Meson-hybrid mixing in $J^{PC} = 1^{++}$ heavy quarkonium from QCD sum rules

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We explore conventional meson-hybrid mixing in $J^{PC} = 1^{++}$ heavy quarkonium using QCD Laplace sum rules. We calculate the cross-correlator between a heavy conventional meson current and heavy hybrid current within the operator product expansion, including terms proportional to the four- and six-dimensional gluon condensates and the six-dimensional quark condensate. Using experimentally determined hadron masses, we construct models of the 1^{++} charmonium and bottomonium mass spectra. These models are used to investigate which resonances couple to both currents and thus exhibit conventional meson-hybrid mixing. In the charmonium sector, we find almost no conventional meson-hybrid mixing in the $\chi_{c1}(1P)$, minimal mixing in the $X(3872)$, and significant mixing in both the $X(4140)$ and $X(4274)$. In the bottomonium sector, we find minimal conventional meson-hybrid mixing in the $\chi_{b1}(1P)$ and significant mixing in both the $\chi_{b1}(2P)$ and $\chi_{b1}(3P)$.

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I. INTRODUCTION

Hybrids are hadrons which consist of a quark-antiquark pair and exhibit explicit gluon degrees of freedom. Hybrids are allowed by QCD as they are color singlets; however, they have not yet been definitively experimentally identified [1].

Hybrids can be classified by J^{PC} , quantum numbers that can be separated into two categories, nonexotic and exotic, depending on whether the quantum numbers are accessible to conventional (quark-antiquark) mesons or not. Hybrids with exotic J^{PC} would not be able to quantum mechanically mix with conventional mesons; however, hybrids with nonexotic quantum numbers can potentially mix with conventional mesons. This mixing would result in hadrons that are superpositions of both conventional meson and hybrid.

In this article, we extend our work from [2] on vector (i.e., 1^{--}) conventional meson-hybrid mixing to axial vector (i.e., 1^{++}) charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$). Of particular interest in the charmonium sector is the $X(3872)$ [3,4], the first of the XYZ resonances [5–9], a collection of charmoniumlike hadrons many of which are not easily accommodated by the constituent quark model. The $X(3872)$ has been studied in the context of conventional meson-tetraquark mixing [10] as well as

tetraquark-hybrid mixing [11] (see also [12,13] for other approaches to mixing). Our analysis complements these two by considering conventional meson-hybrid mixing. At present, the 1^{++} channel is the only channel other than the 1^{--} with enough experimentally observed resonances to allow for the multiresonance analysis methods of [2].

We use the operator product expansion (OPE) [14] to compute the cross-correlator between a heavy conventional meson current and a heavy hybrid current. In this calculation we include leading-order (LO) contributions from perturbation theory and nonperturbative corrections proportional to the four-dimensional (4d) and 6d gluon condensates as well as the 6d quark condensate. Then, using QCD Laplace sum-rules (LSRs) [15–18], we analyze several single and multiresonance models of the 1^{++} charmonium and bottomonium mass spectra. These models take known resonance masses as inputs and allow us to probe the resonances to determine whether they couple to both the conventional meson current and the hybrid current. Resonances which couple to both currents are considered to be quantum mechanical mixtures of conventional meson and hybrid. The QCD sum-rules methodology has been applied to hadron mixing in a number of systems [2,19–22].

We find that multiresonance models which include excited states in addition to the ground state lead to a significant improvement in agreement between QCD and experiment when compared to single resonance models. We show explicitly that the higher mass states make numerically significant contributions to the LSRs despite the LSR's exponential suppression of such resonances. In the charmonium sector, we find very little conventional

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meson-hybrid mixing in the $\chi_{c1}(1P)$, minimal mixing in the $X(3872)$, and large mixing in both the $X(4140)$ and the $X(4274)$. In the bottomonium sector, we find minimal conventional meson-hybrid mixing in the $\chi_{b1}(1P)$ and large mixing in both the $\chi_{b1}(2P)$ and $\chi_{b1}(3P)$.

II. THE CORRELATOR

For the conventional meson current

$$j_\mu^{(m)} = \bar{Q}\gamma_\mu\gamma^5 Q \quad (1)$$

and the hybrid current [23]

$$j_\nu^{(h)} = \frac{g_s}{2} \bar{Q}\gamma^\rho\lambda^a\tilde{G}_{\nu\rho}^a Q \quad (2)$$

where Q is a heavy quark (i.e., charm or bottom) field and

$$\tilde{G}_{\nu\rho}^a = \frac{1}{2}\epsilon_{\nu\rho\omega\zeta}G_{\omega\zeta}^a \quad (3)$$

is the dual gluon field strength tensor, we consider the cross-correlator

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iq\cdot x} \langle \Omega | \tau j_\mu^{(m)}(x) j_\nu^{(h)}(0) | \Omega \rangle \quad (4)$$

$$= \frac{q_\mu q_\nu}{q^2} \Pi_0(q^2) + \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) \Pi_1(q^2). \quad (5)$$

In (5), the function $\Pi_0(q^2)$ probes spin-0 states and $\Pi_1(q^2)$ probes spin-1 states. We focus on $\Pi_1(q^2)$ as we are interested in probing 1^{++} states.

