Killing horizons: Negative temperatures and entropy super-additivity

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Many discussions in the literature of spacetimes with more than one Killing horizon note that some horizons have positive and some have negative surface gravities, but assign to all a positive temperature. However, the first law of thermodynamics then takes a nonstandard form. We show that if one regards the Christodoulou and Ruffini formula for the total energy or enthalpy as defining the Gibbs surface, then the rules of Gibbsian thermodynamics imply that negative temperatures arise inevitably on inner horizons, as does the conventional form of the first law. We provide many new examples of this phenomenon, including black holes in STU supergravity. We also give a discussion of left and right temperatures and entropies, and show that both the left and right temperatures are non-negative. The left-hand sector contributes exactly half the total energy of the system, and the right-hand sector contributes the other half. Both the sectors satisfy conventional first laws and Smarr formulas. For spacetimes with a positive cosmological constant, the cosmological horizon is naturally assigned a negative Gibbsian temperature. We also explore entropy-product formulas and a novel entropy-inversion formula, and we use them to test whether the entropy is a super-additive function of the extensive variables. We find that super-additivity is typically satisfied, but we find a counterexample for dyonic Kaluza-Klein black holes.

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I. INTRODUCTION

Since the early days of black hole thermodynamics there have been suggestions that the thermodynamic of the inner, Cauchy, horizons of charged and or rotating black holes should be taken more seriously than it has been [1–10]. With the development of string theory approaches these suggestions have become more insistent [11–17]. This interest increased considerably with the observation that the product of the areas and hence entropies of the inner and outer horizon takes in many examples a universal form which should be quantized at the quantum level [18–22]. Some of these papers, and others, e.g., Refs. [22–27], encountered the same feature first noticed in [1]: the fact that with a conventional first law of thermodynamics the temperature of the inner horizon would be negative. The authors of [22] chose to resolve this issue by defining the temperature of the inner horizon to be the absolute value of the "thermodynamic" temperature, and proposing an appropriately-modified first law on the inner horizon to compensate for this. In this paper we shall explore the consequences of adhering to the standard first law of thermodynamics for inner horizons, with the inevitable result that the temperature will be negative there.

In the derivation of the first law of black hole dynamics one finds, integrating in the region between the inner and outer horizons, that

$$0 = \frac{\kappa_+}{8\pi} dA_+ - \frac{\kappa_-}{8\pi} dA_- + \cdots, \tag{1.1}$$

where $\kappa_\pm$ are the surface gravities and $A_\pm$ the areas of the outer and inner horizons respectively. (The contributions from the angular momentum and charge(s) are represented by the ellipses in this equation.) If, as turns out to be the case in the examples we consider, the signs of dA_+ and dA_- are opposite for a given change in the black-hole parameters, then the signs of the surface gravities at κ_+ and

κ_- must be opposite too. The surface gravity is defined by evaluating

$$\ell^\mu \nabla_\mu \ell^\nu = \kappa \ell^\nu \tag{1.2}$$

on the horizon, where ℓ^μ is the future-directed null generator of the horizon, which coincides with a Killing vector K^μ on the horizon. One then finds that whilst κ is positive on the outer horizon, it is negative on the inner horizon.[1] Hawking showed that for an isolated event horizon in an asymptotically flat spacetime (for which in fact κ is positive), the temperature is $\kappa/(2\pi)$. We shall discuss the extension of Hawking's calculation to the case of inner horizons in the concluding section of this paper. In what follows, however, we shall frequently make reference to the formula

$$T = \frac{\kappa}{2\pi}, \tag{1.3}$$

with the understanding that T may not be a temperature measured by a physical thermometer, but rather, as we shall explain shortly, a "Gibbsian" temperature.

The occurrence of a negative κ on an inner horizon is somewhat obscured in many discussions in the literature by the fact that the surface gravity is commonly calculated by evaluating

$$\kappa^2 = -\frac{g^{\mu\nu}(\partial_\mu K^2)(\partial_\nu K^2)}{4K^2} \tag{1.4}$$

in the limit on the horizon. This formula is derivable from (1.2), but the information about the sign of κ is lost, and commonly the positive root is assumed when calculating κ from (1.4). A guaranteed correct procedure is to use the formula (1.2), working in a coordinate system that covers the horizon region.

Another situation where one encounters two horizons is when a positive cosmological constant Λ is involved and one has both a black hole event horizon and a cosmological event horizon bounding a static or stationary region [28]. A number of recent studies have pointed out that the surface gravities of the black hole horizon κ_B and the cosmological event horizon κ_C again have opposite signs [29–31]. Most have followed the procedure adopted in [28] and taken the physical temperature to be $\frac{|\kappa|}{2\pi}$ (e.g., see [32]). A similar situation arises in the case of the C-metrics, which contain both a black-hole horizon and an acceleration horizon. Their surface gravities are of opposite signs.

In order to assess the status of these suggestions, in this paper we shall reexamine the foundations of classical black hole thermodynamics from the viewpoint of the approach to classical thermodynamics advocated by Gibbs [33]. The central idea of this approach is that the physical properties of a substance are encoded into the shape of its Gibbs surface, i.e. the surface given by regarding the height of the surface as given by the total energy, regarded as a function of the remaining extensive variables. From this point of view, the temperature is given by the slope of the curve of energy versus entropy. To this end, we shall need explicit Christodoulou-Ruffini formulas, and a major goal of this paper is to obtain these for a variety of black hole solutions. As we shall see, it is a common feature of these examples that the "Gibbsian temperature" thus defined, while positive for black hole event horizons, is negative for inner horizons (i.e. Cauchy horizons) and for cosmological horizons. We shall return to a discussion of the physical consequences for spacetimes with two horizons in the conclusions.

Let us recall that the formalism of thermodynamics, applied to classical black holes, began with two independent discoveries:

(i) Christodoulou's concept of reversible and irreversible transformations such that the energy E of a rotating black hole of angular momentum $\mathbf{J}$ and momentum $\mathbf{P}$ may be expressed as

$$M^2 = M_{\rm irr}^2 + \mathbf{P}^2 + \frac{\mathbf{J}^2}{M_{\rm irr}^2}, \tag{1.5}$$

where the irreducible mass $M_{\rm irr}$ is nondecreasing [34].

(ii) Hawking's theorem [35,36] that the area A of the event horizon is nondecreasing.

In fact

$$A = 16\pi M_{\rm irr}^2, \tag{1.6}$$

and for charged rotating Kerr-Newman black holes and dropping the momentum contribution and setting $J = |\mathbf{J}|$, one has [37] the Christodoulou-Ruffini formula:

$$M^2 = \left(M_{\rm irr} + \frac{Q^2}{4M_{\rm irr}}\right)^2 + \frac{J^2}{M_{\rm irr}^2}. \tag{1.7}$$

The obvious analogy of some multiple of the area of the horizon with entropy became even more striking with the discovery by Smarr [38] of an analogue of the Gibbs-Duhem relation for homogeneous substances. For Kerr-Newman black holes, this reads

[1]For example, in a static metric $ds^2 = -h(r)dt^2 + dr^2/h(r) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ one finds (after changing to a coordinate system that covers the horizon region) from (1.2) that if $K = \partial/\partial t$ then $\kappa = \frac{1}{2}dh/dr$, which is of the form of the negative of the gradient of the gravitational potential, evaluated on the horizon. If $h = (r - r_+)(r - r_-)/r^2$, as in the Reissner-Nordström metric, then $\kappa_+ = (r_+ - r_-)/(2r_+^2) > 0$, while $\kappa_- = -(r_+ - r_-)/(2r_-^2) < 0$. In general, of course, the slope of $h(r)$ must always have opposite signs at two adjacent zeros, and thus the surface gravities must have opposite signs.

$$M = \frac{1}{4\pi}\kappa A + 2\Omega J + \Phi Q, \tag{1.8}$$

where κ is the surface gravity, Ω the angular velocity and Φ the electrostatic potential of the event horizon. The analogy became almost complete with the formulation of three laws of black hole mechanics, including the first law

$$dM = \frac{1}{8\pi}\kappa dA + \Omega dJ + \Phi dQ, \tag{1.9}$$

by Bardeen, Carter and Hawking [39]. Note that the Smarr relation (1.8) follows from the first law (1.9) by differentiating the weighted homogeneity relation

$$M(\lambda^2 A, \lambda^2 J, \lambda Q) = \lambda M \tag{1.10}$$

with respect to λ and then setting $\lambda = 1$.

The existence of a "first law" is not in itself surprising, nor does it, in itself, imply any thermodynamic consequences. Whenever one has a problem involving varying a function subject to some constraints, and considering the value of the function at critical points, one has a formula analogous to (1.9). In the case of black hole solutions of the Einstein equations, they are known to satisfy a variational principle in which the mass is extremized keeping the horizon area, angular momenta and charges fixed (see, e.g., [40–42]). Similar formulae arise in the theory of rotating stars (see, e.g., [43]). The study of these variations is sometimes referred to as comparative statics.

For homogeneous substances with pressure P, volume V and internal energy U, it is well known that the Gibbs-Duhem relation is equivalent to the statement that the Gibbs free energy, or thermodynamic potential,

$$G = U - TS + PV, \tag{1.11}$$

vanishes identically. For black holes the Smarr relation (1.8) implies that

$$G = TS + \Omega J. \tag{1.12}$$

Classically, a number of arguments led to the conclusion that the laws of black hole mechanics were just analogous to the laws of thermodynamics. One argument was that as perfect absorbers, classical black holes should have vanishing temperature and hence the entropy should be infinite (cf. [44,45]). Another was based on dimensional reasoning. In units where Boltzmann's constant is taken to be unity, entropy is dimensionless, but in classical general relativity it is not obvious how to achieve this without introducing a unit of length. The obvious guess for entropy would be some multiple of the area A, but why not some monotonically increasing function of the area? Despite these doubts it was conjectured by Bekenstein [45] that when quantum mechanics is taken into account some multiple of $\frac{A}{l_p^2}$ should correspond to the physical entropy of a black hole. This conjecture was subsequently confirmed at the semiclassical level by Hawking [46,47], using quantum field theory in a curved background. Given this, one recognizes the Christodoulou-Ruffini formula (1.7) in the form

$$M = M(S, J, Q) \tag{1.13}$$

as an explicit expression for the analogue of the Gibbs surface $U = U(S, V)$ for a homogeneous substance.

To summarize, the purpose of the present paper is to reexamine these issues systematically, based on Gibbs's geometric viewpoint of the mathematical formalism of thermodynamics [33]. This starts with a choice of pairs of extensive and intensive variables and an expression for some sort of "energy," which is regarded as a function of the extensive variables. For the black holes in asymptotically flat spacetimes that we shall consider, the energy is taken to be the ADM mass M, and the extensive variables S^μ are usually taken to be $S^\mu = (S, J, Q_i, P^i) = (S, \mathbf{s})$, where $S = \frac{1}{4}A$ and A is the area of the event horizon, J is the total angular momentum and Q_i and P^i are N electric and N magnetic charges.[2] The index μ therefore ranges over $2 + 2N$ values. We have

$$M = M(S, J, Q_i, P^i) = M(S^\mu). \tag{1.14}$$

The intensive variables are taken to be $T_\mu = \frac{\partial M}{\partial S^\mu} = (T, \Omega, \Phi^i, \Psi_i) = (T, \mathbf{t})$ where T is the temperature, Ω is the angular velocity of the horizon, and Φ^i and Ψ_i are the electrostatic and magnetostatic potentials.

The organization of this paper is as follows. In Sec. II, we review the theory of Gibbs surfaces, and the various thermodynamic metrics with which they may be equipped. Section III forms the core of the paper. In it, we give many new results for the thermodynamics of a wide range of asymptotically-flat black holes. We begin in subsections III A, III B and III C by reviewing how the well-known Reissner-Nordström, Kerr and Kerr-Newman black holes fit into the Gibbsian framework. Subsection III D then provides a extensive discussion of the thermodynamics of families of black holes in four-dimensional STU supergravity. (The term STU supergravity was introduced in Ref. [49].) In particular, we give a systematic discussion of the notion of the decomposition of the system into left-handed and right-handed sectors, and their associated thermodynamics. Subsection III E has analogous results for five-dimensional STU supergravity black holes. Subsections III F and III G give similar results for the general family of four-dimensional Einstein-Maxwell-Dilaton (EMD) black holes, and a two-field generalization. Included in the discussion of

[2]We shall not consider scalar charges and moduli [48] in this paper.

these two-field EMD black holes, we exhibit a new area-product formula.

A rather general feature of many asymptoticaly flat black holes with two horizons is that the product of the areas of the two horizons is independent of the mass, and given in terms of conserved charges and angular momenta, which may plausibly be quantized at the quantum level. In Sec. IV, we use this area-product formula to exhibit an intriguing $S \to 1/S$ inversion symmetry of the Christodoulou-Ruffini formulas for such black holes. This symmetry of the Gibbs surface interchanges the positive and negative temperature branches.

In Sec. V we extend our discussion to black holes that are asymptotically AdS, or black holes with positive cosmological constant. In the AdS case the situation for inner and outer horizons is broadly similar to that for the asymptotically flat case. For positive cosmological constant, the black hole event horizon continues to have positive Gibbsian temperature, but that of the cosmological horizon is negative.

In Sec. VI, we revisit an old observation, that the entropy of the Kerr-Newman solution is a super-additive function of the extensive variables, and we its relation to Hawking's area theorem for black-hole mergers. We find that super-additivity holds also for a wide variety of the asymptotically-flat examples that we considered in Sec. III. However, we find that Kaluza-Klein dyonic black holes provide a counterexample, and we speculate on the reason for this.

The paper ends with conclusions and future prospects in Sec. VII.

II. THE GIBBS SURFACE AND THERMODYNAMIC GEOMETRY

A. The Gibbs surface

In this section we shall briefly summarize those aspects of the Gibbs surface which are relevant for the latter part of the paper. If we think of (S^μ, M) as coordinates in $\mathbb{R}^{3+2N}$ then (1.14) defines a hypersurface $\mathcal{G} \subset \mathbb{R}^{3+2N}$ whose conormal is $(T_\mu, -1)$. Since in our case M is a unique function of the extensive variables, the intensive variables are unique functions of the extensive variables: $T_\mu = T_\mu(S^\nu)$. The converse need not be true. If the function $M(S^\mu)$ were convex, then for fixed conormal $(T_\mu, -1)$ the plane

$$T_\mu S^\mu - M = 0 \tag{2.1}$$

would touch the surface at a unique point (S^μ, M). For a smooth Gibbs surface $\mathcal{G}$, convexity requires that the Hessian

$$g^W_{\mu\nu} = \frac{\partial^2 M}{\partial S^\mu \partial S^\nu} \tag{2.2}$$

be positive definite and one may then define a positive definite metric

$$ds^2 = g^W_{\mu\nu} dS^\mu dS^\nu, \tag{2.3}$$

called the Weinhold metric [50]. Because one of the components of the Weinhold metric (2.2) is related to the heat capacity[3] at constant J and Q^i and P^i, namely

$$g^W_{SS} = T C_{\mathbf{s}}^{-1} = \frac{\partial T}{\partial S}\bigg|_{\mathbf{s}}, \tag{2.4}$$

and neutral black holes or black holes with small charges or angular momentum have negative heat capacities, the Gibbs surface $\mathcal{G}$ is typically not convex and the Weinhold metric for black holes is typically Lorentzian [51].

If one defines a totally symmetric co-covariant tensor of rank three by

$$C_{\lambda\mu\nu} = \frac{\partial^3 M}{\partial S^\lambda \partial S^\mu \partial S^\nu}, \tag{2.5}$$

the Riemann and Ricci tensors and the Ricci scalar of the Weinhold metric are given by

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= -\frac{1}{4}[C^\alpha{}_{\mu\lambda} C^\lambda{}_{\nu\beta} - C^\alpha{}_{\nu\lambda} C^\lambda{}_{\mu\beta}], \\ R_{\beta\nu} &= -\frac{1}{4}[C^\alpha{}_{\alpha\lambda} C^\lambda{}_{\beta\nu} - C^{\alpha\lambda}{}_\nu C_{\lambda\alpha\beta}], \\ R &= -\frac{1}{4}[C^\alpha{}_{\alpha\lambda} C^{\lambda\nu}{}_\nu - C^{\alpha\nu\lambda} C_{\alpha\nu\lambda}], \end{aligned} \tag{2.6}$$

all indices being raised with $g^{\mu\nu}_W$, the inverse of $g^W_{\mu\nu}$. Divergences in R are sometimes held to be a diagnostic for phase transitions.

The geometry of the Gibbs surface is essentially the geometry behind the first law of thermodynamics. As we remarked previously, this fits into a pattern that is more general than just the theory of black holes, and arises whenever one is considering a variational problem with constraints. Since this is not as widely known as it deserves to be, we shall pause to describe the general situation, and then we shall restrict attention to its application to black hole theory. Consider a real-valued function $f(x)$ on some space X with coordinates x, subject to the n constraints that certain functions $C^a(x) = c^a$, $1 \le a \le n$, where the c^a are constants. Adopting the method of Lagrange multipliers, we require that

$$\delta f - \lambda_a \delta C^a = 0, \tag{2.7}$$

[3] We use the term "heat capacity" rather than "specific heat" because the latter is defined to be per unit mass.

for all variations in X. Suppose the solutions of these equations lie in an n-dimensional submanifold S of X, parametrized by the values of the constraints, c^a. One may restrict the variations in (2.7) to directions within the solution space S, in which case we obtain the formula

$$\delta f(c) = \lambda_a \delta c^a. \tag{2.8}$$

Geometrically, we can think of this situation as follows. We construct a $(2n+1)$-dimensional space with coordinates (f, λ_a, c^a). Since, locally at least, f and λ_a may be thought of as functions of the c^a, we obtain an n-dimensional surface in this space. From (2.8), it follows that the Lagrange multipliers λ_a are determined by the tangent planes to the surface. From now on, we shall restrict attention to the thermodynamic case, with f being the total energy, or mass, M, and the c^a being (S, J, Q_i, P^i).

The thermodynamic phase space has coordinates (S^μ, T_μ), and is therefore $(4+4N)$-dimensional. We then have a $(2+2N)$-dimensional submanifold $\mathcal{L}A: T_\mu = T_\mu(S^\nu)$ of the thermodynamic phase space, i.e. $\mathcal{L}A \subset \mathbb{R}^{4+4N}$ with coordinates (T_μ, S^ν), equipped with the symplectic form

$$\omega = dT_\mu \wedge dS^\mu. \tag{2.9}$$

Since, when pulled back to $\mathcal{L}A$ we have $T_\mu dS^\mu = dM(S^\mu)$, the pull-back of ω to $\mathcal{L}A$ vanishes,

$$\omega|_{\mathcal{L}A} = 0. \tag{2.10}$$

In other words, $\mathcal{L}A$ is a Lagrangian submanifold of $\mathbb{R}^{4+4N}$.

