Energy evolution of J/ψ production in DIS on nuclei

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In this paper, we show that the J/ψ production in DIS is the main source of information about the events with two-parton shower production. We attempt to develop our theoretical acumen of this process, to a level compatible with the theoretical description of inclusive DIS. We revisit the problem of the linear evolution equation for the double gluon densities and include Bose-Einstein enhancement to these equations. We find that the Bose-Einstein correlations lead to an increase of the anomalous dimension, which turns out to be suppressed as $1/(N_c^2 - 1)^2$, in agreement with the estimates for the twist-four anomalous dimension. We believe that understanding what happens to these contributions at ultra high energies, is a key question for an effective theory, based on high energy QCD. We derive the evolution equation for the scattering amplitude of two dipoles with a nucleus, taking into account the shadowing corrections, and investigate the analytical solutions in two distinct kinematic regions: deep in the saturation region and in the vicinity of the saturation scale. The suggested nonlinear evolution equation is a direct generalization of the Balitsky-Kovchegov equation, which has to be solved with the initial condition that depends on the saturation scale $Q_s(Y = Y_0, b)$. With the goal of finding a new small parameter, it is instructive to compare the solution of the nonlinear equation with the quasi-classical approximation, in which in the initial condition we replace $Q_s(Y = Y_0, b)$ by $Q_s(Y, b)$. Our final result is that the shadowing corrections in the elastic amplitude generate the survival probability, which suppresses the growth of the amplitude with energy, caused by the Bose-Einstein enhancement.

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I. INTRODUCTION

We believe that the process for the production of the J/ψ meson in the fragmentation region in DIS, is the simplest process aside from inclusive DIS, which allows us to investigate the scattering amplitudes in the wide kinematic region, from short distances which are the subject of perturbative QCD, to long distances of about $1/Q_s$ (Q_s is the saturation momentum), which can be described by the color glass condensate (CGC) approach (see Ref. [1] for a review). For distances $> 1/Q_s$, the nonperturbative corrections have to be included. The nucleus target has additional advantages:

a new parameter $\bar{\alpha}_s A^{1/3}$, which results in a more rigorous theoretical approach; and the fact that the dominant mechanism is inelastic J/ψ production, accompanied by the production of two-parton showers, (see Fig. 1), which shows the process at the initial energy in the Born approximation of perturbative QCD, for dipole-nucleon scattering [2,3]). It should be stressed that the production of the one parton shower is suppressed for DIS with nuclei. Indeed, due to C-parity conservation the production of the fastest J/ψ requires an even number (minimal 2) of gluons attached to the $c - \bar{c}$ loop (see Fig. 1). For nucleon-nucleon scattering, we can produce J/ψ + one gluon while the second gluon will generate a single parton shower. However, for a nuclear target the largest cross section stems from the case when both gluons generate parton showers, leading to a cross section which is proportional to $(\bar{\alpha}_s A(1/3))^2$ (more details in Ref. [2]).

Therefore, we believe that the J/ψ production process is the main source of the experimental information about the event with two-parton showers, in the same way as

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FIG. 1. J/ψ production at the initial energy at which the scattering amplitude is taken in the Born approximation of perturbative QCD. Fig. 1(a) the structure of the inelastic event and Fig. 1(b) is J/ψ diffractive production. Helical lines denote the gluons. The black blobs stand for the nucleons. The wavy line corresponds to the virtual photon (γ^*). Due to C-parity conservation the number of gluons attached to $c - \bar{c}$ loop is even, which leads the inelastic interaction of at least two gluons.

inclusive DIS is a source for the structure of the one-parton shower event.¹

The goal of this paper is to develop our theoretical understanding of J/ψ production, to a level compatible with the theoretical description of inclusive DIS. At high energy this description includes the BFKL evolution equation [4] and the CGC/saturation approach for the scattering amplitude [5–10]. As we have eluded, the process of J/ψ production, is intimately related to the double parton densities (double parton distribution functions). The DGLAP [11] evolution equation is known for the double parton distribution functions (see Ref. [12]), also the BFKL evolution is known for these processes [13–15]. In this paper, we write the BFKL equation for the particular double parton distribution that enters the J/ψ production cross sections, and we build the generating functional for all multiparton distributions. We also wish to amend these equations, to include the Bose-Einstein enhancement, resulting from the correlations of identical gluons. This effect has a suppression of the order of $1/(N_c^2 - 1)$, where N_c is the number of colors, and is closely related to the $1/(N_1^2 - 1)$ corrections to the anomalous dimension of the twist-four operator, which is larger than the sum of the anomalous dimensions of two twist-two operators, as has been shown in Refs. [16–18]. Bose-Einstein correlations have drawn considerable attention recently, since they give essential contributions to the azimuthal angle correlations [19–25]. It has been shown (see Ref. [22] for example), that the Bose-Einstein enhancement provides a significant contribution to the measured angular correlations. We believe that this fact calls for a generalization of the evolution equation by also including this enhancement.

Unfortunately, only in the linear approximation, is the cross section for J/ψ production, determined by the double gluon density. The total cross section of DIS, is also

affected by strong shadowing corrections [5–10]. We propose a nonlinear evolution for this cross section, which is a direct generalization of the Balitsky-Kovchegov evolution. We need to solve this nonlinear equation, using the results of Refs. [2,3] as the initial condition for this equation. It should be stressed that all formulas in these papers were derived using the Born approximation of perturbative QCD for the scattering amplitude of a colorless dipole. In making estimates, the so-called quasiclassical approach is widely used, replacing the dipole-proton cross section by $\sigma = r^2 Q_s^2/4$, where Q_s denotes the saturation scale at arbitrary Y , not only at initial $Y = Y_0$, r is the size of the dipole. This is certainly not the correct procedure. For the total DIS cross section, this approximation means that we can use the McLerran-Venugopalan [7] formula with the above replacement, instead of the correct BK nonlinear equation. We can view our equation as a simplification of the general approach in the framework of the CGC, given in Ref. [26], neglecting the $1/N_c$ correction, and using the approximations related to physical nuclei.

In Sec. VI, we show that the saturation for the elastic amplitude leads to the survival probability, which is so large that it suppresses the powerlike increase with energy, which is generated by the Bose-Einstein enhancement. We summarize all our results in the Conclusions.

II. THE FIRST DIAGRAM: TWO BFKL POMERON EXCHANGES

At present, the effective theory for QCD at high energies, exists in two forms: the CGC/saturation approach [5–10] and the BFKL Pomeron calculus [4,5,13,27–32]. It has been proven that in general, these two approaches are equivalent in a limited region of the rapidities [33]. However, for the Balitsky-Kovchegov cascade, which we discuss here for the interaction with a nucleus target, the equivalence holds in the entire kinematic region in rapidity. The interpretation of processes at high energy appear quite different in each approach, since they have different structural elements. The CGC/saturation approach, being more microscopic, describes the high energy interactions in terms of colorless dipoles, their density, distribution over impact parameters, evolution in energy and so on.

¹We would like to clarify what we mean, saying “the production of two -parton showers.” The parton shower is the initial state “cut ladder,” which generates the production of gluons that are almost uniformly distributed in the entire region of rapidities. As one will see below, if we neglect the shadowing corrections, such parton showers correspond to the cut BFKL Pomeron.

FIG. 2. The first diagram in the BFKL Pomeron calculus for J/ψ production. The vertical wavy lines describe the BFKL Pomerons. Helical lines denote the gluons. The horizontal wavy lines correspond to the virtual photon (γ^*). The vertical dashed line shows the cut, which means that all gluons in the Pomeron are produced.

This approach can be easily applied to inclusive processes at high energies, generating a new typical scale: i.e., saturation momentum Q_s .

The BFKL Pomeron calculus which deals with BFKL Pomerons and their interactions, is similar to the old Reggeon theory [34], and is suitable for describing

diffractive physics and correlations in multiparticle production, as we can use the Mueller diagram technique[35]. The relation between different processes at high energy are very often more transparent in this approach, since in addition to the Mueller diagram technique we can use the AGK cutting rules [36], which are useful in spite of the restricted region of their application [37].

In this paper, we use the BFKL Pomeron calculus for the process of interest, since we wish to find the relationship between the J/ψ production and inclusive DIS. This is the main difference to Ref. [26], in which the same process has been treated using the CGC approach.

In the framework of the BFKL Pomeron calculus, the first diagram that describes J/ψ production, is the exchange of two BFKL Pomerons shown in Fig. 2. Since we are interested in inelastic J/ψ production, the Pomerons in Fig. 2 are cut Pomerons, in which all gluons are produced. From the unitarity constraints for the elastic amplitude of the dipole of size r , rapidity Y and at the impact parameter b , we have

$$N_{\text{cut}}^{\text{BFKL}}(Y, r, b) \equiv \sigma_{\text{tot}}^{\text{BFKL}} = 2N^{\text{BFKL}}(Y, r, b). \quad (1)$$

Its contribution to the total cross section for J/ψ production is equal to

$$\sigma(Y, Q^2) = \frac{\bar{\alpha}_s^2}{(2\pi)^6} \int d^2 k_T d^2 q_T d^2 Q_T I(\mathbf{k}_T, \mathbf{q}_T) I(\mathbf{k}'_T, \mathbf{q}_T) N_{\text{cut}}^{\text{BFKL}}\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T\right) N_{\text{cut}}^{\text{BFKL}}\left(-\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T\right), \quad (2)$$

where

$$I(\mathbf{k}_T, \mathbf{q}_T) = \int d^2 r e^{i\frac{1}{2}\mathbf{q}_T \cdot \mathbf{r}} (1 - e^{i\mathbf{k}_T \cdot \mathbf{r}}) \Psi_{\gamma^*}(Q, r) \Psi_{J/\psi}(r) = F(\mathbf{q}_T) - F\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T\right) \quad (3)$$

and $\mathbf{Q}_T = \mathbf{k}_T - \mathbf{k}'_T$. In Eq. (3), $\Psi_{\gamma^*}(Q, r)$ denotes the wave function of the dipole of size r in the photon with virtuality Q , while $\Psi_{J/\psi}(r)$ is the wave function of the same dipole in the J/ψ meson. The amplitude of the BFKL Pomeron $N_{\text{cut}}^{\text{BFKL}}(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T)$ can be rewritten in the case of the interaction of the Pomeron with the nucleus as follows, taking into account Eq. (1).

$$N_{\text{cut}}^{\text{BFKL}}\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T\right) = 2N_N^{\text{BFKL}}\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T\right) S_A(Q_T) \rightarrow 2N_N^{\text{BFKL}}\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T = 0\right) S_A(Q_T), \quad (4)$$

where $N_N^{\text{BFKL}}(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T = 0)$ is the scattering amplitude of the dipole with the nucleon.

$$S_A(Q_T) = \int d^2 b e^{i\mathbf{Q}_T \cdot \mathbf{b}} \int_{-\infty}^{\infty} dz \rho_A(b, z) \equiv \int d^2 b e^{i\mathbf{Q}_T \cdot \mathbf{b}} S_A(b), \quad (5)$$

where $\rho(b, z)$ denotes the nucleon density in the nucleus. $S_A(b)$ is the number of the nucleons at fixed impact parameter b . The last equation in Eq. (4) follows from the fact that the typical value Q_T in the nucleus form factor is about $1/R_A$, where R_A denotes the radius of the nucleus. On the other hand Q_T in N_N^{BFKL} is of the order of the soft scale μ_{soft} , or the saturation scale Q_s and, therefore, turns out to be much larger than $1/R_A$, and can be neglected.

Using Eq. (4), we can rewrite Eq. (2) in the form

$$\begin{aligned}\sigma(Y, Q^2) &= \frac{4\bar{\alpha}_S^2}{(2\pi)^4} \int \frac{d^2 Q_T}{(2\pi)^2} S_A^2(Q_T) \\ &\times \int d^2 k_T d^2 q_T I(\mathbf{k}_T, \mathbf{q}_T) I(\mathbf{k}'_T, \mathbf{q}_T) N_N^{\text{BFKL}}\left(\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T = 0\right) N_N^{\text{BFKL}}\left(-\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T, \mathbf{Q}_T = 0\right) \\ &= \frac{4\bar{\alpha}_S^2(N_c^2 - 1)}{(2\pi)^4} \int \frac{d^2 Q_T}{(2\pi)^2} S_A^2(Q_T) \\ &\times \int d^2 l_T d^2 l'_T I(\mathbf{l}_T, \mathbf{l}_T + \mathbf{l}'_T) I(\mathbf{l}'_T, \mathbf{l}_T + \mathbf{l}'_T) N_N^{\text{BFKL}}(\mathbf{l}_T, \mathbf{Q}_T = 0) N_N^{\text{BFKL}}(\mathbf{l}'_T, \mathbf{Q}_T = 0)\end{aligned}\quad (6)$$

where $\mathbf{l}_T = \mathbf{k}_T + \frac{1}{2}\mathbf{q}_T$ and $\mathbf{l}'_T = -\mathbf{k}_T + \frac{1}{2}\mathbf{q}_T$.