We evaluate the cross-correlator (4) within the OPE where perturbation theory is supplemented by non-perturbative corrections. Each of these nonperturbative corrections is the product of a perturbatively computed Wilson coefficient and a QCD condensate. We include terms proportional to the 4d and 6d gluon condensates and the 6d quark condensate defined respectively as follows:

$$\langle \alpha G^2 \rangle = \alpha_s \langle : G_{\omega\phi}^a G_{\omega\phi}^a : \rangle \quad (6)$$

$$\langle g^3 G^3 \rangle = g_s^3 f^{abc} \langle : G_{\omega\zeta}^a G_{\zeta\rho}^b G_{\rho\omega}^c : \rangle \quad (7)$$

$$\langle J^2 \rangle = \text{Tr}(\langle : J_\nu J_\nu : \rangle) \quad (8)$$

with

$$J_\nu = \frac{-ig_s^2}{4} \lambda^a \sum_A \bar{q}^A \lambda^a \gamma_\nu q^A \quad (9)$$

where, in (9), q is a light quark (i.e., up, down, or strange) field and the sum is over flavors. We use the vacuum saturation hypothesis [15] to express $\langle J^2 \rangle$ in terms of the 3d quark condensate

$$\langle \bar{q}q \rangle = \langle : \bar{q}_i^a q_i^a : \rangle \quad (10)$$

resulting in

$$\langle J^2 \rangle = \frac{2}{3} \kappa g_s^4 \langle \bar{q}q \rangle^2 \quad (11)$$

where κ quantifies deviations from vacuum saturation. As in [2], we set $\kappa = 2$ [18].

The diagrams that contribute to (4) at LO are given in Fig. 1. Each diagram is multiplied by two to account for additional diagrams in which the quark lines run in the opposite directions. Diagram IV gets another factor of two due to symmetry under exchange of the two interaction vertices. We calculate Wilson coefficients via fixed-point gauge methods [24,25], and divergent integrals are dealt with through dimensional regularization in $D = 4 + 2\epsilon$ dimensions at $\overline{\text{MS}}$ -scale μ . The *Mathematica* package TARCER [26] which implements the recurrence relations of [27,28] is used to express results in terms of known master integrals including those of [29,30]. Following [31] we use the γ^5 convention

$$\gamma^5 = \frac{i}{24} \epsilon_{\mu\nu\sigma\rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho. \quad (12)$$

The operator product expansion (OPE) computation of Π_1 from (5), denoted $\Pi^{(\text{OPE})}$, is decomposed as

$$\Pi^{(\text{OPE})} = \Pi^{(\text{I})} + \Pi^{(\text{II})} + \Pi^{(\text{III})} + \Pi^{(\text{IV})} + \Pi^{(\text{V})} + \Pi^{(\text{VI})} \quad (13)$$

where the superscripts in (13) correspond to the labels of the diagrams in Fig. 1. For $\Pi^{(\text{I})}$, the ϵ -dependent result is given by

$$\begin{aligned} \Pi^{(\text{I})}(z; \epsilon) = & \frac{\alpha_s m^{4(1+\epsilon)} (1+\epsilon) \Gamma^2(-1-\epsilon)}{6\pi^3 z (3+2\epsilon)^2 (4+3\epsilon) (4\pi)^{2\epsilon}} \left(-12z(\epsilon+1)(2\epsilon+1)(3\epsilon+4)(z(4\epsilon+5) - 2\epsilon^2 - 7\epsilon - 5)_2 F_1 \left(1, -\epsilon; \frac{3}{2}; z \right) \right. \\ & + (4z^2(\epsilon+1)(2\epsilon+3)(7\epsilon+8) + z(2\epsilon+1)(\epsilon(4\epsilon+7) + 4) + (\epsilon+2)(2\epsilon+1)(4\epsilon+5)) \\ & \times {}_3F_2 \left(1, -2\epsilon-1, -\epsilon; \frac{1}{2}-\epsilon, \epsilon+2; z \right) \\ & \left. + 2(z-1)(2\epsilon+1)(z(\epsilon(8\epsilon+19) + 12) + (\epsilon+2)(4\epsilon+5)) {}_3F_2 \left(1, -2\epsilon, -\epsilon; \frac{1}{2}-\epsilon, \epsilon+2; z \right) \right) \end{aligned} \quad (14)$$

FIG. 1. Feynman diagrams that contribute to the cross-correlator (4) at LO.