One may go a step further and lift $\mathcal{L}A$ to $\mathbb{R}^{5+4N}$ with coordinates (P_μ, S^ν, M), equipped with the contact form

$$\eta = T_\mu dS^\mu - dM, \tag{2.11}$$

as a Legendre submanifold $\mathcal{L}E$, i.e. one for which

$$\eta|_{\mathcal{L}E} = 0. \tag{2.12}$$

In most of the cases we shall be considering, for dimensional reasons $M(S, J, Q_i, P^i)$ satisfies the weighted homogeneity relation

$$M(\lambda^2 A, \lambda^2 J, \lambda Q^i, \lambda P^i) = \lambda M. \tag{2.13}$$

Differentiating with respect to λ and then setting $\lambda = 1$ yields the Smarr relation [38]

$$M = 2TS + 2\Omega J + \Phi^i Q_i + \Psi_i P^i. \tag{2.14}$$

The Gibbs function, or thermodynamic potential, G, is the total Legendre transform of the mass with respect to the extensive variables. It satisfies

$$dG = -S^\mu dT_\mu, \tag{2.15}$$

where

$$\begin{aligned} G(T_\mu) &= M - T_\mu S^\mu \\ &= M - TS - \Omega J - \Phi^i Q_i - \Psi_i P^i \\ &= TS + \Omega J. \end{aligned} \tag{2.16}$$

Note that G is not necessarily a single-valued function of the intensive variables T_μ, unless the Gibbs surface $\mathcal{G}$ is convex. The Hessian of the Gibbs function with respect to the intensive variables is related to the Weinhold metric by the easily verified formula

$$\frac{\partial^2 G}{\partial T_\mu \partial T_\nu} \frac{\partial^2 M}{\partial S^\nu \partial S^\lambda} = -\delta^\mu_\lambda. \tag{2.17}$$

It provides a metric on the space of intensive variables.

It is important to realize that from the point of view of the symplectic and contact structures described above, the coordinates (S^μ, T_μ, M) have a privileged status and it makes little physical sense to consider arbitrary coordinate transformations even if they preserve the symplectic or contact structures. Only a limited number of Legendre transformations are of physical relevance. It is physically meaningful to consider positive linear combinations of the vectors S^μ, i.e. sending $S^\mu \to D^\mu{}_\nu S^\nu$, where $D^\mu{}_\nu$ is a constant diagonal matrix, and also to reverse the sign of any but the first component (i.e. the first diagonal component of $D^\mu{}_\nu$, associated with scaling the entropy itself, should be positive). In other words, physical states are future directed with respect to the first component.

In the literature on thermodynamic metrics, much discussion has focused on whether or not the Ricci scalar is a good indicator of phase transitions. Because, as explained above, general coordinate transformations do not have physical significance, it is not obvious that one should be concerned with a scalar such as the Ricci scalar. In fact, what is more relevant is the behavior of the Hessian, i.e. the thermodynamic metric. If this is not invertible then a divergence of the Ricci scalar will occur, but the value of the Ricci scalar itself does not appear in general to have any physical significance.

B. Thermodynamic metrics

It has been traditional in the literature to focus on the Ruppeiner and Weinhold metrics, and this is especially convenient if one has available an explicit Christrodoulou-Ruffini formula. However, as in standard text books on thermodynamics, it is frequently convenient to introduce a variety of other thermodynamic potentials related by Legendre transformations, depending upon what quantities are being held fixed. In the context of black hole thermodynamics this corresponds to what boundary conditions are being considered. The consequent uniqueness or "No Hair" properties will depend in general on precisely what is to be held fixed. This lack of uniqueness is what is often referred to as a "phase transition," but as in standard thermodynamics it is important to specify the physical conditions under which the phase transition takes place.

From the point of view of the Gibbs surface $\mathcal{G}$, geometrically this should really be thought of as an n-dimensional Legendrian submanifold of the $(2n+1)$-dimensional Legendre manifold whose coordinates consist of the total energy and the n pairs of intensive and extensive variables. Given a choice of n coordinates chosen from these $2n$ variables, one may *locally* describe the surface in terms of the associated thermodynamic potential, and from that compute the associated Hessian metric. But *globally*, it is not in general true that the Gibbs surface equipped with the choice of Hessian is a single-valued nonsingular graph over the n-plane spanned by the chosen set of n coordinates. It should also be remembered that although the Hessian metrics may be thought of as the pull-back to $\mathcal{G}$ of a flat metric on the $2n$-dimensional flat hyperplane spanned by the choice n pairs of intensive and extensive variables, the signature of that flat metric depends upon that choice.

Here we review some key results on the general classes of thermodynamic metrics that were presented in [52]. Consider first the energy $M = M(S^\mu)$, which obeys the first law

$$dM = T_\mu dS^\mu = TdS + \Omega dJ + \Phi^i dQ_i + \cdots. \tag{2.18}$$

One can define from this the metric

$$ds^2(M) = dT_\mu \otimes_s dS^\mu, \tag{2.19}$$

where T_μ are viewed as functions of the S^μ variables, with

$$T_\mu = \frac{\partial M}{\partial S^\mu}, \tag{2.20}$$

and $\otimes_s$ denotes the symmetrized tensor product. In the usual parlance of general relativity we may simply write (2.19) as

$$ds^2(M) = dT_\mu dS^\mu. \tag{2.21}$$

In view of (2.20) we have

$$ds^2(M) = \frac{\partial^2 M}{\partial S^\mu \partial S^\nu} dS^\mu dS^\nu, \tag{2.22}$$

which is nothing but the Weinhold metric.

One can obtain a set of conformally related metrics by dividing (2.18) by any one of the intensive variables T_μ for $\mu = \bar{\mu}$ where $\bar{\mu}$ denotes the associated specific index value of the chosen intensive variable, and then constructing the thermodynamic metric $ds^2(S^{\bar{\mu}})$ for the conjugate extensive variable by using the same procedure as before [52]. Thus, e.g., if we choose $\bar{\mu} = 0$, so that T is the chosen intensive variable and S its conjugate, then we rewrite (2.18) as

$$dS = \frac{dM}{T} - \frac{1}{T} T_a dS^a, \tag{2.23}$$

where we have split the μ index as $\mu = (0, a)$, and then write the associated thermodynamic metric

$$\begin{aligned} ds^2(S) &= -\frac{1}{T^2} dT dM + \frac{T_a}{T^2} dT dS^a - \frac{1}{T} dT_a dS^a \\ &= -\frac{1}{T}(dTdS + dT_a dS^a) \\ &= -\frac{1}{T} ds^2(M). \end{aligned} \tag{2.24}$$

The second line was obtained by using (2.18), and the third line follows from (2.22). Thus $ds^2(S)$, which is the Ruppeiner metric, is conformally related by the factor $-1/T$ to the Weinhold metric. Weinhold and Ruppeiner metrics were introduced into black hole physics in [51,53]. The literature is by now quite extensive. For a recent review see [54]. Other conformally-related metrics can be defined by dividing (2.18) by any other of the intensive variables and the repeating the analogous calculations. For example, if there is a single charge Q and potential Φ, then dividing the first law $dM = TdS + \Phi dQ + \cdots$ by Φ and calculating the metric $ds^2(Q)$, one obtains

$$ds^2(Q) = -\frac{1}{\Phi} ds^2(M). \tag{2.25}$$

Further thermodynamic metrics that are not merely conformally related to the Weinhold metric can be obtained by making Legendre transformations to different energy functions before implementing the above procedure [52]. For example, if one make the Legendre transform to the free energy $F = M - TS$, for which one has the first law

$$dF = -SdT + T_a dS^a, \tag{2.26}$$

then the associated thermodynamic metric will be

$$ds^2(F) = -dTdS + dT_a dS^a, \tag{2.27}$$

where S and T_a, which are now the intensive variables, are viewed as functions of T and S^a. The metric components in $ds^2(F)$ are therefore given by the Hessian of F. As observed in [52], the metric $ds^2(F)$ has the property that, unlike the Weinhold or Ruppeiner metrics, its curvature is singular on the so-called Davies curve where the heat capacity diverges.

Clearly, by making different Legendre transformations, one can construct many different thermodynamic metrics, which take the form

$$ds^2 = \sum_{\mu\geq 0}\eta_\mu dT_\mu dS^\mu, \tag{2.28}$$

where each η_μ can independently be either $+1$ or -1. The overall sign is of no particular importance, and so metrics related by making a complete Legendre transformation of all the intensive/extensive pairs in a given energy definition really yields an equivalent metric. For example, the Gibbs energy $G = M - T_\mu S^\mu$ gives the metric

$$ds^2(G) = -dT_\mu dS^\mu, \tag{2.29}$$

which is just the negative of the Weinhold metric $ds^2(M)$ in (2.22).

One further observation that was emphasized in [52] is that one is not, of course, obliged when writing a thermodynamic metric to use the associated extensive variables as the coordinates. It is sometimes the case, as we shall see in later examples, that although one can calculate the thermodynamic variables in terms of the metric parameters, one cannot explicitly invert these relations. In such cases, one can always choose to use the metric parameters as the coordinates when writing the thermodynamic metrics. Geometric invariants such as the Ricci scalar of the thermodynamic metric will be the same whether written using the thermodynamic variables or the metric parameters, since one is just making a general coordinate transformation. Thus even in cases where the relations between the thermodynamic variables and metric parameters are too complicated to allow one to find an explicit Christodoulou-Ruffini formula to define the Gibbs surface, one can still study the geometrical properties of the various thermodynamic metrics.

III. ASYMPTOTICALLY FLAT BLACK HOLES

In this section, we shall illustrate the issues raised in the previous section by listing the cases of asymptotically-flat black holes for which we have explicit formulae. Whilst the formulae for the Kerr-Newman family of black holes are well known, we first review these in some detail in preparation for our discussion of much less well known black holes, such as those that occur in supergravity or Kaluza-Klein theories.

A. The Gibbs surface for Reissner-Nordström

The Gibbs surface $\mathcal{G}$ for the Reissner-Nordström solution is given by the Christodoulou-Ruffini formula

$$M = \sqrt{\frac{S}{4\pi}} + \frac{Q^2}{4}\sqrt{\frac{4\pi}{S}} = M_{\rm irr} + \frac{Q^2}{4M_{\rm irr}}, \tag{3.1}$$

where $M_{\rm irr} = \sqrt{\frac{S}{4\pi}}$. It is convenient to envisage (M, Q, S) as a right-handed Cartesian coordinate system with $M > 0$ taken vertically and $-\infty < Q < \infty$ and $S > 0$ spanning a horizontal half-plane. In $(M, Q, \sqrt{S})$ coordinates the surface is part of the quadratic cone

$$M^2 = \left(\sqrt{\frac{S}{\pi}} - M\right)^2 + Q^2. \tag{3.2}$$

We have

$$M \geq |Q|, \tag{3.3}$$

with $M > |Q|$ being subextremal black holes. Rewriting (3.2) as

$$S^2 - 2\pi(2M^2 - Q^2)S + \pi^2 Q^4 = 0, \tag{3.4}$$

the two solutions for S at fixed M and Q are given by

$$\frac{S_\pm}{\pi} = 2M^2 - Q^2 \pm 2M\sqrt{M^2 - Q^2}, \tag{3.5}$$

with these corresponding to the entropies (i.e. one quarter the area) of the outer (S_+) and inner (S_-) horizons respectively. It is straightforward to see that the temperature $T = \partial M/\partial S$ is positive on the outer horizon and negative on the inner horizon.

Equality, $M = |Q|$, corresponds to extreme black holes. They lie on the space curve $\gamma_{\rm extreme}$ given by the intersection of the two surfaces

$$M = |Q|, \qquad M = \sqrt{\frac{S}{\pi}}. \tag{3.6}$$

The first is a plane orthogonal to the Q plane, and the second a parabolic cylinder with generators parallel to the Q axis. The projection of $\gamma_{\rm extreme}$ onto the Q–S plane is given by the parabola

$$S = \pi Q^2. \tag{3.7}$$

Roughly speaking, the Gibbs surface $\mathcal{G}$ is folded over the space curve $\gamma_{\rm extreme}$. Now the Weinhold metric, or equivalently the Hessian of $M(S, Q)$, is given by

$$ds_W^2 = \sqrt{\frac{4\pi}{S}}\left\{\frac{1}{2}dQ^2 - \frac{Q}{2S}dQdS + \frac{1}{16S^2}\left(3Q^2 - \frac{S}{\pi}\right)dS^2\right\}. \tag{3.8}$$

Note that $\frac{\partial^2 M}{\partial S^2}$ changes sign, passing through zero, along the space curve $\gamma_{\rm Davies}$, given by

$$S = 3\pi Q^2 = \frac{9}{4}M^2. \tag{3.9}$$

Since the heat capacity at constant charge, C_Q, is given by

$$C_Q = T\left(\frac{\partial^2 M}{\partial S^2}\right)_Q^{-1}, \tag{3.10}$$

it also changes sign across the curve $\gamma_{\rm Davies}$, on which it diverges [55]. This is often taken as a sign of a phase transition. In support of this interpretation, it has been shown [56] that the single negative mode of the Lichnerowicz operator passes through zero and becomes positive as Q is increased across $\gamma_{\rm Davies}$.

The curve $\gamma_{\rm Davies}$ is an example of what, in the literature on phase transitions, is often referred to as a spinodal curve, and is usually defined in terms of the vanishing of a diagonal element of the Hessian of the Gibbs function. In the present case, the Gibbs function is

$$G = M - TS - \Phi Q = \frac{(1-\Phi^2)^2}{16\pi T}, \tag{3.11}$$

and the Hessian is given by

$$\begin{pmatrix} \frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Phi} \\ \frac{\partial^2 G}{\partial \Phi \partial T} & \frac{\partial^2 G}{\partial \Phi^2} \end{pmatrix} = \begin{pmatrix} \frac{(1-\Phi^2)^2}{8\pi T^3} & \frac{(1-\Phi^2)\Phi}{4\pi T^2} \\ \frac{(1-\Phi^2)\Phi}{4\pi T^2} & -\frac{(1-3\Phi^2)}{4\pi T} \end{pmatrix}. \tag{3.12}$$

The spinodal curve is thus given by $\Phi^2 = \pm\frac{1}{3}$, which, in terms of S and Q, coincides with (3.9).

The Weinhold metric may written as

$$ds_W^2 = \sqrt{\frac{4\pi}{S}}\left\{\frac{1}{2}\left(dQ - \frac{Q}{2S}dS\right)^2 - \frac{1}{16S^2}\left(\frac{S}{\pi} - Q^2\right)dS^2\right\}, \tag{3.13}$$

and hence the Gibbs surface for subextremal black holes has a Hessian, or equivalently a Weinhold metric, that is nonsingular but Lorentzian. Moreover the Gibbs surface for nonextreme black holes is nonconvex. Expressed in terms of S and the electrostatic potential

$$\Phi = \Phi(S, Q) = Q\sqrt{\frac{\pi}{S}}, \tag{3.14}$$

the Weinhold metric becomes

$$ds_W^2 = \frac{1}{8\sqrt{\pi}S^{3/2}}[-(1-\Phi^2)dS^2 + 8S^2 d\Phi^2]. \tag{3.15}$$

Note that the metric is nonsingular when either $\Phi^2 < 1$, corresponding to the outer horizon, or $\Phi^2 > 1$, corresponding to the inner horizon. It changes signature from $(-+)$ to $(++)$ as Φ goes from $\Phi^2 < 1$ to $\Phi^2 > 1$. The heat capacity passes through infinity at $\Phi^2 = \frac{1}{3}$.

Expressed in terms of Φ and S, the temperature is given by $T = (1-\Phi^2)/(4\sqrt{\pi S})$, and so the Ruppeiner metric is given by

$$\begin{aligned} ds_R^2 &= -\frac{1}{T}ds_W^2 = -\frac{dS^2}{2S} + 4S\frac{d\Phi^2}{1-\Phi^2} \\ &= -d\tau^2 + \tau^2 d\sigma_+^2, \end{aligned} \tag{3.16}$$

where we have defined, for the outer horizon,

$$S = \frac{1}{2}\tau^2, \qquad \Phi = \sin\frac{\sigma_+}{\sqrt{2}}. \tag{3.17}$$

The metric in the second line of (3.16) is the Milne metric on a wedge of Minkowski spacetime inside the light cone. This is made apparent by introducing new coordinates according to

$$t = \tau\cosh\sigma_+, \qquad x = \tau\sinh\sigma_+, \tag{3.18}$$

in terms of which the Ruppeiner metric becomes

$$ds_R^2 = -dt^2 + dx^2, \qquad S = S_+ = \frac{1}{2}(t^2 - x^2). \tag{3.19}$$

Since the range of σ_+ is $-\frac{\pi}{\sqrt{2}} \le \sigma_+ \le \frac{\pi}{\sqrt{2}}$, the extremal solutions lie on the timelike geodesics $t = \pm\text{arctanh}\frac{\pi}{\sqrt{2}}$. The heat capacity changes sign at $\Phi^2 = \frac{1}{3}$.

If $\Phi^2 > 1$, corresponding to the inner horizon, then, if $Q > 0$, substituting

$$\Phi = \cosh\frac{\sigma_-}{\sqrt{2}} \tag{3.20}$$

(or $\Phi = -\cosh(\sigma_-/\sqrt{2})$ if $Q < 0$) into (3.16) gives

$$ds^2 = -(d\tau^2 + \tau^2 d\sigma_-^2). \tag{3.21}$$

The metric in brackets is the flat metric on Euclidean space in polar coordinates, except that the range of the coordinate σ_- is $0 \le \sigma_- \le \infty$, so we have an infinitely branched covering of the Euclidean plane, with the branch point at the origin. The Weinhold metric is itself positive definite. Thus the Gibbs surface is convex and the entropy surface is concave for the inner horizon.

The flatness of the Ruppeiner metric for Reissner-Nordström has given rise to much comment, because singularities of the Ruppeiner metric are expected to reveal the occurrence of phase transitions. However, the geometrical significance of the change in sign of the heat capacity is that for fixed charge Q, there is a maximum temperature. In fact

$$T = T(S, Q) = \frac{1}{2S}\sqrt{\frac{S}{4\pi}} - \frac{Q^2}{8S}\sqrt{\frac{4\pi}{S}}, \tag{3.22}$$

so for given $|Q|$ and positive T less than $\frac{\sqrt{3}}{8\pi|Q|}$, there are two positive values of S and hence two nonextreme black holes. By contrast, since the electrostatic potential Φ satisfies (3.14), there is a unique positive value of S and hence a unique black hole for given Q and $\Phi^2 < 1$.

Every two-dimensional metric is conformally flat. Therefore it is not surprising that both the Weinhold and Ruppeiner metrics for Reissner-Nordström are conformally flat. It is, however, nontrivial that the Ruppeiner metric is flat. It has recently been pointed out [57] that one can also consider the Hessian of the charge Q, considered as a function of the mass and entropy, as a metric ds^2_Q. In fact $ds^2_Q = -\Phi^{-1} ds^2_W$, as in (2.25). Geometrically, there is no reason to give a preference to any of the metrics ds^2_W, ds^2_R or ds^2_Q. Since T and Φ are both nonsingular on the curve along which the heat capacity diverges, none of the three metrics is capable of detecting the associated "phase transition."

As was shown in [52], and we reviewed in Sec. II B, the thermodynamic metric (2.27) constructed from the free energy $F = M - TS$ does exhibit a singularity on the Davies curve where the heat capacity diverges. For the Reissner-Nordström metric (2.27) is the restriction of $ds^2(F) = -dTdS + d\Phi dQ$ to the Gibbs surface, and hence we find

$$ds^2(F) = \sqrt{\frac{\pi}{S}}dQ^2 + \frac{1}{8\sqrt{\pi}S^{5/2}}(S - 3\pi Q^2)dS^2. \tag{3.23}$$

A straightforward calculation shows that its Ricci scalar is given by

$$R_F = \frac{4\sqrt{\pi}S^{3/2}}{(S - 3\pi Q^2)^2}, \tag{3.24}$$

which does indeed diverge on the Davies curve $S = 3\pi Q^2$.