Introducing the double transverse momentum densities which characterize how many partons with $(x_1, p_{1,T})$ and $(x_2, p_{2,T})$ are in the parton cascade, and which is equal to

$$\begin{aligned}\rho^{(2)}(x_1, \mathbf{p}_{1,T}; x_2, \mathbf{p}_{2,T}, \mathbf{Q}_T) &= \sum_{n=2}^{\infty} \int \prod_{i=1}^n \frac{dx_i}{x_i} d^2 k_{i,T} \Psi^*\left(\{x_i, k_{i,T}\}, x_1, \mathbf{p}_{1,T} + \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} - \frac{1}{2}\mathbf{Q}_T\right) \\ &\times \left\{ a^+\left(x_1, \mathbf{p}_{1,T} + \frac{1}{2}\mathbf{Q}_T; b\right) a^+\left(x_2, \mathbf{p}_{2,T} - \frac{1}{2}\mathbf{Q}_T; c\right) a\left(x_2, \mathbf{p}_{2,T} + \frac{1}{2}\mathbf{Q}_T; c\right) a\left(x_1, \mathbf{p}_{1,T} - \frac{1}{2}\mathbf{Q}_T; b\right) \right\} \\ &\times \Psi\left(\{x_i, k_{i,T}\}, x_1, \mathbf{p}_{1,T} - \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} + \frac{1}{2}\mathbf{Q}_T\right),\end{aligned}\quad (7)$$

where $\Psi(\{x_i, k_{i,T}\}, x_1, \mathbf{p}_{1,T} - \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} + \frac{1}{2}\mathbf{Q}_T)$ denotes the partonic wave function, we can rewrite Eq. (6) in the form

$$\sigma(Y, Q^2) = \frac{4\bar{\alpha}_S^2}{(2\pi)^2} \int d^2 l_T d^2 l'_T I(\mathbf{l}_T, \mathbf{l}_T + \mathbf{l}'_T) I(\mathbf{l}'_T, \mathbf{l}_T + \mathbf{l}'_T) \frac{d^2 Q_T}{(2\pi)^2} \rho_A^{(2)}(x, \mathbf{l}_T; x, \mathbf{l}'_T, \mathbf{Q}_T)$$

where $Y = \ln(1/x)$.

In Eq. (7), a^+ and a denote the creation and annihilation operators for gluons that have longitudinal momentum x_i and transverse momentum $p_{i,T}$.

For the double parton density in the coordinate representation, which is equal to

$$\rho^{(2)}(x, \mathbf{r}; x, \mathbf{r}'; \mathbf{b}) = \int \frac{d^2 l_T}{(2\pi)^2} \frac{d^2 l'_T}{(2\pi)^2} \frac{d^2 Q_T}{(2\pi)^2} e^{i(\mathbf{r} \cdot \mathbf{l}_T + \mathbf{r}' \cdot \mathbf{l}'_T + \mathbf{b} \cdot \mathbf{Q}_T)} \rho_A^{(2)}(x, \mathbf{l}_T; x, \mathbf{l}'_T, \mathbf{Q}_T), \quad (8)$$

Eq. (8) takes the form

$$\sigma(Y, Q^2) = 4\bar{\alpha}_S^2 \int d^2 r d^2 r' \Phi(\mathbf{r}, \mathbf{r}') \int d^2 b \rho_A^{(2)}(x, \mathbf{r}; x, \mathbf{r}'; \mathbf{b}_T), \quad (9)$$

where

$$\Phi(\mathbf{r}, \mathbf{r}') = \int \frac{d^2 l_T}{(2\pi)^2} \frac{d^2 l'_T}{(2\pi)^2} e^{i(\mathbf{l}_T \cdot \mathbf{r} + \mathbf{l}'_T \cdot \mathbf{r}')} I(\mathbf{l}_T, \mathbf{l}_T + \mathbf{l}'_T) I(\mathbf{l}'_T, \mathbf{l}_T + \mathbf{l}'_T). \quad (10)$$

III. GENERATING FUNCTIONAL AND BFKL EVOLUTION FOR THE MULTI-GLUON DENSITIES

The evolution equations for the multiparton densities has been given in Ref. [15]. In this section, we briefly review this proof, and derive the nonlinear evolution equation for the generating functional with two colorless dipoles as the initial condition. This will give us the linear BFKL evolution for all partonic densities $\rho^{(n)}(\{\mathbf{r}_i, \mathbf{b}_i\})$. Following Ref. [8] we consider a colorless dipole of size $\mathbf{r}_i$ at impact parameter $\mathbf{b}_i$, as the parton in the fast projectile which can be described by the partonic wave function $\Psi(\{\mathbf{r}_i, \mathbf{b}_i\})$. For a system of n -partons with coordinates $\{\mathbf{r}_i, \mathbf{b}_i\}$ with $i = 1, \dots, n$ and rapidity Y , we

$$\frac{d P_n(Y, \{r_i, b_i\})}{d \alpha_S Y} = \sum_{i=1}^n - \dots + \dots$$

$$\sum_{i=1}^n - \omega(r_i) P_n(Y, \{r_i, b_i\}) + K(r_i, r_n) P_{n-1}(Y, \{r_i, b_i\})$$

FIG. 3. The graphic form of Eq. (11).

can introduce the probability (probability density) to find them in the projectile wave function: $P_n(Y; \{r_i, b_i\})$. For P_n , we write the classical cascade equation (see Refs. [8,15,31] and Fig. 3):

$$\begin{aligned} & \frac{\partial P_n(Y - Y_0; \{r_i, b_i\})}{\partial \alpha_S \partial Y} \\ &= - \sum_{i=1}^n \omega(r_i) P_n(Y - Y_0; \{r_i, b_i\}) + \sum_{i=1}^{n-1} \frac{(r_i + r_n)^2}{(2\pi) r_i^2 r_n^2} P_{n-1}(Y - Y_0; \{r_i, b_i\}) r_i \rightarrow r_i + r_n, b_i \rightarrow b_{in} \end{aligned} \quad (11)$$

with $b_{in} = b_i + r_n/2 = b_n - r_i/2$. The two terms of Eq. (11) have a simple meaning: the first one describes the decrease in probability to find n dipoles, due to a decay of one dipole into two dipoles of arbitrary size. This probability is equal to

$$\bar{\alpha}_S \omega(r_i) = \frac{\bar{\alpha}_S}{2\pi} \int_{\rho} \frac{r_i^2}{(\mathbf{r}_i - \mathbf{r}')^2 r'^2} d^2 r' = \int d^2 r' K(\mathbf{r} - \mathbf{r}', r') = \bar{\alpha}_S \ln(r_i^2/\rho^2), \quad (12)$$

where ρ is an infrared cutoff. The second term, reflects the increase in probability to find n dipoles due to the creation of a new dipole from $n - 1$ dipoles, with probability

$$K(r_1, r_1) = \frac{\bar{\alpha}_S (r_1 + r_2)^2}{2\pi r_1^2 r_2^2}. \quad (13)$$

To find the probabilities, we need to add the initial condition at $Y = Y_0$ to Eq. (11). We wish to stress that we are only dealing with the decay of the dipoles, neglecting the possibilities for the dipoles to recombine. In terms of the Pomeron calculus, it means that we only take into account “fan” Pomeron diagrams, which give the dominant contribution for DIS with the nuclear target [5,8,9].

Equation (11) can be rewritten in a more elegant form introducing a generating functional Z

$$\begin{aligned} Z(Y - Y_0; [u_i]) &\equiv \sum_{n=1} \int P_n(Y - Y_0; \{r_i, b_i\}) \\ &\times \prod_{i=1}^n u(r_i, b_i) d^2 r_i d^2 b_i, \end{aligned} \quad (14)$$

where $u(r_i, b_i) \equiv u_i$ is an arbitrary function of r_i and b_i . It follows immediately from Eq. (14) that the functional obeys the condition: at $u_i = 1$

$$Z(Y - Y_0; [u_i = 1]) = 1. \quad (15)$$

The physical meaning of (15) is that the sum over all probabilities is one.

Multiplying Eq. (11) by the product $\prod_{i=1}^n u_i$ and integrating over all r_i and b_i , we obtain the following linear equation for the generating functional:

$$\begin{aligned} \frac{\partial Z}{\partial \alpha_S \partial Y} &= - \int d^2 r d^2 b V_{1 \rightarrow 1}(\mathbf{r}, \mathbf{b}, [u]) Z \\ &+ \int d^2 r d^2 r' d^2 b V_{1 \rightarrow 2}(\mathbf{r}, \mathbf{r}', \mathbf{b}, [u]) Z \end{aligned} \quad (16)$$

with the definitions

$$V_{1 \rightarrow 1}(\mathbf{r}, \mathbf{b}, [u]) = \bar{\alpha}_S \omega(r) u(\mathbf{r}, \mathbf{b}) \frac{\delta}{\delta u(\mathbf{r}, \mathbf{b})} \quad (17)$$

and

$$\begin{aligned} V_{1 \rightarrow 2}(\mathbf{r}, \mathbf{r}', \mathbf{b}, [u]) &= \frac{\bar{\alpha}_S}{2\pi} \frac{r^2}{r'^2 (\mathbf{r} - \mathbf{r}')^2} u\left(\mathbf{r}', \mathbf{b} + \frac{1}{2}(\mathbf{r} - \mathbf{r}')\right) \\ &\times u\left(\mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}\right) \frac{\delta}{\delta u(\mathbf{r}, \mathbf{b})}. \end{aligned} \quad (18)$$

The functional derivative with respect to $u(r, b)$, plays the role of an annihilation operator for a dipole of size r , at impact parameter b . The multiplication by $u(r, b)$ corresponds to a creation operator for this dipole. Having this in mind, we can see that the n -dipole densities in the projectile $\rho^{(n)}(Y - Y_0; \{r_i, b_i\})$ are defined as follows:

$$\rho^{(n)}(Y - Y_0; \{\mathbf{r}_i, \mathbf{b}_i\}) = \frac{1}{n!} \prod_{i=1}^n \frac{\delta}{\delta u_i} Z(Y - Y_0; [u_i])|_{u=1}. \quad (19)$$

Using Eq. (19) we obtain the following linear evolution equations for $\rho^{(n)}(Y - Y_0; \{\mathbf{r}_i, \mathbf{b}_i\})$:

$$\begin{aligned} & \frac{\partial \rho^{(n)}(Y - Y_0; \mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_n, \mathbf{b}_n)}{\bar{\alpha}_S \partial Y} \\ &= - \sum_{i=1}^n \omega(r_i) \rho^{(n)}(Y - Y_0; \mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_n, \mathbf{b}_n) \\ &+ 2 \sum_{i=1}^n \int \frac{d^2 r'}{2\pi} \frac{r'^2}{r_i^2 (\mathbf{r}_i - \mathbf{r}')^2} \rho^{(n)}\left(\dots, \mathbf{r}', \mathbf{b}_i - \frac{1}{2} \mathbf{r}', \dots\right) + \sum_{i=1}^{n-1} \frac{(\mathbf{r}_i + \mathbf{r}_n)^2}{(2\pi) r_i^2 r_n^2} \rho^{(n-1)}(\dots, (\mathbf{r}_i + \mathbf{r}_n), \mathbf{b}_{in}, \dots). \end{aligned} \quad (20)$$

For $r_1^2 r_2^2 \int d^2 b \rho_A^{(2)}(x, \mathbf{r}; x, \mathbf{r}'; \mathbf{b}_T) = \tilde{\rho}^{(2)}(Y - Y_0, \mathbf{r}_1, \mathbf{r}_2)$, Eq. (20) can be rewritten in the following form

$$\begin{aligned} & \frac{\partial \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{r}_1, \mathbf{r}_2)}{\bar{\alpha}_S \partial Y} \\ &= \sum_{i=1}^2 \int \frac{d^2 r'}{2\pi} \frac{1}{(\mathbf{r}_i - \mathbf{r}')^2} \left\{ 2 \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{r}', \mathbf{r}_{i+1}) - \frac{r_1^2}{r'^2} \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{r}_1, \mathbf{r}_2) \right\} + \tilde{\rho}^{(1)}(Y - Y_0, \mathbf{r}_1 + \mathbf{r}_2) \end{aligned} \quad (21)$$

where $(\mathbf{r}_1 + \mathbf{r}_2)^2 \int d^2 b \rho^{(1)}(Y - Y_0, \mathbf{r}_1 + \mathbf{r}_2, \mathbf{b}) = \tilde{\rho}^{(1)}(Y - Y_0, \mathbf{r}_1 + \mathbf{r}_2)$.