where

$$z = \frac{q^2}{4m^2}, \quad (15)$$

and all polynomials in z have been omitted as they will not contribute to the LSR. In (14), m is a heavy quark mass, Γ is the Gamma function, and ${}_pF_q(\cdots; \cdots; z)$ are generalized hypergeometric functions [32]. Expanding (14) in ϵ , we find

$$\Pi^{(I)}(z) = \frac{20\alpha_s m^4 z(z-1) {}_2F_1(1, 1; \frac{5}{2}; z)}{27\pi^3} \frac{1}{\epsilon} + \frac{d}{d\epsilon} \Pi^{(I)}(z; \epsilon) \Big|_{\epsilon=0}. \quad (16)$$

We do not include an explicit expression for the derivative term on the right-hand side of (16) as it will be replaced by (27) shortly. Expanding the remaining terms from (13) in ϵ , we find

$$\Pi^{(II)}(z) = \frac{z(3 - {}_2F_1(1, 1; \frac{5}{2}; z))}{36\pi(z-1)} \langle \alpha G^2 \rangle \quad (17)$$

$$\Pi^{(III)}(z) = \frac{(-3(44z^2 - 108z + 73) + (24z^3 - 56z^2 + 38z + 3) {}_2F_1(1, 1; \frac{5}{2}; z))}{13824\pi^2 m^2 (z-1)^3} \langle g^3 G^3 \rangle \quad (18)$$

$$\Pi^{(IV)}(z) = \frac{\langle g^3 G^3 \rangle}{13824\pi^2 m^2 (z-1)^2} \left(132z - 183 + (-24z^2 + 38z + 3) {}_2F_1\left(1, 1; \frac{5}{2}; z\right) \right) \quad (19)$$

$$\Pi^{(V)}(z) = \frac{\langle g^3 G^3 \rangle}{4608\pi^2 m^2 (z-1)^2} \left(12z - 15 - (2z - 3) {}_2F_1\left(1, 1; \frac{5}{2}; z\right) \right) \quad (20)$$

$$\Pi^{(VI)}(z) = \frac{4\alpha_s^2 \langle \bar{q}q \rangle^2}{243m^2 (z-1)^3} \left(3(44z^2 - 108z + 73) - (24z^3 - 56z^2 + 38z + 3) {}_2F_1\left(1, 1; \frac{5}{2}; z\right) \right). \quad (21)$$

The perturbative result (16) contains a nonlocal divergence. We eliminate this nonlocal divergence through operator mixing under renormalization as in [2,21,22]. The meson current (1) is renormalization-group (RG)

invariant, and so we only need to consider the operator mixing of the hybrid current (2). The only operators that can mix with (2) and possibly generate nonzero contributions to the LO renormalized correlator are $j_\nu^{(m)}$ given in (1)

and

$$j_\nu^{(c)} = \bar{Q} i \gamma^5 D_\nu Q \quad (22)$$

where $D_\nu = \partial_\nu - \frac{i}{2} g_s \lambda^a A_\nu^a$ is the covariant derivative. Then, the replacement

$$j_\nu^{(h)} \rightarrow j_\nu^{(h)} + Z_1 j_\nu^{(m)} + Z_2 j_\nu^{(c)} \quad (23)$$

for renormalization constants Z_1 and Z_2 must result in a perturbative contribution free of nonlocal divergences. Substituting (23) into (4) in D dimensions gives

$$\begin{aligned} & i \int d^D x e^{iq \cdot x} \langle \Omega | \tau j_\mu^{(m)} j_\nu^{(h)} | \Omega \rangle \\ & \rightarrow i \int d^D x e^{iq \cdot x} \langle \Omega | \tau j_\mu^{(m)} j_\nu^{(h)} | \Omega \rangle \\ & + i Z_1 \int d^D x e^{iq \cdot x} \langle \Omega | \tau j_\mu^{(m)} j_\nu^{(m)} | \Omega \rangle \\ & + i Z_2 \int d^D x e^{iq \cdot x} \langle \Omega | \tau j_\mu^{(m)} j_\nu^{(c)} | \Omega \rangle. \end{aligned} \quad (24)$$

The two terms in (24) containing Z_1 and Z_2 each generate a renormalization-induced diagram, both of which are shown in Fig. 2. Evaluating these two diagrams and selecting Z_1 and Z_2 such that the right-hand side of (24) is free of nonlocal divergences, we find

$$Z_1 = -\frac{10m^2\alpha_s}{9\pi\epsilon} \quad (25)$$

$$Z_2 = -\frac{4m\alpha_s}{9\pi\epsilon}. \quad (26)$$

Substituting (25) and (26) into (24) and expanding in ϵ gives a renormalized expression

$$\begin{aligned} \Pi^{(1)}(z) &= \frac{m^4\alpha_s}{243\pi^3} \left(9(6z^2 - z - 5)_3 F_2 \left(1, 1, 1; \frac{3}{2}, 3; z \right) \right. \\ &\quad - z(48z^2 + 2z + 5)_3 F_2 \left(1, 1, 2; \frac{5}{2}, 4; z \right) \\ &\quad + 9z \left(20(z-1) \log \left(\frac{m^2}{\mu^2} \right) + 8z + 5 \right) \\ &\quad \left. \times {}_2F_1 \left(1, 1; \frac{5}{2}; z \right) \right) \end{aligned} \quad (27)$$

where, again, we have omitted polynomials in z as they will not contribute to the LSR.

