Large q expansion in the Sachdev-Ye-Kitaev model

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We consider the Sachdev-Ye-Kitaev (SYK) model, where interaction involves q fermions at a time. We find the $1/q^2$ correction to the thermal two-point function in the large q expansion. Using this result, we find the $1/q^3$ correction to the SYK free energy.

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I. INTRODUCTION

The Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical model of N interacting Majorana fermions χ_i , $i = 1, \dots, N$ with the Hamiltonian [1,2],

$$H_{\text{SYK}_q} = (i)^{\frac{q}{2}} \sum_{1 \leq i_1 < i_2 < \dots < i_q \leq N} J_{i_1 \dots i_q} \chi_{i_1} \chi_{i_2} \dots \chi_{i_q}, \quad (1)$$

where $\{\chi_i, \chi_j\} = \delta_{ij}$ and $J_{i_1 \dots i_q}$ are random couplings drawn from a Gaussian distribution with zero mean and a width $\langle J_{i_1 \dots i_q}^2 \rangle = (q-1)! J^2 / N^{q-1}$. One is usually interested in computing correlation functions and, particularly, the two-point function at temperature $T = 1/\beta$:

$$G(\tau) = \langle T \chi(\tau) \chi(0) \rangle_\beta. \quad (2)$$

At the large N limit, only melonic Feynman diagrams contribute to the two-point function in the SYK model. These diagrams can be resummed and one obtains a nonperturbative Schwinger-Dyson equation,

$$G(i\omega_n)^{-1} = -i\omega_n - \Sigma(i\omega_n), \quad \Sigma(\tau) = J^2 G(\tau)^{q-1}, \quad (3)$$

where $G(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} G(\tau)$ and $\omega_n = 2\pi\beta^{-1}(n + 1/2)$. It is not possible to solve this equation analytically, but one can find solution in the infrared limit, where ω is small and the bare $-i\omega_n$ -term in (3) can be neglected [1–4],

$$G_c(\tau) = b \left(\frac{\pi}{\beta \sin \frac{\pi\tau}{\beta}} \right)^{2/q} \text{sgn}(\tau), \quad (4)$$

where $J^2 b^q \pi = (1/2 - 1/q) \tan(\pi/q)$. Nevertheless, it is still interesting to obtain some analytic approximation for $G(\tau)$ which interpolates both UV and IR regions. One way to proceed is to use the large q expansion. The first order in $1/q$ was found in [3]. In this article, we compute the next $1/q^2$ correction and argue that it improves the approximation significantly, such that it agrees with numerical results quite well.

In the next section, we compute the $1/q^2$ correction to the two-point function. Next, we compare the large q results and numerics. At the end, we compute the large q free energy and the coefficient of the Schwarzian action.

II. LARGE q TWO-POINT FUNCTION

We consider the large q ansatz for the two-point function [3]:

$$G(\tau) = \frac{1}{2} \text{sgn}(\tau) \left(1 + \frac{1}{q} g(\tau) + \frac{1}{q^2} h(\tau) + \dots \right). \quad (5)$$

For the self-energy (3), we find (we assume that q is even)

$$\Sigma(\tau) = \frac{\mathcal{J}^2}{q} \text{sgn}(\tau) e^g \left(1 + \frac{1}{q} \left(h - g - \frac{1}{2} g^2 \right) + \dots \right), \quad (6)$$

where a new coupling constant $\mathcal{J}^2 = 2^{1-q} q J^2$ is introduced. From now on, we work on the interval $\tau \in [0, \beta]$ and we can omit $\text{sgn}(\tau)$ in all formulas. Expanding $G(i\omega_n)^{-1}$ in the $1/q$ series up to the $1/q^2$ term using (5), we obtain

$$G(i\omega_n)^{-1} = -i\omega_n + \frac{1}{2q} \omega_n^2 g(i\omega_n) + \frac{\omega_n^2}{2q^2} \left(h(i\omega_n) + \frac{i\omega_n}{2} g * g(i\omega_n) \right), \quad (7)$$

where $g * g(i\omega_n) \equiv \int_0^\beta d\tau e^{i\omega_n \tau} g^2(\tau)$. Then using Eqs. (3) and (6) and going back to the coordinate space, we find differential equations for each order of $1/q$,