In the momentum representation, Eq. (21) takes the form which is similar to the DGLAP evolution (see Ref. [12] and the references therein):

$$\frac{\partial \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{p}_{1,T}, \mathbf{p}_{2,T})}{\bar{\alpha}_S \partial Y} = \sum_{i=1}^2 \int \frac{d^2 k_T}{\pi} K(\mathbf{p}_i, \mathbf{k}) \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{k}_{i,T}, \mathbf{p}_{i+1}) + \tilde{\rho}^{(1)}(Y - Y_0, \mathbf{p}_{1,T}) \delta^{(2)}(\mathbf{p}_{1,T} - \mathbf{p}_{2,T}),$$

where $K(\mathbf{p}_i, \mathbf{k})$ denotes the BFKL kernel that has the form

$$\begin{aligned} & K(\mathbf{p}_i, \mathbf{k}) \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{k}_{i,T}, \mathbf{p}_{i+1,T}) \\ &= \frac{1}{(\mathbf{p}_{i,T} - \mathbf{k}_T)^2} \left\{ \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{k}_{i,T}, \mathbf{p}_{i+1,T}) - \frac{p_{i,T}^2}{k_T^2 + (\mathbf{p}_{i,T} - \mathbf{k}_T)^2} \tilde{\rho}^{(2)}(Y - Y_0; \mathbf{p}_{i,T}, \mathbf{p}_{i+1,T}) \right\}. \end{aligned} \quad (22)$$

Equation (20) can be resolved if we know the initial conditions. Assuming that at $Y = Y_0$, $\rho^{(1)} = \delta^{(2)}(\mathbf{r} - \mathbf{r}_1) \delta^{(2)}(\mathbf{b} - \mathbf{b}_1)$ while $\rho^{(n)} = 0$ for $n > 1$, we see that at $Y = Y_0$ the initial condition for the generating functional implies

$$Z^{(1)}(Y = Y_0, \{\mathbf{r}_i, \mathbf{b}_i\}, [u_i]) = u(\mathbf{r}, \mathbf{b}). \quad (23)$$

The linear equation Eq. (16) has a general solution of the form $Z(Y - Y_0, \{\mathbf{r}_i, \mathbf{b}_i\}, [u_i]) = Z(\{u_i(Y - Y_0, \mathbf{r}_i, \mathbf{b}_i)\})$ and using Eq. (23) we obtain the nonlinear equation of Ref. [8]:

$$\begin{aligned} & \frac{\partial Z^{(1)}(Y, r, b; [u])}{\bar{\alpha}_S \partial Y} = -\omega(r) Z^{(1)}(Y; r, b; [u]) \\ &+ \int \frac{d^2 r'}{2\pi} \frac{r^2}{r'^2 (r - r')^2} Z^{(1)}\left(Y; r', b + \frac{(r - r')}{2}; [u]\right) Z^{(1)}\left(Y; (r - r'), b - \frac{r'}{2}; [u]\right). \end{aligned} \quad (24)$$

As we have demonstrated for the J/ψ production, the initial conditions at $Y = Y_0$ are $\rho^{(2)} = \delta^{(2)}(\mathbf{r} - \mathbf{r}_1) \delta^{(2)}(\mathbf{b} - \mathbf{b}_1) \delta^{(2)}(\mathbf{r}' - \mathbf{r}_2) \delta^{(2)}(\mathbf{b}' - \mathbf{b}_2)$ and all $\rho^{(n)} = 0$ for $n \neq 2$. In other words, for Z we have the initial condition

FIG. 4. The origin of interference diagrams which violate Eq. (27), and should be included in the evolution.

$$Z^{(2)}(Y = Y_0, \{\mathbf{r}_i, \mathbf{b}_i\}, [u_i]) = u(\mathbf{r}_1, \mathbf{b}_1)u(\mathbf{r}_2, \mathbf{b}_2). \quad (25)$$

Using this equation, we obtain

$$\begin{aligned} \frac{\partial Z^{(2)}(Y, \mathbf{r}_1, \mathbf{b}_1; \mathbf{r}_2, \mathbf{b}_2; [u])}{\bar{\alpha}_S \partial Y} = & -(\omega(r_1) + \omega(r_2))Z^{(2)}(Y, \mathbf{r}_1, \mathbf{b}_1; \mathbf{r}_2, \mathbf{b}_2; [u]) \\ & + \int \frac{d^2 r'}{2\pi} \frac{r_1^2}{r'^2 (\mathbf{r}_1 - \mathbf{r}')^2} Z^{(1)}\left(Y; r', b + \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}'); [u]\right) Z^{(2)}\left(Y; \mathbf{r}_1 - \mathbf{r}', \mathbf{b}_1 - \frac{1}{2}\mathbf{r}'; \mathbf{r}_2, \mathbf{b}_2; [u]\right) \\ & + \int \frac{d^2 r'}{2\pi} \frac{r_2^2}{r'^2 (\mathbf{r}_2 - \mathbf{r}')^2} Z^{(1)}\left(Y; r', b + \frac{1}{2}(\mathbf{r}_2 - \mathbf{r}'); [u]\right) Z^{(2)}\left(Y; \mathbf{r}_2 - \mathbf{r}', \mathbf{b}_2 - \frac{1}{2}\mathbf{r}'; \mathbf{r}_1, \mathbf{b}_1; [u]\right). \end{aligned} \quad (26)$$

IV. BOSE-EINSTEIN ENHANCEMENT IN THE EVOLUTION OF DOUBLE PARTON DENSITIES

Comparing Eq. (7) and Eq. (8), one can see that the double gluon density has a general form given by

$$\rho^{(2)}\left(x_1, \mathbf{p}_{1,T} + \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} - \frac{1}{2}\mathbf{Q}_T\right) = F(Q_T)\rho^{(1)}(x_1, p_{1,T})\rho^{(1)}(x_2, p_{2,T}). \quad (27)$$

Equation (27) indicates that two gluons are produced from two different parton showers, and have no correlations between them, at least at large values of rapidities. However, we can see from Fig. 4 that Eq. (27) cannot be correct, due to the interference between gluons coming from different single parton densities, which is possible, and could give sufficiently large contributions for the identical gluons [22].

A. The interference diagram in the DLA

In this section, for the sake of simplicity, we will treat the interference diagrams in double log approximation(DLA) of perturbative QCD, in which we consider $\bar{\alpha}_S Y \geq 1$, $\bar{\alpha}_S \ln(p_T^2) \geq 1$, but $\bar{\alpha}_S \ll 1$. This approximation appears more credible not for the gluon densities, but for double gluon distribution function which is defined as

$$D^{(2)}(x_1, x_2, Q) = \int^Q \frac{d^2 p_{1,T}}{(2\pi)^2} \int^Q \frac{d^2 p_{2,T}}{(2\pi)^2} \int^Q \frac{d^2 Q_T}{(2\pi)^2} \rho^{(2)}\left(x_1, \mathbf{p}_{1,T} + \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} - \frac{1}{2}\mathbf{Q}_T\right). \quad (28)$$

At large Q , the cross section of J/ψ production depends on $D^{(2)}(x, x, Q)$, as can be seen from Eq. (8).

First, we consider the diagram of Fig. 5(a) which shows the emission of an extra gluon in one parton shower. This emission leads to the DLA increase of $\rho^{(1)}(x_2, \mathbf{p}_{2,T}) = \rho^{(1)}(Y_2, \mathbf{p}_{2,T})$. Recalling that $Q_T \propto 1/R_A \ll p_{i,T}$ and (see Ref. [1] for example)

$$1. \rho^{(1)}(Y_2 = 0; \mathbf{p}_{2,T}; \mathbf{p}_{2,T} - \mathbf{Q}_T) = \bar{\alpha}_S \delta_{a,b} \frac{\mathbf{p}_{2,T} \cdot (\mathbf{p}_{2,T} + \mathbf{Q}_T)}{p_{2,T}^2 (\mathbf{p}_{2,T} + \mathbf{Q}_T)^2} S_A(Q_T), \quad (29)$$

$$2. \Gamma_\mu(\mathbf{p}_{2,T}, \mathbf{k}_T) \xrightarrow{p_{2,T} \gg k_T} 2g f^{abc} \mathbf{p}_{2,T}, \quad (30)$$

we see that

FIG. 5. The first interference diagram for double gluon densities [Fig. 5(b)] in DLA of perturbative QCD. Figure 5(a) shows the emission an extra gluon in one of the parton showers. Figure 5(c) shows the calculation of the color coefficient in the diagrams: Fig. 5(c) a for Fig. 5(a) and Fig. 5(c) b for Fig. 5(b).

$$\rho^{(1)}(Y, \mathbf{p}_{2,T}; Y, \mathbf{p}_{2,T} + \mathbf{Q}_T) = \rho^{(1)}(Y, \mathbf{p}_{2,T}; Y, \mathbf{p}_{2,T}) = \int^Y dY' \frac{\bar{\alpha}_S}{p_{2,T}^2} \int^{p_{2,T}^2} dk_T^2 \rho^{(1)}(Y', \mathbf{k}_T; Y', \mathbf{k}_T). \quad (31)$$

For $D^{(1)}(Y, Q)$, Eq. (31) reduces to

$$D^{(1)}(Y, Q) = \bar{\alpha}_S \int^Y dY' \int^Q \frac{d^2 p_{2,T}}{p_{2,T}^2} D^{(1)}(Y', p_{2,T}). \quad (32)$$

For $D^{(2)}$, the sum of the diagrams of Fig. 5(a) give

$$\frac{\partial D^{(2)}(Y, Y, \xi; \text{extra gluon})}{\partial \xi} = \left\{ \int^Y dY' D^{(2)}(Y', Y, \xi) + \int^Y dY' D^{(2)}(Y, Y', \xi) \right\}, \quad (33)$$

where $\xi = \bar{\alpha}_S \ln(Q^2)$.

We wish to stress that $D^{(2)}(Y, Y, \xi; \text{extra gluon})$ can be calculated only using $D^{(2)}(Y', Y, \xi)$ at different values of $Y_1(x_1)$ and $Y_2(x_2)$.

The interference diagram of Fig. 5(b) can be written in the following form:

$$\begin{aligned} D_{\text{int}}^{(2)}(Y, Y, \xi; \text{extra gluon}) &= \frac{\bar{\alpha}_S}{N_c^2 - 1} \int^Y dY' \int \frac{d^2 p_{1,T}}{(2\pi)^2} \int \frac{d^2 p_{2,T}}{(2\pi)^2} \int \frac{d^2 Q'_T}{(2\pi)^2} \int \frac{d^2 Q_T}{(2\pi)^2} \frac{(\mathbf{p}_{2,T} + \mathbf{Q}'_T) \cdot (\mathbf{p}_{1,T} - \mathbf{Q}_T + \mathbf{Q}'_T)}{(\mathbf{p}_{2,T} + \mathbf{Q}'_T)^2 (\mathbf{p}_{1,T} - \mathbf{Q}_T + \mathbf{Q}'_T)^2} \\ &\times \rho^{(1)}(Y', \mathbf{p}_{1,T}; Y', \mathbf{p}_{1,T} - \mathbf{Q}_T) \rho^{(1)}(Y', \mathbf{p}_{2,T} + \mathbf{Q}_T; Y', \mathbf{p}_{2,T}), \end{aligned} \quad (34)$$

where the color coefficient reflects the fact that only identical gluons contribute to this diagram. The calculation of the color coefficients are shown in Fig. 5(c).

One can see that for $Q_T \ll p_{1,T} \leq Q'_T \leq Q$ and for $Q_T \ll p_{2,T} \leq Q'_T \leq Q$ Eq. (34) can be rewritten in the form

$$D_{\text{int}}^{(2)}(Y, Y, \xi; \text{extra gluon}) = \bar{\alpha}_S \int^Y dY' \int^Q \frac{dQ'_T}{Q_T'^2} D_{\text{Fig. 2}}^{(2)}(Y, Y, Q'_T). \quad (35)$$

We would like to draw the reader's attention to the integration over Q_T and Q'_T , since, as has been noticed in Refs. [16–18], without this integration the interference diagrams will not acquire the term proportional to $\ln(Q^2)$.²

²We thank Jochen Bartels for discussion on the importance of Q_T integration.