Finally, collecting (17)–(21) and (27) and then substituting them into (13) gives us the LO expression for $\Pi^{(\text{OPE})}$ up to 6d condensates.

FIG. 2. Renormalization-induced Feynman diagrams resulting from (24). The square insertion denotes the current (22).

III. QCD LAPLACE SUM RULES

For Euclidean momentum $Q^2 = -q^2 > 0$, the quantity Π_1 from (5) satisfies the dispersion relation

$$\Pi(Q^2) = \frac{Q^6}{\pi} \int_{t_0}^{\infty} \frac{\text{Im}\Pi(t)}{t^3(t+Q^2)} dt + \dots \quad (28)$$

where Π on the left-hand side represents the QCD result $\Pi^{(\text{OPE})}$ and $\text{Im}\Pi(t)$ on the right-hand side is the hadronic spectral function. Equation (28) is a statement of quark-hadron duality and allows us to interpret QCD information contained in the cross-correlator in the context of hadrons. In (28), t_0 is the hadron production threshold and the $\dots$ represents unknown subtraction constants (a polynomial in Q^2). To eliminate these subtraction constants, eliminate local divergences in $\Pi^{(\text{OPE})}$, and accentuate the resonance contributions of the hadronic spectral function, we apply to (28) the Borel transform

$$\hat{\mathcal{B}} = \lim_{\substack{N, Q^2 \rightarrow \infty \\ \tau = N/Q^2}} \frac{(-Q^2)^N}{\Gamma(N)} \left(\frac{d}{dQ^2} \right)^N \quad (29)$$

where τ is the Borel parameter. This results in the formation of the 0th-order LSR [15]

$$\mathcal{R}_0(\tau) \equiv \frac{1}{\tau} \hat{\mathcal{B}} \{ \Pi(Q^2) \} = \int_{t_0}^{\infty} e^{-t\tau} \frac{1}{\pi} \text{Im}\Pi(t) dt. \quad (30)$$

We then introduce a “resonance(s) plus continuum” model

$$\frac{1}{\pi} \text{Im}\Pi(t) \rightarrow \rho^{(\text{had})}(t) + \frac{1}{\pi} \text{Im}\Pi^{(\text{OPE})}(t) \theta(t-s_0) \quad (31)$$

where $\rho^{(\text{had})}$ represents the resonance portion of the spectral function, θ is the Heaviside step function, and s_0 is the continuum threshold, and define the continuum-subtracted 0th-order LSR

$$\begin{aligned}\mathcal{R}_0(\tau, s_0) &\equiv \mathcal{R}_0(\tau) - \int_{s_0}^{\infty} e^{-t\tau} \frac{1}{\pi} \text{Im}\Pi^{(\text{OPE})}(t) dt \\ &= \int_{t_0}^{s_0} e^{-t\tau} \rho^{(\text{had})}(t) dt.\end{aligned}\quad (32)$$

To compute $\mathcal{R}_0(\tau, s_0)$, we exploit the following relation between the Borel transform and the inverse Laplace transform $\hat{\mathcal{L}}^{-1}$ [15]:

$$\begin{aligned}\frac{1}{\tau} \hat{\mathcal{B}}\{f(Q^2)\} &= \hat{\mathcal{L}}^{-1}\{f(Q^2)\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(Q^2) e^{Q^2\tau} dQ^2\end{aligned}\quad (33)$$

where $c \in \mathbb{R}$ is selected such that $f(Q^2)$ is analytic for $\text{Re}(Q^2) > c$. Generalized hypergeometric functions of the form ${}_pF_{p-1}(z)$, (such as those appearing in $\Pi^{(\text{OPE})}$) have a branch cut originating at the branch point $z = 1$ that extends along the positive real semiaxis. As such, in the complex Q^2 -plane, $\Pi^{(\text{OPE})}$ is analytic everywhere except along the negative real semiaxis for $z < -Q^2/(4m^2)$. In (33), we let $f \rightarrow \Pi^{(\text{OPE})}$ and warp the contour of integration to that shown in Fig. 3. We then apply definitions (30) and (32) to get

FIG. 3. The contour of integration used in the evaluation of the LSR (34).