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$$\begin{aligned}\partial_\tau^2 g &= 2\mathcal{J}^2 e^g, \\ \partial_\tau^2 h &= 2\mathcal{J}^2 e^g h + \frac{1}{2}\partial_\tau^3(g * g) - 2\mathcal{J}^2 e^g \left(g + \frac{1}{2}g^2\right),\end{aligned}\quad (8)$$

and the functions $g(\tau)$ and $h(\tau)$ satisfy the boundary conditions $g(0) = g(\beta) = 0$ and $h(0) = h(\beta) = 0$. Now we introduce a convenient variable $x = \frac{\pi v}{2} - \frac{\pi v \tau}{\beta}$. Then the first equation has the solution

$$g(x) = \log\left(\frac{\cos \frac{\pi v}{2}}{\cos x}\right)^2, \quad \beta \mathcal{J} = \frac{\pi v}{\cos \frac{\pi v}{2}}. \quad (9)$$

Using this solution, the second equation can be represented as

$$\begin{aligned}\left(\partial_x^2 - \frac{2}{\cos^2 x}\right)h(x) \\ = -\frac{\pi v}{\beta} \frac{1}{2} \partial_x^3(g * g) - \partial_x^2 g(x) \left(g(x) + \frac{1}{2}g^2(x)\right).\end{aligned}\quad (10)$$

The solution to this equation can be written as

$$\begin{aligned}h(x) = -\int_{-\frac{\pi v}{2}}^{\frac{\pi v}{2}} dy \mathcal{G}(x, y) \left(\frac{\pi v}{\beta} \frac{1}{2} \partial_y^3(g * g) \right. \\ \left. + \partial_y^2 g(y) \left(g(y) + \frac{1}{2}g^2(y)\right)\right),\end{aligned}\quad (11)$$

where the Green's function $\mathcal{G}(x, y)$ obeys the equation

$$\left(\partial_x^2 - \frac{2}{\cos^2 x}\right)\mathcal{G}(x, y) = \delta(x - y) \quad (12)$$

with the boundary conditions $\mathcal{G}(-\frac{\pi v}{2}, y) = \mathcal{G}(\frac{\pi v}{2}, y) = 0$. One can solve this equation and obtain an explicit formula for the Green's function,

$$\mathcal{G}(x, y) = \frac{1}{2V} ((V + x_<) \tan x_< + 1)((V - x_>) \tan x_> - 1), \quad (13)$$

where $V \equiv \frac{\pi v}{2} + \cot \frac{\pi v}{2}$ and $x_> \equiv \max(x, y)$ and $x_< \equiv \min(x, y)$. Computing the convolution

$$\begin{aligned}\frac{\pi v}{\beta} \frac{1}{2} \partial_x^3(g * g) \\ = 2\partial_x \left(g(x) \left(\cot\left(\frac{\pi v}{2} + x\right) - \cot\left(\frac{\pi v}{2} - x\right)\right)\right) - 4,\end{aligned}\quad (14)$$

and using the explicit formula for the Green's function (13), we obtain from (11)

$$\begin{aligned}h(x) = \frac{1}{2}g^2(x) - 2\ell(x) - 4\left(\tan x \int_0^x dy \ell(y) + 1\right) \\ + 4\frac{1+x \tan x}{1+\frac{\pi v}{2} \tan \frac{\pi v}{2}} \left(\tan \frac{\pi v}{2} \int_0^{\frac{\pi v}{2}} dy \ell(y) + 1\right),\end{aligned}\quad (15)$$

where $\ell(x) \equiv g(x) - e^{-g(x)} \text{Li}_2(1 - e^{g(x)})$ and $g(x)$ is given in (9). One can compute explicitly the integral (formulas from [5] are useful)

$$\int_0^{\frac{\pi v}{2}} dy \ell(y) = -\frac{\pi^2 v^2}{24 \cos^2 \frac{\pi v}{2}} (\pi v + 3 \sin \pi v). \quad (16)$$

III. COMPARISON WITH NUMERICAL RESULTS

In this section, we compare the large q result with the numerical solution of the Schwinger-Dyson equation (3). In general, we expect the large q formula to work well when $|g(\tau)| \ll q$ and $|h(\tau)| \ll q^2$. These inequalities are fulfilled when $\beta \mathcal{J} \ll \pi e^{q/2}$.