FIG. 6. DGLAP evolution equation for the double parton distribution with the Bose-Einstein enhancement.

B. DGLAP evolution with Bose-Einstein enhancement: Anomalous dimension

The evolution equation for the double parton distribution function can be derived from Eq. (22), and in the DLA and it has the following form:

$$\frac{\partial D^{(2)}(Y, Y, Q)}{\partial \xi} = \int_{Y_0}^Y dY' D^{(2)}(Y', Y, Q) + \int_{Y_0}^Y dY' D^{(2)}(Y, Y', Q) + D^{(1)}(Y, Q). \quad (36)$$

Note that, the last term in Eq. (36) depends on $x_1 + x_2 = 2x$, but in the leading $\log(1/x)$ approximation, we do not take into account the difference between $\tilde{Y} = \ln(1/(2x))$ and $Y = \ln(1/x)$, on which the first two terms depend.

To include the Bose-Einstein enhancement, we need a more general evolution equation, which is shown in Fig. 6.

In this equation, we add to Eq. (36) the term, which is given by Eq. (35), and which describes the contributions of the diagrams of Fig. 6(c) and Fig. 6(d). In the DLA, this equation has the form:

$$\begin{aligned} \frac{\partial D^{(2)}(Y, Y, \xi)}{\partial \xi} = & \int_{Y_0}^Y dY' D^{(2)}(Y', Y, \xi) \\ & + \int_{Y_0}^Y dY' D^{(2)}(Y, Y', \xi) + \frac{2}{N_c^2 - 1} \\ & \times \int_{Y_0}^Y dY' D^{(2)}(Y', Y', \xi) + D^{(1)}(Y, \xi). \end{aligned} \quad (37)$$

Calculating the color coefficient [see Fig. 5(c)], we assumed that the main contribution in the region of small x , comes from the colorless states in the t -channel for the gluons with the same values of x_i , as is shown in Fig. 5. In Eq. (37), we use a new notation $Y = 2N_c \ln(1/x)$.

We first solve the homogeneous equation:

$$\frac{\partial D^{(2)}(Y, Y, \xi)}{\partial \xi} = \int_{Y_0}^Y dY' D^{(2)}(Y', Y, \xi) + \int_{Y_0}^Y dY' D^{(2)}(Y, Y', \xi) + \frac{2}{N_c^2 - 1} \int_{Y_0}^Y dY' D^{(2)}(Y', Y', \xi). \quad (38)$$

We introduce the following Mellin transform to solve this equation:

$$\begin{aligned} D^{(2)}(Y_1, Y_2, \xi) = & \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\gamma\xi + \omega_1 Y_1 + \omega_2 Y_2} d^{(2)}(\omega_1, \omega_2, \gamma) \\ = & 2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_D}{2\pi i} e^{\gamma\xi + \omega_\Sigma(Y_1 + Y_2) + \omega_D(Y_1 - Y_2)} d^{(2)}(\omega_\Sigma, \omega_D, \gamma) \end{aligned} \quad (39)$$

where $\omega_\Sigma = \frac{1}{2}(\omega_1 + \omega_2)$ and $\omega_D = \frac{1}{2}(\omega_1 - \omega_2)$.

Plugging Eq. (39) into Eq. (38), we obtain

$$\gamma d^{(2)}(\omega_\Sigma, \omega_D, \gamma) = \left(\frac{1}{\omega_\Sigma + \omega_D} + \frac{1}{\omega_\Sigma - \omega_D} \right) + \delta \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_D}{2\pi i} d^{(2)}(\omega_\Sigma, \omega_D, \gamma), \quad (40)$$

where $\delta = 2/(N_c^2 - 1)$.

Seeking a solution of the form

$$d^{(2)}(\omega_\Sigma, \omega_D, \gamma; q_T) = \frac{d_{in}^{(2)}(\omega_\Sigma; q_T)}{(\gamma(\omega_\Sigma^2 - \omega_D^2) - 2\omega_\Sigma)}, \quad (41)$$

we obtain for the spectrum of the linear equation:

$$1 = \frac{1}{2} \delta \frac{1}{\sqrt{\gamma\omega_\Sigma(\gamma\omega_\Sigma - 2)}}. \quad (42)$$

Utilizing the smallness of δ , we see that we have an anomalous dimension, whose value is larger than at $\delta = 0$.

This fact has been noted and discussed, for the twist-four anomalous dimension, in Refs. [16–18,38–41]. The solution to Eq. (42) at $\omega_D = 0$ has the following form:

$$\gamma_{an}(\omega_\Sigma, \omega_D = 0) = \frac{2}{\omega_\Sigma} \left(1 + \frac{\delta^2}{16} \right). \quad (43)$$

One can check that the general form of the anomalous dimension is

$$\begin{aligned} \gamma_{an}(\omega_1, \omega_2) &= \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{\delta^2}{8} \frac{2}{\omega_1 + \omega_2} \\ &\equiv \frac{1}{\omega_1} + \frac{1}{\omega_2} + \tilde{\delta} \frac{1}{\omega_1 + \omega_2} \end{aligned} \quad (44)$$

with $\tilde{\delta} = 1/(N_c^2 - 1)^2$.

We wish to stress, that the corrections to the anomalous dimension of the double parton distributions, are of the order $(\frac{1}{N_c^2 - 1})^2$. It has the same dependence on N_c , as the correction to the anomalous dimension of the twist-four operator [16–18].

The numerical value for $\tilde{\delta}$ in Eq. (44) is about 0.016 for $N_c = 3$, which indicates that this correction is rather small. However, these corrections generate the term which increases both with Y and ξ , and could be important in

the region of small x . We will examine them in the next section.

The procedure just described, is a method to find the first corrections with respect to δ , to the value of the anomalous dimension. Replacing Eq. (41) by the following expression,

$$d^{(2)}(\omega_\Sigma, \omega_D, \gamma) = \frac{d_{in}^{(2)}(\omega_\Sigma)}{(\gamma - \gamma_{an}^{(1)}(\omega_1, \omega_2))(\omega_\Sigma^2 - \omega_D^2)}, \quad (45)$$

we obtain an equation for the next order correction to the value of the anomalous dimension. It has the form

$$1 = \frac{1}{2} \delta \frac{\gamma \omega_\Sigma - 1}{\sqrt{\gamma \omega_\Sigma} (\gamma \omega_\Sigma - 2 + \frac{\tilde{\delta}^2}{\gamma \omega_\Sigma})^{3/2}}. \quad (46)$$

Equation (46) leads to the next-order corrections for the value of the anomalous dimension:

$$\gamma_{an}^{(2)}(\omega_1, \omega_2) = \gamma_{an}^{(1)}(\omega_1, \omega_2) - \frac{1}{2} \tilde{\delta}^2 \frac{2}{\omega_1 + \omega_2} + \mathcal{O}(\tilde{\delta}^3). \quad (47)$$

C. Evolution with Bose-Einstein enhancement: Solution

The solution to the nonhomogenous equation has a general form:

$$\begin{aligned} D^{(2)}(Y, Y, \xi) &= \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_D}{2\pi i} e^{2\omega_\Sigma Y + \gamma \xi} \\ &\times \left\{ \underbrace{\frac{d_{in}^{(1)}(\omega_\Sigma) \delta(\omega_D)}{(\gamma - \gamma_{an}(\omega_\Sigma, \omega_D))(\gamma - \gamma_G(\omega_\Sigma))}}_{\text{particular solution of nonhomogenous equation}} + \underbrace{\frac{d_{in}^{(2)}(\omega_1, \omega_2)}{\gamma - \gamma_{an}(\omega_\Sigma, \omega_D)}}_{\text{general solution of homogenous equation}} \right\}. \end{aligned} \quad (48)$$

Taking the integral over γ , by closing the contours around the poles: $\gamma = \gamma_G(\omega_1 + \omega_2)$ and $\gamma = \gamma_{an}(\omega_\Sigma, \omega_D)$, we obtain

$$\begin{aligned} D^{(2)}(Y, Y, \xi) &= \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_D}{2\pi i} e^{\omega_\Sigma Y} \left\{ d_{in}^{(2)}(\omega_\Sigma, \omega_D) e^{\gamma_{an}(\omega_\Sigma, \omega_D) \xi} + \frac{d_{in}^{(1)}(\omega_1 + \omega_2) \delta(\omega_D) (e^{\gamma_{an}(\omega_\Sigma, \omega_D) \xi} - e^{\gamma_G(\omega_1 + \omega_2) \xi})}{\gamma_{an}(\omega_\Sigma, \omega_D) - \gamma_G(\omega_1 + \omega_2)} \right\}. \end{aligned} \quad (49)$$

In the DLA, both Y and ξ are large, and we can use the method of steepest descent to evaluate the integral. The equations for the saddle point values for $\omega_1 = \omega_1^{SP}$ and $\omega_2 = \omega_2^{SP}$ have the form

$$\begin{aligned} Y &= \xi \frac{\partial \gamma_{an}(\omega_\Sigma, \omega_D)}{\partial \omega_1} = \xi \left(\frac{1}{(\omega_1^{SP})^2} + \frac{\tilde{\delta}}{(\omega_1^{SP} + \omega_2^{SP})^2} \right); \\ Y &= \xi \frac{\partial \gamma_{an}(\omega_\Sigma, \omega_D)}{\partial \omega_2} = \xi \left(\frac{1}{(\omega_2^{SP})^2} + \frac{\tilde{\delta}}{(\omega_1^{SP} + \omega_2^{SP})^2} \right). \end{aligned} \quad (50)$$

From Eq. (50), one can see that $\omega_D^{SP} = 0$. Therefore Eq. (50) reduces to

$$2Y = 2 \frac{(1 + \frac{1}{2} \tilde{\delta}) \xi}{(\omega_\Sigma^{SP})^2}; \quad \omega_\Sigma^{SP} = \sqrt{\frac{(1 + \frac{1}{2} \tilde{\delta}) \xi}{Y}}; \quad (51)$$

Integrating over ω_D expanding $\gamma_{an}(\omega_\Sigma, \omega_D) = \gamma_{an}(\omega_\Sigma, \omega_D = 0) + \omega_D^2 / \omega_\Sigma^3$ and neglecting the contributions that are proportional to δ everywhere, except for the anomalous dimension in the exponent, we have the solution in the form

$$D^{(2)}(Y, Y, \xi) = \frac{1}{4\pi} \frac{(\omega_\Sigma^{SP})^{3/2}}{\xi} \left(\frac{4}{3} d_{in}^{(1)}(\omega_\Sigma^{SP}) + d^{(2)}(\omega_\Sigma^{SP}, \omega_D^{SP} = 0) \right) e^{4\sqrt{2N_c(1+\frac{1}{2}\delta)}Y\xi} - \frac{1}{\sqrt{3\pi}} \sqrt{\frac{(\omega_\Sigma^{SP})^3}{\xi}} \omega_\Sigma^{SP} d_{in}^{(1)}(\omega_\Sigma^{SP}) e^{2\sqrt{2N_c}\xi}. \quad (52)$$

Recall, that we have changed our notation, and $Y = \ln(1/x)$ in Eq. (52).

The principle difference of the behavior of the double parton density without Bose-Einstein enhancement, is that we did not have a term which is proportional to the product of two single parton densities. On the other hand, the violation of Eq. (27) is small, and we reach a value larger than 10%, only at $Y\xi \gtrsim 4$.

At the LHC energies, $Y \sim 16$, and $\xi \geq 0.25$ appears to be a reasonable region for measurements.

