$D^*D^*\bar{D}$ and $D^*D^*\bar{D}^*$ three-body systems

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The hidden charm $X(3872)$ resonance is usually thought to be a $D^{*0}\bar{D}^0$ meson-antimeson molecule with quantum numbers $J^{PC} = 1^{++}$. If this is the case, there is the possibility that there might be three body bound states with two charmed mesons and a charmed antimeson. Here we argue that the theoretical existence of this type of three body molecules is expected from heavy quark spin symmetry. If applied to the two body sector, this symmetry implies that the interaction of the $D^{*0}\bar{D}^{*0}$ meson-antimeson pair in the $J^{PC} = 2^{++}$ channel is the same as in the $J^{PC} = 1^{++}$ $D^{*0}\bar{D}^0$ case. From this we can infer that the $J^P = 3^-$ $D^{*0}D^{*0}\bar{D}^{*0}$ molecule will be able to display the Efimov effect if the scattering length of the 2^{++} channel is close enough to the unitary limit. Heavy quark spin symmetry also indicates that the $J^P = 2^-$ $D^{*0}D^{*0}\bar{D}^0$ molecule is analogous to the $J^P = 3^-$ $D^{*0}D^{*0}\bar{D}^{*0}$ one. That is, it can also have a geometric spectrum. If we consider these triply heavy trimers in the isospin symmetric limit, the Efimov effect disappears and we can in principle predict the fundamental state of the 2^- $D^*D^*\bar{D}$ and 3^- $D^*D^*\bar{D}^*$ systems. The same applies to the $B^*B^*\bar{B}^*$ system: if the $Z_b(10650)$ is an isovector $B^*\bar{B}^*$ molecule then the 0^- isodoublet and the 1^- , 2^- isoquartet $B^*B^*\bar{B}^*$ trimers might bind, but do not display Efimov physics. Finally from heavy flavour symmetry it can be argued that scattering in the BD two-body system might be resonant. This would in turn imply the possibility of Efimov physics in the BBD three body system.

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The three boson system in the unitary limit, i.e., when the two-body scattering length goes to infinity, shows a geometric spectrum in which the ratio of the energy of the n th and $(n+1)$ th excited trimer is $E_n/E_{n+1} \simeq 521$. This spectrum, originally theorized by Efimov in the seventies [1], has been confirmed experimentally with cesium atoms a decade ago [2]. The relevance of the Efimov effect [3,4] is not limited to atomic physics, but extends to nuclear physics where it might play an important role in the binding of the triton [5–7], the three-alpha structure of Carbon-12 [8] or two-neutron halo nuclei [9,10]. The Efimov effect also extends to other three-body systems in the unitary limit: relevant to the present investigation is the case of two identical noninteracting boson of species A and a third particle of species B that interacts resonantly with the other two, i.e., the AB scattering length diverges. This system displays a geometric spectrum too [1]. If m_A and m_B are the

masses of particles of species A and B respectively, the larger the mass imbalance m_A/m_B the more conspicuous will it be the geometric spectrum. For $m_A = m_B$ the scaling factor between two consecutive states is $E_n/E_{n+1} \simeq (1986.1)^2$ [11]. The existence of a geometric spectrum can also happen in specific two-body systems that fulfill a series of properties [12]. A remarkable example is the $\bar{D}^*\Sigma_c - \bar{D}^*\Lambda_c$ system, which might be identified with the $P_c(4450)$ pentaquarklike resonance [13] in the molecular picture [14–18].

There is the interesting question of whether there are three hadron systems (besides the three nucleon system) where the Efimov effect might play a role [19–22]. This is particularly relevant in view of the recent renaissance of hadron spectroscopy triggered by the discovery of the $X(3872)$ by the Belle collaboration more than one decade ago [23]. The $X(3872)$ is a $J^{PC} = 1^{++}$ narrow resonance that is located within current experimental uncertainties on top of the D^0D^{*0} threshold. This striking coincidence naturally leads to the interpretation that the $X(3872)$ is a D^0D^{*0} bound state [24–26]. The strongest hint that the $X(3872)$ is molecular lies in the branching ratio of the isospin breaking decays $\Gamma(X \rightarrow J/\Psi 2\pi)$ and $\Gamma(X \rightarrow J/\Psi 3\pi)$ [27], which are easy to explain in the molecular picture [28,29] but are more problematic if the $X(3872)$ is a

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more compact object [30]. This does not preclude the existence of a small $c\bar{c}$ component, negligible at long distances but important for explaining the radiative charmonia decays $X \rightarrow J/\Psi\gamma$ and $X \rightarrow \Psi(2S)\gamma$ [31] which probe the structure of the $X(3872)$ at short distances [32,33].

If the $X(3872)$ is indeed a bound state its binding energy will be extremely small and the scattering length extremely large [26]. That is, a molecular $X(3872)$ will be close to the unitary limit. This has prompted the question of whether there will be an Efimov-like geometric spectrum if we add a third charmed meson. In this regard, Canham, Hammer and Springer [19] considered D^0X and $D^{*0}X$ scattering from effective field theory and pointed out that the system does not fulfill the conditions to have a geometric Efimov-like spectrum. The reason is the existence of several channels in which the interaction is not resonant, which dilutes the interaction strength provided by the $1^{++} D^0\bar{D}^{*0}$ channel. In the present manuscript we update the previous conclusions regarding the $D^{*0}X$ system. We notice that if we consider heavy quark spin symmetry then $D^{*0}\bar{D}^{*0}$ scattering in the $J^{PC} = 2^{++}$ channel might also be resonant. If this additional channel happens to be resonant, it will provide enough attraction as to trigger the Efimov effect in the $2^- D^{*0}X$ system and the $3^- D^{*0}D^{*0}\bar{D}^{*0}$ three body system. Conversely if we turn off the $2^{++} D^{*0}\bar{D}^{*0}$ scattering channel we recover the conclusions of Ref. [19].

In the isospin symmetric limit the $X(3872)$ can be considered to be an isoscalar $1^{++} D^*\bar{D}$ molecule with a binding energy of about 4 MeV. In this limit there is no Efimov effect, independently of the location of the bound state. The reason is the loss of attraction owing to the numerical factors involved in the coupling of isospin. Only if the $1^{++} D^*\bar{D}$ and $2^{++} D^*\bar{D}^*$ interactions were resonant in the isoscalar and isovector channels will the Efimov effect be present. The absence of Efimov physics makes it possible to make predictions for the $2^- D^*D^*\bar{D}$ and $3^- D^*D^*\bar{D}^*$ systems without including a three body force. For these two systems we find a three-body binding energy of about $B_3 \simeq 2\text{--}3$ MeV, where this number is defined with respect to the dimer-particle threshold. If we refer to the three body threshold instead, the 2^- and 3^- states will be located 7–9 MeV below it.

The $Z_b(10610)$ and $Z_b(10650)$ isovector resonances have been speculated to be $I^G(J^{PC}) = 1^+(1^{+-})B^*\bar{B}$ and $B^*\bar{B}^*$ molecules. If this is the case the previous ideas also apply *mutatis mutandis* to the $B^*B^*\bar{B}^*$ system. The molecules that can be predicted from the $Z_b(10650)$ are a trio of degenerate trimers: a 0^- isodoublet and a 1^- and 2^- isoquartet. Their existence is contingent on the location of the $Z_b(10650)$ as a bound state: for a two-body binding energy of $B_2 = 2$ MeV, the trimer binding will be $B_3 \simeq 1$ MeV.

The Efimov effect is more probable for systems with a mass imbalance, as we have already mentioned. This idea

can be combined with heavy flavour symmetry, from which we can theorize the existence of B^+D , $B^{*+}D^0$, B^+D^{*0} and $B^{*+}D^{*0}$ molecules [34]. If these $\bar{b}c$ -like molecules exists and are close to the unitary limit, the $B^+B^+D^0$, $B^+B^{*+}D^0$, $B^{*+}B^{*+}D^0$ and $B^+B^+D^{*0}$ family of three heavy meson system might be a likely candidate for a geometric spectrum. The analysis of the Faddeev equations for these family of trimers indicates that the appearance of the Efimov effect is possible. The discrete scaling factor for the binding energies turns out to be $E_{n+1}/E_n \sim 4000\text{--}5000$, where the exact number depends on the particular system under consideration.