Looking at the explicit formula (15), it is tempting to exponentiate the result and to introduce an exponentiated large q two-point function,

$$G(\tau) = \frac{1}{2} \text{sgn}(\tau) \exp\left(\frac{1}{q}g + \frac{1}{q^2}\left(h - \frac{1}{2}g^2\right)\right), \quad (17)$$

which is equivalent to (5) up to order $1/q^2$. We plot numerical and large q results for $q = 4$ and different values of βJ in Fig. 1. We can see that the exponentiated result works very precisely even for large βJ , whereas the large q answer (5) deviates significantly from numerics at large βJ .

IV. LARGE q FREE ENERGY

The leading large N approximation to the free energy in the SYK model is [2,6]

$$\begin{aligned}-\frac{\beta F}{N} &= \log \text{Pf}(\partial_\tau - \Sigma) \\ &= -\frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \left(\Sigma(\tau_{12})G(\tau_{12}) - \frac{J^2}{q}G(\tau_{12})^q\right).\end{aligned}\quad (18)$$

To avoid evaluating the Pfaffian it is convenient to differentiate the free energy by $J\partial_J$ [3]

$$\begin{aligned}J\partial_J(-\beta F/N) &= -\frac{\beta}{q} \partial_\tau G|_{\tau \rightarrow +0} \\ &= \frac{\pi v}{2q} \left(\frac{1}{q} \partial_x g + \frac{1}{q^2} \partial_x h\right) \Big|_{x \rightarrow \frac{\pi v}{2}},\end{aligned}\quad (19)$$

where from (9) and (15) we find

$$\begin{aligned}\partial_x g|_{x \rightarrow \frac{\pi v}{2}} &= 2 \tan \frac{\pi v}{2}, \\ \partial_x h|_{x \rightarrow \frac{\pi v}{2}} &= \frac{4}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}} \left(\frac{\pi v}{2} - \tan \frac{\pi v}{2} \left(1 + \frac{\pi v}{2} \tan \frac{\pi v}{2} \right) \right. \\ &\quad \left. - \int_0^{\frac{\pi v}{2}} dy \ell(y) \right).\end{aligned}\quad (20)$$

Next, using (16) and

$$J \partial_J = \frac{v \partial_v}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}}, \quad (21)$$

we can integrate back and obtain $-\beta F/N = \frac{1}{2} \log 2 + \frac{1}{q^2} F_{1/q^2} + \frac{1}{q^3} F_{1/q^3} + \dots$, where

$$\begin{aligned}F_{1/q^2}(v) &= \pi v \left(\tan \frac{\pi v}{2} - \frac{\pi v}{4} \right), \\ F_{1/q^3}(v) &= \pi v \left(\pi v - 2 \tan \frac{\pi v}{2} \left(1 - \frac{\pi^2 v^2}{12} \right) \right).\end{aligned}\quad (22)$$

Expanding the free energy at strong coupling by using that

$$v = 1 - \frac{2}{\beta \mathcal{J}} + \frac{4}{(\beta \mathcal{J})^2} - \frac{(24 + \pi^2)}{3(\beta \mathcal{J})^3} + \dots, \quad (23)$$

we find

$$\begin{aligned}-\frac{\beta F}{N} &= \beta \mathcal{J} \left(\frac{1}{q^2} - \frac{12 - \pi^2}{6q^3} \right) + \left(\frac{1}{2} \log 2 - \frac{\pi^2}{4q^2} + \frac{\pi^2}{3q^3} \right) \\ &\quad + \frac{1}{\beta \mathcal{J}} \left(\frac{\pi^2}{2q^2} - \frac{\pi^2(\pi^2 + 12)}{12q^3} \right) \\ &\quad + \frac{1}{(\beta \mathcal{J})^2} \left(-\frac{\pi^2}{q^2} + \frac{\pi^2(5\pi^2 + 24)}{9q^3} \right) + \dots,\end{aligned}\quad (24)$$

where the first three terms are the ground state energy, the zero-temperature entropy and the temperature dependent correction to the entropy. The zero temperature entropy coincides with the large q expansion of the formula [2,7]

$$\frac{S_0}{N} = \frac{1}{2} \log 2 - \int_0^{1/q} dx \pi \left(\frac{1}{2} - x \right) \tan \pi x. \quad (25)$$

The last term in (24) agrees with the formula reported in [8–10].