V. NONLINEAR EVOLUTION EQUATION FOR THE SCATTERING AMPLITUDE

A. The equation

We have discussed the double parton densities, but the cross section for J/ψ production depends on the scattering amplitude for which the two BFKL Pomeron contribution of the diagram of Fig. 2 and the double gluon density in general, only give a contribution in the linear approach. The scattering amplitude depends crucially on the shadowing corrections (see [1] for the review). In this section, we write the generalization of the BK equation [9,15] for the

scattering amplitude for two dipoles with a target. Following Ref. [9], the generating functional for the scattering amplitude N is defined

$$N(Y, [\gamma_n]) = \sum_{n=2}^{\infty} (-1)^n \int \gamma_n(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_n, \mathbf{b}_n; Y_0) \times \rho^{(n)}(\mathbf{r}_1 \mathbf{b}_1, \dots, \mathbf{r}_n, \mathbf{b}_n; Y - Y_0) \prod_{i=1}^n d^2 r_i d^2 b_i, \quad (53)$$

where γ_n denotes the amplitude for simultaneous scattering of n dipoles off the target, and $\rho^{(n)}$ is the n -density of the fast projectile, i.e., the two dipoles. As we saw from Fig. 2 at $Y \rightarrow Y_0$, the scattering amplitude of interest includes the interaction of two dipoles. From Eq. (53), one has that

$$\rho^{(n)} = (-1)^n \frac{\delta N}{\delta \gamma_n}. \quad (54)$$

Using Eq. (19), Eq. (20) and Eq. (21) we obtain the linear functional equation for the amplitude N :

$$\begin{aligned} \frac{\partial N(Y, [\gamma_n])}{\partial Y} &= \sum_{n=1}^{\infty} \int \prod_{i=1}^n d^2 r_i d^2 b_i \gamma_n(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_n, \mathbf{b}_n; Y_0) \\ &\times \left(- \sum_{i=1}^n \omega(r_i) \frac{\delta N}{\delta \gamma_n} + 2 \sum_{i=1}^n \int \frac{d^2 r'}{2\pi} \frac{r_i^2}{r_i^2 (\mathbf{r}_i - \mathbf{r}')^2} \frac{\delta N}{\delta \gamma_n} - \sum_{i=1}^{n-1} \frac{(\mathbf{r}_i + \mathbf{r}_n)^2}{(2\pi) r_i^2 r_n^2} \frac{\delta N}{\delta \gamma_{n-1}} \right). \end{aligned} \quad (55)$$

The equation (55) can be solved using the ansatz: $N(Y, [\gamma]) = N(\gamma_1(Y), \gamma_2(Y) \dots)$ and the initial condition that at $Y = Y_0$:

$$\gamma_2(\mathbf{r}_1, \mathbf{b}, \mathbf{r}_2, \mathbf{b}) = \gamma_1(\mathbf{r}_1, \mathbf{b}) \gamma_1(\mathbf{r}_2, \mathbf{b}) \delta^{(2)}(\mathbf{r} - \mathbf{r}_1) \delta^{(2)}(\mathbf{b} - \mathbf{b}_1) \delta^{(2)}(\mathbf{r}' - \mathbf{r}_2) \delta^{(2)}(\mathbf{b}' - \mathbf{b}_1); \quad \gamma_n = 0 \quad \text{for } n \neq 2; \quad (56)$$

Using Eq. (56) we can reduce Eq. (55) to the following form

$$\begin{aligned} \frac{\partial N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})}{\partial Y} &= \sum_{i=1}^2 \int d^2 r' \frac{r_i^2}{r_i^2 (\mathbf{r}_i - \mathbf{r}')^2} \left\{ 2N^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \right. \\ &\quad \left. - N^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) N^{(1)}\left(Y, \mathbf{r}_1 - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) \right\}, \end{aligned} \quad (57)$$

where $N^{(1)}$ denotes the solution to the Balitsky-Kovchegov equation. Note that in Eq. (57), $r_3 \equiv r_1$.

B. Solution deep in the saturation region

In this section, using the approach developed in Ref. [42], we find the solution to Eq. (57) deep in the saturation region. In this region, both $N^{(2)}$ and $N^{(1)}$ are close to unity, and can be written as

$$N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) = 1 - \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}); \quad N^{(1)}(Y, \mathbf{r}, \mathbf{b}) = 1 - \Delta^{(1)}(Y, \mathbf{r}, \mathbf{b}); \quad (58)$$

with $\Delta^{(2)} \ll 1$ and $\Delta^{(1)} \ll 1$. In this kinematic region, all $\tau_i = r_i^2 Q_s^2(Y, b_i) \gg 1$. Plugging Eq. (58) into Eq. (57), and neglecting the contribution of order $\Delta^{(2)}\Delta^{(1)}$, we obtain the following equation for $\Delta^{(2)}$.

$$\frac{\partial \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})}{\bar{\alpha}_S \partial Y} = \sum_{i=1}^2 \int d^2 r' \frac{r_i^2}{r'^2 (\mathbf{r}_i - \mathbf{r}')^2} \left\{ \Delta^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_2, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \Delta^{(1)}\left(Y, \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \right\}. \quad (59)$$

First, we solve the homogenous equation:

$$\frac{\partial \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})}{\bar{\alpha}_S \partial Y} = \frac{1}{2} \sum_{i=1}^2 \int d^2 r' \frac{r_i^2}{r'^2 (\mathbf{r}_i - \mathbf{r}')^2} \left\{ 2\Delta^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_2, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \right\} - \frac{1}{2}(\omega(r_1) + \omega(r_2))\Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}). \quad (60)$$

Inside the saturation region [42], $\omega(r_i) = \ln(\tau_i) = \ln(r_i^2 Q_s^2(Y, \mathbf{b})) = z_i$ and therefore the last term of the l.h.s. of the equation has a contribution: $\frac{1}{2}(z_1 + z_2)\Delta^{(2)}$. Bearing in mind that we are searching for the solution of Eq. (60) in the form $\Delta^{(2)}(z_1 + z_2)$. To find the solution we go to the Mellin transform given by

$$\Delta^{(2)}(z_1 + z_2) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\gamma(z_1 + z_2)} \tilde{\Delta}^{(2)}(\gamma). \quad (61)$$

The term in $\{\dots\}$ in Eq. (60) gives a contribution which is equal to $\frac{1}{2}\chi(\gamma)\tilde{\Delta}^{(2)}(\gamma)$, where χ is the BFKL kernel[1,4]:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma), \quad (62)$$

where $\psi(x) = d \ln \Gamma(x) / dx$ and Γ is the Euler gamma function [43]. Taking into account that $\ln(Q_s^2(Y, \mathbf{b})) = \kappa Y$ with $\kappa = \frac{\chi(1 - \gamma_{cr})}{1 - \gamma_{cr}}$, where γ_{cr} in the leading log approximation, which we use in this paper, stems from the solution of the equation:

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} = \left| \frac{d\chi(\gamma_{cr})}{d\gamma_{cr}} \right| \quad (63)$$

and $\gamma_{cr} \approx 0.37$.

Finally, Eq. (60) takes the form

$$2\kappa\gamma\tilde{\Delta}^{(2)}(\gamma) = \chi(\gamma)\tilde{\Delta}^{(2)}(\gamma) + \frac{1}{2} \frac{d\tilde{\Delta}^{(2)}(\gamma)}{d\gamma}. \quad (64)$$

Solving this equation, we obtain

$$\tilde{\Delta}^{(2)}(\gamma) = C \exp \left(2\kappa\gamma^2 - 2 \int^\gamma d\gamma' \chi(\gamma') \right), \quad (65)$$

where C is the arbitrary constant. From Eq. (61), we obtain that

$$\Delta^{(2)}(z_1 + z_2) = C \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} \exp \left(\gamma(z_1 + z_2) + 2\kappa\gamma^2 - 2 \int^\gamma d\gamma' \chi(\gamma') \right). \quad (66)$$

Taking the integral over γ using the method of steepest descent, we obtain the following equations for the saddle point γ^{SP} :

$$(z_1 + z_2) + 4\kappa\gamma^{SP} - 2\chi(\gamma^{SP}) = 0. \quad (67)$$

For large values of $z_1 + z_2$, $\gamma^{SP} = -\frac{1}{4\kappa}((z_1 + z_2) + 2\chi(-\frac{1}{2\kappa}(z_1 + z_2)))$ since $\chi(\gamma) \xrightarrow{\gamma \gg 1} \ln(\gamma)$.

Therefore,

$$\Delta^{(2)}(z_1 + z_2) = C \exp \left(-\frac{(z_1 + z_2)^2}{8\kappa} + \frac{1}{2\kappa} \chi^2 \left(-\frac{1}{2\kappa}(z_1 + z_2) \right) - \int^{-\frac{1}{2\kappa}(z_1 + z_2)} d\gamma' \chi(\gamma') \right) \quad (68)$$

Since

$$\int^\gamma d\gamma' \chi(\gamma') = -2\gamma_E \gamma + \ln \left(\frac{\Gamma(1 - \gamma)}{\Gamma(\gamma)} \right) \xrightarrow{\gamma \gg 1} -2\gamma \ln(\gamma) \quad (69)$$

at large $z_1 + z_2$ we see that

$$\Delta_{h.eq.}^{(2)}(z_1 + z_2) \xrightarrow{z_1 + z_2 \gg 1} C e^{-\frac{(z_1 + z_2)^2}{8\kappa} - \frac{(z_1 + z_2)}{2\kappa} \ln(z_1 + z_2)}, \quad (70)$$

where *h.eq.* denotes homogenous equation.

It should be noted that the main contribution to the large $z_1 + z_2$ asymptotic behavior, stems from the term: $\frac{1}{2}(\omega(r_1) + \omega(r_2))\Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})$ in the homogenous equation [see Eq. (60)], which is responsible for the gluon reggeization. Having this in mind, we can simplify the nonhomogenous Eq. (59), replacing it by

$$\frac{\partial \Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})}{\bar{\alpha}_S \partial Y} = \frac{1}{2}(\omega(r_1) + \omega(r_2))\Delta^{(2)}(Y, \mathbf{r}'_1, \mathbf{r}_2, \mathbf{b}) - \frac{1}{2}\omega(r_1)\Delta^{(1)}(Y, \mathbf{r}_1, \mathbf{b}) - \frac{1}{2}\omega(r_2)\Delta^{(1)}(Y, \mathbf{r}_2, \mathbf{b}) \quad (71)$$

Introducing new variables: $z_{12} = z_1 + z_2$ and $\zeta_{12} = z_1 - z_2$, we can rewrite Eq. (71) in the form:

$$2\kappa \frac{\partial \Delta^{(2)}(z_{12}, \zeta_{12})}{\partial z_{12}} = \frac{1}{2}z_{12}\Delta^{(2)}(z_{12}, \zeta_{12}) - \frac{1}{4}(z_{12} + \zeta_{12})\Delta^{(1)}(z_{12} + \zeta_{12}) - \frac{1}{4}(z_{12} - \zeta_{12})\Delta^{(1)}(z_{12} - \zeta_{12}). \quad (72)$$

The solution to this equation has a general form

$$\begin{aligned} \Delta^{(2)}(z_{12}, \zeta_{12}) &= \Delta_{h.eq.}^{(2)}(z_{12}, \zeta_{12}) \\ &- \frac{1}{4}\Delta_{h.eq.}^{(2)}(z_{12}, \zeta_{12}) \int_0^{z_{12}} dz'_{12} \frac{((z'_{12} + \zeta_{12})\Delta^{(1)}(z'_{12} + \zeta_{12}) + (z'_{12} - \zeta_{12})\Delta^{(1)}(z'_{12} - \zeta_{12}))}{\Delta_{h.eq.}^{(2)}(z'_{12}, \zeta_{12})}. \end{aligned} \quad (73)$$

Plugging in Eq. (73) $\Delta^{(1)}(z_{12} + \zeta_{12}) = C^{(1)} \exp(-(z_{12} + \zeta_{12})/(8\kappa))$ [42] we obtain the general solution in the following form:

$$\begin{aligned} \Delta^{(2)}(z_{12}, \zeta_{12}) &= C^{(2)}(\zeta_{12})\Delta_{h.eq.}^{(2)}(z_{12}) - \kappa(\Delta^{(1)}(z'_{12} + \zeta_{12}) + \Delta^{(1)}(z'_{12} - \zeta_{12})) \\ &= C^{(2)}(\zeta_{12}) \exp(-z_{12}^2/(8\kappa)) - \kappa C^{(1)}(\exp(-z_{12}^2/(2\kappa)) + \exp(-z_{12}^2/(2\kappa))), \end{aligned} \quad (74)$$

where $C^{(2)}(z_1 - z_2)$ is an arbitrary function of $z_1 - z_2$, which should be found from the initial conditions.

One can see that the scattering amplitude shows geometric scaling behavior depending on two variable z_1 and z_2 , instead of three variables: Y, r_1, r_2 . For $z_1 \approx z_2$, $\Delta^{(2)} \propto \Delta^{(1)}(z_1 \approx z_2)$, while for $z_1 \gg z_2 \gg 1$, we see that $\Delta^{(2)}(z_1) \propto \exp(-z_1^2/(8\kappa)) \gg \Delta^{(1)}(z_1)$.

