Topological dyonic dilaton black holes in AdS spaces

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We construct a new class of dyonic dilaton black hole solutions in the background of anti-de Sitter (AdS) spacetime. In order to find an analytical solution that satisfies all the field equations, we consider the string case where the dilaton coupling is $\alpha = 1$. The asymptotic behavior of the solution ($r \rightarrow \infty$) is exactly AdS, where the dilaton field becomes zero and the metric function reduces to $f(r) \rightarrow -\Lambda r^2/3$. In this spacetime, black hole horizons and cosmological horizons can be a two-dimensional positive, zero, or negative constant curvature surface. We study the physical properties of the solutions and show that, depending on the metric parameters, these solutions can describe black holes with one or two horizons or a naked singularity. We investigate thermodynamics of the solutions by calculating their charge, mass, temperature, entropy, and electric potential and disclose that these conserved and thermodynamic quantities satisfy the first law of black hole thermodynamics. We also analyze the thermal stability of the solutions and find the conditions for which we have a stable dyonic dilaton black hole.

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I. INTRODUCTION

The first attempt to consider a minimal coupling between the scalar fields and gravity was done by Fisher, who found a static and spherically symmetric solution of the Einstein-massless scalar field equations [1]. A great variety of scalar fields have been examined for different goals. One of the interesting scalar fields in black hole physics is the dilaton field. This scalar field appears in a variety of theories, such as dimensional reductions, low-energy limit of string theory, various models of supergravity, etc. A dilaton field naturally couples to gauge fields and, thus, can influence various physical phenomena. The presence of the dilaton field can change the causal structure and asymptotic behavior of the spacetime as well as the stability and thermodynamic features of the charged black holes. Given the asymptotic behavior of black holes, various theories can be defined for understanding their physics as well as the behavior of the Universe. For instance, a powerful tool to get insight into the strong coupling dynamics of certain field theories by studying classical gravity is AdS/CFT which is a correspondence between the gravity in an AdS

spacetime and conformal field theory (CFT) living on the boundary of the AdS spacetime [2]. The asymptotically flat dilaton black hole solutions in which the dilaton field is coupled to the Maxwell field in an exponential way have been explored in [3]. It was shown that asymptotically nonflat, non-(A)dS solutions can be constructed using one or two Liouville-type dilaton potentials [4]. In order to obtain an asymptotically (A)dS solution in the presence of dilaton field, the combination of three Liouville-type dilaton potentials was proposed [5–7]. In this case, the potential includes three exponential dilaton fields (Φ) and can be written as [8,9]

$$V(\Phi) = \frac{2\Lambda}{3(\alpha^2 + 1)^2} [8\alpha^2 e^{\Phi(\alpha-1/\alpha)} + \alpha^2(3\alpha^2 - 1)e^{-2\Phi/\alpha} - (\alpha^2 - 3)e^{2\alpha\Phi}], \quad (1)$$

where Λ is the cosmological constant, and α is a constant which determines the strength of the coupling between dilaton and gauge fields. A special case occurs when the coupling constant is $\alpha = 1$ which appears in the low-energy limit of the string action in Einstein's frame and corresponds to the bosonic sector of the field theory limit of the superstring model ($N = 4$ in supergravity) which calls the “string” conformal gauge [10]. In addition, the authors of [11] showed that one of the best ways in dilaton gravity to obtain the generalized Toda lattices related to Lie algebra is the string case where $\alpha = 1$. The properties of electrically charged dilaton black holes in the background of AdS spacetime have been investigated [12]. If a black hole

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contains both electric and magnetic charges, it is called a dyonic black hole. This type of black hole is one of the best objects to study the effect of the external magnetic field on superconductors as well as Hall conductance and DC longitudinal conductivity [13]. Also, it has been argued that the dyonic black hole in the AdS background may be a holographic dual of van der Waals fluid with chemical potential [14], and their dual on the conformal boundary of AdS spacetime is the stationary solution of the relativistic magnetohydrodynamics equations [15].

In the present work, we would like to find asymptotically AdS dyonic black hole solutions in the presence of a dilaton field. Many papers are devoted to investigating dilaton dyonic black holes [16]. However, their asymptotic behavior is neither AdS nor dS. Also, in [17], a dyonic black hole in the Kaluza-Klein theory in the presence of an additional scalar potential was explored. In the absence of a dilaton field, dyonic black holes in the background of AdS spacetime have been investigated in [18,19]. The solutions of massive gravity in the presence of electric and magnetic charges were explored in [20].