Using the result (24), one can find the coefficient of the Schwarzian action. The Schwarzian action, which governs the low-energy dynamics of the SYK model is given by the formula [3,11–14]

$$S = -N \frac{\alpha_S}{\mathcal{J}} \int d\tau \{f, \tau\}, \quad \{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2, \quad (26)$$

FIG. 1. Plots of the numerical solution and the large q approximations for $G(\theta)$, $\theta = 2\pi\tau/\beta$ at $\beta J = 20, 50, 100, 1000$ and $q = 4$. The black solid line is the numerical solution for the Schwinger-Dyson equation (3). The blue dash-dotted line is the large q approximation (5) with $1/q^2$ term. The blue dashed line is the exponentiated two-point function (17).

where the coefficient α_S depends on q . This coefficient is related to the finite temperature correction to the free energy, so at large q using (24) we find

FIG. 2. Plot of α_S as a function of q . The black circles correspond to numerical results adapted from [3]. The blue solid line corresponds to the two-sided Pade approximation (29).

$$-\frac{\beta F}{N} \supset \frac{2\pi^2 \alpha_S}{\beta \mathcal{J}} = \frac{1}{\beta \mathcal{J}} \left(\frac{\pi^2}{2q^2} - \frac{\pi^2(\pi^2 + 12)}{12q^3} + \dots \right). \quad (27)$$

At $q = 2$, one has $\alpha_S = \frac{1}{24\pi}$. Using asymptotics for α_S at $q = \infty$ and $q = 2$,

$$\alpha_S(q) = \begin{cases} \frac{1}{24\pi} + \dots, & q \rightarrow 2 \\ \frac{1}{4q^2} - \frac{\pi^2 + 12}{24q^3} + \dots, & q \rightarrow \infty \end{cases} \quad (28)$$

we obtain two-sided Pade approximant:

$$\text{Pade}_{[3,1]}: \alpha_S(q) = \frac{\pi^2 - 18\pi + 24 + 3(3\pi - 2)q}{6q^2(\pi^3 + 8 + 2(3\pi - 2)q)}. \quad (29)$$

We note that one can improve approximation by using more terms near $q = 2$ [9]. We plotted Pade approximation and numerical results adapted from [3] in Fig. 2. We see that the Pade approximation is very close to numerics.

At large $\beta \mathcal{J}$, it is possible to compute correction to the conformal propagator [3]

$$\frac{\delta G}{G_c} = -\frac{\alpha_G(q)}{\beta \mathcal{J}} f_0 \left(\frac{2\pi\tau}{\beta} \right), \quad f_0(x) \equiv 2 + \frac{\pi - |x|}{\tan \frac{|x|}{2}}, \quad (30)$$

where $\alpha_G(q)$ is some numerical coefficient that depends on q . Using the large q result, we find

$$\alpha_G(q) = \frac{2}{q} - \frac{12 + 7\pi^2}{9q^2} + \dots \quad (31)$$

V. CONCLUSIONS

It would be interesting to generalize the result of this article to other SYK-type models, discussed in [15–18]. In particular, it would be interesting to compute the thermalization time using the large q solution for the SYK models discussed in [19].

It is also interesting to develop the $1/q$ expansion for the higher-dimensional SYK models [20–24], where the stability of the large N limit is unclear.

The large q approximation to the two-point function can be used as well in studying tensor models [25–27]. Even though the general q melonic tensor interactions have some ambiguities [27,28], one can formally consider the large q generalization of the Schwinger-Dyson equation.

The discussion of this paper is focused on the leading terms of the large N limit. However, the large q expansion is also expected to be useful for the study of subleading $1/N$ contributions [29,30].

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