$$\begin{aligned}\mathcal{R}_0(\tau, s_0) &= \int_{4m^2(1+\eta)}^{s_0} e^{-t\tau} \frac{1}{\pi} \text{Im}\Pi^{(\text{OPE})}(t) dt \\ &+ \frac{1}{2\pi i} \int_{\Gamma_\eta} e^{Q^2\tau} \Pi^{(\text{OPE})}(Q^2) dQ^2 \quad \text{for } \eta \rightarrow 0^+\end{aligned}\quad (34)$$

where

$$\text{Im}\Pi^{(\text{OPE})}(t) = \sum_{i=1}^{\text{VI}} \text{Im}\Pi^{(i)}(t) \quad (35)$$

and, from (17)–(21) and (27)

$$\begin{aligned}\text{Im}\Pi^{(\text{I})}(t) &= \frac{\alpha_s}{108\pi^2 t^2 \sqrt{t-4m^2}} \left(12m^2 \sqrt{t-4m^2} (20m^6 - 6m^4 t - 6m^2 t^2 + 5t^3) \sinh^{-1} \left(\frac{1}{2m} \sqrt{t-4m^2} \right) \right. \\ &\quad \left. + \sqrt{t(t-4m^2)} \left(60m^6 + 22m^4 t - 7m^2 t^2 + 30m^2 t(t-4m^2) \log \left(\frac{m^2}{\mu^2} \right) - 6t^3 \right) \right)\end{aligned}\quad (36)$$

$$\text{Im}\Pi^{(\text{II})}(t) = \frac{-m^2}{6\sqrt{t(t-4m^2)}} \langle \alpha G^2 \rangle \quad (37)$$

$$\text{Im}\Pi^{(\text{III})}(t) = \frac{24m^6 + 76m^4 t - 28m^2 t^2 + 3t^3}{288\pi t^{3/2} (t-4m^2)^{5/2}} \langle g^3 G^3 \rangle \quad (38)$$

$$\text{Im}\Pi^{(\text{IV})}(t) = \frac{6m^4 + 19m^2 t - 3t^2}{288\pi t^{3/2} (t-4m^2)^{3/2}} \langle g^3 G^3 \rangle \quad (39)$$

$$\text{Im}\Pi^{(\text{V})}(t) = \frac{m^2(6m^2 - t)}{96\pi t^{3/2} (t-4m^2)^{3/2}} \langle g^3 G^3 \rangle \quad (40)$$

$$\text{Im}\Pi^{(\text{VI})}(t) = \frac{-64\pi\alpha_s^2(24m^6 + 76m^4 t - 28m^2 t^2 + 3t^3)}{81t^{3/2}(t-4m^2)^{5/2}} \langle \bar{q}q \rangle^2. \quad (41)$$

Evaluating the integrals on the right-hand side of (34) for all six OPE terms leads to several divergences in η that, when summed, delicately cancel leaving us with a finite LSR. A detailed treatment of the evaluation of (34) for similar inputs is available in [2]. Here, for the sake of brevity, we omit these details and present the LSR:

$$\begin{aligned}
\mathcal{R}_0(\tau, s_0) = & \int_{4m^2}^{s_0} e^{-t\tau} \frac{1}{\pi} (\text{Im}\Pi^{(\text{I})}(t) + \text{Im}\Pi^{(\text{II})}(t) + p(t)) dt \\
& - \frac{me^{-4m^2\tau}}{384\pi^2} \left(\sqrt{\pi\tau}(3 - 8m^2\tau) \text{erf}\left(\sqrt{(s_0 - 4m^2)\tau}\right) + \frac{162e^{-s_0\tau}}{(s_0 - 4m^2)^{3/2}} (-8e^{s_0\tau}m^2\sqrt{\pi}((s_0 - 4m^2)\tau)^{3/2} \right. \\
& + e^{4m^2\tau}(3s_0 + 32m^4\tau - 8m^2(1 + s_0\tau)) + 6e^{s_0\tau}m^2E_{5/2}((s_0 - 4m^2)\tau)) \left. \right) \langle g^3 G^3 \rangle \\
& - \frac{8m\alpha_s^2 e^{-8m^2\tau}}{243} \left(196m^2\tau^{3/2}e^{4m^2\tau} \left(4\sqrt{\pi} - 3\Gamma\left(-\frac{3}{2}, (s_0 - 4m^2)\tau\right) \right) \right. \\
& + \frac{31e^{-s_0\tau}}{(s_0 - 4m^2)^{3/2}} \left(2e^{8m^2\tau}(32m^4\tau - 8m^2(s_0\tau + 1) + 3s_0) \right. \\
& \left. \left. - 2\sqrt{\pi\tau}(s_0 - 4m^2)^{3/2}(8m^2\tau - 3)e^{\tau(4m^2 + s_0)} \text{erf}\left(\sqrt{\tau(s_0 - 4m^2)}\right) \right) \right) \langle \bar{q}q \rangle^2
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
p(t) = & \frac{-1}{20736\pi t^{3/2}(\sqrt{t} + 2m)^2\sqrt{t} - 4m^2} \\
& \times (81m(16m^3 + 16m^2\sqrt{t} + 4mt + t^{3/2})\langle g^3 G^3 \rangle \\
& + 2048\pi^2\alpha_s^2(48m^4 + 48m^3\sqrt{t} + 188m^2t \\
& + 127mt^{3/2} + 24t^2)\langle \bar{q}q \rangle^2)
\end{aligned} \tag{43}$$

and the imaginary parts $\text{Im}\Pi^{(\text{I})}$ and $\text{Im}\Pi^{(\text{II})}$ are given in (36) and (37) respectively. The integral on the right-hand side of (42) can be evaluated analytically; however, we omit the result for the sake of brevity.