B. The Gibbs surface for Kerr

This is qualitatively very similar to the Reissner-Nordström case. To begin with, we shall summarize, in our notation, some results first presented by Curir [1]. One has

$$M^2 = \frac{S}{4\pi} + \frac{\pi J^2}{S}, \tag{3.25}$$

and $M(S, J)$ at fixed J has a minimum value when

$$S = 2\pi|J|, \qquad M = \sqrt{|J|}. \tag{3.26}$$

This is the extreme case and, as before, the inner horizon has a negative temperature, a point made first by Curir [1]. Explicitly one has

$$T = \frac{1}{8\pi M}\left(1 - \frac{4\pi^2 J^2}{S^2}\right). \tag{3.27}$$

For any given values of J and of $M > 0$, there are two positive solutions, S_+ and S_-, of (3.25), where $S_+ \geq 2\pi|J|$ corresponds to one quarter of the area of the outer horizon of a sub-extremal black hole and $S_- \leq 2\pi|J|$ corresponds to one quarter of the area of the inner horizon of a sub-extremal black hole. From (3.25), they obey the entropy product formula

$$S_- S_+ = 4\pi^2 J^2. \tag{3.28}$$

By (3.27), the outer horizon has a positive temperature, which we label T_+, and the inner horizon has a negative temperature, which we label T_-. One has [1]

$$T_\pm = \frac{S_\pm - S_\mp}{8\pi M S_\pm}, \qquad \Omega_\pm = \frac{\pi J}{M S_\pm}, \tag{3.29}$$

where $\Omega_\pm = (\partial M/\partial J)_{S_\pm}$. Note that it follows from the first equation in (3.29) that

$$T_+ S_+ + T_- S_- = 0. \tag{3.30}$$

Note also that M and J, which are conserved quantities defined in terms of integrals at infinity, are universal and do not carry $\pm$ labels.

In terms of S_+ and S_-, one has, from (3.25),

$$M^2 = \frac{S_+}{4\pi} + \frac{S_-}{4\pi}. \tag{3.31}$$

Therefore

$$M = \frac{\sqrt{S_+ + S_-}}{\sqrt{4\pi}} \leq \sqrt{\frac{S_+}{4\pi}} + \sqrt{\frac{S_-}{4\pi}}, \tag{3.32}$$

with equality be attained if $J = 0$. If one varies M, one has

$$dM = T_\pm dS_\pm + \Omega_\pm dJ. \tag{3.33}$$

There is also a modified Smarr formula

$$M = T_+ S_+ + T_- S_- + \Omega_+ J + \Omega_- J = (\Omega_+ + \Omega_-)J, \tag{3.34}$$

where the second equality follows from (3.30). This way of writing the first law of thermodynamics was employed in [58] for deriving a simple formula for holographic complexity. These results were interpreted in [1] as indicating that the total energy of a rotating black hole may be regarded as receiving contributions from two thermodynamic systems; one associated with the outer horizon and the other with the inner horizon. The negative temperature was interpreted in terms of Ramsey's account of the thermodynamics of isolated spin systems [59].

Okamoto and Kaburaki [10] introduced the dimensionless parameter $h = \frac{a}{M+\sqrt{M^2-a^2}}$ in their discussion of the energetics of Kerr black holes and noticed that it satisfies the quadratic equation

$$h^2 - \frac{2hM^2}{|J|} + 1 = 0. \tag{3.35}$$

It was initially assumed that only the solution of (3.35) satisfying $0 \le h \le 1$ has physical significance. However Abramowicz [60] drew their attention to [1,2], and they realized that the other root of (3.35), which satisfies $1 \le h \le \infty$ and is given by $h = \frac{a}{M-\sqrt{M^2-a^2}}$, is associated with the inner horizon [10]. Expressing the thermodynamic variables in terms of h they established (3.30) if T_- is taken to be negative, and they also obtained the formula

$$\frac{\Omega_+}{T_+} + \frac{\Omega_-}{T_-} = 0. \tag{3.36}$$

C. Kerr-Newman black holes

Kerr-Newman black holes may have both electric and magnetic charges. By electric-magnetic duality invariance one may set the magnetic charge P to zero. To restore electric-magnetic duality invariance it suffices to replace Q^2 by $Q^2 + P^2$ in all formulas thus producing a manifestly $O(2)$ invariant Gibbs surface.

The mass of the Kerr-Newman black hole is given by

$$M = \left[\frac{\pi}{4S}\left(\frac{S}{\pi} + Q^2\right)^2 + \frac{\pi J^2}{S}\right]^{\frac{1}{2}}, \tag{3.37}$$

and therefore it satisfies

$$M \ge \sqrt{\sqrt{J^2 + \frac{Q^4}{4}} + \frac{Q^2}{2}}, \tag{3.38}$$

acquiring its least value on the surface γ_{extreme} in the three dimensional space of extensive variables given by

$$S = \pi\sqrt{4J^2 + Q^4}, \tag{3.39}$$

on which the temperature

$$T = \left(\frac{\partial M}{\partial S}\right)_{J,Q} = \frac{1}{8\pi M}\left[1 - \frac{\pi^2}{S^2}(4J^2 + Q^4)\right] \tag{3.40}$$

vanishes. If $J = 0$, then (3.38) is the usual Bogomolnyi bound [61]. One also has

$$\Omega = \frac{\pi J}{MS}, \qquad \Phi = \frac{\pi Q}{2MS}\left(Q^2 + \frac{S}{\pi}\right). \tag{3.41}$$

The explicit formulas (3.37), (3.40) and (3.41) allow a lift of the Gibbs surface $\mathcal{G}$ to a Lagrangian submanifold $\mathcal{L}$ in $\mathbb{R}^6$ and a Legendrian submanifold in $\mathbb{R}^7$. The entropy product law becomes

$$S_+S_- = \pi^2(4J^2 + Q^4), \tag{3.42}$$

where the $-$ refers to the inner and $+$ to outer horizon.

The temperatures and angular velocities of the two horizons are given by

$$T_\pm = \frac{S_\pm - S_\mp}{8\pi M S_\pm}, \qquad \Omega_\pm = \frac{\pi J}{MS_\pm}, \tag{3.43}$$

and one has

$$S_+T_+ + S_-T_- = 0. \tag{3.44}$$

There is a conventional first law for both horizons:

$$dM = T_\pm dS_\pm + \Omega_\pm dJ + \Phi_\pm dQ, \tag{3.45}$$

and a modified Smarr formula

$$\begin{aligned} M &= T_+S_+ + T_-S_- + \Omega_+J + \Omega_-J + \frac{1}{2}\Phi_+Q + \frac{1}{2}\Phi_-Q \\ &= (\Omega_+ + \Omega_-)J + \frac{1}{2}(\Phi_+ + \Phi_-)Q. \end{aligned} \tag{3.46}$$

D. STU black holes

Four-dimensional black holes in string theory or M-theory can be described as solutions of $\mathcal{N} = 8$ supergravity. The most general black holes are supported by just four of the 28 gauge fields, in the Cartan subalgebra of $SO(8)$. The black holes can therefore be described just within the $\mathcal{N} = 2$ STU supergravity theory, which is a consistent truncation of the $\mathcal{N} = 8$ theory whose bosonic sector comprises the metric, the four gauge fields, and six scalar fields. Black holes of the STU model are parametrized by mass M, angular momentum J and four electric Q_i ($i = 1, 2, 3, 4$) and four magnetic charges P^i ($i = 1, 2, 3, 4$). The most general black hole solution was obtained by Chow and Compère [62] by solution generating techniques.

We shall follow the usual conventions for STU supergravity, in which the normalization of the gauge fields $F^{(i)}$ is such that if the scalar fields are turned off, the Lagrangian will take the form $\mathcal{L} = \sqrt{-g}[R - \frac{1}{4}\sum_i (F^{(i)})^2 + \cdots]$ (see Appendix B for a presentation of the bosonic sector of the

STU supergravity Lagrangian). This contrasts with the conventional normalization $\mathcal{L} = \sqrt{-g}(R - F^2)$, in Gaussian units, which we use when describing the pure Einstein-Maxwell theory. Since this means that the charge normalization conventions will be different in the two cases, we shall briefly summarize our definitions here. If we consider the Lagrangian

$$\mathcal{L} = \sqrt{-g}(R - \gamma F^2), \tag{3.47}$$

one can derive by considering variations of the associated Hamiltonian that black holes will obey the first law

$$dM = \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ, \tag{3.48}$$

where κ is the surface gravity, Φ is the potential difference between the horizon and infinity (with the potential being equal to $\xi^\mu A_\mu$, where ξ^μ is the future-directed Killing vector that is null on the horizon and is normalized such that $\xi^\mu \xi_\mu \to -1$ at infinity). The electric charge Q is given by

$$Q = \frac{\gamma}{4\pi} \int *F. \tag{3.49}$$

Thus in Einstein-Maxwell theory, with $\mathcal{L} = \sqrt{-g}(R - F^2)$, we shall have

$$Q = \frac{1}{4\pi} \int *F, \tag{3.50}$$

while in STU supergravity we shall have (neglecting the scalar fields for simplicity[4])

$$Q_i = \frac{1}{16\pi} \int *F^{(i)}. \tag{3.51}$$

The black hole solutions have two horizons, with the product of the horizon entropies quantized:

$$S_+ S_- = 4\pi^2 |J^2 + \Delta|, \tag{3.52}$$

where Δ is the Cayley hyperdeterminant $\Delta(Q_i, P^i)$:

$$\Delta = 16 \left[4(Q_1 Q_2 Q_3 Q_4 + P^1 P^2 P^3 P^4) + 2 \sum_{i<j} Q_i Q_j P^i P^j - \sum_i (Q_i)^2 (P^i)^2 \right]. \tag{3.53}$$

Note that Eq. (3.52) has previously appeared in the literature without the absolute value symbol (e.g., in [62]). We have written (3.52) with an absolute value sign since Δ, and hence $\Delta + J^2$, can be negative; for example for a static Kaluza-Klein dyonic black hole. (In [62] it was proposed that S_- is negative when $\Delta + J^2 < 0$, but this would contradict the fact that, e.g., the area of the inner horizon of the static Kaluza-Klein dyonic black hole is positive.)

It should be noted that if J vanishes and $\Delta = 0$, then S_- will vanish also. In this case there is no nonsingular inner horizon.

The entropy formulas (3.52) can be cast in the form

$$S_+ = S_L + S_R, \qquad S_- = |S_L - S_R|, \tag{3.54}$$

with

$$S_L = 2\pi\sqrt{F + \Delta}, \qquad S_R = 2\pi\sqrt{F - J^2}, \tag{3.55}$$

where F is another complicated expression that is a function of M, Q_i and P^i only [62]. Note that it follows from (3.54) that $S_+ \geq S_-$. Unlike [62], we have put an absolute value sign around $(S_L - S_R)$ in the expression for S_-, since, for the reasons discussed above, there can be circumstances where $S_L < S_R$, but S_- should be non-negative. Note that $F + \Delta$ is always non-negative, and $F - J^2$ is non-negative provided that the black hole is not overrotating [62]. The quantities S_L and S_R are both non-negative. In the extremal limit $F - J^2 = 0$, one gets the extremal value for the entropy $S_+ = S_- = 2\pi\sqrt{|\Delta|}$. This was seen for the BPS (Bogomol'nyi-Pasad-Sommerfield) solutions ($F = 0$ and $J^2 = 0$) in [13].

Note from (3.55) that while the right-moving entropy S_R is a function of all the extensive variables (M, Q_i, P_i, J), the left-moving entropy S_L is a function of (M, Q_i, P^i) but not J [62]. This was noted previously in the special case of the four-charge black holes characterized by (M, Q_i, J) in [11,63]. The expressions (3.55) may in principle be inverted to give *two* different Christodoulou-Ruffini formulas:

$$M = M(S_L, Q_i, P^i), \text{ and } M = M(S_R, Q_i, P^i, J). \tag{3.56}$$

The structure (3.55) ensures that the two entropies S_+ and S_- are solutions of the quadratic equation

[4] In general, including the scalar fields, and writing the Lagrangian as a 4-form, we shall have $\mathcal{L} = R * \mathbb{1} - \frac{1}{2} M_{ij}(\Phi) * F^{(i)} \wedge F^{(j)} - \frac{1}{2} N_{ij}(\Phi) F^{(i)} \wedge F^{(j)} + \cdots$, where $F^{(i)} = dA^{(i)}$. The electric charges can be written as

$$Q_i = -\frac{1}{16\pi} \int \frac{\delta \mathcal{L}}{\delta F^{(i)}}.$$

(Here the variational derivative is defined by $\delta X = (\delta X / \delta F) \wedge \delta F$. For example if $X = u * F \wedge F + v F \wedge F$ then $\delta X / \delta F = 2u * F + 2vF$.) The magnetic charges are given by $P^i = \frac{1}{16\pi} \int F^{(i)}$.

$$S^2 - S\Sigma + 4\pi^2|J^2 + \Delta| = 0, \tag{3.57}$$

where $\Sigma = S_L + S_R + |S_L - S_R|$, and we employed (3.54), (3.55) and (3.52). Note that $\Sigma = 2S_L$ if $S_L > S_R$, which corresponds to $J^2 + \Delta > 0$, whilst $\Sigma = 2S_R$ if $S_L < S_R$, corresponding to $J^2 + \Delta < 0$. From (3.57) we can deduce

$$\frac{\partial M}{\partial S}\frac{\partial \Sigma}{\partial M}\Big|_{(Q_i,P^i,J)} = \left[1 - \frac{4\pi^2|J^2+\Delta|}{S^2}\right] = \frac{1}{S}\left[S - \frac{S_+S_-}{S}\right]. \tag{3.58}$$

Since $S_+ \geq S_-$, the final expression in (3.58) is non-negative for $S = S_+$, and nonpositive for $S = S_-$. Since $\frac{\partial M}{\partial S}|_{(Q_i,P^i,J)} = T$, and since $\frac{\partial \Sigma}{\partial M}|_{(Q_i,P^i,J)}$ is independent of whether one takes $S = S_+$ or $S = S_-$, it then follows that

$$S_+T_+ + S_-T_- = 0. \tag{3.59}$$

In particular, this implies that T_+ and T_- must have opposite signs.

As well as considering the left-moving and right-moving entropies S_L and S_R, one can also introduce left-moving and right-moving temperatures T_L and T_R, defined by [15]

$$\frac{1}{T_L} = \frac{1}{T_+} + \frac{1}{T_-}, \qquad \frac{1}{T_R} = \frac{1}{T_+} - \frac{1}{T_-}. \tag{3.60}$$

These definitions are motivated by the fact that when one calculates scattering amplitudes for test fields propagating in the black-hole background, one finds that they factorize into the product of thermal Boltzmann factors for the temperatures T_L and T_R respectively [15]. Using (3.59), together with the expressions for S_+ and S_- in terms of S_L and S_R in (3.54), it follows from (3.60) that

$$\begin{aligned} S_L \geq S_R&:\ \frac{S_L}{T_L} = \frac{S_R}{T_R}, \\ S_L \leq S_R&:\ \frac{S_R}{T_L} = \frac{S_L}{T_R}, \end{aligned} \tag{3.61}$$

for the two cases that we described previously. From its definition, T_R is obviously non-negative since $T_+ \geq 0$ and $T_- \leq 0$. It is then evident from (3.61) that T_L is non-negative also, since we already know that S_L and S_R are non-negative.

We can also derive, from

$$\Omega = \frac{\partial M}{\partial J}\Big|_{(Q_i,P^i,S)} = \frac{\partial M}{\partial S}\frac{\partial S}{\partial J}\Big|_{(Q_i,P^i,S)}, \tag{3.62}$$

and using either (3.57) or else simply writing S_+ and S_- in terms of S_L and $|J^2 + \Delta|$ by using (3.52), that in the two cases $S_L \geq S_R$ and $S_L \leq S_R$ we have

$$\begin{aligned} S_L \geq S_R&:\ \Omega_+S_+ = \Omega_-S_-, \qquad \frac{\Omega_+}{T_+} = -\frac{\Omega_-}{T_-}, \\ S_L \leq S_R&:\ \Omega_+S_+ = -\Omega_-S_-, \qquad \frac{\Omega_+}{T_+} = \frac{\Omega_-}{T_-}. \end{aligned} \tag{3.63}$$

Note that when $S_L < S_R$, i.e. when $J^2 + \Delta < 0$, the angular velocities of the inner and outer horizons are opposite. Note also that the two cases in (3.63) can be expressed in the single universal formula

$$(S_L + S_R)\Omega_+ = (S_L - S_R)\Omega_-. \tag{3.64}$$

1. Thermodynamics of the left-moving and right-moving sectors

The introduction of the left and right temperatures and entropies suggested the possibility of viewing the black hole as being composed of excitations in left-moving and right-moving sectors in a string or D-brane description, associated with degrees of freedom of a weakly coupled two-dimensional conformal quantum field theory.

The total entropy S_+ of the outer horizon is viewed as the sum of the entropies S_L and S_R of the left-moving and right-moving sectors. It is then natural to expect that there should exist thermodynamic descriptions for these sectors, with first laws of the form[5]

$$\begin{aligned} dE_L &= T_L dS_L + \Omega_L dJ + \Phi_L^i dQ_i + \Psi_{L,i} dP^i, \\ dE_R &= T_R dS_R + \Omega_R dJ + \Phi_R^i dQ_i + \Psi_{R,i} dP^i. \end{aligned} \tag{3.65}$$

For now, we shall focus for simplicity on the regime where $S_L \geq S_R$, i.e. $(J^2 + \Delta) \geq 0$.

Let us first consider processes where $dJ = 0$ and $dQ_i = dP^i = 0$. From the definitions of T_L, T_R, S_L and S_R given in (3.54) and (3.60), it straightforward to see from the first laws

$$dM = T_\pm dS_\pm + \Phi_\pm^i dQ_i + \Psi_{\pm,i} dP^i + \Omega_\pm dJ \tag{3.66}$$

on the outer and inner horizons that we must have

$$E_L = E_R = \frac{1}{2}M. \tag{3.67}$$

In other words, the left-moving and right-moving sectors contribute equally to the mass of the black hole. (This was observed in the case of Kerr-Newman black holes in [23,24].) Dividing the first laws (3.66) by $T_\pm$ respectively and then taking the plus and minus combinations, one finds

[5]The analysis the thermodynamics of asymptotically-flat black holes in terms of left-moving and right-moving degrees of freedom was first addressed in [15] for general STU black holes in five dimensions, and briefly in [16] for four charge STU black holes.

that these match with (3.65) provided that we define the left-moving and right-moving quantities as

$$\Phi^i_L = T_L\left(\frac{\Phi^i_+}{2T_+}+\frac{\Phi^i_-}{2T_-}\right), \qquad \Psi_{L,i} = T_L\left(\frac{\Psi_{+,i}}{2T_+}+\frac{\Psi_{-,i}}{2T_-}\right),$$
$$\Omega_L = T_L\left(\frac{\Omega_+}{2T_+}+\frac{\Omega_-}{2T_-}\right), \qquad \Phi^i_R = T_R\left(\frac{\Phi^i_+}{2T_+}-\frac{\Phi^i_-}{2T_-}\right),$$
$$\Psi_{R,i} = T_R\left(\frac{\Psi_{+,i}}{2T_+}-\frac{\Psi_{-,i}}{2T_-}\right), \qquad \Omega_R = T_R\left(\frac{\Omega_+}{2T_+}-\frac{\Omega_-}{2T_-}\right), \tag{3.68}$$

and so we have the first laws

$$\frac{1}{2}dM = T_L dS_L + \Omega_L dJ + \Phi^i_L dQ_i + \Psi_{L,i} dP^i,$$
$$\frac{1}{2}dM = T_R dS_R + \Omega_R dJ + \Phi^i_R dQ_i + \Psi_{R,i} dP^i \tag{3.69}$$

for the left-moving and right-moving sectors.