This paper is organized as follows. In Sec. II, we introduce the action and find the basic field equations by varying the action. In Sec. III, we study thermodynamical properties of the solutions. In Sec. IV, we investigate thermal stability of the obtained dyonic dilaton black hole solutions. We finish our paper with closing remarks in Sec. V.

II. FIELD EQUATIONS AND SOLUTIONS

The action of Einstein-Maxwell-dilaton gravity in the presence of a dilaton potential is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (\mathcal{R} - 2g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) - e^{-2\Phi} F^2), \quad (2)$$

where the Ricci scalar curvature is displayed by $\mathcal{R}$ and $F^2 = F_{\mu\nu} F^{\mu\nu}$ is the usual Maxwell Lagrangian. Also, $V(\Phi)$ is the potential for Φ which is given by Eq. (1). Since we consider the string case where $\alpha = 1$, the potential reduces to

$$V(\Phi) = \frac{\Lambda}{3} (4 + e^{2\Phi} + e^{-2\Phi}) = \frac{2}{3} \Lambda (2 + \cosh(2\Phi)). \quad (3)$$

The reason for considering the string case ($\alpha = 1$) comes from the fact that we could only find an analytical solution in this case, and it is impossible to find an analytical solution for an arbitrary α that fully satisfies the field equations. One may ignore the dilaton field, so the potential reduces to $V(\Phi = 0) = 2\Lambda$, which is the famous term in the action of AdS spaces. Variation of the action (2) with respect to the metric, Maxwell, and dilaton fields, respectively, results in

$$\mathcal{R}_{\mu\nu} = 2\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} g_{\mu\nu} V(\Phi) + 2e^{-2\Phi} \left(F_{\mu\beta} F_\nu^\beta - \frac{1}{4} g_{\mu\nu} F^2 \right), \quad (4)$$

$$\partial_\mu (\sqrt{-g} e^{-2\Phi} F^{\mu\nu}) = 0, \quad (5)$$

$$\nabla^2 \Phi = \frac{1}{4} \frac{\partial V(\Phi)}{\partial \Phi} - \frac{1}{2} e^{-2\Phi} F^2. \quad (6)$$

Since we are going to construct topological static and spherically symmetric black holes in four dimensions, we take the following form for the metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) d\Omega_k^2, \quad (7)$$

where $f(r)$ and $R(r)$ are functions of r which should be determined and $d\Omega_k^2$ is the metric of a two-dimensional hypersurface with area ω_2 and constant curvature k . Hereafter, we take $k = \pm 1, 0$, without loss of generality. The explicit form of $d\Omega_k^2$ is

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2 \theta d\phi^2, & \text{for } k = 1, \\ d\theta^2 + \theta^2 d\phi^2, & \text{for } k = 0, \\ d\theta^2 + \sinh^2 \theta d\phi^2, & \text{for } k = -1. \end{cases} \quad (8)$$

In order to construct the dyonic dilaton black hole, we first find the electric and magnetic fields using the Maxwell equation (5). The only nonzero components of the electromagnetic field tensor are

$$F_{tr} = -F_{rt} = \frac{q e^{2\Phi(r)}}{r^2 R^2(r)}, \quad (9)$$

$$F_{\theta\phi} = -F_{\phi\theta} = \begin{cases} p \sin \theta, & \text{for } k = 1, \\ p\theta, & \text{for } k = 0, \\ p \sinh \theta, & \text{for } k = -1. \end{cases} \quad (10)$$

Here, we denote the electric and magnetic charge parameters by q and p , respectively. Using the Maxwell electromagnetic tensor, one can find that F_{tr} is the electric field and $F_{\theta\phi}$ is the magnetic field. It is worth noting that the topological structure of the event horizon does not affect the electric field (9), while it affects the functional form of the magnetic field (10). Also, with respect to the values of r , the electric field can change, but the magnetic field is independent of r in all three topologies. In order to find the metric functions, we try to solve two remaining field equations (4) and (6) by considering the metric function (7) and the nonzero components of the electromagnetic tensor (9) and (10). The nonzero components of the Einstein equations (4) are

$$Eq'_t = \frac{f''}{2} + \left(\frac{R'}{R} + \frac{1}{r}\right)f' - \frac{q^2 e^{2\Phi} + p^2 e^{-2\Phi}}{r^4 R^4} + \frac{V(\Phi)}{2} = 0, \quad (11)$$