Renormalization-group improvement [33] requires that the strong coupling and quark mass get replaced by their corresponding running quantities evaluated at renormalization scale μ . At one-loop in the $\overline{\text{MS}}$ renormalization scheme, for charmonium we have

$$\alpha_s \rightarrow \alpha_s(\mu) = \frac{\alpha_s(M_\tau)}{1 + \frac{25\alpha_s(M_\tau)}{12\pi} \log\left(\frac{\mu^2}{M_\tau^2}\right)} \tag{44}$$

$$m \rightarrow m_c(\mu) = \bar{m}_c \left(\frac{\alpha_s(\mu)}{\alpha_s(\bar{m}_c)} \right)^{12/25} \tag{45}$$

and for bottomonium

$$\alpha_s \rightarrow \alpha_s(\mu) = \frac{\alpha_s(M_Z)}{1 + \frac{23\alpha_s(M_Z)}{12\pi} \log\left(\frac{\mu^2}{M_Z^2}\right)} \tag{46}$$

$$m \rightarrow m_b(\mu) = \bar{m}_b \left(\frac{\alpha_s(\mu)}{\alpha_s(\bar{m}_b)} \right)^{12/23} \tag{47}$$

where [4]

$$\alpha_s(M_\tau) = 0.330 \pm 0.014 \tag{48}$$

$$\alpha_s(M_Z) = 0.1185 \pm 0.0006 \tag{49}$$

$$\bar{m}_c = (1.275 \pm 0.025) \text{ GeV} \tag{50}$$

$$\bar{m}_b = (4.18 \pm 0.03) \text{ GeV}. \tag{51}$$

For charmonium, $\mu \rightarrow \bar{m}_c$ and for bottomonium, $\mu \rightarrow \bar{m}_b$. Finally, the following values are used for the gluon and quark condensates [34–36]:

$$\langle \alpha G^2 \rangle = (0.075 \pm 0.02) \text{ GeV}^4 \tag{52}$$

$$\langle g^3 G^3 \rangle = ((8.2 \pm 1.0) \text{ GeV}^2) \langle \alpha G^2 \rangle \tag{53}$$

$$\langle \bar{q}q \rangle = -(0.23 \pm 0.03)^3 \text{ GeV}^3. \tag{54}$$

IV. ANALYSIS AND RESULTS

We now turn our attention to $\rho^{(\text{had})}$ [recall (31)] which represents the resonance portion of the hadronic spectral function and contains the experimentally determined resonances we wish to probe for conventional meson-hybrid mixing. Resonances in $\rho^{(\text{had})}$ which couple to both the conventional meson current (1) and the hybrid current (2) can be thought of as meson-hybrid mixtures.

Our analysis approach is to build a variety of models of the 1^{++} heavy quarkonium mass spectra (i.e., a variety of choices for $\rho^{(\text{had})}$) that take known resonance masses as inputs, and test them for conventional meson-hybrid mixing. In Table I, we list all 1^{++} charmonium resonances that have a Particle Data Group entry in [4], and in Table II, we do the same for bottomonium. Note that, in Tables I and II, all entries have $I^G = 0^+$.

TABLE I. Particle Data Group masses of 1^{++} charmonium resonances [4].

Name	Mass (GeV)
$\chi_{c1}(1P)$	3.51
$X(3872)$	3.87
$X(4140)$	4.15
$X(4274)$	4.27

TABLE II. Particle Data Group masses of 1^{++} bottomonium resonances [4].

Name	Mass (GeV)
$\chi_{b1}(1P)$	9.89
$\chi_{b1}(2P)$	10.26
$\chi_{b1}(3P)$	10.51

Laplace sum rules are generally insensitive to resonance widths, and so we consider $\rho^{(\text{had})}$ to be a sum of narrow resonances, i.e.,

$$\rho^{(\text{had})}(t) = \sum_{i=1}^n \xi_i \delta(t - m_i^2) \quad (55)$$

where n is the number of resonances in the model. The $\{\xi_i\}_{i=1}^n$ are mixing parameters (products of hadron masses, signed hadronic couplings, and mixing angle factors) which are a measure of the combined coupling to both the conventional meson current and hybrid current. For example, in a simple case of two-state mixing, we would have $\xi_1 = m_H^2 m_M^2 f_H f_M \sin^2 \theta$ and $\xi_2 = -m_H^2 m_M^2 f_H f_M \cos^2 \theta$ where θ is a mixing angle between pure hybrid and meson states with corresponding couplings f_H and f_M . A state with both conventional meson and hybrid components has $\xi_i \neq 0$. A pure conventional meson state or pure hybrid state has $\xi_i = 0$. The specific models for which we present results are given for the charmonium and bottomonium sectors in Tables III and IV respectively.