In a similar fashion, we can then see that the Smarr relations

$$M = 2T_\pm S_\pm + 2\Omega_\pm J + \Phi^i_\pm Q_i + \Psi_{\pm,i} P^i \tag{3.70}$$

on the outer and inner horizons imply the Smarr relations

$$\frac{1}{2}M = 2T_L S_L + 2\Omega_L J + \Phi^i_L Q_i + \Psi_{L,i} P^i,$$
$$\frac{1}{2}M = 2T_R S_R + 2\Omega_R J + \Phi^i_R Q_i + \Psi_{R,i} P^i \tag{3.71}$$

for the left-moving and right-moving sectors.

It should be noted that, from (3.63) and (3.68), the left-moving angular velocity is in fact zero:

$$\Omega_L = 0, \qquad \Omega_R = \frac{T_R}{T_+}\Omega_+. \tag{3.72}$$

If we now turn to the regime where $S_L < S_R$, we find that the roles of S_L and S_R are exchanged in both the first laws and the Smarr relations for the left-moving and right-moving sectors, so that we have

$$S_L < S_R\colon\ \frac{1}{2}dM = T_L dS_R + \Omega_L dJ + \Phi^i_L dQ_i + \Psi_{L,i} dP^i,$$
$$\frac{1}{2}dM = T_R dS_L + \Omega_R dJ + \Phi^i_R dQ_i + \Psi_{R,i} dP^i, \tag{3.73}$$

$$\frac{1}{2}M = 2T_L S_R + 2\Omega_L J + \Phi^i_L Q_i + \Psi_{L,i} P^i,$$
$$\frac{1}{2}M = 2T_R S_L + 2\Omega_R J + \Phi^i_R Q_i + \Psi_{R,i}. P^i \tag{3.74}$$

Furthermore, it follows from (3.63) and (3.68) that it is now Ω_R, rather than Ω_L, that vanishes. One possible way to make the formulae more uniform for the $S_L < S_R$ regime would be exchange the L and R labels in the definitions of all the intensive thermodynamic variables, $T, \Phi^i, \Psi_i, \Omega$, when $S_L < S_R$. This would have the merit that, with the relabeling, the left-moving angular velocity would vanish in all cases, while still retaining the property that S_L is independent of J in all cases. The left-moving and right-moving first laws and Smarr relations would then take the same forms as in (3.69) and (3.71) for both $S_L \geq S_R$ and $S_L < S_R$, in terms of the relabeled variables.

2. Four-charge STU black holes

The prospects for obtaining an explicit Christodoulou-Ruffini formula for the general 8-charge black hole solutions are not good. The main problem is the F-invariant that appears in the expressions for S_L and S_R in Eq. (3.55), whose evaluation in terms of physical charges and mass appears to be quite intractable [64]. In order to obtain more explicit, concrete expressions, we shall now focus on the specialization to black-hole solutions carrying just four electric charges, which were found in [11].

These black holes are parametrized in terms of the nonextremality parameter $m \geq 0$ (Kerr mass parameter), the "bare" angular momentum a (Kerr rotation parameter) and four boost parameters $\delta_i \geq 0$ ($i = 1, 2, 3, 4$) [11] (see also [65] for compact expressions for the metric and the other fields). In terms of these, the physical mass, charges and angular momentum are given by

$$M = \frac{m}{4}\sum_i \cosh 2\delta_i,$$
$$Q_i = \frac{1}{4} m \sinh 2\delta_i,$$
$$J = ma(\Pi_c - \Pi_s). \tag{3.75}$$

The black hole entropies, associated with the inner and the outer horizon, are given by [11,16]:

$$S_\pm \equiv \frac{A_\pm}{4} = 2\pi m\left[m(\Pi_c+\Pi_s) \pm (\Pi_c-\Pi_s)\sqrt{m^2-a^2}\right] \tag{3.76}$$

$$= 2\pi\left[m^2(\Pi_c+\Pi_s) \pm \sqrt{m^4(\Pi_c-\Pi_s)^2 - J^2}\right]. \tag{3.77}$$

The temperatures $T_\pm$, related to surface gravities $\kappa_\pm$ by $T_\pm = \frac{\kappa_\pm}{2\pi}$, and angular velocities $\Omega_\pm$, which are associated with the inner and out horizon respectively, are given by [16]:

$$\frac{1}{T_\pm}=\frac{2\pi}{\kappa_\pm}$$
$$=\frac{4\pi m}{\sqrt{m^2-a^2}}\left[\pm m(\Pi_c+\Pi_s)+(\Pi_c-\Pi_s)\sqrt{m^2-a^2}\right], \tag{3.78}$$

$$\Omega_\pm=\pm\frac{2\pi a T_\pm}{\sqrt{m^2-a^2}}, \tag{3.79}$$

where

$$\Pi_c=\prod_i\cosh\delta_i, \qquad \Pi_s=\prod_i\sinh\delta_i. \tag{3.80}$$

Note that T_- is negative.[6] From the above expressions one also finds

$$S_\pm=\pm\frac{\sqrt{m^2-a^2}}{2T_\pm}. \tag{3.81}$$

It can easily be verified that the entropies $S_\pm$, temperatures $T_\pm$ and angular velocities $\Omega_\pm$ satisfy equation (3.59) and the $S_L\geq S_R$ equations in (3.63).

The entropies and the inverses of the surface gravities, associated with the outer and inner horizons, have a suggestive form in terms of the left-moving and right-moving entropy and inverse temperature excitations of a weakly coupled 2-dimensional conformal field theory (2D CFT), given in [16]:

$$S_L=\frac{1}{2}(S_++S_-)=2\pi m^2(\Pi_c+\Pi_s),$$
$$S_R=\frac{1}{2}(S_+-S_-)=2\pi m\sqrt{m^2-a^2}(\Pi_c-\Pi_s), \tag{3.82}$$

$$\frac{1}{T_L}=\frac{1}{T_+}+\frac{1}{T_-}=8\pi m(\Pi_c-\Pi_s), \tag{3.83}$$

$$\frac{1}{T_R}=\frac{1}{T_+}-\frac{1}{T_-}=\frac{8\pi m^2}{\sqrt{m^2-a^2}}(\Pi_c+\Pi_s). \tag{3.84}$$

Note that these solutions with four electric charges have $\Delta\geq 0$, as can be seen from (3.53), and so they have $S_L\geq S_R$, as is evident from (3.82). In this suggestive form the central charges $C_{L,R}$ of the left-moving and right-moving sector of the 2D CFT, related to $S_{L,R}$ and $T_{L,R}$ via Cardy relation $S_L=\frac{\pi^2}{3}C_LT_L$ and $S_R=\frac{\pi^2}{3}C_RT_R$, respectively, turn out to be the same and equal to:

$$C_L=\frac{3S_L}{\pi^2T_L}=48m^3(\Pi_c^2-\Pi_s^2)=\frac{3S_R}{\pi^2T_R}=C_R. \tag{3.85}$$

relations:

Again the product of outer and inner horizon entropies is quantized in terms of J and Q_i ($i=1,2,3,4$) only [18]:

$$S_+S_-=S_L^2-S_R^2=4\pi^2\left(J^2+64\prod_i Q_i\right), \tag{3.86}$$

which agrees with the result for Kerr-Newman black hole after equating $Q_1=Q_2=Q_3=Q_4=\frac{1}{4}Q$:

$$S_+S_-=4\pi^2\left(J^2+\frac{1}{4}Q^4\right). \tag{3.87}$$

The main challenge here is to obtain the formulas $M=M(S,J,Q_i)$ and $S=S(M,J,Q_i)$. As an initial step, we observe the solutions for $S_\pm$, due to relation (3.82), satisfy a quadratic equation:

$$S^2-2SS_L+4\pi^2\left(J^2+64\prod_{i=1}^4 Q_i\right)=0, \tag{3.88}$$

where S_L, defined in (3.82), depends on M and Q_i ($i=1,2,3,4$) only. Furthermore as $S_L\geq S_R$, $S_+\geq S_-\geq 0$, where the extremal value $S_+=S_-$ is achieved for $S_R=0$. The extremal the case either corresponds to the BPS solution $\delta_i\to\infty$, $m\sim a\to 0$ and $Q_i=\frac{m}{2}\exp(2\delta_i)$—finite, or to the extremal rotating solution with $m=a$.

Equation (3.88) (which is a special case of (3.57) implies again that T_+ and T_- have opposite signature. By having an explicit expression for S_L we can actually obtain an explicit expression for the temperatures. Namely, we can express S_L in terms of m and Q_i, by employing:

$$4m^2(\Pi_c\pm\Pi_s)=\left(\prod_{i=1}^4\sqrt{\sqrt{m^2+16Q_i^2}+m}\right.$$
$$\left.\pm\prod_{i=1}^4\sqrt{\sqrt{m^2+16Q_i^2}-m}\right), \tag{3.89}$$

and

$$M=\frac{1}{4}\sum_{i=1}^4\sqrt{m^2+16Q_i^2}. \tag{3.90}$$

From (3.88) we obtain:

$$\frac{\partial S_L}{\partial S}=\frac{1}{2}\left[1-\frac{4\pi^2(J^2+64\prod_{i=1}^4 Q_i)}{S^2}\right], \tag{3.91}$$

[6]Note that in [16] the value of T_- was taken to be positive, and equal to the absolute value of the T_- given in (3.79).

Furthermore, employing (3.89) and (3.90) we obtain:

$$\left.\frac{\partial S_L}{\partial S}\right|_{Q_i} = \frac{\partial S_L}{\partial m}\frac{\partial m}{\partial M}\frac{\partial M}{\partial S} = 4\pi m(\Pi_c - \Pi_s)\frac{\partial M}{\partial S}, \tag{3.92}$$

which leads to the explicit expression for the temperature:

$$T = \frac{\partial M}{\partial S} = \frac{1}{8\pi m(\Pi_c - \Pi_s)}\left[1 - \frac{4\pi^2(J^2 + 64\prod_{i=1}^4 Q_i)}{S^2}\right], \tag{3.93}$$

and angular velocity:

$$\Omega = \frac{\partial M}{\partial J} = \frac{1}{m(\Pi_c - \Pi_s)}\frac{\pi J}{S} = \frac{a\pi}{S}. \tag{3.94}$$

These expressions are in agreement (3.59) and (3.63), and explicitly determine $T_+ > 0, T_- < 0$ and $\Omega_\pm$, in agreement with direct calculations at the horizons (3.79).

The technical difficulty in obtaining an explicit Christodoulou-Ruffini mass expression is due to the fact that an explicit expression for S_L in terms of M and Q_i is cumbersome, in general. However, we succeeded in the following special cases.

3. Pairwise-equal charges

The four-charge black-hole solutions simplify considerably in the special case of pair-wise equal charges (see, e.g., Ref. [65]) $Q_1 = Q_3$ and $Q_2 = Q_4$ where (3.88) can be solved explicitly for M:

$$M^2 = \frac{\pi}{4S}\left[\left(\frac{S}{\pi} + 16Q_1^2\right)\left(\frac{S}{\pi} + 16Q_2^2\right) + 4J^2\right]$$
$$\cdot\sqrt{\frac{\pi}{S_\pm}q_1^2})\left(\sqrt{\frac{S_\pm}{\pi}} + \sqrt{\frac{\pi}{S_\pm}q_2^2}\right) + \frac{\pi J^2}{S_\pm}. \tag{3.95}$$

Furthermore (3.95) and (3.86) implies:

$$M^2 = \frac{S_+}{4\pi} + \frac{S_-}{4\pi} + 4Q_1^2 + 4Q_2^2. \tag{3.96}$$

For $Q_2 = 0$ the result reduces to the example of rotating dilatonic black hole with the dilaton coupling $a = 1$.[7] The result reduces to the Kerr-Newman (or Reissner-Nordström) black hole expression when $Q_1 = Q_2 = \frac{1}{4}Q$.

[7]Note, however, that when the black hole is rotating, an axion in the STU supergravity is also turned on when Q_1 and/or Q_2 is nonzero (except in the case $Q_1 = Q_2$).

It becomes straightforward that the differentiation of (3.95) with respect to $S_\pm$ (with J and $Q_{1,2}$ fixed), produces the expected expressions for $T_\pm$, including the sign.

4. Three equal nonzero charges

It turns out that for the example of three equal nonzero charges, i.e. $Q_1 = Q_2 = Q_3 = q$ and $Q_4 = 0$, which corresponds to the rotating dilatonic black hole with the dilaton coupling $a = \frac{1}{\sqrt{3}}$, one can again obtain an explicit expression for the Christodoulou-Ruffini mass:

$$M^2 = \frac{\left[16q^2 + \sqrt{64q^4 + (\frac{S_\pm}{\pi} + \frac{4\pi}{S_\pm}J^2)^2}\right]^2}{32q^2 + 4\sqrt{64q^4 + (\frac{S_\pm}{\pi} + \frac{4\pi}{S_\pm}J^2)^2}}. \tag{3.97}$$

(As in the pairwise-equal charge case above, here too an axion is also turned on if the black hole is rotating.)

5. One nonzero charge

We also note in the case of only one nonzero charge (say, $Q_1 = q = \frac{1}{4}m\sinh 2\delta$), which corresponds to the rotating dilatonic black hole with the dilaton coupling $a = \sqrt{3}$, the Christodoulou-Ruffini mass can be expressed in the following form:

$$M^2 = \frac{S_L}{8\pi}\left(3\cosh\delta + \frac{1}{\cosh\delta} + y\right), \tag{3.98}$$

where $y = \frac{32\pi}{S_L}q^2$, $S_L = \frac{1}{2}(S_\pm + \frac{4\pi^2 J^2}{S_\pm})$, and $\cosh\delta$ is a solution of the cubic equation $\cosh^3\delta - \cosh\delta - y = 0$:

$$\cosh\delta = A^{\frac{1}{3}} + \frac{1}{3A^{\frac{1}{3}}}, \qquad A = \frac{y}{2} + \sqrt{\frac{y^2}{4} - \frac{1}{27}}. \tag{3.99}$$

6. Dyonic Kaluza-Klein black hole

In all the explicit STU supergravity black holes we have discussed so far, each of the four field strengths carries a charge of a single complexion (which could be pure electric or pure magnetic). The most general possibility is where each field strength carries independent electric and magnetic charges, as described in the general 8-charge case that was constructed by Chow and Compère. Although explicit, these general solutions are rather unwieldy. Here, we discuss a much simpler case, which is still rather nontrivial, and that goes beyond what we have explicitly presented so far. We consider the case where just one of the four field strengths is nonvanishing, but it carries independent electric and magnetic charges. For simplicity we shall restrict attention to the case of static black holes.

The Lagrangian (in the normalization we are using for the STU supergravities) is given by[8]

$$\mathcal{L}_4 = \sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}F^2\right], \qquad (3.100)$$

and a convenient way [66] to present the static dyonic black hole solutions is

$$\begin{aligned} ds_4^2 &= -(H_1H_2)^{-\frac{1}{2}} f dt^2 + (H_1H_2)^{\frac{1}{2}}(f^{-1}dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\varphi^2)), \\ \phi &= \frac{\sqrt{3}}{2}\log\frac{H_2}{H_1}, \qquad f = 1 - \frac{2\mu}{r}, \\ A &= \sqrt{2}\left[\frac{(1-\beta_1 f)}{\sqrt{\beta_1\gamma_2}H_1}dt + \frac{2\mu\sqrt{\beta_2\gamma_1}}{\gamma_2}\cos\theta d\varphi\right], \\ H_1 &= \gamma_1^{-1}(1 - 2\beta_1 f + \beta_1\beta_2 f^2), \\ H_2 &= \gamma_2^{-1}(1 - 2\beta_2 f + \beta_1\beta_2 f^2), \\ \gamma_1 &= 1 - 2\beta_1 + \beta_1\beta_2, \\ \gamma_2 &= 1 - 2\beta_2 + \beta_1\beta_2, \end{aligned} \qquad (3.101)$$

where m, β_1 and β_2 are constants that parametrize the physical mass M, electric charge Q and magnetic charge P, with

$$\begin{aligned} M &= \frac{(1-\beta_1)(1-\beta_2)(1-\beta_1\beta_2)\mu}{\gamma_1\gamma_2}, \\ Q &= \frac{\sqrt{\beta_1\gamma_2}\mu}{\sqrt{2}\gamma_1}, \qquad P = \frac{\sqrt{\beta_2\gamma_1}\mu}{\sqrt{2}\gamma_2}. \end{aligned} \qquad (3.102)$$

A necessary condition for regularity of the black hole is $0 \le \beta_i \le 1$. The entropy of the outer horizon, located at $r = 2\mu$, is given by

$$S_+ = \frac{4\pi\mu^2}{\sqrt{\gamma_1\gamma_2}}, \qquad (3.103)$$

whilst the entropy of the inner horizon, located at $r = 0$, is given by

$$S_- = \frac{4\pi\beta_1\beta_2\mu^2}{\sqrt{\gamma_1\gamma_2}}. \qquad (3.104)$$

The product of the entropies on the outer and inner horizons is given by

$$S_+S_- = 64\pi^2P^2Q^2. \qquad (3.105)$$

Note that S_- vanishes if Q or P vanishes. Note also that the dyonic black hole is an example where the invariant Δ, defined in (3.53), is negative. Of course since the solutions we are considering here are static, $(J^2 + \Delta)$ is negative too, and so we are in the regime where $S_L < S_R$ for these black holes, and in fact we have

$$\begin{aligned} S_L &= \frac{2\pi\mu^2(1-\beta_1\beta_2)}{\sqrt{\gamma_1\gamma_2}}, \\ S_R &= \frac{2\pi\mu^2(1+\beta_1\beta_2)}{\sqrt{\gamma_1\gamma_2}}. \end{aligned} \qquad (3.106)$$

One can straightforwardly calculate the temperatures on the outer and inner horizons, finding as usual that the temperature T_+ is positive and T_- is negative. The left-moving and right-moving temperatures, defined by (3.60), then turn out to be

$$\begin{aligned} T_L &= \frac{\sqrt{\gamma_1\gamma_2}}{8\pi\mu(1-\beta_1\beta_2)}, \\ T_R &= \frac{\sqrt{\gamma_1\gamma_2}}{8\pi\mu(1+\beta_1\beta_2)}. \end{aligned} \qquad (3.107)$$

These are both non-negative.