$$Eq'_r = \frac{f''}{2} + \left(\frac{R'}{R} + \frac{1}{r}\right)f' + 2\left(\Phi'^2 + \frac{rR'' + 2R'}{rR}\right)f - \frac{q^2 e^{2\Phi} + p^2 e^{-2\Phi}}{r^4 R^4} + \frac{V(\Phi)}{2} = 0, \quad (12)$$

$$Eq^\phi_\phi = Eq^\theta_\theta = \left(\frac{R'}{R} + \frac{1}{r}\right)f' + \left(\frac{R''}{R} + \frac{R'^2}{R^2} + \frac{4R'}{rR} + \frac{1}{r^2}\right)f + \frac{q^2 e^{2\Phi} + p^2 e^{-2\Phi}}{r^4 R^4} + \frac{V(\Phi)}{2} - \frac{k}{r^2 R^2} = 0. \quad (13)$$

where the prime denotes derivative with respect to r . It is a matter of calculations to show that the subtraction of Eqs. (11) and (12) leads to the following equation,

$$R'' + \frac{2}{r}R' + \Phi'^2 R = 0. \quad (14)$$

Now, we make the ansatz [21]

$$R(r) = e^{\Phi(r)}. \quad (15)$$

Using this ansatz, one can easily show that Eq. (14) admits a solution of the form

$$\Phi(r) = \frac{1}{2} \ln \left(1 - \frac{b}{r}\right), \quad (16)$$

where b is an integration constant which is related to the dilaton field so that if we set $b = 0$, the dilaton field will disappear. In the asymptotic region where $r \rightarrow \infty$, the dilaton field goes to zero. This is an expected result, since our solutions are asymptotically AdS and, thus, the effects of the dilaton field in the asymptotic region should be removed. Besides, from (16), we see that the solutions only exist for $r > b$. Thus, one may exclude the region $0 < r < b$ from the spacetime by redefinition of the coordinate $r \rightarrow \rho$ such that $\rho = \sqrt{r^2 - b^2}$. Thus, in the new coordinate, $\rho = 0$ corresponds to $r = b$, the spacetime is regular for $\rho > 0$, and the line elements of the metric may be written as

$$ds^2 = -f(\rho)dt^2 + \frac{\rho^2 d\rho^2}{(\rho^2 + b^2)f(\rho)} + (\rho^2 + b^2)R^2(\rho)d\Omega_k^2. \quad (17)$$

In order to find the metric function $f(r)$, we use Eq. (13). Substituting potential (3), the ansatz (15) and the dilaton field (16) in Eq. (13), we can easily obtain

$$f(r) = -\frac{\Lambda r^2}{3} \left(1 - \frac{b}{r}\right) + \frac{2p^2}{(b-2r)(b-r)} - \frac{2q^2}{(b-2r)r} + \frac{2(c-kr)}{b-2r}, \quad (18)$$

where c is an integration constant. One can also rewrite the above solution in term of the new coordinate ρ by replacing $r = \sqrt{\rho^2 + b^2}$. In the asymptotic region ($r \rightarrow \infty$), the dominant term in (18) is the cosmological term, $f(r \rightarrow \infty) = -\Lambda r^2/3$ which confirms that our solutions are asymptotically AdS. In the absence of the magnetic charge ($p = 0$), our solution reduces to the topological dilaton black hole in AdS spaces for $\alpha = 1$ [5,6,8]. Also, in the absence of the dilaton field ($b = 0$), the metric (18) reduces to the dyonic black hole in the AdS spacetime [19,22],

$$f(r)|_{b=0} = k - \frac{c}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{r^2} + \frac{p^2}{r^2}. \quad (19)$$

Next, we must check whether the obtained solutions are fully satisfied all the field equations. For this purpose, we first consider the equation for the dilaton field (6) which can be rewritten as

$$\Phi'' f + \left[f' + 2f\left(\frac{1}{r} + \frac{R'}{R}\right)\right]\Phi' + \frac{p^2 e^{-2\Phi} - q^2 e^{2\Phi}}{r^4 R^4} - \frac{1}{4} \frac{\partial V(\Phi)}{\partial \Phi} = 0. \quad (20)$$

Substituting the obtained functions $f(r)$, $\Phi(r)$ and $R(r)$ in Eq. (20), it reduces to

$$\frac{4(q^2 - p^2) + b^2 k - 2cb}{r(b-r)(b-2r)^2} = 0 \quad (21)$$

Thus, this equation will be satisfied provided the constants of integration b and c are related together via the following condition,

$$c = \frac{b}{2}k + \frac{2}{b}(q^2 - p^2). \quad (22)$$

Inserting condition (22) in solution (18), it can be written as

$$f(r) = k - \frac{\Lambda r^2}{3} \left(1 - \frac{b}{r}\right) - \frac{2q^2}{br} + \frac{2p^2}{b(r-b)}. \quad (23)$$