The manuscript is structured as follows: after the introduction, we briefly comment on the heavy-light spin decomposition of the three heavy meson states in Sec. II. After that we will present the Faddeev equations for these systems in Sec. III. We will analyze the conditions for the existence of the Efimov effect in Sec. IV. We will present the results for the $2^- D^*D^*\bar{D}$ and $3^- D^*D^*\bar{D}^*$ systems in Sec. V. The isodoublet 0^- and isoquartet 1^- and $2^- B^*B^*\bar{B}^*$ trimers are studied in Sec. VI. Then we will extend the formalism to the BBD , BB^*D , B^*B^*D and BBD^* molecules in Sec. VII. Finally we will present the conclusions of this work at the end.

II. HEAVY-LIGHT SPIN DECOMPOSITION OF THE $HH\bar{H}$ SYSTEM

From heavy quark spin symmetry (HQSS) we expect the spectrum of hadron systems containing a mixture of heavy ($Q = c, b$) and light ($q = u, d, s$) degrees of freedom to be independent of the spin of the heavy quarks [35–39]. Conversely, the spectrum of these systems mostly depends on the total angular momentum of the light quarks. We can state this idea in more concrete terms by considering the heavy-light decomposition of a system of N heavy hadrons $H_1, H_2, \dots, H_N$ with total angular momentum J

$$|H_1H_2\dots H_N(J)\rangle = \sum_{J_H, J_L} c(J_H, J_L) J_H \otimes J_L |J\rangle, \quad (1)$$

where the $c(J_H, J_L)$ are the coefficients involved in this angular momentum decomposition.¹ For this system *loosely speaking* the energy levels can be determined in terms of the total light quark angular momenta

$$E_N = \sum d(J_L)E(J_L), \quad (2)$$

¹Actually the decomposition is more involved than stated here: each J_H and J_L can be further decomposed in orthogonal contributions corresponding to different intermediate couplings of the angular momenta. For example, if we have three heavy hadrons $J_H = \frac{1}{2}$ can be further decomposed into $0_{12} \otimes \frac{1}{2}$ and $1_{12} \otimes \frac{1}{2}$, where 0_{12} and 1_{12} refer to coupling of heavy hadrons 1 and 2. Yet for the current illustrative purposes this distinction will not be made.

with

$$d(J_L) = \sum_{J_H} |c(J_H, J_L)|^2. \quad (3)$$

The reason why we say *loosely speaking* is because for a nonrelativistic system of N heavy hadrons what is determined from this decomposition is the potential energy rather than the binding energy, yet the idea can be illuminating.

If we consider for instance the charmed meson-antimeson system, the spectrum is determined by the fact that the total light spin is either $J_L = 0$ or $J_L = 1$. For instance, the $D^*\bar{D}^*$ system with $J^{PC} = 2^{++}$ can be trivially decomposed as

$$|D^*\bar{D}^*(2^{++})\rangle = 1_H \otimes 1_L|_{J=2}. \quad (4)$$

For the $DD^*/D^*\bar{D}$ system we have exactly the same decomposition for the $J^{PC} = 1^{++}$ state²

$$\begin{aligned} |DD^*(1^{++})\rangle &= \frac{1}{\sqrt{2}}|D\bar{D}^*(1^+)\rangle + \frac{1}{\sqrt{2}}|D^*\bar{D}(1^+)\rangle, \\ &= 1_H \otimes 1_L|_{J=1}, \end{aligned} \quad (5)$$

which corresponds to the $X(3872)$. From this we deduce that the interaction in the $J_L = 1$ channel is attractive and leads to the formation of a bound state. We also deduce that if the 1^{++} system binds, the same should be true for the 2^{++} system as the interaction is identical. Of course this conclusion is subjected to the limitation that HQSS is violated at the $\Lambda_{\text{QCD}}/m_c \sim 10\text{--}15\%$ level in the charm sector plus the fact that the $X(3872)$ is barely bound. If we take this into account we can still expect the 2^{++} interaction to be strong, but not necessarily resonant or leading to a bound state.

This idea can be trivially extended to the three charmed meson system, in which case the total light angular momentum is either $J_L = \frac{1}{2}$ or $J_L = \frac{3}{2}$. For the quantum numbers $J^P = 3^-$ the decomposition is indeed trivial

$$|D^*D^*\bar{D}^*(3^-)\rangle = \frac{3}{2_H} \otimes \frac{3}{2_L} \Big|_{J=3}. \quad (6)$$

There is also a $J^P = 2^-$ system for which the decomposition is identical

$$\begin{aligned} &\frac{1}{\sqrt{3}}|D^*D^*\bar{D}(2^-)\rangle + \frac{1}{\sqrt{3}}|D^*D\bar{D}^*(2^-)\rangle \\ &+ \frac{1}{\sqrt{3}}|DD^*\bar{D}^*(2^-)\rangle = \frac{3}{2_H} \otimes \frac{3}{2_L} \Big|_{J=2}, \end{aligned} \quad (7)$$

²Notice that here we are implicitly taking the C-parity convention $C|D^*\rangle = |\bar{D}^*\rangle$. This will lead to a far simpler analysis in the three body sector.

and this implies that if the 3^- trimer binds, the 2^- trimer should also bind.

III. FADDEEV EQUATIONS FOR THE $HH\bar{H}$ SYSTEM

In this section we write the Faddeev decomposition and equations for the 3^- $D^*D^*\bar{D}^*$, 2^- $D^*D^*\bar{D}^*$ and 1^- $DD\bar{D}^*$ charmed meson-meson-antimeson trimers. We will consistently assume that the DD , DD^* and D^*D^* charmed meson-meson pairs do not interact. Equivalently, we consider that their interaction is weak³ and will only provide a small subleading correction to the three body binding energy.

A. The equations for $D^*D^*\bar{D}^*$

We begin by writing the three body wave function in terms of Faddeev components for the $J^P = 3^-$ $D^{*0}D^{*0}\bar{D}^{*0}$ system

$$\Psi_{3B} = [\phi(\vec{k}_{23}, \vec{p}_1) + \phi(\vec{k}_{31}, \vec{p}_2)]|D^{*0}D^{*0}\bar{D}^{*0}\rangle, \quad (8)$$

where we assume that the $D^{*0}D^{*0}$ subsystem is not interacting, from which we can ignore the third component of the Faddeev decomposition. The Jacobi momenta are defined as usual

$$\vec{k}_{ij} = \frac{m_j\vec{k}_i - m_i\vec{k}_j}{m_i + m_j}, \quad (9)$$

$$\vec{p}_k = \frac{1}{M_T}[(m_i + m_j)\vec{k}_k - m_k(\vec{k}_i + \vec{k}_j)], \quad (10)$$

with m_1, m_2, m_3 the masses of particles 1, 2, 3, $M_T = m_1 + m_2 + m_3$ the total mass and ijk an even permutation of 123. In this case we have $m_1 = m_2 = m_3 = m_{D^{*0}}$. If the charmed mesons interact via a potential of the type

$$V_{D^*\bar{D}^*}(J=2) = C_0 g(p)g(p'), \quad (11)$$

where $g(p)$ is a regulator function then the $D^*\bar{D}^*$ T-matrix can be written as

$$T_{23} = \tau(Z)g(p)g(p'), \quad (12)$$

while the Faddeev component ϕ admits the ansatz

$$\phi(k, p) = \frac{g(k)}{Z - \frac{k^2}{2\mu_{23}} - \frac{p^2}{2\mu_1}} a(p), \quad (13)$$

³If we adapt the OBE model of Ref. [40] to the case at hand, the isovector 0^+ DD , 1^+ DD^* and 2^+ D^*D^* scattering lengths happen to be $a_0 = -0.1$ fm, -0.6 fm and -0.7 fm. That is, these channels are slightly attractive and might increase the trimer binding energy by a very small amount.

with the reduced masses defined as

$$\frac{1}{\mu_{ij}} = \frac{1}{m_i} + \frac{1}{m_j}, \quad (14)$$

$$\frac{1}{\mu_k} = \frac{1}{m_k} + \frac{1}{m_i + m_j}. \quad (15)$$

From the previous, we find that $a(p)$ follows the integral equation

$$a(p_1) = \tau_2(Z_{23}) \int \frac{d^3 p_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a(p_2), \quad (16)$$

where the driving term B_{12}^0 is given by

$$B_{ij}^0(\vec{p}_i, \vec{p}_j) = \frac{g(q_i)g(q_j)}{Z - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} - \frac{p_3^2}{2m_3}}, \quad (17)$$

with $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$ and

$$\vec{q}_k = \frac{m_j \vec{p}_i - m_i \vec{p}_j}{m_j + m_i}. \quad (18)$$

The integral equation can be discretized, in which case finding the bound state solution reduces to an eigenvalue problem.