A special case is when the black hole is extremal, which is achieved in this parameterization by taking a limit in which m goes to zero and the β_i go to 1. The result is that in the extremal case

$$M_{\rm ext} = (Q^{\frac{2}{3}} + P^{\frac{2}{3}})^{\frac{3}{2}}, \qquad S_{\rm ext} = 8\pi QP. \qquad (3.108)$$

By a straightforward, although somewhat intricate, procedure, one can eliminate the metric parameters m, β_1 and β_2 from the four equations (3.102) and (3.103) that define the physical mass, charges and entropy, thereby arriving at a Christodoulou-Ruffini type formula relating these quantities. If we first define

$$\tilde{S} = \frac{S}{\pi}, \qquad (3.109)$$

we find that they and M obey the relation $W(\tilde{S}, M, Q, P) = 0$ where

[8]This Lagrangian can also be obtained by means of a circle reduction of five-dimensional pure Einstein gravity. For this reason, the black hole solutions are sometimes referred to as Kaluza-Klein dyons.

$$
\begin{aligned}
W(\tilde{S},M,Q,P) = 4096M^8 &+ \frac{16M^6(P^2+Q^2)(P^2Q^2-8PQ\tilde{S}+4\tilde{S}^2)(P^2Q^2+8PQ\tilde{S}+4\tilde{S}^2)}{P^2Q^2\tilde{S}^2} \\
&+ \frac{M^4}{16P^2Q^2\tilde{S}^4}(P^8Q^8 - 48P^8Q^4\tilde{S}^2 - 400P^6Q^6\tilde{S}^2 + 1152P^6Q^2\tilde{S}^4 - 48P^4Q^8\tilde{S}^2 \\
&- 2208P^4Q^4\tilde{S}^4 - 768P^4\tilde{S}^6 + 1152P^2Q^6\tilde{S}^4 - 6400P^2Q^2\tilde{S}^6 - 768Q^4\tilde{S}^6 + 256\tilde{S}^8) \\
&- \frac{M^2(P^2+Q^2)}{64P^2Q^2\tilde{S}^4}(5P^8Q^8 - 12P^8Q^4\tilde{S}^2 + 40P^6Q^6\tilde{S}^2 + 160P^6Q^2\tilde{S}^4 - 12P^4Q^8\tilde{S}^2 \\
&- 352P^4Q^4\tilde{S}^4 - 192P^4\tilde{S}^6 + 160P^2Q^6\tilde{S}^4 + 640P^2Q^2\tilde{S}^6 - 192Q^4\tilde{S}^6 + 1280\tilde{S}^8) \\
&- \frac{(P^4+4\tilde{S}^2)^2(Q^4+4\tilde{S}^2)^2(P^2Q^2-4\tilde{S}^4)^2}{4096P^2Q^2\tilde{S}^6}. \qquad (3.110)
\end{aligned}
$$

This defines a multinomial of 12th order in $\tilde{S}$, and W is invariant under the inversion transformation $\tilde{S} \to Q^2P^2/(4\tilde{S})$. Note that because M is invariant under the inversion, the coefficients of each separate power of M in (3.110) are invariant under the inversion.

E. Five-dimensional STU supergravity

Here, we consider black hole solutions in five-dimensional STU supergravity. General solutions with mass M, two angular momenta J_ϕ and J_ψ, and three charges Q_i were constructed in [12] by employing solution generating techniques. We use principally the conventions of [15], except that we shall use the labels $\uparrow$ and $\downarrow$ to denote the sum and difference combinations of the angular momenta and angular velocities associated with the ϕ and ψ azimuthal coordinates, reserving L and R to denote the combinations of inner and outer horizon quantities, analogous to the definitions used previously for the four-dimensional STU black holes. The physical mass, charges and angular momenta are given by [15]

$$
\begin{aligned}
&M = m\sum_{i=1}^{i=3}\cosh 2\delta_i, \qquad Q_i = m\sinh 2\delta_i, \\
&J_\downarrow = m(l_1-l_2)(\Pi_c+\Pi_s), \quad J_\uparrow = m(l_1+l_2)(\Pi_c-\Pi_s), \qquad (3.111)
\end{aligned}
$$

where $\Pi_c = \prod_{i=1}^{i=3}\cosh\delta_i$, $\Pi_s = \prod_{i=1}^{i=3}\sinh\delta_i$, and $J_\downarrow = \frac{1}{2}(J_\phi - J_\psi)$, $J_\uparrow = \frac{1}{2}(J_\phi + J_\psi)$. Here the five-dimensional Newton constant is taken to be $G_5 = \frac{\pi}{4}$. We shall, without loss of generality, take the rotation parameters l_1 and l_2 and the charge boost parameters δ_i to be non-negative in what follows.

These black holes have many analogous properties to those of the four-dimensional STU black holes, except, of course, that they can carry only electric charges but not magnetic. In particular, they have two horizons, with the inner and outer horizon entropies expressed as [15]:

$$S_+ = S_L + S_R, \qquad S_- = S_L - S_R, \qquad (3.112)$$

where

$$S_L = 2\pi\sqrt{2m^3(\Pi_c+\Pi_s)^2 - J_\downarrow^2}, \qquad (3.113)$$

$$S_R = 2\pi\sqrt{2m^3(\Pi_c-\Pi_s)^2 - J_\uparrow^2}. \qquad (3.114)$$

The product of the inner and outer horizon entropies is again quantized as:

$$S_+S_- = 4\pi^2\left(J_\phi J_\psi + \prod_{i=1}^{i=3}Q_i\right) = 4\pi^2\left(J_\uparrow^2 - J_\downarrow^2 + \prod_{i=1}^{i=3}Q_i\right). \qquad (3.115)$$

Note that as in the case of the four-dimensional STU black holes, here it would in general be necessary to use an absolute value in the expression for S_- in (3.112), and on the right-hand side of (3.115), since S_- must be non-negative while S_L and S_R, which are both non-negative, could obey either $S_L > S_R$ or $S_L < S_R$ depending on the relative values of the charge and angular momentum parameters. However, our non-negativity assumptions stated above for the charge and rotation parameters imply that in fact $S_L \geq S_R$ in this case, and so we can omit the absolute value in the expression for S_-, as we have done in (3.112), and in (3.115).

From the above expressions it follows that S (either S_+ or S_-) again obeys a quadratic equation,

$$S^2 - 2SS_L + 4\pi^2\left(J_\uparrow^2 - J_\downarrow^2 + \prod_{i=1}^{i=3}Q_i\right) = 0. \qquad (3.116)$$

Furthermore one can analogously derive the general result that T_+ and T_- have opposite signs, with:

$$S_+T_+ + S_-T_- = 0, \qquad (3.117)$$

and similarly

$$\frac{\Omega_+^\uparrow}{T_+} + \frac{\Omega_-^\uparrow}{T_-} = 0, \qquad \frac{\Omega_+^\downarrow}{T_+} - \frac{\Omega_-^\downarrow}{T_-} = 0, \tag{3.118}$$

where $\Omega_\pm^\uparrow = \frac{1}{2}(\Omega_\pm^\phi + \Omega_\pm^\psi)$ and $\Omega_\pm^\downarrow = \frac{1}{2}(\Omega_\pm^\phi - \Omega_\pm^\psi)$. (The relative signs between the terms in these two equations in (3.118) are the opposite of those given in [15], because in that paper κ_- was taken to be positive.)

The black holes obey the usual first laws on the outer and inner horizons:

$$dM = T_\pm dS_\pm + \Omega_\pm^\uparrow dJ_\uparrow + \Omega_\pm^\downarrow dJ_\downarrow + \Phi_\pm^i dQ_i. \tag{3.119}$$

As in the four-dimensional case, the calculation of scattering amplitudes in the black-hole background shows that they factorize into left and right sectors with Boltzman factors corresponding to temperatures T_L and T_R given by (3.60) [15]. Together with the normalization of S_L and S_R, such that $S_+ = S_L + S_R$ in accordance with the interpretation of the entropy as the sum of left-moving and right-moving contributions, one can then establish by rewriting the first laws $dM = T_\pm dS_\pm + \cdots$ in terms of left and right-moving quantities that $\frac{1}{2}dM = T_L dS_L + \cdots$ and $\frac{1}{2}dM = T_R dS_R + \cdots$, and so each of the sectors contributes one half the total mass of the black hole. Matching the first laws for arbitrary variations of the parameters then allows one to read off the appropriate definitions of the left-moving and right-moving angular momenta and electric potentials. Thus one finds the first laws

$$\begin{aligned} \frac{1}{2}dM &= T_L dS_L + \Omega_L^\uparrow dJ_\uparrow + \Omega_L^\downarrow dJ_\downarrow + \Phi_L^i dQ_i, \\ \frac{1}{2}dM &= T_R dS_R + \Omega_R^\uparrow dJ_\uparrow + \Omega_R^\downarrow dJ_\downarrow + \Phi_R^i dQ_i, \end{aligned} \tag{3.120}$$

where

$$\begin{aligned} \Phi_L^i &= T_L\left(\frac{\Phi_+^i}{2T_+} + \frac{\Phi_-^i}{2T_-}\right), \qquad & \Omega_L^\uparrow &= T_L\left(\frac{\Omega_+^\uparrow}{2T_+} + \frac{\Omega_-^\uparrow}{2T_-}\right), \\ \Omega_L^\downarrow &= T_L\left(\frac{\Omega_+^\downarrow}{2T_+} + \frac{\Omega_-^\downarrow}{2T_-}\right), \qquad & \Phi_R^i &= T_R\left(\frac{\Phi_+^i}{2T_+} - \frac{\Phi_-^i}{2T_-}\right), \\ \Omega_R^\uparrow &= T_R\left(\frac{\Omega_+^\uparrow}{2T_+} - \frac{\Omega_-^\uparrow}{2T_-}\right), \qquad & \Omega_R^\downarrow &= T_R\left(\frac{\Omega_+^\downarrow}{2T_+} - \frac{\Omega_-^\downarrow}{2T_-}\right). \end{aligned} \tag{3.121}$$

In view of the relations (3.118), one finds

$$\Omega_L^\uparrow = 0, \qquad \Omega_L^\downarrow = \frac{T_L}{T_+}\Omega_+^\downarrow; \qquad \Omega_R^\uparrow = \frac{T_R}{T_+}\Omega_+^\uparrow, \qquad \Omega_R^\downarrow = 0. \tag{3.122}$$

Thus we see that the angular momentum $J_\uparrow$ and the associated angular velocity $\Omega^\uparrow$ enters only in the right-moving first law and in S_R, while the angular momentum $J_\downarrow$ and associated angular velocity $\Omega^\downarrow$ enters only in the left-moving first law and in S_L. Note that as in four dimensions, T_L and T_R are both non-negative.

The Smarr formulas for the left-moving and right-moving sectors agree with the ones derived in [15]:

$$\begin{aligned} \frac{1}{2}M &= \frac{3}{2}T_L S_L + \frac{3}{2}\Omega_L^\downarrow J_\downarrow + \Phi_L^i Q, \\ \frac{1}{2}M &= \frac{3}{2}T_R S_R + \frac{3}{2}\Omega_R^\uparrow J_\uparrow + \Phi_R^i Q. \end{aligned} \tag{3.123}$$

The expression for the Christodoulou-Ruffini formula in terms solely of the conserved charges, angular momenta, mass and entropy are too cumbersome to present explicitly. Even in the case of three equal charges, the mass is determined by a cubic equation.

F. Einstein-Maxwell-dilaton black holes

There exists a more general class of black holes in the theory of Einstein-Maxwell gravity with an additional dilatonic scalar field, which is coupled to the Maxwell field with a dimensionless coupling constant a, with the Lagrangian

$$\mathcal{L} = \sqrt{-g}(R - 2(\partial\phi)^2 - e^{-2a\phi}F^2), \tag{3.124}$$

The electrically-charged black-hole solution can be written as [67–69]

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^b dt^2 \\ &\quad + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-b} dr^2 \\ &\quad + r^2\left(1-\frac{r_-}{r}\right)^{1-b} d\Omega^2, \\ e^{2a\phi} &= \left(1-\frac{r_-}{r}\right)^{1-b}, \\ A &= \frac{Q}{r}dt, \end{aligned} \tag{3.125}$$

where

$$b = \frac{1-a^2}{1+a^2}. \tag{3.126}$$

The relevant thermodynamic quantities for these black holes in this theory are given by

$$S=\pi r_+^2\left(1-\frac{r_-}{r_+}\right)^{1-b},\quad T=\frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{b},$$
$$Q=\sqrt{\frac{r_+r_-}{1+a^2}},\ M=\frac{1}{2}(r_++br_-),\ \Phi=\frac{1}{\sqrt{1+a^2}}\sqrt{\frac{r_-}{r_+}},\tag{3.127}$$

where r_+ is the radius of the outer horizon, and r_- is a singular surface unless $a=0$. Since by assumption $r_+\geq r_-$, it follows that

$$M>\frac{|Q|}{\sqrt{1+a^2}}.\tag{3.128}$$

This is consistent with the BPS bound derived in [70] using "fake supersymmetry."

The Smarr relations continue to hold and the Gibbs free energy is again given by

$$G=TS=\frac{1}{4}(r_+-r_-).\tag{3.129}$$

The coordinates $\{r_+,r_-\}$ are now related to the coordinates $\{T,\Phi\}$ by

$$r_+=\frac{1}{4\pi T}(1-(1+a^2)\Phi^2)^b\tag{3.130}$$

and

$$r_-=\frac{(1+a^2)\Phi^2}{4\pi T}(1-(1+a^2)\Phi^2)^b.\tag{3.131}$$

Thus the Gibbs energy as a function of $\{T,\Phi\}$ is given by

$$G=\frac{1}{16\pi T}(1-(1+a^2)\Phi^2)^{1+b}.\tag{3.132}$$

As discussed in Sec. II B, the Ricci scalar of the Helmholtz free energy metric $ds^2(F)=-dSdT+d\Phi dQ$ will be singular on the Davies curve where the heat capacity at constant charge changes sign. It is easiest to use r_+ and r_- as the coordinate variables in this calculation, which gives

$$R=\frac{4(1+a^2)^2r_+}{[(1+a^2)r_+-(3-a^2)r_-]^2}.\tag{3.133}$$

Thus the Davies curve is given by

$$\frac{r_-}{r_+}=\frac{1+a^2}{3-a^2},\tag{3.134}$$

which implies

$$\frac{Q^2}{M^2}=\frac{3-a^2}{(2-a^2)^2}.\tag{3.135}$$

Since we must have $r_-<r_+$, a solution for (3.134) exists only for $a^2<1$. The spinodal curve thus projects down to the parabola in the S–Q plane given by

$$S=(3-a^2)^{\frac{1-a^2}{1+a^2}}2^{\frac{2a^2}{1+a^2}}(1-a^2)^{\frac{2a^2}{1+a^2}}\pi Q^2.\tag{3.136}$$

From (3.127), one can in general solve for r_+ and r_- in terms of M and Q, obtaining

$$r_+=M+\sqrt{M^2-(1-a^2)Q^2},$$
$$r_-=\frac{1}{b}\left(M-\sqrt{M^2-(1-a^2)Q^2}\right),\tag{3.137}$$

and hence express S in terms of M and Q [71]:

$$\frac{S}{\pi}=\left(M+\sqrt{M^2-(1-a^2)Q^2}\right)^2$$
$$\times\left(1-\frac{(1+a^2)Q^2}{(M+\sqrt{M^2-(1-a^2)Q^2})^2}\right)^{\frac{2a^2}{1+a^2}}.\tag{3.138}$$

If $a^2>0$ the entropy vanishes at extremality, namely $r_+=r_-$ and hence $|Q|=\sqrt{1+a^2}M$. Then $r=r_+=r_-$ is a point-like singularity and there is no inner horizon. One can also, in general, express the entropy in terms of r_+ and Q, using

$$\left(\frac{S}{\pi r_+^2}\right)^{\frac{1+a^2}{2a^2}}=1-\frac{(1+a^2)Q^2}{r_+^2}.\tag{3.139}$$

Particular cases include the following, which also arise as special cases of STU Black holes:

(i) $a=0$ is the Reissner-Nordström case.
(ii) $a^2=\frac{1}{3}$ is a reduction of Einstein-Maxwell in 5 dimensions.
(iii) $a^2=1$ is the so-called string case. We have

$$\frac{S}{\pi}=4M^2-2Q^2,\qquad M=\frac{1}{2}\sqrt{\frac{S}{\pi}+2Q^2}.\tag{3.140}$$

The spinodal curve coincides with the Q-axis and the Gibbs surface is *nowhere* convex. It is a hyperbolic paraboloid for which the Ruppeiner metric is flat [71]. The temperature is given by

$$T=\frac{1}{4\pi\sqrt{\frac{S}{\pi}+2Q^2}}=\frac{1}{8\pi M},\tag{3.141}$$

and is always positive. It goes to a nonvanishing value at extremality. The heat capacity at constant charge is given by

$$C_Q = -\frac{1}{8\pi^2(\frac{S}{\pi}+2Q^2)^{\frac{3}{2}}} = -\frac{1}{64\pi^2 M^3} \qquad (3.142)$$

and is always negative, and is also nonvanishing at extremality [68].

(iv) $a^2 = 3$ is the Kaluza-Klein black hole.

G. Two-field dilatonic black holes

Here we review a class of theories [72] which are similar to the Einstein-Maxwell-dilaton (EMD) theory of the previous subsection, but with two field strengths rather than just one. The Lagrangian, in an arbitrary dimension D, is given by

$$\mathcal{L}_D = \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{a_1\phi}F_1^2 - \frac{1}{4}e^{a_2\phi}F_2^2\right). \qquad (3.143)$$

The advantage of considering this extension of EMD theory is that by choosing the coupling constants a_1 and a_2 appropriately, we can find general classes of static black hole solutions with two horizons, and one can study the thermodynamic properties at both the outer and inner horizon.

If we turn on both the gauge fields A_i independently, the theory for general (a_1, a_2) does not admit explicit black hole solutions. We shall determine the condition on (a_1, a_2) so that the system will give such explicit solutions. It is advantageous for later purpose that we reparameterize these dilaton coupling constants as

$$a_1^2 = \frac{4}{N_1} - \frac{2(D-3)}{D-2}, \quad a_2^2 = \frac{4}{N_2} - \frac{2(D-3)}{D-2}. \qquad (3.144)$$

(Note that N_1 and N_2 are not necessarily integers.) For the a_i to be real, we must have

$$0 < N_i \le \frac{2(D-2)}{D-3}. \qquad (3.145)$$

(If both N_i are outside the range, the Lagrangian could still be made real by sending $\phi \to \mathrm{i}\phi$, corresponding to having a ghostlike dilaton. We shall not consider this possibility here.)

Here we shall consider the case where a_1 and a_2 obey the constraint

$$a_1 a_2 = -\frac{2(D-3)}{D-2}, \qquad (3.146)$$

which implies the identities

$$N_1 a_1 + N_2 a_2 = 0, \qquad N_1 + N_2 = \frac{2(D-2)}{D-3}. \qquad (3.147)$$

It follows from the second identity in (3.147) that both N_i can take integer values only in four and five dimensions, with $N_1 + N_2 = 3$ and 4 respectively. The solutions with positive integers for N_i are known black holes in relevant supergravities.