C. Solution in the vicinity of the saturation scale

As has been discussed in Refs. [5,44], one does not need to know the exact form of the shadowing corrections to find the behavior of the scattering amplitude ($N^{(1)}(Y, r, b)$), in the vicinity of the saturation scale. One should find the solution to the linear equation and the equation for the saturation momentum has the form: $N(Y, r^2 = 1/Q_s^2(Y, b), b) = N_0 < 1$, where N_0 is an arbitrary constant, which is numerically small (say $N_0 = 1/3$).

We can find the solution to the linear equation using the Mellin transform:

$$N^{(2)}(Y, \xi_1, Y, \xi_2; b) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma_2}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} e^{(1-\gamma_1)\xi_1 + (1-\gamma_2)\xi_2 + \omega Y} d^{(2)}(\omega, \gamma_1, \gamma_2) \quad (75)$$

where $\xi_i = \ln(r_i^2 \mu^2)$ where μ is the soft scale.

For $d^{(2)}(\omega, \gamma_1, \gamma_2)$, the equation takes the form

$$\omega d^{(2)}(\omega, \gamma_1, \gamma_2) = \bar{\alpha}_S \{\chi(\gamma_1) + \chi(\gamma_2)\} d^{(2)}(\omega, \gamma_1, \gamma_2). \quad (76)$$

Therefore, the amplitude $N^{(2)}(Y, \xi_1, Y, \xi_2; b)$ is equal to

$$N^{(2)}(Y, \xi_1, Y, \xi_2; b) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma_2}{2\pi i} d_{in}^{(2)}(\gamma_1, \gamma_2) \exp(\{\chi(\gamma_1) + \chi(\gamma_2)\}Y + (1-\gamma_1)\xi_1 + (1-\gamma_2)\xi_2). \quad (77)$$

Taking the integral using the method of steepest descent, we obtain the following equations for the saddle points γ_1^{SP} and γ_2^{SP} :

$$\frac{d\chi(\gamma_1^{\text{SP}})}{d\gamma_1^{\text{SP}}} Y - \xi_1 = 0; \quad \frac{d\chi(\gamma_2^{\text{SP}})}{d\gamma_2^{\text{SP}}} Y - \xi_2 = 0. \quad (78)$$

The line, on which $N^{(2)}(Y, \xi_1, Y, \xi_2; b)$ is constant, is given by the equation:

$$\{\chi(\gamma_1^{\text{SP}}) + \chi(\gamma_2^{\text{SP}})\}Y + (1-\gamma_1^{\text{SP}})\xi_1 + (1-\gamma_2^{\text{SP}})\xi_2 = 0. \quad (79)$$

FIG. 7. $d^2\chi(\gamma)/(d\gamma)^2$ [Fig. 7(a)] and the solution to Eq. (79) [Fig. 7(b)]. In Fig. 7(b), dotted, dashed and dot-dashed lines corresponds to $(\xi_1 - \xi_2)/Y = 0, 1, 10$, respectively. The solid line corresponds to the term $\{\chi(\gamma_1^{\text{SP}}) + \chi(\gamma_2^{\text{SP}})\}$ in Eq. (79).

One can see that the difference $\Delta\gamma^{\text{SP}} = \gamma_1^{\text{SP}} - \gamma_2^{\text{SP}}$ turns out to be small. Indeed,

$$\left(\frac{d\chi(\gamma_1^{\text{SP}})}{d\gamma_1^{\text{SP}}} - \frac{d\chi(\gamma_2^{\text{SP}})}{d\gamma_2^{\text{SP}}}\right)Y = \frac{d^2\chi(\frac{1}{2}(\gamma_1^{\text{SP}} + \gamma_2^{\text{SP}}))}{(d(\frac{1}{2}(\gamma_1^{\text{SP}} + \gamma_2^{\text{SP}}))^2)}Y\Delta\gamma^{\text{SP}} = \xi_1 - \xi_2. \quad (80)$$

In Fig. 7(a), we show that $d^2\chi(\gamma)/d\gamma^2$ turns out to be large, as well as the value of Y . Therefore, for reasonable values of $\xi_1 - \xi_2 \leq Y$, Eq. (80) leads to a large value of $\Delta\gamma^{\text{SP}}$. Bearing this in mind we see that the solution of Eq. (79) coincides with the solution of Eq. (63). Figure 7(b) gives the solution of Eq. (79) when $\gamma_1^{\text{SP}} \neq \gamma_2^{\text{SP}}$. One can see that a significant difference for $\Delta\gamma^{\text{SP}}$ can be obtained for $(\xi_1 - \xi_2)/Y \approx 10$.

Generally, the solution to Eq. (77) has the following dependence on r_1 and r_2 :

$$N^{(2)}(Y, r_1, Y, r_2; b) = \text{Const}(r_1^2 Q_1^2(Y, b))^{1-\gamma_1^{\text{cr}}}(r_2^2 Q_2^2(Y, b))^{1-\gamma_2^{\text{cr}}}, \quad (81)$$

where γ_1^{cr} and γ_2^{cr} are solutions to Eq. (79) [see Fig. 7(b)] and

$$\begin{aligned} \ln(Q_1^2(Y, b)/Q_1^2(Y = Y_0, b)) &= \frac{\chi(\gamma_1^{\text{cr}})}{1 - \gamma_1^{\text{cr}}} Y; \\ \ln(Q_2^2(Y, b)/Q_1^2(Y = Y_0, b)) &= \frac{\chi(\gamma_2^{\text{cr}})}{1 - \gamma_2^{\text{cr}}} Y. \end{aligned} \quad (82)$$

D. Initial conditions

Taking the shadowing corrections into account for the scattering amplitude, we can rewrite Eq. (9) for the cross section of J/ψ production in the form:

$$\sigma(Y, Q^2) = 4\bar{\alpha}_S^2 \int d^2 r d^2 r' \Phi(\mathbf{r}, \mathbf{r}') \int d^2 b N_A^{(2)}(x, \mathbf{r}; x, \mathbf{r}'; \mathbf{b}_T). \quad (83)$$

From Eq. (3) and Eq. (10) we find that Eq. (83) takes the following form:

$$\begin{aligned} \sigma(Y, Q^2) &= 4\bar{\alpha}_S^2 \int \frac{d^2 r_1}{4\pi} \Psi_{\gamma^*}(Q, r_1) \Psi_{J/\psi}(r_1) \int \frac{d^2 r_2}{4\pi} \Psi_{\gamma^*}(Q, r_2) \Psi_{J/\psi}(r_2) \\ &\quad \times \left\{ N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) - N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) \right. \\ &\quad \left. - N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) + N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); Y, -\frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) \right\} \\ &= 4\bar{\alpha}_S^2 \int \frac{d^2 r_1}{4\pi} \Psi_{\gamma^*}(Q, r_1) \Psi_{J/\psi}(r_1) \int \frac{d^2 r_2}{4\pi} \Psi_{\gamma^*}(Q, r_2) \Psi_{J/\psi}(r_2) \\ &\quad \times 2 \left\{ N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) - N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) \right\}. \end{aligned} \quad (84)$$

To solve Eq. (57), we need to put in the initial conditions. First, we need to find the solution to the BK equation, and the initial conditions for $N^{(1)}(Y, r, b)$ for dipole scattering with nuclei, this is given by the McLerran-Venugopalan formula [7]

$$N^{(1)}(Y = Y_0, r, b) = 1 - \exp(-\sigma_{\text{dn}}(r)S_A(b)) \xrightarrow{\text{QCA}} 1 - \exp\left(-\frac{1}{8}r^2 Q_s^2(Y_0, b)\right), \quad (85)$$

where the dipole-nucleon cross section (σ_{dn}) in the Born approximation of perturbative QCD is equal to

$\sigma_{\text{dn}} = 4\pi\alpha_s^2 r^2 \ln(1/(r^2\mu^2))$ where μ is the soft scale. In the quasiclassical approach (QCA), we replace this cross section by $\sigma_{\text{dn}} = \frac{1}{8}r^2 Q_s^2(Y = Y_0, b)$.

The initial condition for $\{\dots\}$ in Eq. (84) can be found using the approach of Refs. [2,3]. For J/ψ production at low energy, the $q\bar{q}$ pair propagate from first to the last inelastic interaction (see Fig. 1(a) points z_1 and z_2 , respectively), undergoing inelastic interactions with the cross section $\frac{1}{16}(\mathbf{r}_1 - \mathbf{r}_2)^2 Q_s^2(Y_0, b)$. Before the first inelastic collisions and after the last one, the quark-antiquark pair has only elastic rescatterings. Plugging in the expression for the first and the last inelastic interaction, given in Refs. [2,3], we obtain

$$\left\{ N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) - N_A^{(2)}\left(Y, \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2); Y, \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2); \mathbf{b}_T\right) \right\}_{Y=Y_0} = Q_s^2(Y = Y_0, b) \frac{(\mathbf{r}_1 \cdot \mathbf{r}_2)^2}{(\mathbf{r}_1 + \mathbf{r}_2)^2} \left(\exp\left(-\frac{1}{16}Q_s^2(Y_0, b)(\mathbf{r}_1 - \mathbf{r}_2)^2\right) - \exp\left(-\frac{1}{8}Q_s^2(Y_0, b)(r_1^2 + r_2^2)\right) \right). \quad (86)$$

In Ref. [26] (see also Refs. [45,46]), it is shown how Eq. (86) stems from the scattering amplitude of two dipoles with the nucleus target in the large N_c limit.

1. Wave functions for virtual photon and J/ψ

For completeness of presentation, we need to specify what expression we use for the wave functions in Eq. (57). We adopt the wave functions from Refs. [47,48] which have the following form, Denoting $\Psi_{\gamma^*}(Q, r_1)\Psi_{J/\psi}(r_1) = \int dz \Phi_{L,T}^{J/\psi}(z, \mathbf{r}, Q^2)$

$$\begin{aligned} \Phi_L^{J/\psi}(z, \mathbf{r}, Q^2) &= \hat{e}_f \sqrt{\frac{\alpha_e}{4\pi}} N_c 2Q K_0(r\bar{Q}_f) M_{J/\psi} z(1-z) \phi_L(r, z), \\ \Phi_T^{J/\psi}(z, \mathbf{r}, Q^2) &= \hat{e}_f \sqrt{\frac{\alpha_e}{4\pi}} N_c \frac{\alpha_e N_c}{2\pi^2} \{ m_f^2 K_0(r\bar{Q}_f) \phi_T(r, z) - [z^2 + (1-z)^2] \bar{Q}_f K_1(r\bar{Q}_f) \partial_r \phi_T(r, z) \}, \end{aligned} \quad (87)$$

where e_f denotes the charge of the c -quark and m_c is its mass. $\bar{Q}_f^2 = m_c^2 z(1-z) + Q^2$. L and T denote the polarizations of the photon. For $\phi_{L,T}(r, z)$, we suggest to use the simplest Gaussian parametrization.

$$\phi_L = N_L \exp[-r^2/(2R_L^2)]; \quad \phi_T = N_T z(1-z) \exp[-r^2/(2R_T^2)]. \quad (88)$$

All numerical parameters can be found in Refs. [47,48].

VI. BOSE-EINSTEIN ENHANCEMENT AND SHADOWING

At first sight, the increase of the double parton density, which is stronger than the product of two single parton densities, is potentially harmful, and could lead to a violation of the unitarity constraints in the framework of the BFKL Pomeron calculus. On the other hand the JIMWLK equation [10] which includes all corrections of the order of $1/(N_c^2 - 1)$ does not exhibit any problem with unitarity. The goal of this section is to show that it is sufficient that $N^{(1)} \rightarrow 1 - \Delta^{(1)}$ for large Y , with $\exp(\Delta_P Y) \Delta^{(1)} \times (Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \ll 1$, to ensure unitarity at high energies. The reason for this feature can be illustrated using Eq. (57) in which we insert $N^{(1)} = 1$. Such an equation determines the asymptotic solution for $N^{(2)}$. The resulting equation takes the form:

$$\begin{aligned} \frac{\partial N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b})}{\bar{\alpha}_S \partial Y} &= \sum_{i=1}^2 \int d^2 r' \frac{r_i^2}{r'^2 (\mathbf{r}_i - \mathbf{r}')^2} \left\{ N^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \right\} \\ &= \frac{1}{2} \sum_{i=1}^2 \int d^2 r' \frac{r_i^2}{r'^2 (\mathbf{r}_i - \mathbf{r}')^2} \left\{ 2N^{(2)}\left(Y, \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \right\} \\ &\quad - \frac{1}{2} \bar{\alpha}_S (\omega(r_1) + \omega(r_2)) N^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}). \end{aligned} \quad (89)$$

FIG. 8. The exchange of the extra BFKL Pomeron for DGLAP [Fig. 8(a)] and interference [Fig. 8(b)] diagrams in DLA of perturbative QCD. Figure 8(a) and Fig. 8(b) show the emission an extra gluon in one of the parton showers and its interaction with the target due to the BFKL Pomeron exchange.

