Comment on “Nucleon spin-averaged forward virtual Compton tensor at large Q^2 ”

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In recent work, Hill and Paz apply the operator product expansion to forward doubly virtual Compton scattering. The resulting large- Q^2 form of the amplitude $W_1(0, Q^2)$ is compatible with the one we obtain by extrapolation of low- Q^2 results from a chiral effective field theory, providing support for our approach. That paper also presents a result for the two-photon contribution to the Lamb shift in muonic hydrogen that has a much larger uncertainty than in previous work. We show that this is an overestimate arising from the inclusion of the proton pole term in the subtracted dispersion relation for W_1 .

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In Ref. [1], Hill and Paz use the operator product expansion (OPE) to determine the large- Q^2 behavior of the amplitude for forward doubly virtual Compton scattering (VVCS) from a nucleon. This work extends and corrects an older treatment by Collins [2]. The result is important because it can help to constrain the uncertainty in the proton radius extracted from measurements of the Lamb shift of muonic hydrogen [3,4].

In particular, the result provides the large- Q^2 form of the subtraction term in one of the dispersion relations used in that analysis [5]. (See Ref. [6] for a summary of the theory of the Lamb shift in muonic hydrogen.) Previous work by the current authors [7] used chiral effective field theory (χ EFT) to determine the low- Q^2 form of the subtraction term and then extrapolated this to higher Q^2 by matching it on to the asymptotic $1/Q^2$ behavior of the OPE. The resulting values of the coefficient of the $1/Q^2$ term are compatible with those from the QCD calculations of Ref. [1], providing support for the approach taken in Ref. [7].

Ref. [1] also presents results for the two-photon contribution to the muonic Lamb shift obtained with a simple interpolation between their OPE result at large Q^2 and the low-energy theorem (LET) governing the behavior as $Q^2 \rightarrow 0$. This gives an energy that is compatible with the previous result, but with an uncertainty that is an order of magnitude larger. As we discuss below, this is an overestimate, arising in the most part from the way that the

authors of Ref. [1] treat the pole terms of the Born amplitude.

To recap, most of the two-photon contributions to the muonic Lamb shift can be determined from empirical input using the methods outlined by Pachucki [5]. The Born (elastic) pieces can be written as integrals over proton electromagnetic form factors. Dispersion relations can be used to express the inelastic pieces in terms of integrals over structure functions. However the dispersion relation for $W_1(\nu, Q^2)$ (the amplitude multiplying $-g_{\mu\nu} + q_\mu q_\nu/q^2$ in the usual tensor decomposition) requires a subtraction. The resulting term in the energy has the form of an integral over $W_1(0, Q^2)$. This zero-energy limit of the amplitude for forward VVCS cannot be measured directly and is one of the main sources of uncertainty in extractions of the proton radius. Its small Q^2 behavior is constrained by a LET [8–10] and can be calculated in χ EFTs [7,11–13].

The large- Q^2 behavior of W_1 can be obtained from QCD using the OPE [2]. The calculation in Ref. [1] shows that $Q^2 W_1(0, Q^2)/(2M_p^2) \sim 0.27\text{--}0.37$ for $Q \gtrsim 5 \text{ GeV}^2$. For comparison, the extrapolation of χ EFT in Ref. [7] gives a central value of 1.3, but with a wide uncertainty band: 0.2–23. Although the central value differs from the OPE by a factor of 3 to 4, these results are consistent within that large uncertainty. Moreover, in the corresponding contribution to the Lamb shift, the integral over W_1 is heavily weighted to low Q^2 so that differences at high- Q^2 have only a small effect. This indicates that that a smooth interpolation between χ EFT and the OPE should lead to results that lie within the uncertainties of Ref. [7]. For example, even reducing the entire contribution from $Q^2 > 0.3 \text{ GeV}^2$ by a factor of 4 would alter the Lamb shift by only about $0.3 \mu\text{eV}$. This is not to imply that the extrapolation is an alternative determination of the high- Q^2 behavior

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in QCD, merely that it provides a good enough estimate in the context of the muonic Lamb shift.

The bottom line of Ref. [1] for the two-photon contribution to the muonic Lamb shift is $\Delta E(2P - 2S) = +30 \pm 13 \mu\text{eV}$. For comparison, the result of Ref. [7] is $+33 \pm 2 \mu\text{eV}$. The central values are similar¹ but the uncertainty in Ref. [1] is a factor of 6 larger.

The main reason for this large uncertainty is that the subtraction is applied to a dispersion relation that includes the proton poles [1,17], in contrast to other treatments that use a dispersion relation for the amplitude with the Born terms removed [5,7,16]. If the proton form factors were known with sufficient accuracy, the results from both approaches would be the same, but this is not the case in practice. The version used in Ref. [1] leads to integrals that are much more sensitive to proton form factors that are not currently well determined.

At low ν and Q^2 , the forward VVCS amplitude can be written in the form [7–10]²

$$W_1(\nu, Q^2) = \frac{2Q^4}{Q^4 - 4M^2\nu^2} G_M^2(Q^2) - 2F_D^2(Q^2) + \frac{2M}{\alpha} [Q^2\beta_M + \nu^2(\alpha_E + \beta_M)] + \dots, \quad (1)$$

up to terms of fourth order in ν and Q . Here α_E and β_M are the electric and magnetic polarizabilities of the proton [15], and

$$F_D(Q^2) = \frac{G_E(Q^2) + \frac{Q^2}{4M^2} G_M(Q^2)}{1 + \frac{Q^2}{4M^2}} \quad (2)$$

is its Dirac form factor. The nonpole term involving F_D^2 [the second term in Eq. (1)] has been the subject of some controversy and we return to it below; for now we focus on the pole term.