B. The equations for $D^* D^* \bar{D}$

The Faddeev decomposition of the three body wave function for the $J^P = 2^- D^* D^* \bar{D}$ system involves three different particle channels

$$\begin{aligned} \Psi_{3B} = & [\phi_1(\vec{k}_{23}, \vec{p}_1) + \phi_1(\vec{k}_{31}, \vec{p}_2)] |D^{*0} D^{*0} \bar{D}^0\rangle \\ & + [\phi_2(\vec{k}_{23}, \vec{p}_1) + \phi_2(\vec{k}_{31}, \vec{p}_2)] |D^{*0} D^0 \bar{D}^{*0}\rangle \\ & + [\phi_2(\vec{k}_{23}, \vec{p}_1) + \phi_2(\vec{k}_{31}, \vec{p}_2)] |D^0 D^{*0} \bar{D}^{*0}\rangle, \end{aligned} \quad (19)$$

but it is essentially analogous to that of the $J^P = 2^- D^* D^* \bar{D}^*$ system. Notice that we do not explicitly write the spin wave functions. The choice of wave functions is constrained by the symmetry of the first particle channel, i.e., $|D^{*0} D^{*0} \bar{D}^0\rangle$. Particles 1 and 2 are considered to be noninteracting, while the interaction with particle 3 (the charmed antimeson) can be written as

$$\langle D\bar{D}^* | T_{23}(Z) | D\bar{D}^* \rangle = \tau_{1D}(Z) g(p) g(p'), \quad (20)$$

$$\langle D\bar{D}^* | T_{23}(Z) | D^* \bar{D} \rangle = \tau_{1E}(Z) g(p) g(p'), \quad (21)$$

for the $D\bar{D}^*$ system, while for the $D^* \bar{D}^*$ case we write

$$\langle D^* \bar{D}^* | T_{23}(Z) | D^* \bar{D}^* \rangle = \tau_2(Z) g(p) g(p'). \quad (22)$$

That is, for the $D\bar{D}^*$ we are distinguishing between a direct scattering process, where the D and D^* are not exchanged, and an exchange scattering process, where the D and D^* are flipped.

The three independent Faddeev components ϕ_1 , ϕ_2 , and φ_2 can be written as

$$\phi_1(k, p) = \frac{g(k)}{Z - \frac{k^2}{2\mu_{23}} - \frac{p^2}{2\mu_1}} a_1(p), \quad (23)$$

$$\phi_2(k, p) = \frac{g(k)}{Z - \frac{k^2}{2\mu_{23}} - \frac{p^2}{2\mu_1}} a_2(p), \quad (24)$$

$$\varphi_2(k, p) = \frac{g(k)}{Z - \frac{k^2}{2\mu_{31}} - \frac{p^2}{2\mu_2}} b_2(p), \quad (25)$$

where for ϕ_1 the ordering of particles is $|123\rangle = |D^{*0} D^{*0} \bar{D}^0\rangle$, while for ϕ_2 and φ_2 is $|123\rangle = |D^{*0} D^0 \bar{D}^{*0}\rangle$. The corresponding Faddeev equations are

$$\begin{aligned} a_1(p_1) = & \tau_{1D}(Z_{23}) \int \frac{d^3 \vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(\vec{p}_2) \\ & + \tau_{1E}(Z_{23}) \int \frac{d^3 \vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) b_2(\vec{p}_2), \end{aligned} \quad (26)$$

$$\begin{aligned} a_2(p_1) = & \tau_{1E}(Z_{23}) \int \frac{d^3 \vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2) \\ & + \tau_{1D}(Z_{23}) \int \frac{d^3 \vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) b_2(p_2), \end{aligned} \quad (27)$$

$$b_2(p_1) = \tau_2(Z_{13}) \int \frac{d^3 \vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_2(p_2). \quad (28)$$

These equations can be simplified if we consider a series of properties of the system. First, we can write the direct and exchange T-matrices as combinations of the T-matrices in the $D\bar{D}^*$ positive and negative C-parity channels

$$\tau_{1D} = \frac{1}{2} \tau_1^+ + \frac{1}{2} \tau_1^-, \quad (29)$$

$$\tau_{1E} = \frac{1}{2} \tau_1^+ - \frac{1}{2} \tau_1^-, \quad (30)$$

where $\tau_1^\pm$ refers to the scattering in the $C = \pm 1$ channel. Second, we will assume that the interaction in the positive C-parity $X(3872)$ channel is strong while the interaction in the negative C-parity channel is weak and can be ignored. In this case we end up with

$$\tau_{1D} = \tau_{1E} = \frac{1}{2} \tau_1^+. \quad (31)$$

from which it also follows that

$$a_1 = a_2. \quad (32)$$

Third, from HQSS we also expect that

$$\tau_1^+ = \tau_2, \quad (33)$$

from which we obtain that

$$b_2 = a_1. \quad (34)$$

With all these relations and if we ignore the mass difference between the D^0 and D^{*0} mesons, the Faddeev equations reduce to

$$a_1(p_1) = \tau_2(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2), \quad (35)$$

that is, exactly the same Faddeev equation that we had for the $J^P = 3^- D^{*0}D^{*0}\bar{D}^{*0}$ system.

Instead of the constructive approach that we have followed before, alternatively we could have simply begun with the HQSS three body wave function with $J_L = \frac{3}{2}$, that is

$$\begin{aligned} \Psi_{3B} = & [\phi_1(\vec{k}_{23}, \vec{p}_1) + \phi_1(\vec{k}_{31}, \vec{p}_2)] |D^{*0}D^{*0}\bar{D}^0\rangle \\ & + [\phi_1(\vec{k}_{23}, \vec{p}_1) + \phi_1(\vec{k}_{31}, \vec{p}_2)] |D^{*0}D^0\bar{D}^{*0}\rangle \\ & + [\phi_1(\vec{k}_{23}, \vec{p}_1) + \phi_1(\vec{k}_{31}, \vec{p}_2)] |D^0D^{*0}\bar{D}^{*0}\rangle, \end{aligned} \quad (36)$$

in which case we will have obtained the same eigenvalue equation, only in a more direct manner.

C. The equations for $D^0D^0\bar{D}^{*0}$

For the $J^P = 1^- D^0D^0\bar{D}^{*0}$ system the nontrivial Faddeev components can be written as

$$\begin{aligned} \Psi_{3B} = & [\phi_1(\vec{k}_{23}, \vec{p}_1) + \phi_1(\vec{k}_{31}, \vec{p}_2)] |D^0D^0\bar{D}^{*0}\rangle \\ & + \phi_2(\vec{k}_{23}, \vec{p}_1) |D^0D^{*0}\bar{D}^0\rangle \\ & + \phi_2(\vec{k}_{31}, \vec{p}_2) |D^{*0}D^0\bar{D}^0\rangle, \end{aligned} \quad (37)$$

where we are assuming that the charmed meson-meson and the $D^0\bar{D}^0$ interactions are both trivial. The ansatz for the ϕ_1 and ϕ_2 wave functions is identical to the $J^P = 2^- D^{*0}D^{*0}\bar{D}^0$ system, from which we derive the equations

$$a_1(p_1) = \tau_{1D}(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2), \quad (38)$$

$$a_2(p_1) = \tau_{1E}(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2). \quad (39)$$

We can see that the equation in the first line is homogeneous while the one in second line is inhomogeneous, i.e., $a_1(p)$ can be determined by itself, while $a_2(p)$ can be determined from $a_1(p)$. In other words, the binding energy of the charmed meson trimers can be determined exclusively from the homogeneous equation.

If we now take into account that we only expect the interaction to be resonant in the positive C-parity channel, we can rewrite the equations as

$$a_1(p_1) = \frac{1}{2} \tau^+(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2), \quad (40)$$

$$a_2(p_1) = \frac{1}{2} \tau^+(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2). \quad (41)$$

This will be later very useful to understand why we do not expect the $J^P = 1^- D^0D^0\bar{D}^{*0}$ system to have a geometric Efimov-like spectrum.