Finally, substituting solution (23) together with functions (9), (10), (15) and (16) in the field Eqs. (11)–(13), we find out that they are fully satisfied. This implies that the obtained solutions here are indeed the solutions of the full field equations. Next, we look for the singularity and horizon(s) of the spacetime. The divergency in the Ricci and Kretschmann scalars confirm the existence of the essential singularity. The Ricci scalar is calculated as

$$\text{Ricci} = -\frac{(b^2 - 8br + 8r^2)\Lambda}{2(b-r)r} - \frac{bq^2}{r^3(b-r)^2} - \frac{p^2b}{r^2(b-r)^3} + \frac{kb^2}{2r^2(b-r)^2}. \quad (24)$$

As one can see from Eq. (24), the surface $r = b$ ($\rho = 0$) as well as $r = 0$ are curvature singularities. Since we have excluded $r < b$ from our spacetime, in our new coordinates, the essential singularity is located at $\rho = 0$ ($r = b$). As we already mentioned our solution only exist for $r > b$. The explicit form of the Kretschmann invariant is very complicated and for the economic reasons we do not present it here. It is easy to show that

$$\lim_{\rho \rightarrow 0} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \rightarrow \infty. \quad (25)$$

From Eqs. (24) and (25), we conclude that there is an essential singularity at $\rho = 0$. The behavior of these scalars in the large ρ limit is interesting, too. At this limit, the

dominant term in the Ricci and Kretschmann scalars are given by

$$\lim_{\rho \rightarrow \infty} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \rightarrow \frac{8\Lambda^2}{3},$$

$$\lim_{\rho \rightarrow \infty} \text{Ricci} \rightarrow 4\Lambda, \quad (26)$$

which are the characterization of the (A)dS spaces. Now we look for the horizon(s). Black hole horizon(s) is indeed the largest positive real root of Eq. $f(r) = 0$. Unfortunately, it is not easy to find the location of the horizons, analytically. However, we can find the location of horizons by plotting the function $f(r)$ versus r . In all figures, we set $\Lambda = -3$. According to Fig. 1 the number of horizons depend on the values of the metric parameters. Our solutions may have one or two horizons and in some cases, it may describe a naked singularity depending on the parameters. In summary, our solutions can describe black holes in the case $b < r_+$, where r_+ is the outer horizon of the black hole.

(a) $p = 0.5$, $q = 0.61$ and $b = 0.2$

(b) $k = 1$, $q = 0.7$ and $b = 0.2$

(c) $k = 1$, $p = 0.5$ and $b = 0.2$

(d) $k = 1$, $p = 0.4$ and $q = 0.56$

FIG. 1. The behavior of the metric function $f(r)$ versus r .

In Fig. 1(a) we consider the effects of the horizon topology on the number of horizons. By fixing other metric parameters, the number of horizons decreases by increasing k . Figures 1(b) and 1(c) show that with increasing the value of the magnetic charge, the number of horizons decrease, while by increasing the electric charge the number of horizons can increase, too.

III. THERMODYNAMICAL PROPERTIES

The Hawking temperature is attributed to the black hole horizon (surface gravity κ) and is defined as

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)}, \quad (27)$$

where $\chi = \partial/\partial t$ is null Killing vector of the horizon. It is a matter of calculation to show

$$T = \frac{f'(r)|_{r=r_+}}{4\pi} = \frac{1}{2\pi} \left[\frac{k}{2r_+} - \frac{p^2}{(r_+)^3} \left(1 - \frac{b}{r_+} \right)^{-2} + \frac{(2b-3r_+)\Lambda}{6} \right], \quad (28)$$

where $r_+ = \sqrt{\rho_+^2 + b^2}$, is the outer horizon radius. It is also interesting to note that in the absence of magnetic charge ($p = 0$), it reduces to the temperature of charged AdS dilaton black hole [6] for $\alpha = 1$. The entropy of the dilaton black hole typically obeys the area law of the entropy which is a quarter of the event horizon area [23]. For our solution the entropy per unit area ω_2 is obtained as

$$S = \frac{r_+^2}{4} \left(1 - \frac{b}{r_+} \right), \quad (29)$$

The above expression is exactly the entropy of charged AdS dilaton black hole [6]. Using Brown and York formalism we calculate the mass of the asymptotically AdS dyonic dilaton black hole [6]. We find the mass per unit area ω_2 of the horizon as