The first line of this equation shows that $N^{(2)} = \text{Const}$ is the solution of the equation. The second line illustrates that $N^{(2)}$ approaches the asymptotic solution with the function $\Delta^{(2)}(Y, \mathbf{r}_1, \mathbf{r}_2, \mathbf{b}) \propto \exp(-\frac{1}{2}\bar{\alpha}_S(\omega(r_1) + \omega(r_2)Y))$ ($N^{(2)} = \text{Const} - \Delta^{(2)}$). It should be stressed that the fact that $N^{(2)}$

approaches a constant, does not depend on the value of $\bar{\alpha}_S$ which determines the intercept of the energy (Y) dependence of the solution to the linear equation. This screening is provided by the elastic amplitude, which is equal to unity at high energy. Hence, the elastic rescattering of patrons creates a survival probability [49,50], which suppresses the powerlike increase of the double parton amplitudes. We wish to show that $1/(N_c^2 - 1)$ corrections to the evolution equation have the same structure as Eq. (89). For the sake of simplicity, we discuss the shadowing corrections in the framework of the DLA of perturbative QCD (see Eq. (37) and Fig. 6). Figure 8 illustrates that the emission of an extra gluon can be screened by the exchange of the BFKL Pomeron, both for normal DGLAP evolution, as well as for the interference diagram.

The advantage of the evolution equation (see Eq. (57)), is that we can take into account the shadowing corrections in the same way for both terms of the equation for the double gluon structure function. The homogenous part of the resulting equation has the form:

$$\begin{aligned} \frac{\partial \mathcal{N}^{(2)}(Y, Y, \xi)}{\partial \xi} = & \int_{Y_0}^Y dY' \mathcal{N}^{(2)}(Y', Y, \xi) \left\{ 1 - \frac{\bar{\alpha}_S}{\pi R_A^2 Q^2} \mathcal{N}^{(1)}(Y', \xi) \right\} \\ & + \int_{Y_0}^Y dY' \mathcal{N}^{(2)}(Y, Y', \xi) \left\{ 1 - \frac{\bar{\alpha}_S}{\pi R_A^2 Q^2} \mathcal{N}^{(1)}(Y', \xi) \right\} \\ & + \frac{2}{N_c^2 - 1} \int_{Y_0}^Y dY' \mathcal{N}^{(2)}(Y', Y', \xi) \left\{ 1 - \frac{\bar{\alpha}_S}{\pi R_A^2 Q^2} \mathcal{N}^{(1)}(Y', \xi) \right\}, \end{aligned} \quad (90)$$

where

$$\frac{1}{\pi R_A^2} \equiv \int d^2 q_T S_A(q_T), \quad (91)$$

and $\mathcal{N}^{(2)}(Y, Y, \xi)$ denotes the scattering amplitude, which in the linear approximation coincides with the double parton distribution function. However, this amplitude, as can be seen from Eq. (90) and Fig. 8, contains shadowing corrections, while the double parton distribution function cannot be affected by the shadowing, as we have seen above. This screening is provided by the elastic amplitude which is equal to unity at high energy. Hence, the elastic

rescattering of patrons creates a survival probability [49,50], which suppress the powerlike increase of the double parton amplitudes Eq. (90), and has a typical form for the shadowing correction in DLA of perturbative QCD [1,5,6]. However, to see all numerical coefficients and to discuss the influence of shadowing on the asymptotic behavior of the scattering amplitude, it is desirable to rewrite Eq. (90) in the coordinate representation.

The amplitude $\mathcal{N}^{(2)}(Y, Y, \xi)$ in the coordinate representation is intimately related to the scattering amplitude of the dipole of size x_{01} (see Fig. 9) with the nucleus which interacts with the target two or more times. In the Born approximation, such an amplitude is equal to

$$\tilde{N}_{\text{B.A.}}^{(2)}(Y, Y, \xi; \mathbf{b}) = \int d^2 b (N_{\text{B.A.}}^{(1)}(Y, x_{01}, b))^2 / x_{01}^4 = \left(\int d^2 b N_{\text{B.A.}}^{(1)}(Y, x_{01}, b) \right)^2 / (x_{01}^4 \pi R_A^2) \propto \ln^2(1/x_{01}^2) \propto \xi^2, \quad (92)$$

where $\xi = \ln(1/r^2)$. In DLA, it is sufficient to change ξ to $\tilde{\xi}$ for transforming from momentum to coordinate representation.

The energy evolution of this amplitude is given by the emission of the extra gluon [see Fig. 9(c)] and replaces ξ in Eq. (95) by the function of $\tilde{\xi}$. The contribution of the

interference diagram is shown in Fig. 9(a). One can see that gluons with transverse momenta $\mathbf{p}_{2T}$ and $-\mathbf{p}_{2T}$ interacts with the dipole x_{12} . However, gluons with momenta $\mathbf{p}_{1T}$ and $-\mathbf{p}_{1T}$ interact with the dipoles of the different sizes: x_{12} and x_{01} . In the DLA, $x_{12} \gg x_{01}$. The interaction with these two gluons can be written as follows:

FIG. 9. The interference diagram for the scattering of one dipole with two nucleons. The blocks show the amplitude: the exchange of two BFKL Pomerons [Fig. 9(a)] and the contribution of the shadowing corrections to this diagram [Fig. 9(b)]. Figure 9(c) shows the diagrams that correspond to the square of the amplitude. The blobs show the amplitudes of the gluon-nucleon interactions.

$$\int d^2 b N^{(1)}(Y, x_{12}, x_{01}, b) \propto \int d^2 p_{1T} (1 + e^{i\vec{p}_{1T} \cdot \vec{x}_{12}}) (1 - e^{i\vec{x}_{01} \cdot \vec{p}_{1T}}) \frac{N_G(Y, p_{1T})}{p_{1T}^4} \xrightarrow{p_{1T}^2 x_{12}^2 \ll 1; p_{1T}^2 x_{01}^2 \ll 1, x_{12}^2 \gg x_{01}^2} \int_{1/x_{12}^2} d^2 p_{1T} p_{1T}^2 x_{01}^2 \frac{N_G(Y, p_{1T})}{p_{1T}^4} \rightarrow \frac{x_{01}^2}{x_{12}^2} \int d^2 b N^{(1)}(Y, x_{12}, b). \quad (93)$$

Using the result of Eq. (96), we have the following expression for the diagram of Fig. 9(a):

$$\frac{\partial \int d^2 b \tilde{N}^{(2)}(\text{Fig. 9-a}; Y, Y, \tilde{\xi}; \mathbf{b})}{\partial \tilde{\alpha}_S Y} = \frac{1}{x_{01}^4 \pi R_A^2} \left\{ \int d^2 x_{12} \int \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \int d^2 b N^{(1)}(Y, x_{12}, b) \right\} \frac{x_{01}^2}{x_{12}^2} \int d^2 b N^{(1)}(Y, x_{12}, b) \xrightarrow{\text{DLA}} \frac{1}{R_A^2} \int_{x_{01}^2} \frac{dx_{12}^2}{x_{12}^2} \left(\frac{\int d^2 b N^{(1)}(Y, x_{12}, b)}{x_{12}^2} \right)^2. \quad (94)$$

From Eq. (94), we can rewrite Eq. (90) in the coordinate representation in a more transparent form. It has the structure

$$\frac{\partial \int d^2 b \tilde{N}^{(2)}(Y, Y, \tilde{\xi}; \mathbf{b})}{\partial \tilde{\xi}} = \int d^2 b \left(\int_{Y_0}^Y dY' \tilde{N}^{(2)}(Y', Y, \tilde{\xi}, \mathbf{b}) \{2 - e^{-\tilde{\xi} n^{(1)}(Y', \tilde{\xi}, \mathbf{b})}\} + \int_{Y_0}^Y dY' \tilde{N}^{(2)}(Y, Y', \tilde{\xi}, \mathbf{b}) \{2 - e^{-\tilde{\xi} n^{(1)}(Y', \tilde{\xi}, \mathbf{b})}\} + \frac{2}{N_c^2 - 1} \int_{Y_0}^Y dY' \tilde{N}^{(2)}(Y', Y', \tilde{\xi}, \mathbf{b}) \{2 - e^{-\tilde{\xi} n^{(1)}(Y', \tilde{\xi}, \mathbf{b})}\} \right), \quad (95)$$

where $n^{(1)}(Y', \tilde{\xi}, \mathbf{b}) = N^{(1)}(Y', \tilde{\xi}, \mathbf{b})/r^2$. All factors in Eq. (95) can be clarified using the Born Approximation for $N^{(1)} \propto \tilde{\alpha}_S \ln Q^2$. Indeed, $\tilde{\alpha}_S^2 (1/Q^2) \ln(Q^2) \xrightarrow{\text{coordinate representation}} \tilde{\alpha}_S^2 r^2 \ln(r^2)$. The ratio $\tilde{\alpha}_S^2 r^2 \ln(r^2)/(\pi R_A^2) = N^{(1)}(Y', \tilde{\xi}, \mathbf{b}) = r^2 n^{(1)}(Y', \tilde{\xi}, \mathbf{b})$. Equation (95) is the double log limit of the general nonlinear equation [42] [see Eq. (57)], for the terms without Bose-Einstein enhancement). This general equation has the following form for $\tilde{N}^{(2)}(Y, Y, \tilde{\xi}; \mathbf{b})$:

$$\begin{aligned} \int d^2 b \tilde{N}^{(2)}(Y, Y, \mathbf{r}, \mathbf{b}) &= \tilde{\alpha}_S \int_{Y_0}^Y dY' \int d^2 r' \frac{r^2}{r'^2 (\mathbf{r} - \mathbf{r}')^2} \times \left(\left\{ 2\tilde{N}^{(2)}\left(Y', Y, \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \tilde{N}^{(2)}(Y', Y\mathbf{r}, \mathbf{b}) \right. \right. \\ &\quad \left. \left. - \tilde{N}^{(2)}\left(Y', Y, \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) N^{(1)}\left(Y', \mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) \right\} \right. \\ &\quad \left. + \left\{ 2\tilde{N}^{(2)}\left(Y, Y', \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \tilde{N}^{(2)}(Y, Y', \mathbf{r}, \mathbf{b}) \right. \right. \\ &\quad \left. \left. - \tilde{N}^{(2)}\left(Y, Y', \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) N^{(1)}\left(Y', \mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) \right\} \right. \\ &\quad \left. + \frac{2}{N_c^2 - 1} \left\{ 2\tilde{N}^{(2)}\left(Y', Y', \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) - \tilde{N}^{(2)}(Y', Y', \mathbf{r}, \mathbf{b}) \right. \right. \\ &\quad \left. \left. - \tilde{N}^{(2)}\left(Y', Y', \mathbf{r}', \mathbf{r}_{i+1}, \mathbf{b} - \frac{1}{2}(\mathbf{r}_i - \mathbf{r}')\right) N^{(1)}\left(Y', \mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) \right\} \right). \quad (96) \end{aligned}$$

Each term of this equation for $N^{(1)} = 1$ has the same form as Eq. (89), including the term proportional to $\bar{\alpha}_S/(N^2 - 1)$ and, therefore, the asymptotic solution gives $\tilde{N}^{(2)}(Y, Y, \mathbf{r}, \mathbf{b}) \rightarrow \text{Const}$. In other words, the fast increase of the double gluon densities, which is generated by the Bose-Einstein enhancement, turned out to be damped by the screening. It happens in spite of the powerlike increase of this amplitude with energy in the linear evolution, which is faster, than the increase for the elastic amplitude of dipole-target interaction.

VII. CONCLUSIONS

In this paper, we discussed several facets of the energy evolution of the J/ψ inelastic production cross section in deep inelastic processes. We showed that this cross section in the linear approximation, can be written by means of the double gluon density, for which we can use the evolution equation in the BFKL kinematic region. This equation has been proposed previously in Refs. [13–15] (see Ref. [12] for a review of the equations in the DGLAP kinematic region). We completed this study by suggesting an equation for the generating functional which allows us to calculate multigluon density, that can be generated in the process of J/ψ production.