D. The isospin symmetric limit

Finally we consider the previous three body molecules in the isospin symmetric limit. For the $J^P = 3^- D^*D^*\bar{D}^*$ molecule the Faddeev decomposition is

$$\Psi_{3B} = [\phi(\vec{k}_{23}, \vec{p}_1) + \phi(\vec{k}_{31}, \vec{p}_2)] |D^*D^*\bar{D}^*\rangle |1 \otimes \frac{1}{2}\rangle_{\frac{1}{2}}, \quad (42)$$

where in addition to the particle wave function we have included the isospin wave function

$$|I_{12} \otimes I_3\rangle_{I_T}, \quad (43)$$

which means that particles 1 and 2 couple to isospin I_{12} , particle 3 has isospin I_3 and the total isospin of the system is I_T . The choice $I_{12} = 1$ together with the fact that the total spin is $S = 3$ (and hence $S_{12} = 2$) implies that the spin and isospin wave functions are symmetric. In the isospin symmetric limit we expect the $1^{++} D\bar{D}^*$ and $2^{++} D^*\bar{D}^*$ interactions to be resonant in the isoscalar $I = 0$ channel and weak in the isovector channel. If we ignore the isovector interaction, the Faddeev equation reduces to

$$a(p_1) = \frac{3}{4} \tau_2^{\text{IS}}(Z_{23}) \int \frac{d^3\vec{p}_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a(p_2), \quad (44)$$

where the $3/4$ is an isospin factor which is derived from overlapping the three body isospin wave function with the fact that we consider the isoscalar channel for particles 2 and 3.

The same Faddeev decomposition and equations apply for the $J^P = 2^-$ molecule after making the substitution

$$|D^*D^*\bar{D}^*\rangle \rightarrow \frac{1}{\sqrt{3}}[|D^*D^*\bar{D}\rangle + |D^*D\bar{D}^*\rangle + |DD^*\bar{D}^*\rangle]. \quad (45)$$

For the $J^P = 1^-$ $|DD\bar{D}^*\rangle$ molecule the inclusion of isospin follows the same pattern as in the $J^P = 3^-$ $|D^*D^*\bar{D}^*\rangle$ case. That is, we end up with the same equations as in the neutral charmed meson case

$$a_1(p_1) = \frac{3}{8}\tau_1^{\text{IS}(+)}(Z_{23}) \int \frac{d^3p_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2), \quad (46)$$

$$a_2(p_1) = \frac{3}{8}\tau_1^{\text{IS}(+)}(Z_{23}) \int \frac{d^3p_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a_1(p_2), \quad (47)$$

modulo the $\frac{3}{4}$ factor from the isospin projection and the fact that we are using the isoscalar T-matrix.

IV. THE EFIMOV EFFECT IN THE $HH\bar{H}$ SYSTEM

Now we will consider the previous set of Faddeev equations in the unitary limit. In all cases the eigenvalue equation reduces to

$$a(p_1) = \lambda \tau(Z_{23}) \int \frac{d^3p_2}{(2\pi)^3} B_{12}^0(\vec{p}_1, \vec{p}_2) a(p_2), \quad (48)$$

with $\lambda = 1, \frac{3}{4}, \frac{1}{2}$ and $\frac{3}{8}$ depending on the situation. If we make the simplification $m_D = m_D^* = m_H$, which is compatible with the heavy quark limit, and we consider the unitary limit then we have

$$\tau(Z_{23}) \rightarrow -\frac{2\pi\sqrt{3}}{m_H} \frac{1}{p_1}, \quad (49)$$

$$\int \frac{d^2p_2}{4\pi} B_{12}^0 \rightarrow -\frac{m_H}{2p_1 p_2} \log \left[\frac{p_1^2 + p_2^2 + p_1 p_2}{p_1^2 + p_2^2 - p_1 p_2} \right]. \quad (50)$$

From this we arrive to the equation

$$p_1 a(p_1) = \frac{\lambda}{\pi} \sqrt{\frac{3}{4}} \frac{1}{p_1} \int_0^\infty dp_2 p_2 a(p_2) \times \log \left[\frac{p_1^2 + p_2^2 + p_1 p_2}{p_1^2 + p_2^2 - p_1 p_2} \right], \quad (51)$$

which after the change of variable $p^2 a(p) = b(p)$ transforms into

$$b(p) = \frac{\lambda}{\pi} \sqrt{\frac{3}{4}} \int_0^\infty dx \frac{b(xp)}{x} \log \left[\frac{1+x^2+x}{1+x^2-x} \right]. \quad (52)$$

This equations admits power-law solutions of the type $b(p) = p^s$, in which case we end up with an eigenvalue equation for s

$$1 = \frac{\lambda}{\pi} \sqrt{\frac{3}{4}} \int_0^\infty dx x^{s-1} \log \left[\frac{1+x^2+x}{1+x^2-x} \right], \quad (53)$$

where the integral above can be evaluated analytically. From this we can rewrite the eigenvalue equation into the more familiar form

$$1 = \lambda I_{\text{Efimov}}(s) = \lambda \frac{4}{\sqrt{3}s} \frac{\sin \frac{\pi s}{6}}{\cos \frac{\pi s}{2}}. \quad (54)$$

The eigenvalue equation has complex solutions of the type $s = \pm i s_0$ for

$$\lambda \geq \lambda_c = \frac{3\sqrt{3}}{2\pi} \simeq 0.826993. \quad (55)$$

From this we can see that the $J^P = 2^-, 3^-$ three body states might display the Efimov effect if we consider the neutral components only, i.e., $D^{*0}D^{*0}\bar{D}^0$ and $D^{*0}D^{*0}\bar{D}^{*0}$. For this configurations the solution of the previous eigenvalue equation yields $s_0 = 0.413697$, which implies that the system shows discrete scale invariance under transformations $p \rightarrow \mu_0 p$ with $\mu_0 = e^{\pi/s_0} \simeq 1986.1$. If we consider binding energies instead of momenta the scaling transformation becomes $E_B \rightarrow \mu_0^2 E_B$ where $\mu_0^2 = 3.9447 \times 10^6$. The sheer size of this number implies that the existence of the Efimov effect in these systems is more of a theoretical curiosity than something that could ever be hoped to be observed. The energy of the first excited Efimov state of a $D^{*0}D^{*0}\bar{D}^0$ or $D^{*0}D^{*0}\bar{D}^{*0}$ system is in fact orders of magnitude smaller than the width of the D^{0*} meson. It is nonetheless interesting in the sense that it provides an example of a hadronic system where this type of spectrum could be realized.

If we consider the isospin symmetric limit we have $\lambda = \frac{3}{4} < \lambda_c$. This is interesting in the following sense: the existence of the Efimov effect in a three body system implies the requirement of a three-body force to properly renormalize the Faddeev equations [5,6]. That is, Faddeev calculations in the isospin symmetric limit have predictive power. Here it is also curious to notice that even if isospin symmetry is broken at the level of the masses of the charmed mesons, if the interactions are isospin symmetric there will be no requirement of three-body forces at short distances.

For the $J^P = 1^-$ state we have $\lambda = \frac{1}{2}$ or $\frac{3}{8}$ depending on whether we are considering the long range neutral component or the isospin symmetric limit. In both cases this is insufficient to trigger the Efimov geometric scaling. This situation is indeed equivalent to the one considered in Ref. [19] for $D^0 X$ scattering. Here a few comments are in order: if we consider the $D^{0*} X$ system but do not take into account that $2^{++} D^{*0}\bar{D}^{*0}$ might be resonant too, then the numerical factor for the $J^P = 2^- D^{*0}D^{*0}\bar{D}^{*0}$ system is also

$\lambda = \frac{1}{2}$ and we end up agreeing with the conclusions of Ref. [19]. Other interesting observation is that if in the $1^{++} D^0\bar{D}^{*0}$ system was also resonant, then we will end up with $\lambda = 1$ for the $J^P = 1^-$ state. Yet there is no evidence of the existence of a negative C-parity partner of the $X(3872)$.