With a_1 and a_2 obeying (3.146), one can find black hole solutions, given by [72]

$$\begin{aligned} ds^2 &= -(H_1^{N_1}H_2^{N_2})^{-\frac{(D-3)}{D-2}} f dt^2 \\ &\quad + (H_1^{N_1}H_2^{N_2})^{\frac{1}{D-2}}(f^{-1}dr^2 + r^2 d\Omega_{D-2}^2) \\ A_1 &= \frac{\sqrt{N_1}c_1}{s_1}H_1^{-1}dt, \qquad A_2 = \frac{\sqrt{N_2}c_2}{s_2}H_2^{-1}dt, \\ \phi &= \frac{1}{2}N_1 a_1 \log H_1 + \frac{1}{2}N_2 a_2 \log H_2, \qquad f = 1 - \frac{\mu}{r^{D-3}}, \\ H_1 &= 1 + \frac{\mu s_1^2}{r^{D-3}}, \qquad H_2 = 1 + \frac{\mu s_2^2}{r^{D-3}}, \end{aligned} \qquad (3.148)$$

where we are using the standard notation where $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$. The mass and charges are given by

$$\begin{aligned} M &= \frac{(D-2)\mu\omega_{D-2}}{16\pi}\left(1 + \frac{D-3}{D-2}(N_1 s_1^2 + N_2 s_2^2)\right), \\ Q_i &= \frac{(D-3)\mu\omega_{D-2}}{16\pi}\sqrt{N_i}c_i s_i, \end{aligned} \qquad (3.149)$$

where ω_{D-2} is the volume of the unit $(D-2)$-sphere. The outer horizon is located at $r_0 = \mu^{1/(D-3)}$, and the entropy is given by

$$S = S_+ \equiv \frac{1}{4}\omega_{D-2}\mu^{\frac{D-2}{D-3}}c_1^{N_1}c_2^{N_2}. \qquad (3.150)$$

The inner horizon is located at $r = 0$, and we have

$$S_- \equiv \frac{1}{4}\omega_{D-2}\mu^{\frac{D-2}{D-3}}s_1^{N_1}s_2^{N_2}. \qquad (3.151)$$

Multiplying the two entropies gives the product formula

$$S_+ S_- = S_{\rm ext}^2, \qquad (3.152)$$

where

$$S_{\rm ext} = 4^{\frac{D-1}{D-3}}\left(\frac{\pi}{D-3}\right)^{\frac{D-2}{D-3}}\omega_{D-2}^{-\frac{1}{D-3}}\left(\frac{Q_1}{\sqrt{N_1}}\right)^{\frac{1}{2}N_1}\left(\frac{Q_2}{\sqrt{N_2}}\right)^{\frac{1}{2}N_2}. \qquad (3.153)$$

Thus the entropy product is independent of the mass.

There exists an extremal limit in which we send $\mu \to 0$ while keeping the charges Q_i nonvanishing. In this limit, the inner and outer horizons coalesce and the near-horizon geometry becomes $\mathrm{AdS}_{D-2} \times S^2$. The mass now depend only on the charges, and is given by

$$M_{\mathrm{ext}} = \sqrt{N_1} Q_1 + \sqrt{N_2} Q_2. \tag{3.154}$$

It is useful to define

$$\begin{aligned} \tilde{M} &= \frac{16\pi}{(D-2)\omega_{D-2}} M, \\ \tilde{Q}_i &= \frac{8\pi}{(D-3)\omega_{D-2}\sqrt{N_i}} Q_i, \\ \tilde{S} &= \frac{1}{\omega_{D-2}} S, \end{aligned} \tag{3.155}$$

and then we have

$$s_i^2 = \frac{\sqrt{\tilde{Q}_i^2 + 16\mu^2}}{2\mu} - \frac{1}{2}. \tag{3.156}$$

Some specific examples are as follows:
Case 1: $D = 4$, $N_1 = N_2 = 2$:

$$\tilde{M}^2 - \frac{4(\tilde{Q}_1^2 + \tilde{S})(Q_2^2 + \tilde{S})}{\tilde{S}} = 0. \tag{3.157}$$

We can define

$$\hat{S} = \frac{\tilde{S}}{\tilde{Q}_1 \tilde{Q}_2}, \tag{3.158}$$

and then

$$\tilde{M}^2 - 4(\tilde{Q}_1^2 + \tilde{Q}_2^2) - 4\tilde{Q}_1\tilde{Q}_2\left(\hat{S} + \frac{1}{\hat{S}}\right) = 0. \tag{3.159}$$

Case 2: $D = 4$, $N_1 = 1$, $N_2 = 3$:

$$\begin{aligned} \tilde{M}^6 &+ \frac{\tilde{M}^4(\tilde{S}^4 - 3\tilde{S}^2\tilde{Q}_1^4 - 15\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^2 + \tilde{Q}_1^2\tilde{Q}_2^6)}{\tilde{S}^2\tilde{Q}_1^2} \\ &- \frac{(4\tilde{S}^4 + \tilde{S}^2\tilde{Q}_1^4 - 6\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^2 - 3\tilde{S}^2\tilde{Q}_2^4 + 4\tilde{Q}_1^2\tilde{Q}_2^6)^2}{\tilde{S}^4\tilde{Q}_1^2} \\ &- \frac{\tilde{M}^2}{\tilde{S}^2\tilde{Q}_1^2}(20\tilde{S}^4\tilde{Q}_1^2 + 12\tilde{S}^4\tilde{Q}_2^2 - 3\tilde{S}^2\tilde{Q}_1^6 - 3\tilde{S}^2\tilde{Q}_1^4\tilde{Q}_2^2 \\ &- 57\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^4 - \tilde{S}^2\tilde{Q}_2^6 + 20\tilde{Q}_1^4\tilde{Q}_2^6 + 12\tilde{Q}_1^2\tilde{Q}_2^8) = 0. \end{aligned} \tag{3.160}$$

Case 3: $D = 5$, $N_1 = 1$, $N_2 = 2$:

$$\begin{aligned} 0 = \tilde{M}^4 &+ \frac{\tilde{M}^3(4\tilde{S}^4 + \tilde{Q}_1^2\tilde{Q}_2^4)}{3\tilde{S}^2\tilde{Q}_1^2} - \frac{4\tilde{M}^2(8\tilde{Q}_1^4 + 20\tilde{Q}_1^2\tilde{Q}_2^2 - \tilde{Q}_2^4)}{9\tilde{Q}_1^2} - \frac{8\tilde{M}(2\tilde{Q}_1^2 + \tilde{Q}_2^2)(4\tilde{S}^4 + \tilde{Q}_1^2\tilde{Q}_2^4)}{3\tilde{S}^2\tilde{Q}_1^2} \\ &- \frac{4(432\tilde{S}^8 - 64\tilde{S}^4{\tilde{Q}_1}^6 + 192\tilde{S}^4\tilde{Q}_1^4{\tilde{Q}_2}^2 + 24\tilde{S}^4\tilde{Q}_1^2\tilde{Q}_2^4 + 64\tilde{S}^4\tilde{Q}_2^6 + 27\tilde{Q}_1^4\tilde{Q}_2^8)}{81\tilde{S}^4\tilde{Q}_1^2}. \end{aligned} \tag{3.161}$$

Case 4: *General* D, *but with* $N_1 = N_2 = (D-2)/(D-3)$

These cases lie, in general, outside the realm of supergravity theories. We have

$$\tilde{M}^2 - 4(\tilde{Q}_1^2 + \tilde{Q}_2^2) - (16^{\frac{1}{D-2}}\tilde{Q}_1^2\tilde{Q}_2^2\tilde{S}^{\frac{2}{D-2}-2} + 16^{\frac{D-3}{D-2}}\tilde{S}^{2-\frac{2}{D-2}}) = 0. \tag{3.162}$$

Entropy super-additivity is difficult to prove in general, but we can at least look at the case of extremal black holes, for which

$$S_{\mathrm{ext}} \sim \sqrt{Q_1^{N_1} Q_2^{N_2}}. \tag{3.163}$$

It seems that super-additivity will be satisfied if $N_1 + N_2 \geq 2$, and in fact, from (3.147), we have $N_1 + N_2 >$ in all dimensions.

IV. ENTROPY PRODUCT AND INVERSION LAWS

It is well known from many examples that if a black hole has two horizons then the product of the areas, or equivalently entropies, of these horizons is equal to an expression written purely in terms of the conserved charges and angular momenta [18,20]. Thus we may write

$$S_+ S_- = K(\mathbf{Q}, \mathbf{J}), \tag{4.1}$$

where $\mathbf{Q}$ represents the complete set of charges carried by the black hole, and $\mathbf{J}$ represents the set of angular momenta. [Generalizations arise also if there are more than two horizons or "pseudohorizons" (see, e.g., Ref. [18].)] We also saw various examples in the previous section where there is a Christodoulou-Ruffini formula relating the entropy to the mass, charges and angular momenta, of the form

$$W(S, M, \mathbf{Q}, \mathbf{J}) = 0, \tag{4.2}$$

for which there was a symmetry under a certain inversion of the entropy, $S \to S' \sim 1/S$.

Here, we make some observations about the relation between these properties of the black hole entropy. First, we note that when one derives a Christodoulou-Ruffini formula of the form (4.2), one uses properties of the metric functions that determine the horizon radius in terms of the metric parameters, and hence implicitly they determine the horizon radius in terms of M, $\mathbf{Q}$ and $\mathbf{J}$. This means that when one arrives at the Christodoulou-Ruffini relation (4.2), the expression will necessarily be valid not only when $S = S_+$, but also when instead $S = S_-$. Since S_+ and S_- are related by the product formula (4.1), this means that if S, the entropy of the outer horizon, obeys (4.2) then we will also have

$$W\left(\frac{K(\mathbf{Q},\mathbf{J})}{S},\mathbf{Q},\mathbf{J}\right) = 0. \tag{4.3}$$

In other words, the Christodoulou-Ruffini formula will be invariant[9] under the inversion symmetry

$$S \to \frac{K(\mathbf{Q},\mathbf{J})}{S}, \tag{4.4}$$

where $K(\mathbf{Q},\mathbf{J})$ is the right-hand side of the entropy-product formula (4.1).

In some cases, for example in the case of STU black holes where $J = 0$ and insufficiently many charges are turned on, there is only one horizon and so there is no entropy-product formula. In such cases the argument above demonstrating the existence of an inversion symmetry of the Christodoulou-Ruffini relation breaks down. Indeed, in Sec. III F we saw examples where, for this reason, the Christodoulou-Ruffini relation had no inversion symmetry.

One important consequence of the inversion symmetry of the Christodoulou-Ruffini relation $M = M(S,\mathbf{Q},\mathbf{J})$ is that the relation $S_+T_+ + S_-T_- = 0$, seen, e.g., for the STU black holes in (3.59), is true quite generally. Since the temperature is given by $\partial M/\partial S$ at fixed $\mathbf{Q}$ and $\mathbf{J}$ we have

$$\begin{aligned} T &= \frac{\partial M(S,\mathbf{Q},\mathbf{J})}{\partial S} = \frac{\partial}{\partial S} M\left(\frac{K}{S},\mathbf{Q},\mathbf{J}\right) \\ &= -\frac{K}{S^2}\frac{\partial M(S',\mathbf{Q},\mathbf{J})}{\partial S'}\bigg|_{S'=K/S}, \end{aligned} \tag{4.5}$$

where $K = K(\mathbf{Q},\mathbf{J})$ is the numerator in the inversion formula (4.4). Taking $S = S_+$ we therefore have $S' = S_-$, and so we find from (4.5) that

$$T_+S_+ + T_-S_- = 0 \tag{4.6}$$

whenever there is an entropy-product rule of the form (4.1) and the related inversion symmetry under (4.4).

[9]Or *conformally* invariant, depending on how one chooses the overall multiplicative factor when defining $W(S,M,\mathbf{Q},\mathbf{J})$.

V. ASYMPTOTICALLY AdS AND DS BLACK HOLES

In this section we shall extend the previous discussion to the case of a nonvanishing cosmological constant. If the cosmological constant is negative, the situation is similar to the case when it vanishes. However, if the cosmological constant is positive a new feature arises, namely the occurrence of an additional "cosmological" horizon outside the black hole event horizon. Typically the surface gravity at the cosmological horizon is negative.

A. Kottler

Either we regard Λ as a fixed constant or as an intensive variable which may be varied, in which case we obtain an analogy with a gas with positive pressure

$$P = -\frac{\Lambda}{8\pi}. \tag{5.1}$$

In the first case we should think of the Abbott-Deser mass M as the total energy. In the second case, we should instead think of it as the total enthalpy [73,74]. In both cases we have

$$2M = \left(\frac{S}{\pi}\right)^{\frac{1}{2}} - \frac{\Lambda}{3}\left(\frac{S}{\pi}\right)^{\frac{3}{2}}, \tag{5.2}$$

and in both cases

$$T = \frac{\partial M}{\partial S}\bigg|_{\Lambda} = \frac{1}{4\pi}\left[\sqrt{\frac{\pi}{S}} - \Lambda\sqrt{\frac{S}{\pi}}\right] \tag{5.3}$$

and the heat capacity at constant pressure is given by

$$C_\Lambda = T\left(\frac{\partial T}{\partial S}\bigg|_{\Lambda}\right)^{-1} = \frac{2S(\Lambda S - \pi)}{\Lambda S + \pi}. \tag{5.4}$$

We now consider the two cases where $\Lambda < 0$ and $\Lambda > 0$.

1. $\Lambda < 0$

The temperature T is a positive, monotonic-increasing function of entropy S at fixed pressure P. The isobaric curve in the S–M plane has a point of inflection at which the heat capacity changes sign when

$$\frac{S}{\pi} = -\frac{1}{\Lambda}, \qquad M = \frac{2}{3\sqrt{-\Lambda}}, \tag{5.5}$$

where the slope, and hence the temperature, has a minimum value;

$$T = T_{\min} = \frac{1}{2\pi}\sqrt{-\Lambda}. \qquad (5.6)$$

It follows that for fixed negative Λ there are no black holes with temperatures less than $T_{\min}$. For temperatures above $T_{\min}$ there are two black holes, one with a mass smaller than $\frac{2}{3\sqrt{-\Lambda}}$ and the other with a mass greater than $\frac{2}{3\sqrt{-\Lambda}}$.

The radius r_H of the critical black hole, where the two branches coalesce, is given by

$$r_H = \frac{3}{2}M. \qquad (5.7)$$

This is the location where the heat capacity diverges. It is connected with the Hawking-Page phase transition [75,76]. There is actually a region of masses $M_{HP} > M > M_{cr}$ where the AdS_4 space is entropically favored; however the black hole still has a positive heat capacity. As with the Reissner-Nordström black hole, it has been shown that the sign of the lowest eigenvalue of the Lichnerowicz operator changes sign as the heat capacity changes sign [77].

2. $\Lambda > 0$

We have a negative pressure, $P < 0$. If M is assumed positive we have two horizons, a black hole horizon with

$$0 < S \leq \frac{\pi}{\Lambda}, \qquad (5.8)$$

and positive temperature $T = \partial M/\partial S$, and a cosmological horizon with

$$\frac{\pi}{\Lambda} \leq S \leq \frac{3\pi}{\Lambda}, \qquad (5.9)$$

for which $T = \partial M/\partial S < 0$, and hence the temperature is negative. The heat capacity is therefore always negative. The temperature vanishes when the two horizons coincide, that is if

$$\frac{S}{\pi} = \Lambda, \qquad (5.10)$$

at which the mass has a maximum of

$$M = \frac{1}{3\sqrt{\Lambda}}. \qquad (5.11)$$

In summary, we have two horizons; a black hole horizon and a cosmological horizon. The entropy of the former is smaller then or equal to the entropy of the latter. It seems most appropriate to regard M as the enthalpy. In this case the black hole horizon has positive temperature and the cosmological horizon has negative temperature. This differs from the usual interpretation in which both temperatures are taken to be positive. In effect one takes $T_C = \frac{|\kappa_C|}{2\pi}$ where κ_C, where κ_C is the surface gravity of the event horizon [28–31]. However, even if one follows the conventional interpretation it should be borne in mind that it is not an equilibrium system and there is no period in imaginary time which would produce an everywhere nonsingular gravitational instanton, except when the black hole is absent as in [28,78].

B. Reissner-Nordström-de Sitter

1. $\Lambda < 0$

If $r = \sqrt{\frac{S}{\pi}}$ is the radius in the area coordinate, we have

$$2M = r + \frac{Q^2}{r} + g^2 r^3. \qquad (5.12)$$

where $\frac{\Lambda}{3} = -g^2$. using the fact that

$$\frac{\partial}{\partial S} = \frac{1}{2\pi r}\frac{\partial}{\partial r} \qquad (5.13)$$

one finds that

$$T = \frac{\partial M}{\partial S} = \frac{1}{4\pi r}\left(1 - \frac{Q^2}{r^2} + 3g^2 r^2\right) \qquad (5.14)$$

and thus T vanishes at $r = r_{\mathrm{extreme}}$ where

$$r^2_{\mathrm{extreme}} = \frac{1}{6g^2}\left(\sqrt{1 + 12Q^2 g^2} - 1\right). \qquad (5.15)$$

One has

$$\frac{\partial^2 M}{\partial S^2} = \frac{1}{4\pi^2}\left(-\frac{1}{r^3} + \frac{Q^2}{r^5} + \frac{3g^2}{r}\right) \qquad (5.16)$$

If $6|gQ| < 1$ there are two inflection points at which the heat capacity changes sign at $r = r_{\mathrm{inflection}}$ where

$$r^2_{\mathrm{inflection}} = \frac{1}{6g^2}\left(1 + \pm\sqrt{1 - 36Q^2 g^2}\right). \qquad (5.17)$$

If we take the limit that $Q^2 \to 0$ we obtain the spinodal curve of the Hawking-Page phase transition [75] and if we take the limit $g^2 \to 0$ we obtain the spinodal curve of the Davies phase transition [55]. The two curves meet at the critical point $6|gQ| = 1$.

2. $\Lambda > 0$

This case admits new qualitatively different phenomena since both a black hole and a cosmological horizon are present. This was extensively investigated in 1989 [79–84]. In all these references the absolute value of the surface gravity was taken and the and so the temperature of both horizons was take to be positive. For the choice $M = |Q|$ the temperatures of the black hole and cosmological horizon were observed to be equal. This allowed the construction of a gravitational instanton. To ensure that the electromagnetic field is real on the Euclidean section it is most convenient to assume that the electro-magnetic field is purely magnetic which can be arranged by a duality rotation. In order to avoid confusion with pressure in what follows we replace Q by Z and take Z to be real and positive. We have

$$-r^2 g_{tt} = (r-M)^2 + Z^2 - M^2 - \frac{r^4}{l^2}, \qquad (5.18)$$

and

$$2M = r + \frac{Z^2}{r} - \frac{r^3}{l^2}, \qquad (5.19)$$

with $l^2 = \frac{3}{\Lambda}$.

If $M^2 = Z^2$ there are three positive values of r for which $g_{tt} = 0$:

$$r_1 = \frac{l}{2}\left(1 + \sqrt{1 - 4\frac{M}{l}}\right), \qquad (5.20)$$

$$r_2 = \frac{l}{2}\left(1 - \sqrt{1 - 4\frac{M}{l}}\right), \qquad (5.21)$$

$$r_3 = \frac{l}{2}\left(\sqrt{1 + 4\frac{M}{l}} - 1\right). \qquad (5.22)$$

which correspond to the cosmological event horizon, the black hole horizon and its inner horizon respectively. From the Gibbsian point of view one has $T = \frac{\kappa}{2\pi}$ and therefore

$$T_1 = -\frac{1}{2\pi l}\sqrt{1 - 4Ml}, \qquad (5.23)$$

$$T_2 = \frac{1}{2\pi l}\sqrt{1 - 4Ml}, \qquad (5.24)$$

$$T_3 = -\frac{1}{2\pi l}\sqrt{1 + 4Ml}. \qquad (5.25)$$

Because $|T_1| = T_2$ we obtain a gravitational instanton by setting $t = i\tau$ and identifying τ modulo $\beta = \frac{1}{T_2}$ [80]. The sign used for the period appears to have no geometrical significance and proceeding in the standard way one may argue that the two horizons are in equilibrium with respect to the exchange of thermal Hawking quanta.