Substituting (55) into (32) gives

$$\mathcal{R}_0(\tau, s_0) = \sum_{i=1}^n \xi_i e^{-m_i^2 \tau}. \quad (56)$$

TABLE III. A representative collection of hadron models analyzed in the charmonium sector.

Model	m_1 (GeV)	m_2 (GeV)	m_3 (GeV)	m_4 (GeV)
C1	3.51			
C2	3.51	3.87		
C3	3.51	3.87	4.15	
C4	3.51	3.87	4.15	4.27

TABLE IV. A representative collection of hadron models analyzed in the bottomonium sector.

Model	m_1 (GeV)	m_2 (GeV)	m_3 (GeV)
B1	9.89		
B2	9.89	10.26	
B3	9.89	10.26	10.51

To extract hadronic properties from (56) together with LSR (42), we must first, for each model, select an acceptable range of τ values, i.e., a Borel window $(\tau_{\min}, \tau_{\max})$. To determine the Borel window, we follow the same methodology as in [21,22,37,38]. To select $\tau_{\min}$, we consider

$$\frac{\mathcal{R}_0(\tau, s_0)}{\mathcal{R}_0(\tau, \infty)}, \quad (57)$$

i.e., the ratio of the LSR's hadron contribution to its hadron plus continuum contribution. We demand that this ratio be at least 10%. To select $\tau_{\max}$, we demand that the LSR converge where convergence is taken to mean that the magnitude of successive OPE terms be at most one-third that of any previous term. This means that we require the magnitude of the 4d gluon condensate contribution be less than one-third that of the perturbative contribution. We also require that the magnitude of the sum of the 6d gluon and quark condensate contributions be less than one-third that of 4d gluon condensate contribution.

For particular choices of $\{m_i\}_{i=1}^n$, the quantities $\{\xi_i\}_{i=1}^n$ and s_0 are extracted as best fit parameters to (56). To do so, we partition the Borel window into $N = 20$ equal length subintervals with $\{\tau_j\}_{j=0}^N$, and define

$$\chi^2(\xi_1, \dots, \xi_n, s_0) = \sum_{j=0}^N \left(\mathcal{R}_0(\tau_j, s_0) - \sum_{i=1}^n \xi_i e^{-m_i^2 \tau_j} \right)^2. \quad (58)$$

Minimizing (58) gives predictions for $\{\xi_i\}_{i=1}^n$ and s_0 corresponding to the best fit agreement between QCD and the hadronic model in question.

The procedure described above for selecting a Borel window depends on s_0 . However, s_0 is not known at the outset. It is one of the parameters that emerges from the minimization of (58). But the definition of (58) requires a Borel window. Hence, we determine both the Borel window and s_0 iteratively. We start with a seed value of $s_0 = 2m_{\max}^2$ where $m_{\max}$ is the mass of the heaviest resonance in the model. This seed value separates the continuum from the resonances by a generous margin. We generate a Borel window for this s_0 value according to the criteria outlined above. Minimization of (58) then yields an updated value for s_0 . This process is iteratively repeated until s_0 and the Borel window settle. For all the models examined in the charmonium sector, we found that the Borel window settled to $\tau_{\min} = 0.17 \text{ GeV}^{-2}$ to $\tau_{\max} = 0.41 \text{ GeV}^{-2}$, and, in the

TABLE V. Continuum thresholds and χ^2 values for hadron models defined in Table III and their resulting extracted mixing parameters with their theoretical uncertainties.

Model	s_0 (GeV ²)	$\chi^2 \times 10^9$ (GeV ¹²)	ζ (GeV ⁶)	$\frac{\xi_1}{\zeta}$	$\frac{\xi_2}{\zeta}$	$\frac{\xi_3}{\zeta}$	$\frac{\xi_4}{\zeta}$
C1	18.8	7990	0.18(1)	1			
C2	28.8	76.3	0.83(7)	0.47(2)	-0.53(2)		
C3	18.8	27.4	2.6(4)	0.21(2)	-0.45(1)	0.34(2)	
C4	31.7	0.0586	44(6)	0.03(1)	-0.16(1)	0.46(1)	-0.35(1)

TABLE VI. Continuum thresholds and χ^2 values for hadron models defined in Table IV and their resulting extracted mixing parameters with their theoretical uncertainties.

Model	s_0 (GeV ²)	$\chi^2 \times 10^6$ (GeV ¹²)	ζ (GeV ⁶)	$\frac{\xi_1}{\zeta}$	$\frac{\xi_2}{\zeta}$	$\frac{\xi_3}{\zeta}$
B1	128	2580	49(1)	1		
B2	282	1980	70(4)	0.30(1)	0.70(1)	
B3	241	0.832	1905(28)	0.16(1)	-0.48(1)	0.36(1)

bottomonium sector, all of the models have Borel windows that settled to $\tau_{\min} = 0.02 \text{ GeV}^{-2}$ to $\tau_{\max} = 0.12 \text{ GeV}^{-2}$. These persistent values for the Borel window across different models in each sector demonstrates the LSR's insensitivity to changes in s_0 and is consistent with our findings in [2].