V. THE $J^P = 2^-, 3^-$ THREE BODY STATES

Here we will consider the 2^- and 3^- three body molecules in the isospin symmetric limit. In this limit we simply take the masses of the D and D^* charmed mesons to be their isospin average $m_D = (m_{D^0} + m_{D^-})/2 = 1867$ MeV and $m_{D^*} = (m_{D^{*0}} + m_{D^{*-}})/2 = 2009$ MeV. As a consequence the $X(3872)$ is bound by about 4 MeV. We describe the $X(3872)$ potential in terms of a contact interaction of the type

$$\langle p'|V|p\rangle = C_0(\Lambda)g_\Lambda(p')g_\Lambda(p), \quad (56)$$

with $C_0(\Lambda)$ a coupling constant that depends on a cut-off Λ and g_Λ a regulator function. Following Ref. [39] we will take $\Lambda = 0.5\text{--}1.0$ GeV and a Gaussian regulator $g_\Lambda(p) = \exp(-p^2/\Lambda^2)$. If we now consider HQSS in the line of Refs. [38,39], the $X(3872)$ implies a 2^{++} $X(4012)$ state with a binding energy of

$$B_2 = 5_{-3}^{+3} \text{ MeV} \quad (B_2 = 6_{-5}^{+8} \text{ MeV}), \quad (57)$$

for $\Lambda = 0.5$ GeV ($\Lambda = 1$ GeV). If we take into account isospin violation for the thresholds of the 2^{++} state, $D^{*0}\bar{D}^{*0}$ and $D^{*+}D^{*-}$, then in a first approximation the binding energy with respect to the neutral component is $B_2(D^{*0}\bar{D}^{*0}) \simeq B_2 - 3.3$ MeV. From this we can write the condition

$$B_2 \geq 3.3 \text{ MeV}, \quad (58)$$

for the neutral component to be bound and the 2^{++} state to be safe. Taking into account the uncertainty in the binding energy of the 2^{++} state, there is a sizable probability that its neutral component will be unbound.

For the discussion of the three body states, we will begin with the $3^- D^*D^*\bar{D}^*$ molecule. A three body calculation indicates that the binding of the 3^- state is

$$B_3 = 3_{-2}^{+2} \text{ MeV} \quad (B_3 = 3_{-3}^{+4} \text{ MeV}), \quad (59)$$

below the two-body binding threshold, by which we mean that the location of the $D^*D^*\bar{D}^*$ is $m(3^-) = 3m_{D^*} - B_2 - B_3$. In this case safety with respect to the isospin breaking thresholds implies the condition

$$(B_2 + B_3) \geq 3.3 \text{ MeV}, \quad (60)$$

for the state with $|IM_I\rangle = |\frac{1}{2} - \frac{1}{2}\rangle$, which contains the $D^{*0}D^{*0}\bar{D}^{*0}$ and $D^{*0}D^{*-}D^{*+}$ thresholds. For comparison we have $(B_2 + B_3) = 8_{-4}^{+4}$ and 9_{-5}^{+9} MeV for $\Lambda = 0.5$ and 1 GeV (adding the errors in quadrature). This is less stringent than in the two-body case and suggest that there is the possibility of a Borromean configuration, in which the two-body 2^{++} state is unbound but the three body 3^- state is bound. Yet checking these conclusions will require a full calculation including isospin breaking, which is beyond the exploratory scope of the present manuscript.

The $2^- D^*D^*\bar{D}$ is more complex for the following reasons: while the interaction in the 1^{++} channel is fixed, the interaction in the 2^{++} channel is derived from HQSS and subjected to a $\Lambda_{\text{QCD}}/m_c \sim 15\%$ uncertainty. Besides, there is the technical complication that the masses of the three charmed mesons are not identical. This mismatch between the 1^{++} and 2^{++} channels has a curious consequence: if the interaction in the 2^{++} channel is stronger than expected, it happens that the two-body binding energy of the $2^{++} D^*\bar{D}^*$ system will grow quicker than the three-body binding of the $2^- D^*D^*\bar{D}$ system. The reason is that the attraction increases only in one of the channels, but not in the other. In particular it can happen that the $2^- D^*D^*\bar{D}$ bound state can end up above the DX_2 threshold if the X_2 is too deep, which happens for $B_2(X_2) \geq 16.6$ and 12.5 MeV for $\Lambda = 0.5$ and 1 GeV, respectively. How this happens can be seen in Fig. 1, where the binding energy of the 2^- trimer

TABLE I. Predictions for the three body binding energies of the $cc\bar{c}$ - and $bb\bar{b}$ -type molecular trimers considered in this work. In the upper part of the table we have the $J^P = 2^- D^*D^*\bar{D}$ and $3^- D^*D^*\bar{D}^*$ trimers, which have been deduced from HQSS and the assumption that the $X(3872)$ is molecular. These are the most solid predictions in this work. In the bottom part we have the isodoublet 0^- and isoquartet 1^- , $2^- B^*B^*\bar{B}^*$ trimers. The binding energies of these trimers depend on the hypothesis that the $Z_b(10650)$ is indeed molecular and located below the $B^*\bar{B}^+$ threshold. The calculations have been made in the isospin symmetric limit.

3B System	2B System	J^P	I	B_2 (0.5 GeV)	B_3 (0.5 GeV)	B_2 (1.0 GeV)	B_3 (1.0 GeV)
$D^*D^*\bar{D}$	$D^*\bar{D}/D^*\bar{D}^*$	2^-	$\frac{1}{2}$	Input/ 5_{-3}^{+3}	$1.9_{-1.4}^{+0}$	Input/ 6_{-4}^{+8}	$1.3_{-1.4}^{+0}$
$D^*D^*\bar{D}^*$	$D^*\bar{D}^*$	3^-	$\frac{1}{2}$	5_{-3}^{+3}	3_{-2}^{+2}	6_{-4}^{+8}	3_{-3}^{+4}
		0^-	$\frac{1}{2}$				
		1^-	$\frac{3}{2}$	2 ± 2 (Input)	$1.1_{-1.2}^{+1.3}$	2 ± 2 (Input)	$1.0_{-1.0}^{+1.1}$
		2^-	$\frac{3}{2}$

FIG. 1. Binding energy B_3 of the $J^P = 2^- D^* D^* \bar{D}$ trimer versus the binding energy of the theorized $J^{PC} = 2^{++} D^* \bar{D}^*$ X_2 partner of the $X(3872)$. The dots indicate the dimer binding energy for which the trimer is above the $X_2 D$ threshold and becomes unstable, which is $B_2 = 16.6$ and 12.5 MeV for $\Lambda = 0.5$ and 1 GeV respectively. The vertical line indicates $B_2 = 3.3$ MeV: for $B_2 \leq 3.3$ MeV the neutral $D^{*0} \bar{D}^{*0}$ component of the X_2 wave function will not bind.

is shown as a function of the binding energy of the X_2 . In Fig. 1 we can also notice that the most tightly bound configuration for the trimer happens when the binding of the X_2 is identical to that of the $X(3872)$. That is not the case in the $3^- D^* D^* \bar{D}^*$ molecule because the binding in the three body system grows a bit faster than in the two body one.

The predictions for the $D^* D^* \bar{D}$ and $D^* D^* \bar{D}^*$ trimers are summarized in Table I. We stress the theoretical nature of the present work. The 2^{++} partner of the $X(3872)$ has not been observed yet. This could mean that it has simply not been detected or it could mean that it does not exist. Probably the best chance for its detection is $e^+ e^- \rightarrow \psi(nS) \rightarrow \gamma X_2$ (with ψ a 1^{--} charmonium) in the 4.4–4.5 GeV region [41]. Among the theoretical reasons for the X_2 not to exist the most prosaic is that HQSS is not exact: if the interaction is weaker than expected the X_2 will simply become a virtual state [39], which might be difficult to detect. Other possibility is that part of the attraction that binds the $X(3872)$ might come from the coupling to a nearby 1^{++} charmonium: the X_2 does not benefit from these extra attraction, resulting in an interaction too weak to bind [42]. In these two scenarios the three body bound states might still survive contingent on how much attraction is lost owing to these effects. A third option is the one pion exchange potential [43], which could lead to a much more bound and broad X_2 .⁴ In this third scenario the $3^- D^* D^* \bar{D}^*$

⁴This in turn might suggest the identification of the X_2 with the $X(3915)$ hidden charm resonance discovered by Belle [44]. This was proposed for instance in Ref. [45], where a detailed discussion of this scenario can be found in Ref. [46]. A more standard interpretation of the $X(3915)$ is that of the $\chi_{c2}(2P)$ charmonium [47].

molecule will be also much more bound, while the $2^- D^* D^* \bar{D}$ trimer will decay into DX_2 . Yet we consider this scenario less realistic because of the large cutoffs required for the X_2 to be tightly bound [43], while other theoretical studies with pion exchanges and a softer cutoff find the X_2 to be much more shallow [39] and narrow [48]. Be it as it may, unless the X_2 is detected the discussion of the previous effects will remain theoretical.