It was also argued that if $|\kappa_3| \geq |\kappa_1|$, then the Cauchy horizon should be stable.

C. Kerr-Newman-de Sitter black holes

From [85] we take the formula

$$M = \frac{1}{2}\sqrt{\tilde{S}}\sqrt{\left(1 - \frac{\Lambda\tilde{S}}{3} + \frac{Q^2}{\tilde{S}}\right)^2 + \frac{4J^2}{\tilde{S}^2}\left(1 - \frac{\Lambda\tilde{S}}{3}\right)} \qquad (5.26)$$

where $\tilde{S} = \frac{S}{\pi}$. Writing $\Lambda = -3g^2$, the formula takes the form

$$M^2 = \frac{\pi}{4S}\left\{\left[\frac{S}{\pi}\left(1 + g^2\frac{S}{\pi}\right) + Q^2\right]^2 + 4J^2\left(1 + g^2\frac{S}{\pi}\right)\right\}. \qquad (5.27)$$

For $\Lambda = 0$ the result reduces to that of the Kerr-Newman black hole.

D. Pairwise-equal charge anti-de Sitter black hole

These solutions were obtained in [65], and they are special cases of solutions in the gauged STU supergravity model. (Those are also solutions of maximally supersymmetric four-dimensional theory, which is a consistent truncation of a Kaluza-Klein compactified eleven-dimensional supergravity on S^7.) The theory is specified by mass M, angular momentum J, two charges, i.e., equating $Q_1 = Q_3$ and $Q_2 = Q_4$, and cosmological constant $\Lambda = -3g^2$. In [65] the solution was parametrized by the nonextremality parameter m, rotational parameter a, two boost parameters $\delta_{1,2}$ and g^2. The thermodynamic quantities are of the following form:

$$M = \frac{m(1 + s_1^2 + s_2^2)}{\Xi^2}, \qquad (5.28)$$

$$J = \frac{am(1 + s_1^2 + s_2^2)}{\Xi^2}, \qquad (5.29)$$

$$Q_i = \frac{m s_i c_i}{2\Xi}, \qquad i = 1, 2, \qquad (5.30)$$

where $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$ $(i = 1, 2)$. and $\Xi = 1 - g^2 a^2$. The entropy is of the form:

$$S = \frac{\pi}{\Xi}(r_1 r_2 + a^2), \qquad (5.31)$$

where $r_i = r_+ + m s_i^2$ $(i = 1, 2)$ and r_+ is a location of a horizon, which is a solution of the equation:

$$r^2 - 2mr + a^2 + g^2 r_1 r_2 (r_1 r_2 + a^2) = 0. \qquad (5.32)$$

Manipulation of the horizon equation, along with the expressions for the M, J, Q_i and S, allows one to derive the following explicit Christodoulou-Ruffini mass:

$$M^2 = \frac{\pi}{4S}\left\{\left[\frac{S}{\pi}\left(1+g^2\frac{S}{\pi}\right)+16Q_1^2\right]\left[\frac{S}{\pi}\left(1+g^2\frac{S}{\pi}\right)+16Q_2^2\right]+4J^2\left(1+g^2\frac{S}{\pi}\right)\right\}. \tag{5.33}$$

E. Wu black hole

The Wu black hole [86] is 5D, three charge rotating solution with negative cosmological constant ($\propto g^2$). Employing expressions from [87] for a product of the entropy and temperature of this black hole, associated with all three horizons we obtain the following interesting expression:

$$n_1+n_2+n_3+\frac{1}{2}\left(\frac{n_1n_2}{n_3}+\frac{n_1n_3}{n_2}+\frac{n_2n_3}{n_1}\right)=0, \tag{5.34}$$

where

$$n_1=\frac{4\xi_a\xi_b}{g^2\pi}T_1S_1=(u_1-u_2)(u_2-u_3)$$
$$\text{\& cyclic permutations.} \tag{5.35}$$

Here $\xi_a=1-g^2a^2$, $\xi_b=1-g^2b^2$ and u_i is the root of the horizon equation $X=g^2(u-u_1)(u-u_2)(u-u_3)$. Note that as $g^2\to 0$, $u_3\to -1/g^2\to -\infty$, and in this case the above equation reduces to the standard equation $T_1S_1+T_2S_2=0$.

VI. ENTROPY AND SUPER-ADDITIVITY

The thermodynamics of equilibrium systems with a substantial contribution to the total energy from their gravitational self-energy differs significantly from that of ordinary substances encountered in the laboratory. This is because of the long range nature of the Newtonian gravitational force, which cannot be screened. As a consequence the total entropy S of a gravitating system need not be proportional to the total energy M. A consequence of this is that *negative* heat capacities are possible, and indeed these have long been encountered in the theory of stellar structure [88].

In the case of black holes, the long range nature of gravitational interaction expresses itself in the fact that while the individual extensive variables may be added, they do not necessarily scale. Even if they do, they do not scale with the same power as the total energy M. In the case of ungauged supergravity black holes, the scaling behavior is guaranteed, but the fact that the scaling behaviour is not *homogeneous*, that is, not the same for all extensive variables, leads to a modification of the standard form of the Gibbs-Duhem relation for ordinary homogeneous substances

$$G=M-TS-PV=0, \tag{6.1}$$

where G is the Gibbs free energy, V the volume and P the pressure. By contrast, for black holes the Smarr relation (2.14) gives rise to the Gibbs function (2.16).

The requirement of homogeneous scaling plays such an important role in the thermodynamics of ordinary substances that it has been suggested that it be called the *fourth law of thermodynamics* [89,90]. It certainly fails for systems with significant self-gravitation and, *a fortiori*, for black holes. In fact if the matter sector is sufficiently nonlinear such as in Einstein's theory coupled to nonlinear electrodynamics, even the property of weighted homogeneity ceases to hold.[10] As a consequence, while the first law of black hole thermodynamics holds there is no analogue of a Smarr formula [91].

In the thermodynamics of ordinary substances it is usually assumed that the total energy M is a convex function[11] of the extensive variables or that the S is a concave function of the other extensive variables. This guarantees that the heat capacity and other susceptibilities are positive, and that the Hessians have the correct signs to render the Weinhold and Ruppeiner metrics positive definite.

Now if the extensive quantities scale in a uniform fashion, the property of concavity is equivalent to that of super-additivity,[12] but not necessarily if uniform scaling ceases to hold [92–95]. Remarkably, it was shown long ago in a little noticed paper by Tranah and Landsberg [94][13] that while concavity fails for the entropy of Kerr-Newman black holes, super-additivity remains true. In other words

$$S(M_1+M_2,J_1+J_2,Q_1+Q_2)\geq S(M_1,J_1,Q_1)+S(M_2,J_2,Q_3). \tag{6.2}$$

[10] A function $f(x_1,x_2,\ldots,x_n)$ of n variables is said to be weighted homogeneous of weights $w_1,w_2,\ldots,w_n$ if $f(\lambda^{w_1}x_1,\lambda^{w_2}x_2,\ldots,\lambda^{w_n}x_n)=\lambda f(x_1,x_2,\ldots,x_n)$. If $w_i=1$ for all i, the function is said to be homogeneous of weight one. The Fourth Law is the statement that all extensive variables have weight one and thus all intensive variables have weight zero.

[11] A function $f(\mathbf{x})$ is said to be *convex* if $f(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2)\leq\lambda f(\mathbf{x}_1)+(1-\lambda)f(\mathbf{x}_2)\ \forall\ 1\leq\lambda\leq 1$ and *concave* if $\leq$ is changed to $\geq$. Subject to suitable differentiability this is equivalent to negative (positive) definiteness of the Hessian $\frac{\partial^2 f}{\partial x^i\partial x^j}$. In other words, if M is the total energy then the graph of the Gibbs surface along a straight line joining two equilibrium states $\mathbf{x}_1$ and $\mathbf{x}_2$ never lies above the straight line joining these points on the Gibbs surface.

[12] A function $f(\mathbf{x})$ is super-additive if $f(\mathbf{x}_1+\mathbf{x}_2)\geq f(\mathbf{x}_1)+f(\mathbf{x}_2)$ and subadditive if we replace $\geq$ by $\leq$.

[13] Apparently not accessible on-line. The only paper we know of that has followed up on this is [8].

The super-additivity inequality (6.2) is related to Hawking's area theorem [35,36]. If two black holes of areas A_1 and A_2 can merge to form a single black hole of area A_3, then, subject to the assumption of cosmic censorship,

$$A_3 \geq A_1 + A_2. \tag{6.3}$$

If the angular momentum and charge of the final black hole are equal to the sums of the angular momenta and charges of the initial black holes, one has in addition

$$S(M_3, J_1 + J_2, Q_1 + Q_2) \geq S(M_1, J_1, Q_1) + S(M_2, J_2, Q_3), \tag{6.4}$$

where M_3, the mass of the black hole final state after the merger, obeys

$$M_3 < M_1 + M_2, \tag{6.5}$$

since energy will be lost by gravitational radiation. It follows from the first law that at fixed charge and angular momentum, $dM = TdS$ and so provided that the temperature is positive,

$$S(M_1 + M_2, J_1 + J_2, Q_1 + Q_2) > S(M_3, J_1 + J_2, Q_1 + Q_2). \tag{6.6}$$

The assumption that $Q_3 = Q_1 + Q_2$ is reasonable for theories like Einstein-Maxwell or ungauged supergravity, where there are no particles that carry charge. The assumption that $J_3 = J_1 + J_2$, however, is less reasonable, because both electromagnetic and gravitational waves can carry angular momentum.

In the following subsections we shall obtain generalizations of the Kerr-Newman super-additivity result of Tranah and Landsberg for various more complicated black hole solutions. We also obtain a counterexample in the case of dyonic Kaluza-Klein black holes.

A. STU black holes with pairwise-equal charges

From the formula expressing M in terms of S, Q_1, Q_2 and J for pairwise-equal charged STU black holes, we have

$$\frac{1}{\pi} S(M, Q_1, Q_2, J) = Y + \sqrt{X}, \quad Y = 2M^2 - \frac{1}{2}(Q_1^2 + Q_2^2),$$
$$X = Y^2 - Q_1^2 Q_2^2 - 4J^2. \tag{6.7}$$

For regular black holes we must have $X \geq 0$ and hence $Y \geq \sqrt{4Q_1^2 Q_2^2 + 16J^2}$, thus implying

$$4M^2 \geq Q_1^2 + Q_2^2 + \sqrt{4Q_1^2 Q_2^2 + 16J^2}. \tag{6.8}$$

Without loss of generality, we shall assume Q_1, Q_2 and J are all non-negative. Note that we also have the weaker inequality

$$M \geq \frac{1}{2}(Q_1 + Q_2), \tag{6.9}$$

which we shall use frequently in the following.

We wish to check whether the entropy of these pairwise-equal charged black holes obey the super-additivity inequality

$$S_{\rm tot} \geq S + S', \tag{6.10}$$

where

$$S_{\rm tot} \equiv S(M + M', Q_1 + Q_1', Q_2 + Q_2', J + J'),$$
$$S \equiv S(M, Q_1, Q_2, J), \quad S' \equiv S(M', Q_1', Q_2', J'). \tag{6.11}$$

With analogous definitions for the quantities X and Y, proving super-additivity requires proving that

$$Y_{\rm tot} - Y - Y' + \sqrt{X_{\rm tot}} - \sqrt{X} - \sqrt{X'} \geq 0. \tag{6.12}$$

We first note that the Y functions are non-negative, and that they obey

$$\begin{aligned} Y_{\rm tot} - Y - Y' &= 4MM' - Q_1 Q_1' - Q_2 Q_2' \\ &\geq \ (Q_1 + Q_2)(Q_1' + Q_2') - Q_1 Q_1' - Q_2 Q_2' \\ &= Q_1 Q_2' + Q_2 Q_1' \\ &\geq 0. \end{aligned} \tag{6.13}$$

Thus, if we can show that

$$\sqrt{X_{\rm tot}} - \sqrt{X} - \sqrt{X'} \geq 0 \tag{6.14}$$

then the super-additivity inequality (6.10) will be established. To prove this, we first note that is can be reexpressed as

$$X_{\rm tot} - (\sqrt{X} + \sqrt{X'})^2 \geq 0. \tag{6.15}$$

We now observe that the following identity holds:

$$\begin{aligned} P &:= \left(c\sqrt{X} - \frac{1}{c}\sqrt{X'}\right)^2 + 4\left(cJ - \frac{1}{c}J'\right)^2 \\ &= -2\sqrt{X}\sqrt{X'} - 8JJ' + 8M^2 M'^2 - 2M^2({Q_1'}^2 + {Q_2'}^2) \\ &\quad - 2M'^2(Q_1^2 + Q_2^2) - 2Q_1 Q_2 Q_1' Q_2' \\ &\quad + \frac{1}{2}(Q_1^2 + Q_2^2)({Q_1'}^2 + {Q_2'}^2), \end{aligned} \tag{6.16}$$

where we have defined

$$c^2 = \frac{4M'^2 - (Q'_1 - Q'_2)^2}{4M^2 - (Q_1 - Q_2)^2}. \tag{6.17}$$

We can use (6.16) to substitute for $\sqrt{X}\sqrt{X'}$ in (6.15), thus yielding

$$\begin{aligned} X_{\rm tot} - X - X' - 2\sqrt{X}\sqrt{X'} &= P + 8(MM' - Q_- Q'_-)(M^2 + M'^2 - Q_+^2 - Q'_+{}^2) \\ &\quad + 8[(M+M')^2 - (Q_- + Q'_-)^2](MM' - Q_+ Q'_+), \end{aligned} \tag{6.18}$$

where we have defined

$$Q_\pm = \frac{1}{2}(Q_1 \pm Q_2), \qquad Q'_\pm = \frac{1}{2}(Q'_1 \pm Q'_2). \tag{6.19}$$

The inequality (6.9) implies $M \geq Q_+$ and $M' \geq Q'_+$, and *a fortiori* $M \geq |Q_-|$ and $M' \geq |Q'_-|$ (recall that we are taking all charges to be non-negative). Since P, defined in (6.16), is manifestly non-negative it follows from (6.18) that the left-hand side must be non-negative, and hence the required inequality (6.14) is satisfied. Thus we have proven that the super-additivity property (6.10) is indeed obeyed by the entropy of the pairwise-equal charged black holes of STU supergravity.

B. STU black holes with three equal nonzero charges

One can also show analytically that the super-additivity property of the entropy is true for the case of STU black holes with three equal nonzero charges, say, $Q_1 = Q_2 = Q_3 = q$, with $Q_4 = 0$. In this case $S = \pi(Y + \sqrt{X})$ with:

$$Y^2 = \frac{1}{64}(3z - 2M)(z + 2M)^3, \tag{6.20}$$

where

$$z = \sqrt{4M^2 - 2q^2}, \tag{6.21}$$

and

$$X = Y^2 - J^2. \tag{6.22}$$

It is straightforward to show that

$$z_{\rm tot}^2 - (z + z')^2 = 8MM'(1 - ww' - \sqrt{1-w^2}\sqrt{1-w'^2}) \geq 0, \tag{6.23}$$

where $w = \frac{q}{\sqrt{2}M}$ and $w' = \frac{q'}{\sqrt{2}M'}$. The second inequality in (6.23) is true for any value of $\{w, w'\} \leq 1$. This result implies

$$Y_{\rm tot} - Y - Y' \geq 0. \tag{6.24}$$

It is now straightforward to show that

$$\sqrt{X_{\rm tot}} - \sqrt{X} - \sqrt{X'} \geq 0, \tag{6.25}$$

thus proving the super-additivity of the entropy in this case as well.

An analytic proof of the super-additivity of the entropy for the case of one nonzero charge follows analogous steps.

While a numerical analysis indicates that the super-additivity is true for the STU black holes with four arbitrary electric charges, it would be interesting to prove this result analytically.

C. Dyonic Reissner-Nordström

In the explicit examples we have studied so far, the black hole is supported by one or more field strengths that each carry a single complexion of field (pure electric charge, or instead and equivalently, one could consider pure magnetic charge). The details of the entropy super-additivity inequality are different if we consider a case where one or more field strengths carries both electric and magnetic charge. In this subsection, we shall study the dyonic Reissner-Nordström black hole, and show that in this case too the super-additivity property is satisfied. This case, where the Lagrangian is just that of the pure Einstein-Maxwell system, can be view as STU black holes where all four field strengths are equal. By contrast, in the next subsection we shall see that in the case of STU black holes where only a single field strength is nonzero, the dyonic black holes have an entropy that violates the super-additivity property.

The Einstein-Maxwell Lagrangian $\mathcal{L} = \sqrt{-g}(R - F^2)$ admits static dyonic black hole solutions given by

$$\begin{aligned} ds^2 &= -hdt^2 + \frac{dr^2}{h} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ A &= \frac{Q}{r}dt + P\sin\theta d\varphi, \qquad h = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}, \end{aligned} \tag{6.26}$$

with mass M, electric charge Q and magnetic charge P. To have a black hole, these quantities must obey the inequality

$$M \geq \sqrt{Q^2 + P^2}, \tag{6.27}$$

with extremality being attained when the inequality is saturated. The entropy is given by

$$S(M, Q, P) = \pi[2M^2 - Q^2 - P^2 + 2M\sqrt{M^2 - Q^2 - P^2}]. \tag{6.28}$$

For super-additivity, one must have

$$S(M+M', Q+Q', P+P') - S(M,Q,P) - S(M',Q',P') \geq 0, \tag{6.29}$$

where, as usual, we assume, without loss of generality, that the charges are all non-negative. Substituting (6.28) into this, we see that super-additivity is satisfied if

$$4MM' - 2QQ' - 2PP' + (M+M')\sqrt{(M+M')^2 - (Q+Q')^2 - (P+P')^2} - M\sqrt{M^2 - Q^2 - P^2} - M'\sqrt{M'^2 - Q'^2 - P'^2} \geq 0. \tag{6.30}$$

First, we note that the argument of the first square root is non-negative, since, after using (6.27) for the unprimed and primed quantities we have

$$(M+M')^2 - (Q+Q')^2 - (P+P')^2 \geq 2(MM' - QQ' - PP'), \tag{6.31}$$

and since

$$(MM')^2 - (QQ' + PP')^2 \geq (Q^2 + P^2)(Q'^2 + P'^2) - (QQ' + PP')^2 = (QP' - PQ')^2 \geq 0, \tag{6.32}$$

the non-negativity is proven.

Returning to the inequality (6.30) that we wish to establish, we see that the terms $4MM' - 2QQ' - 2PP'$ are themselves certainly non-negative, since $2MM' - 2QQ' - 2PP' \geq 0$ as we just demonstrated. The inequality is therefore established if we can show that

$$M\left(\sqrt{(M+M')^2 - (Q+Q')^2 - (P+P')^2} - \sqrt{M^2 - Q^2 - P^2}\right) \geq 0, \tag{6.33}$$

together with the analogous expression with the primes and unprimed variables exchanged. The expression in parentheses is non-negative if

$$(M+M')^2 - (Q+Q')^2 - (P+P')^2 - (M^2 - Q^2 - P^2) \tag{6.34}$$

is non-negative. After using (6.27) again we see that (6.34) is greater than or equal to $2(MM' - QQ' - PP')$, and we have already shown that this is non-negative. Thus the super-additivity property (6.29) is established for the dyonic Reissner-Nordström black holes.