$$M = \frac{q^2 - p^2}{4b\pi}. \quad (30)$$

In the absence of magnetic charge ($p = 0$), it recovers the mass of the AdS dilaton black hole [6], while in the absence of dilaton field ($b = 0$), it reduces to the mass of topological dyonic AdS black holes. One may use the Gauss's law to calculate the total electric and magnetic charge of the black hole. According to the Gauss theorem, the electric charge of the black hole per unit area ω_2 is

$$Q = \frac{1}{4\pi} \int_{r \rightarrow \infty} \sqrt{-g} F_{tr} d^2x = \frac{q}{4\pi}. \quad (31)$$

Similarly, we can obtain the total magnetic charge of the dyonic black hole per unit area ω_2 as

$$P = \frac{p}{4\pi}. \quad (32)$$

Also, one can obtain U_Q and U_P which are, respectively, the electric and magnetic potential by using the free energy, which is given as [22]

$$W = \frac{I_{\text{onshell}}}{\beta} \quad (33)$$

where I_{onshell} is the on shell action and β is the inverse of temperature. Multiplying both sides of Eq. (4) by $g^{\mu\nu}$, we arrive at

$$\mathcal{R} = 2\partial^\mu \phi \partial_\mu \phi + 2V(\Phi). \quad (34)$$

Substituting Eq. (34) in Eq. (2), we find

$$\begin{aligned} I_{\text{onshell}} &= \frac{1}{16\pi} \int d^4x \sqrt{-g} (V(\Phi) - e^{-2\Phi} F^2) \\ &= -\frac{1}{16\pi} \int d^4x \left[\frac{2 \sin(\theta)}{r^2(r-b)^2} \left(\frac{-r^2 \Lambda}{6} (r^2(b-r)^2(6r^2 + b^2 - 6br)) + r^2(P^2 - Q^2) + Q^2(2br - b^2) \right) \right] \end{aligned} \quad (35)$$

As r goes to infinity, the above integration diverges. In order to remove the divergency, one may add some counterterm to the original action or subtracting the contribution of the background spacetime. It is a matter of calculation to show that the electric and magnetic potential, at the horizon, per unit area ω_2 are given by

$$U_Q = -\frac{\partial W}{\partial Q} = \frac{Q}{r_+}, \quad (36)$$

$$U_P = \frac{\partial W}{\partial P} = \frac{P}{r_+ - b}. \quad (37)$$

It is important to note that the dilaton field does not affect the electric potential, while it changes the magnetic potential. In the absence of the dilaton field ($b = 0$), magnetic potential is the same as that in [19,22]. In the thermodynamics consideration, the satisfaction of the first law of thermodynamics implies the correctness of conserved and thermodynamic quantities. In order to check this, we obtain the mass M per unit area ω_2 as a function of extensive quantities S , Q and P . We find

$$M(S, Q, P) = \frac{4\pi}{b} (Q^2 - P^2). \quad (38)$$

Considering the fact that all thermodynamic quantities should be defined on the horizon, we write $b = b(Q, r, P)$. Taking into account the fact that the entropy

is a function of $S = S(r, b)$ and the mass is a function of extensive parameters $M = M(S, Q, P)$, we define the temperature, electric and magnetic potentials as

$$T = \left(\frac{\partial M}{\partial S} \right)_{Q,P} = \frac{\left(\frac{\partial M}{\partial b} \right)_{Q,P} \times dbr}{\left(\frac{\partial S}{\partial r_+} \right)_{Q,P} + \left(\frac{\partial S}{\partial b} \right)_{Q,P} \times dbr}, \quad (39)$$

$$U_Q = \left(\frac{\partial M}{\partial Q} \right)_{S,P} + \left(\frac{\partial M}{\partial b} \right)_{S,P} \left(\frac{\partial b}{\partial Q} \right)_{S,P}, \quad (40)$$

$$U_P = \left(\frac{\partial M}{\partial P} \right)_{S,Q} + \left(\frac{\partial M}{\partial b} \right)_{S,Q} \left(\frac{\partial b}{\partial P} \right)_{S,Q}, \quad (41)$$

where we can compute dbr as

$$dbr = \left(\frac{db}{dr_+} \right)_{Q,P} = - \frac{\left(\frac{\partial f}{\partial r_+} \right)_{Q,P}}{\left(\frac{\partial f}{\partial b} \right)_{Q,P}} = \frac{\frac{2Q^2}{br_+^2} - \frac{2P^2}{b(b-r_+)^2} - \frac{\Lambda(b-2r_+)}{3}}{\frac{2Q^2}{b^2r_+} + \frac{2P^2(2b-r_+)}{b^2(b-r_+)^2} + \frac{\Lambda r_+}{3}}. \quad (42)$$