Setting $\nu = 0$ and expanding W_1 to order Q^2 gives the LET in the form in Refs. [1,17]:

$$W_1(0, Q^2) = 2\kappa(1 + \kappa) - \frac{2}{3}(1 + \kappa)^2 r_M^2 Q^2 + \frac{2}{3} r_E^2 Q^2 - \frac{\kappa}{M^2} Q^2 + \frac{2M}{\alpha} \beta_M Q^2 + \dots, \quad (3)$$

¹There are minor differences in the input data. Ref. [1] takes the magnetic polarizability β_M from the Particle Data Group [14] and a dipole ansatz for the elastic form factors. Ref. [7] takes β_M from a χEFT analysis [15] and the elastic contributions from the work of Carlson and Vanderhaeghen [16], who considered several empirical parametrizations of the form factors. Both works take the inelastic contribution from Ref. [16].

²Note that W_1 as defined in Ref. [1] differs from T_1 in Ref. [7] by a factor of $2M/e^2$.

where r_E and r_M are the charge and magnetic radii of the proton and κ is its anomalous magnetic moment. The slope of this with respect to Q^2 controls the low- Q^2 contribution to the subtraction term (cf. Eq. (37) of Ref. [1]). Using the same input as in Ref. [1], the various pieces of the LET contribute to the slope as follows: Born pole [second term of Eq. (3)] $-80.4 \pm 7.9 \text{ GeV}^{-2}$, Born nonpole (third and fourth terms) $+11.1 \pm 0.2 \text{ GeV}^{-2}$, magnetic polarizability, $+8.4 \pm 1.3 \text{ GeV}^{-2}$.

As noted above, the dispersion relation used in Ref. [1] includes the Born pole. All three pieces listed above contribute to its subtraction. In this case, both the slope of W_1 and its error are dominated by the subtraction of the pole term. The rather poorly determined magnetic radius of the proton appears multiplied by a large factor containing the square of the magnetic moment. It thus contributes significantly to the large uncertainty found in Ref. [1].

In addition, the subtraction of the Born pole leads to a second enhancement in the uncertainty. The slope of the corresponding term in Eq. (3) is an order of magnitude larger than that arising from the magnetic polarizability. Including it leads to a large factor multiplying the poorly known form factor of the subtraction term. Even with the constraints from χEFT , this form factor is one of the main contributions to the uncertainty in Ref. [7]. In their interpolation between the low- Q^2 regime and their OPE, the authors of Ref. [1] use no theoretical input on terms of higher-order in Q^2 , assuming that the order- Q^4 term has a typical hadronic scale but leaving even its sign unfixed. This leads to a somewhat larger relative error on their result for the subtraction term than that in Ref. [7]. Multiplying this by the slope of W_1 including the pole contribution enhances the absolute error by nearly an order of magnitude. Combined with the contribution from the magnetic radius just discussed, this leads to the large overall uncertainty in the two-photon energy found in Ref. [1]. In contrast, if the Born pole is removed from the dispersion relation, the magnetic radius does not appear separately from the magnetic form factor in the Born contribution to the Lamb shift. This form factor has a much better determined dependence on Q^2 and so leads to a smaller uncertainty.

The Born pole is a well defined nonanalytic structure in the VVCS amplitude, with a residue that is given in terms of on-shell proton form factors. It is thus straightforward to remove it from the amplitude W_1 and apply a subtracted dispersion relation to the remainder [5,16]. The large uncertainties in Ref. [1] associated with the pole contribution to $W_1(0, Q^2)$ are absent in approaches that use a dispersion relation with the pole removed [5,7,16]. In addition, the low- Q^2 form factor for the subtraction has been calculated using χEFT in Ref. [7], including the order- Q^4 term that is left undetermined in Ref. [1]. This further

constrains the subtraction term in the region that makes an important contribution to Lamb shift.

As just discussed, the main uncertainty in their two-photon exchange energy is driven by the way the pole term in the VVCS amplitude is treated in Ref. [1]. There is one quite separate final issue, which concerns the treatment of the nonpole Born term [the F_D^2 term in Eq. (1)]. This can be generated from a Dirac equation with proton form factors—the procedure that the authors of Ref. [1] refer to as the “sticking-in form factors” ansatz. However, as argued in Ref. [7], the nonpole Born term follows from Lorentz invariance and so it is natural to treat it together with the pole term. Up to order Q^2 , the terms in $F_D(Q^2)^2$ are required by low-energy theorems, but those of order Q^4 and above can get contributions from low-energy constants that are beyond the order of the χ EFT calculation in Ref. [7]. Using the empirical form factor for $F_D(Q^2)^2$ instead of the one from the χ EFT means that some higher-order terms have been included in $W_1(0, Q^2)$. Provided they are of natural size, such terms have already been accounted for in the estimated uncertainty on the χ EFT result [7] and the effects of this choice should fall within that error estimate.

In Appendix B of Ref. [1], it is suggested that there may be larger uncertainties associated with this treatment. This is illustrated by a comparison of results for the subtraction term from the third order χ EFT result of Alarcón *et al.* [12] and the fourth-order EFT-informed result of Ref. [7]. This is misleading since the subtraction term of Ref. [12] cannot not be compared with those from other approaches—as Alarcón *et al.* point out. This is because they find a cancellation between contributions of the Δ to the subtraction and inelastic terms, and hence they omit these entirely from both terms [12]. In particular, their result—if taken as complete—would correspond to a magnetic polarizability with the wrong sign. At fourth order the χ EFT has the freedom to reproduce the experimental value of β , and so the subtraction term has the correct slope in the small Q^2 region that is crucial for the Lamb shift [7].

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