D. A counterexample: The dyonic Kaluza-Klein black hole

Here, we demonstrate that dyonic Kaluza-Klein black holes that we discussed in Sec. III D 6 provide counterexamples where entropy super-additivity breaks down. The phase space for checking entropy super-additivity for these dyonic black holes is rather large, so we shall just focus on a restricted subspace within which we are able to exhibit violations. Specifically, we shall consider two black holes with the following (M,Q,P) values:

$$(P,0,P) \quad \text{and} \quad (M',Q',0), \tag{6.35}$$

so the unprimed case is an extremal black hole with purely magnetic charge,[14] and the primed case is a (subextremal) black hole with purely electric charge. The masses and charges will be chosen so that the black hole with the summed mass and charges will be an extremal dyonic black hole, for which $M_{\text{tot}} = (Q_{\text{tot}}^{2/3} + P_{\text{tot}}^{2/3})^{3/2}$. Thus

$$M_{\text{tot}} = M + M' = P + M', \qquad Q_{\text{tot}} = Q', \qquad P_{\text{tot}} = P, \tag{6.36}$$

with

$$P + M' = (Q'^{\frac{2}{3}} + P^{\frac{2}{3}})^{\frac{3}{2}}. \tag{6.37}$$

We shall characterize the ratio P/Q' by means of a constant x, such that

$$P = x^{\frac{3}{2}}Q'. \tag{6.38}$$

We therefore have

$$S = 0, \qquad S' = \frac{\pi m^2}{\sqrt{1-2\beta_1}}, \qquad S_{\text{tot}} = 8\pi x^{\frac{3}{2}}Q'^2, \tag{6.39}$$

where the primed black hole defined above has metric parameters m and β_1, with $\beta_2 = 0$. This means that

$$M' = \frac{(1-\beta_1)m}{2(1-2\beta_1)}, \qquad Q' = \frac{\sqrt{2\beta_1}m}{4(1-2\beta_1)}, \tag{6.40}$$

the entropy is given by

[14]Strictly speaking, the extremal configuration $(P,0,P)$ is not a black hole, but rather a naked singularity. However, one can make an infinitesimal deformation away from extremality, to a configuration with parameters $(P+\delta,0,P)$, and this will describe a genuine black hole. The results that we shall derive here, including the bound (6.46) on P versus Q' for obtaining violations of entropy super-additivity, are thus valid.

$$S' = 8\pi \frac{(1-2\beta_1)^{\frac{3}{2}}}{\beta_1} Q'^2, \tag{6.41}$$

and from (6.37) β_1 is given in terms of x by

$$\frac{2(1-\beta_1)}{\sqrt{2\beta_1}} = (1+x)^{\frac{3}{2}} - x^{\frac{3}{2}}. \tag{6.42}$$

Let us first consider the case where x is very small, $x = \epsilon^{\frac{2}{3}}$. From (6.42) we find at leading order $\beta_1 = \frac{1}{2}(1-\epsilon^{\frac{2}{3}})$, and so $S' = 16\pi\epsilon Q'^2$. Thus we have

$$S_{\rm tot} - S - S' = 8\pi\epsilon Q'^2 - 0 - 16\pi\epsilon Q'^2 = -8\pi\epsilon Q'^2, \tag{6.43}$$

and so super-additivity does not hold in this region of the parameter space.

When x becomes larger, we find from numerical analysis that the ratio $S_{\rm tot}/(S+S')$, which equals 2 in the limit as x goes to zero, falls monotonically. The ratio reaches unity when $S' = S_{\rm tot}$, which implies

$$x = (1-2\beta_1)\beta_1^{-\frac{2}{3}}. \tag{6.44}$$

Substituting into (6.42), we find that this occurs when $\beta_1 = y^3$ and y is the single real root of the 9th-order polynomial

$$17y^9 - 12y^8 + 42y^7 - 80y^6 + 39y^5 - 48y^4 + 54y^3 - 12y^2 + 9y - 8 = 0. \tag{6.45}$$

This root is given approximately by $y = 0.698234$, implying $\beta_1 = 0.340411$, and hence $x = 0.654681$. Thus the parameter range where we find a violation of entropy super-additivity is when

$$0 < P < 0.529718Q'. \tag{6.46}$$

In other words, we have found super-additivity violation when we add an extremal purely magnetic black hole and a nonextremal purely electric black hole, with parameters arranged such that the "total" dyonic black hole is extremal, provided that the magnetic charge of the original extremal black hole is sufficiently small in comparison to the electric charge of the original nonextremal black hole.

We can give a more complete treatment by choosing two black holes with parameters (M,Q,P) of the form $(M,0,P)$ and $(M',Q',0)$, subject to the assumption that the total black hole $(M_{\rm tot},Q_{\rm tot},P_{\rm tot})$ is again extremal, obeying

$$M_{\rm tot} = [Q_{\rm tot}^{2/3} + P_{\rm tot}^{2/3}]^{3/2}. \tag{6.47}$$

Thus

$$M_{\rm tot} = M + M', \qquad Q_{\rm tot} = Q', \qquad P_{\rm tot} = P. \tag{6.48}$$

It is straightforward to show from the formulae in Sec. III D 5 that for the individual black holes that carry purely electric or purely magnetic charge, one has

$$S = \sqrt{8}\pi\sqrt{M^4 - 20M^2P^2 - 8P^4 + M(M^2+8P^2)^{3/2}},$$
$$S' = \sqrt{8}\pi\sqrt{M'^4 - 20M'^2Q'^2 - 8Q'^4 + M'(M'^2+8Q'^2)^{3/2}}. \tag{6.49}$$

One can then use (6.47), together with (6.48), to solve for M', and hence one can express $Y \equiv S_{\rm tot} - S - S'$, where $S_{\rm tot} = 8\pi P_{\rm tot} Q_{\rm tot}$, as a function of M, P and Q'. One can then explore the regions in the space of these parameters for which Y is negative, signifying a violation of entropy super-additivity.

Of course, by continuity we expect that super-additivity violations will occur at least in some neighborhood of the region found above when all the masses and charges are allowed to be adjusted. In other words, there will also be super-additivity violations if we consider cases where all three black holes are nonextremal, for appropriate ranges of the various masses and charges.

In our earlier remarks relating super-additivity to the Hawking area theorem, we assumed not only cosmic censorship but also that the coalescence of the two black holes was allowed physically. In the case of dyons, it should be recalled that they carry angular momentum, and moreover it is not localized within the event horizon. This, as suggested in [96], may lead to restrictions on what coalescences are allowed, and thus the nonsuper-additivity of the entropy in this counterexample need not imply any conflict with Hawking's area theorem. This is an interesting problem worthy of further study.

VII. CONCLUSIONS AND FUTURE PROSPECTS

We shall turn in this section to a consideration of the significance of negative surface gravities, and negative Gibbsian temperatures. We shall begin by recalling the most physically convincing argument that Schwarzschild black holes have a temperature, and hence entropy. This was given by Hawking [46,47], who coupled a collapsing black-hole metric in an asymptotically-flat spacetime to a quantum field, and showed that if the quantum field was initially in its vacuum state, then at late times it would emit particles with a thermal spectrum and temperature given by (1.3). The term "vacuum state" implied that it contained no particles having positive frequency with respect to the standard retarded time coordinate on past null infinity. This required his considering the behavior of the quantum field as it passed through the time-dependent spacetime generated by the collapse. However, one may dispense with that region, and work with the exact vacuum Schwarzschild solution, obtaining the same result, by choosing an appropriate boundary condition for the quantum field on the past horizon. The appropriate boundary condition, which reproduces Hawking's result, in the exterior region of the Schwarzschild solution, corresponds to requiring that the state contains no particles defined as having positive frequency with respect to a Kruskal null coordinate on the past horizon. This state is now referred to as the Unruh vacuum state. This is obviously not a state in thermal equilibrium. A different state, introduced by Hartle and Hawking, is defined on the past horizon in the same way, but at past null infinity the definition of positive frequency is such that it describes an ingoing flux of particles at the Hawking temperature. Thus the Hartle-Hawking state should be regarded as a state in thermal equilibrium.

The situation with two event horizons is more complicated. In order to discuss quantum fields between the horizons, one needs to specify a notion of positive frequency on each past horizon. If the region is static, and one interprets positive frequency as being with respect to a local Kruskal coordinate on the horizons, the resulting quantum state will describe thermal radiation entering the static region at temperatures given by $\frac{1}{2\pi}|\kappa_\pm|$. This is not a state in thermal equilibrium. If the region between the two horizons is not static, as for example in the Reissner-Nordström solution, one may define a similar state which would also not be in thermal equilibrium. If, on the other hand, one considers the static region behind the inner horizon in the Reissner-Nordström, one needs to specify boundary conditions on the singularity at $r=0$. If one chose the notation of positive frequency on the past inner horizon, then whatever boundary conditions were chosen on the singularity, the quantum state would contain radiation coming from the inner horizon with a temperature $\frac{1}{2\pi}|\kappa_\pm|$. Thus if we adopt this procedure, we see in all cases that the temperature we associate with particles coming from the horizons is given by the absolute value of the surface gravity, divided by 2π.

An alternative way of establishing the temperature and entropy of an asymptotically flat black hole is to follow the procedure of [78,97], in which one analytically continues the metric to imaginary time, and discovers that the metric is periodic in imaginary time with a period given by $2\pi/|\kappa|$, which is what one expects for a state in thermal equilibrium at temperature $\frac{1}{2\pi}|\kappa_\pm|$. Of course, the period itself can have either sign, but the quantum state would not necessarily exist if one chose a negative sign for the temperature. This procedure will work when one has a single horizon, including an asymptotically anti-de Sitter spacetime [75,76]. However, this procedure will not work for a spacetime with two horizons having differing values of $|\kappa|$. The conclusion seems to be that classically, the sign of the temperature can only be determined by appealing to the first law, and this provides us with a Gibbsian temperature. Quantum mechanically, which seems to be the only physically reliable argument provided one is prepared to contemplate nonequilibrium situations, the temperature should be taken to be positive. In other words, the temperature is not uniquely defined by the metric, a conclusion also reached in [25].

The original suggestion that inner horizons should be assigned a negative temperature [1] was based not quantum field theoretic considerations, but rather on a consideration of quantum mechanical systems, such as spin systems, exhibiting population inversion [59]. Thus one might regard the total energy of a black hole as receiving contributions both from the outer and inner horizons. The inner system would then be thought of as the analogue of a spin system. This viewpoint was supported by the existence for the Kerr-Newman black hole of the modified Smarr formula (3.34), and its variation, which may be written as

$$dM = \frac{1}{2}(T_+ dS_+ + \Omega_+ dJ + \Phi_+ dQ) + \frac{1}{2}(T_- dS_- + \Omega_+ dJ + \Phi_+ dQ). \tag{7.1}$$

As we saw, these formulas generalize to the case of STU black holes with four electric charges. The addition of electric charges, which were not included in the discussion in [1], suggest that the posited spin system inverted population should be supplemented by the inclusion of charged states.

In the case of four-dimensional STU black holes, the generalization of equation (7.1) may be rewitten in terms of the left-moving and right-moving sectors [see (3.69)] as

$$dM = (T_L dS_L + \Omega_L dJ + \Phi_L^i dQ_i + \Psi_{L,i} dP^i) + (T_R dS_R + \Omega_R dJ + \Phi_R^i dQ_i + \Psi_{R,i} dP^i), \tag{7.2}$$

with each sector contributing equally to dM. In contrast to the proposal in [1], which attempted to give a microscopic interpretation to the negative temperature on the inner horizon, here the left-moving and right-moving sectors both have positive temperatures, consistent with the proposed microscopic interpretation in terms of D-brane states [11,63]. An analogous interpretation for five-dimensional STU black holes has also been given [16].

This paper has been concerned exclusively with time-independent solutions; we have not discussed what happens to inner horizons when perturbations are considered. There is a widespread belief that in classical general relativity, generic perturbations will render Cauchy horizons, of the sort one finds inside black holes, singular. This is referred to as the cosmic censorship hypothesis. There are various forms of this hypothesis, and the literature is at present rather inconclusive. A recent discussion can be found in [98]. Our motivation is largely quantum mechanical, and the relevance of these classical results to a full quantum gravitational treatment is unclear.

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APPENDIX A: CARTER-PENROSE DIAGRAM FOR TWO HORIZONS

In this Appendix, we summarize some facts about the Carter-Penrose diagram of asymptotically-flat spherically symmetric spacetimes with an inner and outer horizon. Consider a suitable metric of the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{f^2(r)A(r)} + R(r)^2 d\Omega^2. \quad \text{(A1)}$$

Introducing an advanced time coordinate v by defining

$$dv = dt + \frac{dr}{fA}, \quad \text{(A2)}$$

the metric takes the Eddington-Finkelstein form

$$ds^2 = -Adv^2 + 2f^{-1}drdv + R^2 d\Omega^2. \quad \text{(A3)}$$

The metric will be regular as long as A, f and R^2 are real, bounded, and twice differentiable, and in addition f and R are nonzero. We may take f, without loss of generality, to be positive. In particular, the metric is well-behaved regardless of whether A is positive, zero or negative. Asymptotic flatness requires that A and f tend to 1 as R^2 tends to infinity. In the cases we shall consider, R tends to r at infinity. We shall assume that A is positive in the interval $r_+ < r \le \infty$, and negative in the interval $r_- < r < r_+$, and that it vanishes on the outer horizon $r = r_+$ and the inner horizon $r = r_-$. We shall also assume that A has a smooth positive extension for values of $r < r_-$. The Killing vector $K = \partial/\partial v$ is thus timelike for $r_+ < r < \infty$, lightlike at $r = r_+$, spacelike for $r_- < r < r_+$, lightlike at $r = r_-$ and timelike for $r < r_-$. It becomes lightlike as v tends to $\pm\infty$, and also as r tends to infinity.

If $r_+ < r < \infty$, then as v tends to $+\infty$ we obtain future null infinity, $\mathcal{I}^+$. For v instead tending to $-\infty$, we obtain past null infinity $\mathcal{I}^-$. As v tends to $-\infty$ and r tends to r_+ we obtain the past null horizon. The Killing vector K is future-directed inside and on the boundary of this region. The inner region is bounded by a past Cauchy horizon at $v = -\infty$ and $r = r_+$, and a future Cauchy horizon at $v = +\infty$ and $r = r_-$. It has a further boundary on the inner horizon at $r = r_-$, with $-\infty < v < +\infty$. Thus the Killing vector K is future directed both on this inner horizon and on the outer horizon.

If one looks at radial geodesics in this spacetime, there are two conserved quantities p_v and k, where

$$p_v = A\dot{v} - f^{-1}\dot{r}, \qquad -A\dot{v}^2 + 2f^{-1}\dot{r}\,\dot{v} = -k, \quad \text{(A4)}$$

and a dot denotes a derivative with respect to an affine parameter λ. Thus radially-infalling geodesics obey

$$\dot{r} = -f\sqrt{p_v^2 - kA}, \quad \text{(A5)}$$

with $k > 0$ and $p_v^2 > k$ for timlike geodesics that originate at large r. The constant p_v is positive. The infalling particle passes through the outer and the inner horizons before reaching a turning point at a radius $\bar{r} < r_-$ at which $p_v^2 = kA(\bar{r})$.

Solving for $\dot{v}$ one finds

$$\dot{v} = \frac{p_v - \sqrt{p_v^2 - kA}}{A}, \quad \text{(A6)}$$

and so

$$\frac{dv}{dr} = \frac{1}{fA}\left[1 - \frac{p_v}{\sqrt{p_v^2 - kA}}\right]. \quad \text{(A7)}$$

Thus one finds that $\dot{v}$, dv/dr and v all remain finite as the particle falls in from infinity to $\bar{r}$. Note that $\dot{v}$ is always positive.

In conclusion, we note that the Killing vector $K = \partial/\partial v$ is future directed and lightlike on both the future event horizon of the exterior region, $r = r_+$ with $-\infty < v < +\infty$, and on the inner horizon, $r = r_-$ with $-\infty < v < +\infty$.

For the four-charge STU black holes considered in this paper, the situation when they are nonrotating is qualitatively similar to that for the Reissner-Nordström solution. The metric takes the form

$$ds^2 = -(H_1H_2H_3H_4)^{-1/2} W dt^2 + (H_1H_2H_3H_4)^{1/2}(W^{-1}dr^2 + r^2 d\Omega^2), \quad \text{(A8)}$$

where

$$H_i = 1 + \frac{\mu \sinh^2\delta_i}{r}, \qquad W = 1 - \frac{\mu}{r}. \quad \text{(A9)}$$

The outer horizon is located at $r_+ = \mu$, and the inner horizon at $r_- = 0$. There are curvature singularities at the four locations $r = -\mu \sinh^2 \delta_i$, and the Carter-Penrose diagram will be similar to that for Reissner-Nordström, with the curvature singularity in the diagram occurring at the least negative of the four locations.

APPENDIX B: STU SUPERGRAVITY

The Lagrangian of the bosonic sector of four-dimensional ungauged STU supergravity can be written in the relatively simple form

$$\begin{aligned}\mathcal{L}_4 = {}& R * \mathbb{1} - \frac{1}{2} * d\varphi_i \wedge d\varphi_i - \frac{1}{2} e^{2\varphi_i} * d\chi_i \wedge d\chi_i \\ & - \frac{1}{2} e^{-\varphi_1} (e^{\varphi_2 - \varphi_3} * F_{(2)1} \wedge F_{(2)1} \\ & + e^{\varphi_2 + \varphi_3} * F_{(2)2} \wedge F_{(2)2} + e^{-\varphi_2 + \varphi_3} * \mathcal{F}^1_{(2)} \wedge \mathcal{F}^1_{(2)} \\ & + e^{-\varphi_2 - \varphi_3} * \mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)}) \\ & - \chi_1 (F_{(2)1} \wedge \mathcal{F}^1_{(2)} + \mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)}),\end{aligned} \tag{B1}$$

where the index i labeling the dilatons φ_i and axions χ_i ranges over $1 \leq i \leq 3$. The four field strengths can be written in terms of potentials as

$$\begin{aligned} F_{(2)1} &= dA_{(1)1} - \chi_2 d\mathcal{A}^2_{(1)}, \\ F_{(2)2} &= dA_{(1)2} + \chi_2 d\mathcal{A}^1_{(1)} - \chi_3 dA_{(1)1} + \chi_2 \chi_3 d\mathcal{A}^2_{(1)}, \\ \mathcal{F}^1_{(2)} &= d\mathcal{A}^1_{(1)} + \chi_3 d\mathcal{A}^2_{(1)}, \\ \mathcal{F}^2_{(2)} &= d\mathcal{A}^2_{(1)}. \end{aligned} \tag{B2}$$

The field strengths here are not in the same duality frame as the one we have assumed in our discussions in this paper however. To convert from (B1) and (B2) to the frame we are using, one would need to dualize the field strengths $\mathcal{F}^1_{(2)}$ and $\mathcal{F}^2_{(2)}$, and if then written explicitly, the resulting Lagrangian would be rather cumbersome. Alternatively, one could simply exchange the roles of the electric and magnetic charges for the field strengths $\mathcal{F}^1_{(2)}$ and $\mathcal{F}^2_{(2)}$, and work with (B1) without performing any dualizations. For example, the 4-charge black hole solutions that we refer to in this paper as having four electric charges would, as solutions in terms of the fields in (B1), instead comprise two electric and two magnetic charges. (As for example, in the presentation of these solution in [65].)

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