Finally, our calculations show that the intensive quantities given in Eqs. (28), (36), and (37) coincide with

Eqs. (39)–(41). This implies that these quantities satisfy the first law of black hole thermodynamics,

$$dM = TdS + U_Q dQ + U_P dP. \quad (43)$$

IV. THERMAL STABILITY

In this section, we would like to study the stability of the dyonic dilaton black holes. Since we consider three extensive variables X_i , the best way to check the stable range of our solutions is to work in the grand-canonical ensemble and study the Hessian matrix [24] which is defined as

$$H_{X_i X_j}^M = \frac{\partial^2 M}{\partial X_i \partial X_j}. \quad (44)$$

Positivity of determinant of the Hessian matrix is sufficient to ensure thermal stability in the grand-canonical ensemble. As mentioned before, we may select $X_i = (S, Q, P)$. Since the mass is not an explicit function of X_i , we use the following definitions to obtain the components of the Hessian matrix,

$$\begin{aligned} H_{11} &= \left(\frac{d^2 M}{d^2 S} \right)_{P,Q} = \left(\frac{dT}{dS} \right)_{P,Q} = \frac{\left(\frac{\partial T}{\partial r_+} \right)_{P,Q} + \left(\frac{\partial T}{\partial b} \right)_{P,Q} \times dbr}{\left(\frac{\partial S}{\partial r_+} \right)_{P,Q} + \left(\frac{\partial S}{\partial b} \right)_{P,Q} \times dbr}, \\ H_{22} &= \left(\frac{d^2 M}{d^2 P} \right)_{S,Q} = \left(\frac{dM_P}{dP} \right)_{S,Q} = \left(\frac{\partial M_P}{\partial P} \right)_{S,Q} + \left[\left(\frac{\partial M_P}{\partial r_+} \right)_{S,Q} \left(\frac{dr_+}{db} \right)_{S,Q} + \left(\frac{\partial M_P}{\partial b} \right)_{S,Q} \right] \left(\frac{dP}{db} \right)_{S,Q}^{-1}, \\ H_{33} &= \left(\frac{d^2 M}{d^2 Q} \right)_{S,P} = \left(\frac{dM_Q}{dQ} \right)_{S,P} = \left(\frac{\partial M_Q}{\partial Q} \right)_{S,P} + \left[\left(\frac{\partial M_Q}{\partial r_+} \right)_{S,P} \left(\frac{dr_+}{db} \right)_{S,Q} + \left(\frac{\partial M_Q}{\partial b} \right)_{S,P} \right] \left(\frac{dQ}{db} \right)_{S,P}^{-1}, \\ H_{12} &= H_{21} = \left(\frac{d^2 M}{dSdP} \right)_Q = \left(\frac{dT}{dP} \right)_{S,Q} = \left(\frac{\partial T}{\partial P} \right)_{S,Q} + \left[\left(\frac{\partial T}{\partial r_+} \right)_{S,Q} \left(\frac{dr_+}{db} \right)_{S,Q} + \left(\frac{\partial T}{\partial b} \right)_{S,Q} \right] \left(\frac{dP}{db} \right)_{S,Q}^{-1}, \\ H_{13} &= H_{31} = \left(\frac{d^2 M}{dSdQ} \right)_P = \left(\frac{dT}{dQ} \right)_{S,P} = \left[\left(\frac{\partial T}{\partial r_+} \right)_{S,P} \left(\frac{dr_+}{db} \right)_{S,P} + \left(\frac{\partial T}{\partial b} \right)_{S,P} \right] \left(\frac{dQ}{db} \right)_{S,P}^{-1}, \\ H_{23} &= H_{32} = \left(\frac{d^2 M}{dQdP} \right)_S = \left(\frac{dM_Q}{dP} \right)_{S,Q} \\ &= \left(\frac{\partial M_Q}{\partial P} \right)_{S,Q} + \left[\left(\frac{\partial M_Q}{\partial r_+} \right)_{S,Q} \left(\frac{dr_+}{db} \right)_{S,Q} + \left(\frac{\partial M_Q}{\partial b} \right)_{S,Q} \right] \left(\frac{dP}{db} \right)_{S,Q}^{-1}, \end{aligned} \quad (45)$$