We include in the evolution equation the Bose-Einstein enhancement which occurs at all values of energy (for example Ref. [22]). We did this for the simplest case: DGLAP evolution in DLA approximation of perturbative QCD. Indeed, the particular form of the double gluon distribution function on which the cross section of J/ψ production depends: $\int d^2Q_T D^{(2)}(x_1, \mathbf{p}_{1,T} + \frac{1}{2}\mathbf{Q}_T; x_2, \mathbf{p}_{2,T} - \frac{1}{2}\mathbf{Q}_T)$, has a DLA limit. We found that the Bose-Einstein enhancement leads to the double gluon distribution function which increases faster than the product of two single gluon distribution functions. We discussed the value of the anomalous dimension as well as the solution of the resulting evolution equations. It turns out that the contribution of the Bose-Einstein enhancement is rather small, and proportional to $\bar{\alpha}_S/(N_c^2 - 1)^2$, in accord with the estimates for twist-four anomalous dimension [16–18]. However, we believe that understanding what happens to these contributions at ultra high energies, is a principal question for the effective theory, based on high energy QCD.

Bearing this in mind, we derived the evolution equation for the scattering amplitude of two dipoles with a nucleus, taking into account the shadowing corrections. We investigated the analytical solutions in two distinct kinematic regions: deep in the saturation region, and in the vicinity of the saturation scale. Therefore, we have prepared the ground for building saturation models to describe the J/ψ production cross section. Most data exists for the J/ψ production, not in DIS but for the proton-nucleus

interaction. As has been demonstrated in Refs. [2,3,26] this process is closely related to the production of J/ψ in DIS. Hence, we need to rewrite our formulas, since the distribution over transverse momentum of the produced J/ψ was measured, which has not been discussed in this paper. We plan to discuss this process elsewhere.

The suggested nonlinear evolution equation is a direct generalization of the BK equation, and has to be solved using the formulas of Refs. [2,3], at the initial energy ($Y = Y_0$). The initial conditions depend on the saturation scale $Q_s(Y = Y_0, b)$. We need to compare the solution to the nonlinear equation in quasiclassical approach, in which we replace $Q_s(Y = Y_0, b)$ by $Q_s(Y, b)$ at arbitrary values of rapidities Y . Such substitution looks attractive from the physical point of view, since it takes into account the fact that the main properties of the dense system of patrons depend on the saturation scale. However, we do not observe a small parameter that allows us to do this. Consequently, it is instructive to compare the solution of the nonlinear equation with this approximation, with the goal to find a new small parameter.

Our last result is that the shadowing corrections in the elastic amplitude, generate the survival probability that suppressed the energy behavior of the double gluon densities with Bose-Einstein enhancement. We found that the cross section reaches a constant at ultra high energies. More precisely, we found that a constant is the asymptotic solution for the nonlinear equation. Nevertheless, this constant could be equal to zero. In this case, the cross section decreases as $\exp(-\frac{z_1+z_2}{8k})$ being the solution of Eq. (89). If quasiclassical approach reflects the main property of the solution to the nonlinear equation, we can expect from Eq. (86) that, indeed, the cross section vanishes at high energies.

In the paper, we used the BFKL Pomeron calculus which is not as general as the CGC approach, being equivalent to this approach only, in the limited range of rapidity. On the other hand, the BFKL Pomeron calculus has two advantages: the possibility to treat on the same footing elastic (diffractive) and inelastic processes, and to consider processes with different multiplicities.

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- [1] Y. V. Kovchegov and E. Levin, *Quantum Chromodynamics at High Energy*, Vol. 33 (Cambridge University Press, Cambridge, England, 2012).
- [2] D. Kharzeev and K. Tuchin, *Nucl. Phys.* **A770**, 40 (2006); D. Kharzeev, E. Levin, M. Nardi, and K. Tuchin, *Phys. Rev. Lett.* **102**, 152301 (2009); *Nucl. Phys.* **A826**, 230 (2009); **A924**, 47 (2014).
- [3] F. Dominguez, D. E. Kharzeev, E. Levin, A. H. Mueller, and K. Tuchin, *Phys. Lett. B* **710**, 182 (2012).
- [4] V. S. Fadin, E. A. Kuraev, and L. N. Lipatov, *Phys. Lett. B* **60**, 50 (1975); E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Sov. Phys. JETP* **45**, 199 (1977); I. I. Balitsky and L. N. Lipatov, *Sov. J. Nucl. Phys.* **28**, 822 (1978).
- [5] L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Phys. Rep.* **100**, 1 (1983).
- [6] A. H. Mueller and J. Qiu, *Nucl. Phys.* **B268**, 427 (1986).
- [7] L. McLerran and R. Venugopalan, *Phys. Rev. D* **49**, 2233 (1994); **50**, 2225 (1994); **53**, 458 (1996); **59**, 094002 (1999).
- [8] A. H. Mueller, *Nucl. Phys.* **B415**, 373 (1994); **B437**, 107 (1995).
- [9] I. Balitsky, *Nucl. Phys.* **B463**, 99 (1996); *Phys. Rev. D* **60**, 014020 (1999); Y. V. Kovchegov, *Phys. Rev. D* **60**, 034008 (1999).
- [10] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, *Phys. Rev. D* **59**, 014014 (1998); *Nucl. Phys.* **B504**, 415 (1997); J. Jalilian-Marian, A. Kovner, and H. Weigert, *Phys. Rev. D* **59**, 014015 (1998); A. Kovner, J. G. Milhano, and H. Weigert, *Phys. Rev. D* **62**, 114005 (2000); E. Iancu, A. Leonidov, and L. D. McLerran, *Phys. Lett. B* **510**, 133 (2001); *Nucl. Phys.* **A692**, 583 (2001); E. Ferreira, E. Iancu, A. Leonidov, and L. McLerran, *Nucl. Phys.* **A703**, 489 (2002); H. Weigert, *Nucl. Phys.* **A703**, 823 (2002).
- [11] V. N. Gribov and L. N. Lipatov, *Sov. J. Nucl. Phys.* **15**, 438 (1972); G. Altarelli and G. Parisi, *Nucl. Phys.* **B126**, 298 (1977); Y. L. Dokshitzer, *Sov. Phys. JETP* **46**, 641 (1977).
- [12] M. Diehl, D. Ostermeier, and A. Schafer, *J. High Energy Phys.* **03** (2012) 089; **03** (2016) 001(E).
- [13] A. H. Mueller and B. Patel, *Nucl. Phys.* **B425**, 471 (1994).
- [14] R. B. Peschanski, *Phys. Lett. B* **409**, 491 (1997).
- [15] E. Levin and M. Lublinsky, *Phys. Lett. B* **607**, 131 (2005).
- [16] J. Bartels, *Z. Phys. C* **60**, 471 (1993).
- [17] J. Bartels, *Phys. Lett. B* **298**, 204 (1993).
- [18] E. M. Levin, M. G. Ryskin, and A. G. Shuvaev, *Nucl. Phys.* **B387**, 589 (1992).
- [19] A. Kovner and M. Lublinsky, *Phys. Rev. D* **83**, 034017 (2011).
- [20] Y. V. Kovchegov and D. E. Wertepny, *Nucl. Phys.* **A906**, 50 (2013).
- [21] T. Altinoluk, N. Armesto, G. Beuf, A. Kovner, and M. Lublinsky, *Phys. Lett. B* **752**, 113 (2016).
- [22] T. Altinoluk, N. Armesto, G. Beuf, A. Kovner, and M. Lublinsky, *Phys. Lett. B* **751**, 448 (2015).
- [23] E. Gotsman and E. Levin, *Phys. Rev. D* **95**, 014034 (2017).
- [24] E. Gotsman and E. Levin, *Phys. Rev. D* **96**, 074011 (2017).
- [25] B. Blok, C. D. Jäkel, M. Strikman, and U. A. Wiedemann, *J. High Energy Phys.* **12** (2017) 074.
- [26] Z. B. Kang, Y. Q. Ma, and R. Venugopalan, *J. High Energy Phys.* **01** (2014) 056.
- [27] L. N. Lipatov, *Phys. Rep.* **286** (1997) 131; *Sov. Phys. JETP* **63**, 904 (1986) and references therein.
- [28] J. Bartels, M. Braun, and G. P. Vacca, *Eur. Phys. J. C* **40**, 419 (2005); J. Bartels and C. Ewerz, *J. High Energy Phys.* **09** (1999) 026; J. Bartels and M. Wusthoff, *Z. Phys. C* **66**, 157 (1995); J. Bartels, *Z. Phys. C* **60**, 471 (1993).
- [29] M. A. Braun, *Phys. Lett. B* **632**, 297 (2006); *Eur. Phys. J. C* **16**, 337 (2000); *Phys. Lett. B* **483**, 115 (2000); *Eur. Phys. J. C* **33**, 113 (2004); **6**, 321 (1999); M. A. Braun and G. P. Vacca, *Eur. Phys. J. C* **6**, 147 (1999).
- [30] Y. V. Kovchegov and E. Levin, *Nucl. Phys.* **B577**, 221 (2000).
- [31] E. Levin and M. Lublinsky, *Nucl. Phys.* **A763**, 172 (2005); **A730**, 191 (2004).
- [32] E. Levin, J. Miller, and A. Prygarin, *Nucl. Phys.* **A806**, 245 (2008).
- [33] T. Altinoluk, C. Contreras, A. Kovner, E. Levin, M. Lublinsky, and A. Shulkim, *Int. J. Mod. Phys. Conf. Ser.* **25**, 1460025 (2014); T. Altinoluk, N. Armesto, A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **08** (2014) 007; T. Altinoluk, A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **04** (2014) 075; T. Altinoluk, C. Contreras, A. Kovner, E. Levin, M. Lublinsky, and A. Shulkim, *J. High Energy Phys.* **09** (2013) 115.
- [34] P. D. B. Collins, *An Introduction to Regge Theory and High Energy Physics* (Cambridge University Press, Cambridge, England, 1977).
- [35] A. H. Mueller, *Phys. Rev. D* **2**, 2963 (1970).
- [36] V. A. Abramovsky, V. N. Gribov, and O. V. Kancheli, *Yad. Fiz.* **18**, 595 (1973) [*Sov. J. Nucl. Phys.* **18**, 308 (1974)].
- [37] Y. V. Kovchegov, *Phys. Rev. D* **64**, 114016 (2001); **68**, 039901(E) (2003); Y. V. Kovchegov and K. Tuchin, *Phys. Rev. D* **65**, 074026 (2002); J. Jalilian-Marian and Y. V. Kovchegov, *Phys. Rev. D* **70**, 114017 (2004); **71**, 079901(E) (2005); A. Kovner and M. Lublinsky, *J. High Energy Phys.* **11** (2006) 083; E. Levin and A. Prygarin, *Phys. Rev. C* **78**, 065202 (2008).
- [38] J. Bartels and M. G. Ryskin, *Z. Phys. C* **60**, 751 (1993); **62**, 425 (1994).
- [39] E. Laenen, E. Levin, and A. G. Shuvaev, *Nucl. Phys.* **B419**, 39 (1994).
- [40] E. Laenen and E. Levin, *Nucl. Phys.* **B451**, 207 (1995).
- [41] J. Bartels and M. G. Ryskin, arXiv:1105.1638.
- [42] E. Levin and K. Tuchin, *Nucl. Phys.* **B573**, 833 (2000); **A691**, 779 (2001); **693**, 787 (2001).
- [43] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition (Academic Press, London, 1994).
- [44] A. H. Mueller and D. N. Triantafyllopoulos, *Nucl. Phys.* **B640**, 331 (2002).
- [45] F. Dominguez, C. Marquet, B. W. Xiao, and F. Yuan, *Phys. Rev. D* **83**, 105005 (2011).
- [46] J. P. Blaizot, F. Gelis, and R. Venugopalan, *Nucl. Phys.* **A743**, 57 (2004).
- [47] H. Kowalski and D. Teaney, *Phys. Rev. D* **68**, 114005 (2003).
- [48] C. Marquet, R. B. Peschanski, and G. Soyez, *Phys. Rev. D* **76**, 034011 (2007).
- [49] J. D. Bjorken, *Int. J. Mod. Phys. A* **07**, 4189 (1992); *Phys. Rev. D* **47**, 101 (1993).
- [50] E. Gotsman, E. M. Levin, and U. Maor, *Phys. Lett. B* **309**, 199 (1993); *Phys. Rev. D* **49**, R4321 (1994).