VI. THE $J^P = 0^-, 1^-, 2^- B^* B^* \bar{B}^*$ THREE BODY STATES

The $Z_b(10610)$ and $Z_b(10650)$ isovector hidden-bottom resonances, which we will also call Z_b and Z'_b for short, are also strong candidates to be molecular. If this is the case, they might be $B^* \bar{B}$ and $B^* \bar{B}^*$ bound states with quantum numbers $I^G = 1^+$ and $J^{PC} = 1^{+-}$. From this assumption it is easy to adapt the previous formalism to the $B^* B^* \bar{B}^*$ system. If we assume a noninteracting $B^* B^*$ pair, the ansatz to the Faddeev decomposition of the wave function is

$$\Psi_{3B} = [\phi(\vec{k}_{23}, \vec{p}_1) + \phi(\vec{k}_{31}, \vec{p}_2)] |B^* B^* \bar{B}^*\rangle, \quad (61)$$

where the $|B^* B^* \bar{B}^*\rangle$ will be in a given spin and isospin configuration that we have not indicated yet. If we consider the Z'_b channel to be the only nontrivial scattering channel, the eigenvalue equation will be given by Eq. (48). In turn this equation depends on the numerical factor λ , which we can write as

$$\lambda(B^* B^* \bar{B}^*) = \lambda_S \lambda_I, \quad (62)$$

with λ_S and λ_I a spin and isospin factor. The largest this factor can be is $\lambda = \frac{3}{4}$, for which there exist three configurations of the $B^* B^* \bar{B}^*$ system. The first is the isodoublet $J^P = 0^-$ configuration, with the spin and isospin wave functions

$$|B^* B^* \bar{B}^*\left(J^P = 0^-, I = \frac{1}{2}\right)\rangle = |1_{12} \otimes 1\rangle_J |0_{12} \otimes \frac{1}{2}\rangle_I, \quad (63)$$

which lead to $\lambda_S = 1$ and $\lambda_I = \frac{3}{4}$. The second and third are the isoquartet $J^P = 1^-$ and 2^- configurations

$$\begin{aligned} &|B^* B^* \bar{B}^*\left(J^P = 1^-, I = \frac{3}{2}\right)\rangle \\ &= \left(-\frac{2}{3}|0_{12} \otimes 1\rangle_J + \frac{\sqrt{5}}{3}|2_{12} \otimes 1\rangle_J\right) \left|1_{12} \otimes \frac{1}{2}\right\rangle_I, \end{aligned} \quad (64)$$

$$|B^* B^* \bar{B}^*\left(J^P = 2^-, I = \frac{3}{2}\right)\rangle = |2_{12} \otimes 1\rangle_J \left|1_{12} \otimes \frac{1}{2}\right\rangle_I, \quad (65)$$

for which $\lambda_S = \frac{3}{4}$, $\lambda_I = 1$. For these three configurations the eigenvalue equations are identical, as far as we are only considering the interaction in the Z'_b channel. The predictions depend however on what is the binding energy of the isovector 1^{+-} $B^*\bar{B}^*$ molecule. In Ref. [49] the binding energy was estimated to be $B_2 = 4.7^{+2.3}_{-2.2}$ MeV and $0.11^{+0.14}_{-0.06}$ MeV for the Z_b and Z'_b states respectively. Here we will simply assume

$$B_2 = 2 \pm 2 \text{ MeV}, \quad (66)$$

(the average of the previous two values) for both the Z_b and Z'_b , from which we deduce a three body binding energy of

$$B_3 = 1.1^{+1.3}_{-1.1} \text{ MeV} \quad (1.0^{+1.1}_{-1.0} \text{ MeV}), \quad (67)$$

for $\Lambda = 0.5 \text{ GeV}$ ($\Lambda = 1 \text{ GeV}$).

The dependence of the trimer B_3 binding energy in the dimer B_2 binding energy is shown in Fig. 2. The dependence is not linear, but can be roughly approximated by $B_3 \sim (0.45-0.55)B_2$ for $B_2 \geq 1 \text{ MeV}$. The system is not Borromean: if $B_2 = 0$ there will be no three body bound states. For this reason the existence of the isodoublet 0^- $B^*B^*\bar{B}^*$ and isoquartet $1^-, 2^-$ $B^*B^*\bar{B}^*$ molecules is contingent to the exact nature of the Z'_b . If the Z'_b is a virtual state or if its binding is too close to the unitary limit, it will not necessarily bind. This happens for instance in Ref. [20], where they consider $B^*Z'_b$ scattering in the dibaryon formalism: this work finds an extremely large scattering length for the states we are considering here, but not a bound state, where the reason lies in their choice of the Z'_b binding energy, $0.11^{+0.14}_{-0.06}$ MeV, consistent with the extraction of Ref. [49]. If we consider the most recent analysis of Ref. [50], the Z'_b would be either on top of the $B^*\bar{B}^*$ threshold or slightly above. In the theoretical model of Ref. [51] the Z_b 's pole is also above the threshold. From this the previous three $B^*B^*\bar{B}^*$ trimers should not bind,

FIG. 2. Binding energy B_3 of the isodoublet $J = 0^-$ $B^*B^*\bar{B}$ and the isoquartet $J = 1^-, 2^-$ $B^*B^*\bar{B}$ trimers versus the binding energy of the $Z_b(10650)$ in the molecular picture, where it is an isovector 1^{+-} $B^*\bar{B}^*$ bound state.

unless there is some missing attraction not accounted for from the channels we have not considered. If we ignore the B^*B^* interaction, this missing attraction might come from $B^*\bar{B}^*$ scattering in spin and isospin channels different from the Z'_b . Their effect can be accounted for by changing the prefactor of the eigenvalue equation, Eq. (48), as follows

$$\lambda\tau(Z_{23}) \rightarrow \sum_{\alpha} \lambda_{\alpha}\tau_{\alpha}(Z_{23}), \quad (68)$$

where α refers to the other possible $B^*\bar{B}^*$ scattering channels. The missing channels for each of the trimers are

- (i) For the isodoublet $J^P = 0^-$ trimer we have the $I^G(J^{PC}) = 0^-(1^{+-})$ channel with $\lambda = 1/4$.
- (ii) For the isoquartet $J^P = 1^-$ trimer we have the $I^G(J^{PC}) = 1^-(0^{++})$ and $1^-(2^{++})$ channels with the factors $\lambda = 1/9$ and $5/36$, respectively.
- (iii) For the isoquartet $J^P = 2^-$ trimer we have the $I^G(J^{PC}) = 1^-(2^{++})$ channel with $\lambda = 1/4$.

We will merely mention the existence of these channels, but will not take them into account. We do not have any experimental information about them and their treatment will require a phenomenological model of their interactions. However we will mention that bound states have been theorized in a few of these channels [37,52,53].

VII. THE $J^P = 0^-$ BBD , 1^- BBD^* AND 2^- B^*B^*D THREE BODY SYSTEM

The heavy-light spin decomposition of the 1^{++} $D\bar{D}^*$ and the 2^{++} $D^*\bar{D}^*$ states is $1_H \otimes 1_L$, as previously explained. This leads to a potential of the type

$$V(D\bar{D}^*, 1^{++}) = V(D^*\bar{D}^*, 2^{++}) = V_1, \quad (69)$$

where V_1 indicates the potential for $S_L = 1$, with S_L the total light spin. The particular decomposition depends on the symmetries and quantum numbers of the heavy-light system under consideration. If we consider the BD , B^*D , BD^* systems the heavy-light decomposition implies a potential of the type [37,38]

$$V(BD) = V(B^*D) = V(D^*B) = \frac{1}{4}V_0 + \frac{3}{4}V_1, \quad (70)$$

where now there is V_0 , the potential for $S_L = 0$. For the B^*D^* system the decomposition will depend on the J^P quantum numbers

$$V(B^*D^*, J = 0^+) = \frac{3}{4}V_0 + \frac{1}{4}V_1, \quad (71)$$

$$V(B^*D^*, J = 1^+) = \frac{1}{2}V_0 + \frac{1}{2}V_1, \quad (72)$$

$$V(B^*D^*, J = 2^+) = V_1. \quad (73)$$

According to heavy flavor symmetry (HFS), the potentials V_0 and V_1 are identical for the $c\bar{c}$ and the $c\bar{b}$ sectors [34]. Hence we can relate the $BD/B^*D/\dots$ potentials with the $\bar{D}D/\bar{D}^*D/\dots$ ones.

