi_3-\phi_4)} \\ &\times \left(\mathbf{1} \quad \frac{1}{N_c} \quad \frac{1}{N_c} \quad \frac{1}{N_c^2} \right) [e^{M_{(1,2),(3,4)}} + e^{M_{(1,4),(3,2)}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}}, \end{aligned} \quad (41)$$

where

$$\left(\mathbf{1} \quad \frac{1}{N_c} \quad \frac{1}{N_c} \quad \frac{1}{N_c^2} \right) = \left(\mathbf{1} \quad \frac{1}{N_c} \right) \otimes \left(\mathbf{1} \quad \frac{1}{N_c} \right), \quad (42)$$

$$e^{M_{(1,2),(3,4)}} = e^{M_{(1,2)} \oplus M_{(3,4)}} = e^{M_{(1,2)}} \otimes e^{M_{(3,4)}}, \quad (43)$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (44)$$

which gives

$$\begin{aligned} \left(1 \quad \frac{1}{N_c} \quad \frac{1}{N_c} \quad \frac{1}{N_c}\right) e^{M_{(1,2),(3,4)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \left(1 \quad \frac{1}{N_c}\right) e^{M_{(1,2)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \left(1 \quad \frac{1}{N_c}\right) e^{M_{(3,4)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left\langle D\left(\mathbf{b}_1 + \frac{\mathbf{r}_1}{2}\right) D\left(\mathbf{b}_2 - \frac{\mathbf{r}_2}{2}\right) \right\rangle \left\langle D\left(\mathbf{b}_3 + \frac{\mathbf{r}_3}{2}\right) D\left(\mathbf{b}_4 - \frac{\mathbf{r}_4}{2}\right) \right\rangle. \end{aligned} \quad (45)$$

Then substituting Eq. (45) into Eq. (41) gives

$$\begin{aligned} &\langle e^{in(\phi_1 - \phi_2 + \phi_3 - \phi_4)} \rangle \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} \\ &= \langle e^{in(\phi_1 - \phi_2)} \rangle \langle e^{in(\phi_3 - \phi_4)} \rangle \\ &\quad + \langle e^{in(\phi_1 - \phi_4)} \rangle \langle e^{in(\phi_3 - \phi_2)} \rangle \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}} \\ &= 2 \langle e^{in(\phi_1 - \phi_2)} \rangle^2 \Big|_{\text{to the } \frac{1}{N_c^4} \text{ order}}, \end{aligned} \quad (46)$$

which proves that $c_n\{4\}$ up to the $\frac{1}{N_c^4}$ order is 0. Nevertheless, the $\frac{1}{N_c^6}$ order involves new configurations that all of the four dipoles connect with each other, and they

cannot be subtracted by term $2 \langle e^{in(\phi_1 - \phi_2)} \rangle^2$ in the definition of $c_n\{4\}$.

By the same way, for the $c_n\{m\}$ case, the configurations that can be divided into unconnected parts [and the corresponding color transition matrices can be decomposed into the Kronecker sum of individual matrices as Eq. (40)] in $\frac{\kappa_n\{m\}}{\kappa_0\{m\}}$, is totally subtracted by the minus terms in $c_n\{m\}$. Therefore the conclusion of the N_c scaling in Eq. (39) is proved.

Finally, we can calculate the small Q s expansions of $c_2\{2\}$ and $c_2\{4\}$ with the color transition matrices $M_{2\text{-dipole}}$ and $M_{4\text{-dipole}}$. With a conjectured $2m$ -particle cumulant formula

$$c_n\{2m\} \equiv \frac{\kappa_n\{2m\}}{\kappa_0\{2m\}} - \sum_{j=2}^m \left\{ \left[\prod_{i=1}^{j-1} \sum_{a_i=1}^{a_{i+1}-1} \right] \left[\frac{1}{j!} \prod_{i=1}^j c_n\{2(a_i - a_{i-1})\} \binom{a_i - a_{i-1}}{a_i}^2 \right] \Big|_{a_0=0, a_j=m} \right\}, \quad (47)$$

one obtains the definitions [49]

$$c_n\{2\} \equiv \frac{\kappa_n\{2\}}{\kappa_0\{2\}}, \quad c_n\{4\} \equiv \frac{\kappa_n\{4\}}{\kappa_0\{4\}} - 2 \left(\frac{\kappa_n\{2\}}{\kappa_0\{2\}} \right)^2, \quad c_n\{6\} \equiv \frac{\kappa_n\{6\}}{\kappa_0\{6\}} - 9 \frac{\kappa_n\{2\}}{\kappa_0\{2\}} \frac{\kappa_n\{4\}}{\kappa_0\{4\}} + 12 \left(\frac{\kappa_n\{2\}}{\kappa_0\{2\}} \right)^3, \dots \quad (48)$$

Substituting $M_{2\text{-dipole}}$ and $M_{4\text{-dipole}}$ into the above definitions gives the small Q s expansions in the leading N_c order as

$$\begin{aligned} c_2\{2\} &= \frac{1}{N_c^2} \left[\frac{1}{2} Q_s^4 B_p^2 - Q_s^6 B_p^3 + \frac{23}{16} Q_s^8 B_p^4 - \frac{437}{240} Q_s^{10} B_p^5 + \mathcal{O}(Q_s^{12} B_p^6) \right] + \mathcal{O}\left(\frac{1}{N_c^4}\right), \\ c_2\{4\} &= \frac{1}{N_c^6} \left[Q_s^8 B_p^4 - \frac{20}{3} Q_s^{10} B_p^5 + \frac{1781}{72} Q_s^{12} B_p^6 - \frac{686351}{10080} Q_s^{14} B_p^7 + \mathcal{O}(Q_s^{16} B_p^8) \right] + \mathcal{O}\left(\frac{1}{N_c^8}\right), \end{aligned} \quad (49)$$

as well as the 3-particle cumulants:

$$c_{123} = \frac{-\pi}{N_c^4} \left[\frac{3}{64} Q_s^8 B_p^4 - \frac{109}{512} Q_s^{10} B_p^5 + \frac{7091}{12288} Q_s^{12} B_p^6 + \mathcal{O}(Q_s^{14} B_p^7) \right] + \mathcal{O}\left(\frac{1}{N_c^6}\right), \quad (50)$$

which finally helps to crosscheck the N_c scaling of $c_n\{m\}$ in Eq. (39).

V. CONCLUSION

In this paper, we evaluated the 3-dipole correlator up to the $\frac{1}{N_c^4}$ order in the CGC formalism, and conjectured an n -dipole formula as a $\frac{1}{N_c}$ power expansion. A clear pattern was observed that each term of the n -dipole correlator corresponds to individual color transitions between configurations. We calculated the azimuthal cumulant c_{123} by using the 3-dipole correlator, and proved the N_c scaling of

$c_n\{m\}$, which was cross-checked by our calculation of $c_2\{2\}$, $c_2\{4\}$, and c_{123} in the dilute limit.

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