together with the following definitions:

$$\begin{aligned} M_Q &= \left(\frac{dM}{dQ} \right)_{S,P} = \left(\frac{\partial M}{\partial Q} \right)_{S,P} + \left(\frac{\partial M}{\partial b} \right)_{S,P} \left(\frac{dQ}{db} \right)_{S,P}^{-1}, \\ M_P &= \left(\frac{dM}{dP} \right)_{S,Q} = \left(\frac{\partial M}{\partial P} \right)_{S,Q} + \left(\frac{\partial M}{\partial b} \right)_{S,Q} \left(\frac{dP}{db} \right)_{S,Q}^{-1}. \end{aligned} \quad (46)$$

Using the above definitions, we can calculate the determinant of Eq. (44). The final result is very complex, so we use diagrams to analyze the stability. In all figures, we set $\Lambda = -3$.

In Fig. 2, we plot the determinate for all three values of topological constant k respect to r_+ . Since a real black hole should have a nonzero value for temperature, so we should have a careful look at the temperature in all considerations. In this figure, the three bold lines refer to the determinate of

 FIG. 2. $b = 0.2$, $P = 1$ and $Q = 0.6$.

Hessian matrix which is displayed by $\det(H)$,¹ and the three thin lines relate to the temperature. According to this figure, when we fix the metric parameters, all three temperature functions enjoy a minimum at radius $r_{\min_T}$ for which $r < r_{\min_T}$, this function is negative. Also, one may see a minimum radius for the determinate functions $r_{\min_d}$ which is negative for small black holes ($r_+ < r_{\min_d}$). Black holes with the horizon in the interval $r_{\min_T} < r_+ < r_{\min_d}$, are not thermally stable. This range increases as the values of k decreases. Our calculations show that another special radius exists for the determinate function that we call it ($r_{\max_d}$) which is the final point in which this function is displayed. We see that when $r_+ > r_{\max_d}$, the values of the determinate functions are imaginary, and we cannot discuss in this interval using the Hessian matrix. It is important to note that the imaginary values for the Hessian matrix are due to the presence of the dilaton field and such imaginary values are not seen for dyonic black holes in the absence of the dilaton field [14]. However, this issue needs further investigation, so we leave it to future work. As a result, for special values of the metric parameters, black holes with the horizon between $r_{\min_d} < r_+ < r_{\max_d}$ experience a stable state. This interval changes as metric parameters change; for example, by increasing k , this stable range increases, too.

In order to better understand the effects of magnetic charge on the stability, we plot Fig. 3.² In this diagram, we fix the radius and allow the magnetic charge P to change. By fixing other metric parameters, one can see that in order to have a real black hole which has positive values of temperature, there exists a maximum for the magnetic charge ($P_{\max_T}$). Also, a divergency in $\det(H)$ is observed which corresponds to the equality of charges ($P = Q$). As

mentioned before, the positivity of both temperature and Hessian determinate is necessary for stability. Based on the figures, when $P < P_{\max_T}$ the temperature is positive. The curve of $\det(H)$ has some special point (P_d^1) in this interval which shows that the curve goes to zero at this point and its sign will change. Let us note that, based on Fig. 3(b), one may find four roots for the determinate. The first one occurs in the singularity, so this point is not physical. The third one equals the divergency in $\det(H)$, and we do not care about it. Thus, there are two roots for $\det(H)$ with physical importance which occur around $P \approx 0.3$ and $P \approx 0.9$. The numbers of these points increase as we consider the larger black hole. For instance, when $r_+ = 0.6$, one special point exists (P_d^1) and the stable interval places at $0 < P < P_d^1$. But for $r_+ = 1$, $\det(H)$ becomes zero at two points. We call them P_d^1 and P_d^2 . Therefore, thermal stability occurs for two smaller intervals ($0 < P < P_d^1$ and $P_d^2 < P < P_{\max_T}$).