From the $c\bar{c}$ sector we know that V_1 is strong, but we do not know that much about V_0 . However in the isoscalar sector there is a conspicuous experimental absence of other hidden charm molecular $D\bar{D}/D^*\bar{D}/D^*\bar{D}^*$ candidates besides the $X(3872)$. This points out to a V_0 that is either repulsive or weaker than V_1 . From this we will make the assumption that $|V_0| \ll |V_1|$ and explore the consequences. We warn that this assumption is based on very partial information. As a matter of fact there are theoretical models in which the V_0 interaction is indeed as attractive as, if not more attractive than, the V_1 , leading to the prediction of multiplets of six hidden charm molecular states [39,54,55]. But here we will simply explore the consequences of $V_0 = 0$.

Independently of V_0 , there is the trivial conclusion that the $2^+ B^*D^*$ molecule should bind [34]. The rationale is that attraction in nonrelativistic bound states depends on the reduced potential $U = 2\mu V$, with μ the reduced mass and V the potential. For the $2^+ B^*D^*$ molecule the potential is identical to the one for the $1^{++} D\bar{D}$ molecule, i.e., the $X(3872)$, and the reduced mass is 1.51 times larger. Binding is expected and concrete calculations indicate a bound state at $B = 12 \pm 5$ MeV (26_{-13}^{+14} MeV) for $\Lambda = 0.5$ GeV (1.0 GeV) [34]. The BD , B^*D and BD^* cases are more interesting. If $V_0 = 0$ the strength of the BD , B^*D and BD^* potentials is 3/4 of the one in the $X(3872)$, while their reduced masses is a bit above 4/3 of the one in the $D^*\bar{D}$ system. Concrete numbers show

$$\frac{3}{4} \frac{2\mu_{DB}}{2\mu_X} \simeq 1.07, \quad (74)$$

indicating that the nonrelativistic description of the two systems should be similar. In particular we expect the D^0B^+ , D^0B^{*+} and $D^{*0}B^+$ scattering lengths to be unnaturally large.

This opens the possibility of having the Efimov effect in the three body case. If we consider the BBD , B^*BD/BB^*D , BBD^* and B^*B^*D systems, the Faddeev equations are analogous to the ones for the $D^*D^*\bar{D}$ and $D^*D^*\bar{D}^*$ cases. We begin with the ansatz

$$\Psi_{3B} = [\phi(\vec{k}_{23}, \vec{p}_1) + \phi(\vec{k}_{31}, \vec{p}_2)] |H_b H_b H_c\rangle, \quad (75)$$

where the heavy meson wave function refers to one of these possibilities

$$|H_b H_b H_c\rangle = |B^+ B^+ D^0\rangle, \quad (76)$$

$$|H_b H_b H_c\rangle = \frac{1}{\sqrt{2}} |B^{*+} B^+ D^0\rangle + \frac{1}{\sqrt{2}} |B^+ B^{*+} D^0\rangle, \quad (77)$$

$$|H_b H_b H_c\rangle = |B^{*+} B^{*+} D^0\rangle, \quad (78)$$

$$|H_b H_b H_c\rangle = |B^+ B^+ D^{*0}\rangle, \quad (79)$$

depending on the molecule considered. In either case, following the same steps as before, we end up with the equation

$$a(p_1) = \tau(Z_{23}) \int \frac{d^3 \vec{p}_1}{(2\pi)^3} B_{12}(\vec{p}_1, \vec{p}_2) a(p_2). \quad (80)$$

The difference is that there is now a mass imbalance. If we consider the unitary limit, particularizing for the present case, we now obtain the equation

$$1 = \lambda J_{\text{Efimov}}(s, \alpha), \quad (81)$$

where J_{Efimov} is known [11,56]

$$J_{\text{Efimov}}(s, \alpha) = \frac{1}{\sin 2\alpha} \frac{2 \sin \alpha s}{s \cos \frac{\pi}{2} s}, \quad (82)$$

and with the angle α determined as

$$\alpha = \arcsin\left(\frac{1}{1 + \frac{m_3}{M}}\right), \quad (83)$$

with m_3 the mass of the charmed meson and M the mass of the bottom meson. We find that the condition for having the Efimov geometric spectrum is

$$\lambda \geq \lambda_c = \frac{\sin 2\alpha}{2\alpha}, \quad (84)$$

which for the BBD case give us $\lambda_c \simeq 0.599$ plus similar values for the other cases under consideration. It is also interesting to notice that for $m_3 = M$ we have $\alpha = \pi/6$ and we recover the original Efimov integral

$$I_{\text{Efimov}}(s) = J_{\text{Efimov}}\left(s, \frac{\pi}{6}\right). \quad (85)$$

If we go back to the $B^+ B^+ D^0$ three body system, we have $\lambda = 1$ and there should be discrete scale invariance in the unitary limit. If we define $\mu_0 = e^{\pi/s_0}$ we find the value $\mu_0 = 65.6$ and $\mu_0^2 = 4310$ for the $B^+ B^+ D^0$ molecule, plus similar numbers for the other cases, see Table III for details. In the isospin symmetric limit we have $\lambda = \frac{3}{4}$, which is now strong enough as to trigger the Efimov effect, though in this case the effect will be really weak. However in the isospin symmetric limit we do not expect the BD system to be close enough to the unitary limit as to display the Efimov effect. Yet it is important because this involves the existence of a

TABLE II. Predictions for the three body binding energies in the isospin symmetric limit for the mass-imbalanced $\bar{b}\bar{b}c$ -type molecular trimers. In the upper part of the table we have the isodoublet $J^P = 0^- B\bar{B}D$, $1^- B\bar{B}^*D$, $2^- B^*B^*D$ and $1^- B\bar{B}D^*$ molecules. Their binding is derived from the assumption that the $X(3872)$ is molecular, heavy quark symmetry (HQSS and HFS) and the hypothesis that the two-body BD interaction in the light-spin $S_L = 0$ channel is negligible in comparison with the $S_L = 1$ one. In the bottom part there are the predictions for the $B^*B^*D^*$ partners of the $B^*B^*\bar{B}^*$ trimers, for which we have used HFS. The binding of these trimers is however dependent on the cutoff used in the calculations.

3B System	2B System	J^P	I	B_2 (0.5 GeV)	B_3 (0.5 GeV)	B_2 (1.0 GeV)	B_3 (1.0 GeV)
$B\bar{B}D$	BD	0^-	$\frac{1}{2}$				
$B\bar{B}^*D$	BD/B^*D	1^-	$\frac{1}{2}$	4_{-2}^{+3}	4_{-2}^{+2}	5_{-4}^{+6}	6_{-3}^{+5}
B^*B^*D	B^*D	2^-	$\frac{1}{2}$				
$B\bar{B}D^*$	BD^*	1^-	$\frac{1}{2}$	5_{-3}^{+3}	4_{-2}^{+2}	7_{-5}^{+7}	7_{-5}^{+6}
		0^-	$\frac{1}{2}$				
$B^*B^*D^*$	B^*D^*	1^-	$\frac{3}{2}$	$\dagger_{-0.6}$	$0.2_{-0.6}^{+\dagger}$	$\dots$	$\dots$
		2^-	$\frac{3}{2}$				

three body force at least in the $\Lambda \rightarrow \infty$ limit [5,6]. This however will happen at ridiculously high cutoffs (clearly beyond the breakdown scale of the theory), which means that we can effectively estimate the binding energy of the $0^- B\bar{B}D$ to be

$$B_3 = 4 \pm 2 \text{ MeV} \quad (B_3 = 6_{-3}^{+5} \text{ MeV}), \quad (86)$$

for $\Lambda = 0.5 \text{ GeV}$ ($\Lambda = 1.0 \text{ GeV}$). The predictions for the $1^- B\bar{B}^*D$ and $2^- B^*B^*D$ are identical, as a consequence of