We extract $\{\xi_i\}_{i=1}^n$ and s_0 for each of the models defined in Tables III and IV, and present our results in Tables V and VI respectively. Instead of presenting each ξ_i , we present ζ and $\frac{\xi_i}{\zeta}$ where

$$\zeta = \sum_{i=1}^n |\xi_i|. \quad (59)$$

The errors included are associated with the strong coupling values (48)–(49), the quark masses (50)–(51), the condensates (52)–(54), and an allowed $\pm 0.1 \text{ GeV}$ variability in

FIG. 4. Relative residuals (60) for models C2–C4 using the optimized values of s_0 and ξ_i from Table V. Residuals for model C1 are not shown because they are much larger than the scales of the figure.

the renormalization scale [39]. We also allow for the endpoints of the Borel window to vary by 0.1 GeV^{-2} in the charmonium sector and 0.01 GeV^{-2} in the bottomonium sector. The vacuum saturation parameter κ from (11) is not varied because the numerical contribution to the LSR (42) stemming from the 6d quark condensate diagram is negligible. Our results are most sensitive to varying the value of $\tau_{\min}$ and varying the value of the quark masses (50) and (51). In Fig. 4, we plot relative residuals representing the difference between the QCD prediction and the resonance plus continuum hadronic model,

$$r(\tau) = \frac{\mathcal{R}_0(\tau, s_0) - \sum_{i=1}^n \xi_i e^{-m_i^2 \tau}}{\mathcal{R}_0(\tau, s_0)}, \quad (60)$$

[the numerator in (60) is the difference between the left- and right-hand sides of (56)] for models C2–C4 using the optimized values of s_0 and $\{\xi_i\}$ from Table V. In Fig. 5, we

FIG. 5. Relative residuals (60) for models B1–B3 using the optimized values of s_0 and ξ_i from Table VI.

do the same for models B1–B3 using the optimized values from Table VI.

V. DISCUSSION

As shown in Tables V and VI, the inclusion of heavy resonances beyond the ground state significantly improves agreement between QCD and experiment in both the charmonium and bottomonium sectors. In particular, in the charmonium sector, going from three to four resonances (i.e., from model C3 to C4 in Table V) decreases the value of the χ^2 [recall (58)] by a factor of 468 while in the bottomonium sector, going from two to three resonances (i.e., from model B2 to B3 in Table VI) decreases the χ^2 by a factor of 2380. This improvement can also be seen from the trend of decreasing magnitude of relative residuals with increasing number of resonances depicted in Figs. 4 and 5. For the highest mass resonance in a given model we define a measure of its contribution to the LSR as in [2]:

$$\frac{|\int_{\tau_{\min}}^{\tau_{\max}} \xi_n e^{-m_n^2 \tau} d\tau|}{\sum_{i=1}^n |\int_{\tau_{\min}}^{\tau_{\max}} \xi_i e^{-m_i^2 \tau} d\tau|} \quad (61)$$

where n is the number of resonances in the model. The highest mass resonances make substantial contributions to the LSRs in spite of the exponential suppression inherent in LSRs: in the charmonium sector, evaluating (61) for model C4 gives 0.25, and in the bottomonium sector, evaluating (61) for model B3 gives 0.30. These results, coupled with the dramatic improvement in χ^2 -values when compared to models containing less resonances, indicate the significant impact that the highest mass resonances have on the LSRs, and cause us to favour models C4 and B3.

In the charmonium sector, model C4 indicates that there is almost no conventional meson-hybrid mixing in the $\chi_{c1}(1P)$, minimal mixing in the $X(3872)$, and significant mixing in both the $X(4140)$ and $X(4274)$. Assuming the $\chi_{c1}(1P)$, the lightest known resonance in this sector, has a large conventional meson component [40], then our results indicate that it has very little hybrid component. Regarding the interpretation of the $X(3872)$, if the $X(3872)$ does have a significant hybrid component, a possibility put forth in [11], then our results indicate that it does not have a significant conventional meson component. However, if the $X(3872)$ does have a large conventional meson component as argued in [10], then our results indicate that it does not have a large hybrid component. In addition, our results imply that the $X(4140)$ and the $X(4274)$ both contain significant conventional meson and hybrid components.

In the bottomonium sector, model B3 indicates that there is minimal conventional meson-hybrid mixing in the $\chi_{b1}(1P)$ and significant mixing in both the $\chi_{b1}(2P)$ and the $\chi_{b1}(3P)$. Thus, assuming the $\chi_{b1}(1P)$, the lightest observed resonance in this sector, contains a significant conventional meson component [40], our results imply that it does not have a large hybrid component. Also, our results indicate that the $\chi_{b1}(2P)$ and the $\chi_{b1}(3P)$ each contain significant conventional meson and hybrid components.

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