In order to determine thermal stability and possible phase transition of the black holes, one may check the heat capacity for fixed values of the electric charge. In this case, we work in the canonical ensemble,

$$C_Q = T \left(\frac{dS}{dT} \right)_Q = \frac{1}{A} [3(-(2b - 3r_+)r_+(b - r_+)^2 \Lambda - 3(b - r_+)^2 k + 6P^2)r_+^2(b - r_+)^2 \times (P^2 - Q^2)], \quad (47)$$

where one can calculate the denominator as

$$A = [-3r_+^2 b^2 (b - r_+)^4 \Lambda - 18r_+ (b - r_+)^2 (2b - r_+) P^2 - 18(b - r_+)^4 Q^2] k - r_+^3 (b - r_+)^4 (2b - r_+) b^2 \Lambda^2 + [-6r_+^2 (b^2 + 4br_+ - 3r_+^2) (b - r_+)^2 P^2 - 6r_+ (b - r_+)^4 (2b + 3r_+) Q^2] \Lambda + 36r_+ (2b - 3r_+) P^4 - 36(b + 3r_+) (b - r_+) Q^2 P^2. \quad (48)$$

As we know, only the positive values of the thermal capacity of a thermodynamic object are acceptable, and the negative values show that the object is not real. Also, in black hole physics in the presence of positive temperature, divergency of heat capacity shows that the black hole is not stable and needs a thermodynamical phase transition. We call this transition type one. Also, zero values of C_Q are not acceptable and usually occur for extremal black holes (which have zero value for temperature). The black hole may experience another type of phase transition at this point which we call type two. If the values of heat capacity change to positive ones, it means that the extremal black hole becomes a black hole with two horizons.

Looking at (47), one may find that, regardless of other metric parameters, by choosing equal values for both charges ($Q = P$), this function vanishes. We plot heat

¹We use scale 10^{-1} for determinate function.

²To be more clear, we use scale 10^2 for the temperature.

FIG. 3. $b = 0.2$, $k = 1$ and $Q = 0.5$. FIG. 4. $b = 0.1$, $k = 1$ and $P = 0.4$

capacity and the temperature in Fig. 4 for better analysis.³ We consider $\Lambda = -3$ in this figure.

By fixing other metric parameters, Fig. 4(a) shows that when the magnetic charge is larger than the electric charge ($P > Q$), the type two phase transition occurs. The divergency in C_Q happens at positive temperature interval. As we consider larger black holes, the heat capacity changes from positive to negative values, and thermal phase transition makes the black hole smaller (transition between small and large black hole). That's because, for a large black hole, heat capacity is negative and the black hole cannot exist in this range. Choosing smaller values for the magnetic charge ($P < Q$), we only observe the type one phase transition [see Fig. 4(b)]. Since the temperature values are also becoming positive, we can deduce that, after this point, our black hole will have two horizons. Note that for extremal black holes,

the temperature is zero. Finally, we conclude that black hole stability is crucially dependent on the metric parameters.

V. CLOSING REMARKS

In this paper, we have constructed a new class of topological dyonic dilaton black hole solutions in the background of AdS spacetime. The string case, where the dilaton coupling constant is $\alpha = 1$, is the only case we found that fully satisfies all components of the field equations. We have taken the dilaton potential in the form of the combination of three Liouville types introduced in [8,9]. By solving the equations of motion, we obtained the solutions that depend on both magnetic and electric charges. Based on the metric parameters, our solutions can represent a black hole with two horizons, an extremal black hole, or a naked singularity. Then, we explored the thermodynamics of these topological dilaton black holes and verified that the conserved and thermodynamic quantities satisfy the first law of thermodynamics.

³We multiplied both functions at 100 for better illustration.

Finally, we disclosed the effects of metric parameters on the stability of the solutions.

There are three extensive parameters (electric charge Q , magnetic charge P , and entropy S) in our theory, so it is recommended to use the Hessian matrix to check the stability. The positivity of the temperature and the determinate of the Hessian matrix simultaneously ensure that the solution is stable. We found that the stability of our solutions is closely related to the values of the metric parameters, and it is not stable at all intervals of r_+ . For example, when other metric parameters are fixed, by increasing the value of the topological constant, k , the stable interval decreases. Also, for the equal values of electric and magnetic charges, the Hessian method did not work since the determinate diverges.

We also checked the stability in the canonical ensemble, where the electric charge is a fixed parameter. Thus, the positivity of the thermal capacity C_Q is sufficient to ensure the local stability. In our theory, we observed that, as other metric parameters are fixed, when $Q < P$, the heat capacity diverges at a special radius and a type two

phase transition occurs which coincides with the divergence of C_Q (while the temperature is positive). It is important to note that C_Q vanishes by considering equal values for both electric and magnetic charges ($P = Q$) for all radii. Also, we see that the type one phase transition coincides with vanishing of the heat capacity and temperature, and we interpret it as a transition from a non-physical black hole to a physical one or a transition from an extremal black hole to a black hole with two horizons. By choosing $Q > P$, this transition is observed. As a result, all figures show that the stability is quite dependent on the metric parameters and changing them can alter the black hole stability.

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