TABLE III. Candidate $\bar{b}\bar{b}c$ -like three heavy meson molecules for which the Efimov effect might be possible. In the top part of the table we consider systems for which B^+D^0 , B^+D^{*0} and $B^{*+}D^0$ scattering might be resonant if the interaction in the isoscalar $S_L = 0$ channel is considerably weaker than in the $S_L = 1$ one. In the middle part we consider the same systems in the isospin symmetric limit. In the bottom part we consider systems for which the Efimov effect could be present if the isovector $J^P = 1^+ B^*D^*$ scattering is resonant, where this channel is the $\bar{b}c$ heavy flavor partner of the Z_b and Z'_b hidden bottom states. In the table above J^P is the spin and parity of the state, I the isospin (if well-defined), M/m is the mass imbalance of the system, λ_c the critical coupling for the Efimov effect to appear, s_0 the power-law scaling, e^{π/s_0} the discrete scaling factor for the momenta and $e^{2\pi/s_0}$ the discrete scaling factor for the three-body binding energies.

System	J^P	I	M/m	λ_c	s_0	e^{π/s_0}	$e^{2\pi/s_0}$
$B^+B^+D^0$	0^-	$\dots$	2.83	0.599	0.7513	65.48	4287.3
$B^{*+}B^{*+}D^0$	2^-	$\dots$	2.86	0.597	0.7546	64.27	4131.3
$B^+B^+D^{*0}$	1^-	$\dots$	2.63	0.614	0.7263	75.61	5716.8
$B\bar{B}D$	0^-	$\frac{1}{2}$	2.83	0.599	0.4691	810.0	6.561×10^5
B^*B^*D	2^-	$\frac{1}{2}$	2.85	0.597	0.4731	765.2	5.885×10^5
$B\bar{B}D^*$	1^-	$\frac{1}{2}$	2.63	0.614	0.4345	1381.8	1.909×10^6
$B^*B^*D^*$	0^-	$\frac{1}{2}$	2.65	0.614	0.4385	1291.9	1.669×10^6
$B^*B^*D^*$	2^-	$\frac{3}{2}$	2.65	0.614	0.4385	1291.9	1.669×10^6

the small mass difference between the B and B^* mesons. Meanwhile the predictions for the $1^- B\bar{B}D^*$ trimer are slightly more bound for the $\Lambda = 1.0 \text{ GeV}$ case. In Table II we include a list of these states, their quantum numbers and their properties.

If we consider the heavy meson-antimeson interaction in the isovector channel, then it appears that the existence of the $Z_c(3900)$ and $Z'_c(4020)$ in the hidden charm sector can indeed be deduced from HFS and the $Z_b(10610)$ and $Z'_b(10610)$ in the hidden bottom sector [34]. This argument also predicts the existence of an isovector B^*D^* bound or virtual state near the threshold with $J^P = 1^+$. If this prediction were to be confirmed, it would open the possibility of Efimov physics for the $B^*B^*D^*$ system. We end up with the same configurations, equations and isospin factors than in the previous section: 0^- isodoublet and 1^- , 2^- isoquartet. The only difference is the mass imbalance from the $\bar{B}^* \rightarrow D^*$ substitution. The discrete scaling factor is $\mu_0 = e^{\pi/s_0} = 1291.9$ in this case, leading to a geometric factor of 1.669×10^6 for the spacing of the excited Efimov states, see Table III. As in the previous case, concrete calculations require a three body force. If we ignore this requirement there is the possibility of a very shallow trimer for $\Lambda = 0.5 \text{ GeV}$ which disappears at larger cutoffs, see Table II.

Finally we stress that the conclusions of this section are rather theoretical. They depend on a series of assumptions to be correct and on a series of theoretical uncertainties to lean into the right direction.

VIII. CONCLUSIONS

In this work we have considered the $J^P = 2^- D^*D^*\bar{D}$ and $3^- D^*D^*\bar{D}^*$ molecules. From HQSS we expect the binding energy and properties of these two systems to be identical. Calculations in the isospin symmetric limit indicate a three body binding energy of $B_3 \sim 1.5 \text{ MeV}$ and 3 MeV for the 2^- and 3^- trimers, respectively.

The rationale behind this prediction is straightforward: the application of HQSS in the charmed meson-antimeson system implies that the potential for the $1^{++} D\bar{D}^*$ and $2^{++} D^* \bar{D}^*$ systems is the same. The $1^{++} D\bar{D}^*$ can be identified with the $X(3872)$, from which we can deduce the strength of the potential. In the isospin-symmetric limit $2^- D^* \bar{D}^*$ and $3^- D^* \bar{D}^*$ molecules can be computed in terms of this two body potential, provided that the DD^* and $D^* \bar{D}^*$ interaction is weak, as seems to be the case from phenomenological considerations. The 2^- and 3^- molecules do bind indeed, leading to the previous predictions. If instead of the $X(3872)$ and the $D^* \bar{D}^*$ and $D^* \bar{D}^*$ systems we consider the $Z_b(10650)$ as an isovector $1^{+-} B^* \bar{B}^*$ molecule, then we can determine whether there are isodoublet 0^- and isoquartet 1^- and $2^- B^* \bar{B}^*$ trimers. The existence of these trimers depends on the $Z_b(10650)$ being a bound state instead of a virtual state or resonance.

We have also investigated the conditions for the existence of the Efimov effect. If we consider the neutral components of these systems, i.e., the $D^{0*} \bar{D}^{0*} \bar{D}^0$ and $D^{0*} \bar{D}^{0*} \bar{D}^{0*}$ molecules, and if the $D^{0*} \bar{D}^{0*}$ scattering length is unnaturally large, then the Efimov geometric spectrum is in principle possible. The relevance of this possibility is mostly theoretical though: the discrete scaling factor is about 1986.1 and the first Efimov state should be four millions times less bound than the fundamental state. This makes the existence of a geometric spectrum more of a theoretical nicety than a phenomenon that we could realistically ever expect to observe, except in the lattice perhaps. The Efimov effect is absent in the isospin symmetric limit.

A more promising candidate for the Efimov effect are the family of $0^- B^+ B^+ D^0$, $1^- B^{*+} B^+ D^0$, $2^- B^{*+} B^{*+} D^0$ and $1^- B^+ B^+ D^{*0}$ molecules. In these systems there is a marked mass imbalance between the bottom and charmed mesons that favors the appearance of Efimov physics. From HFS

there is the possibility that $B^+ D^0$ scattering might be resonant, which in turn will imply the existence of an Efimov spectrum for the aforementioned molecules. The discrete scaling factor is between 65–70, indicating that the binding of the excited Efimov state should be 4000–5000 times shallower than the fundamental state. Independently of this, it is worth noticing that three body systems with a large mass imbalance are more likely to bind, as illustrated with the $B^* B^* \bar{K}$ [22], $D^* \bar{D}^* \rho$ [57], $D^* \bar{D}^* \rho$ [58], $B^* \bar{B}^* \rho$ [59] and $B\bar{B}D$ [60] systems.

We stress the theoretical nature of the present work. The 2^{++} partner of the $X(3872)$ has not been observed yet neither ruled out. Its existence has been extensively discussed in the literature [38,39,42,43,48,55]. Without knowing whether this state exists or its binding, it is difficult to make definite predictions about prospective three body states. Nonetheless the possibility of a Borromean $3^- D^* \bar{D}^* \bar{D}^*$ molecule is there, which if detected will provide relevant information about the $D^* \bar{D}^*$ interaction. The predictions for the $\bar{b} \bar{b} b$ - and $\bar{b} \bar{b} c$ -type molecules are more hypothetical. The $\bar{b} \bar{b} b$ trimers are conditional to the location of the $Z_b(10650)$ or the existence of HQSS partners, while the $\bar{b} \bar{b} c$ trimers rely on the assumption that the heavy meson-antimeson potential for the $S_L = 0$ configuration is weaker than for the $S_L = 1$ case, where S_L is the total light spin. Yet the eventual observation of a shallow $B^+ D^0$ molecule will be really exciting owing to its connection with Efimov physics.

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