Remarks on BMS₃ invariant field theories: Correlation functions and nonunitary CFTs

Thiago Araujo*

Asia Pacific Center for Theoretical Physics, POSTECH, Pohang 37673, Korea

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We use the isomorphism between the BMS₃ and the $W(2, 2)$ algebras to reconsider some generic aspects of conformal field theories (CFTs) with the BMS₃ algebra defined as a chiral symmetry. For unitarity theories, it is known that the extended symmetry generator acts trivially, and the resulting theory is equivalent to a CFT with a Virasoro symmetry only. For nonunitary CFTs, we define an operator depending on a nilpotent variable, and we organize the Verma module through the action of this new operator. Finally, we find the conditions imposed by the modified Ward identity.

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I. INTRODUCTION

In a harbinger of the AdS/CFT correspondence, Brown and Henneaux [1] showed that the naive asymptotic symmetry $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ of the AdS₃ space is enhanced to two copies of the Virasoro algebra $\text{Vir} \oplus \overline{\text{Vir}}$. These insights gave the necessary framework for the modern developments in the AdS₃/CFT₂ duality, e.g., Refs. [2–7]. Recently, after the observations relating the memory effect, soft theorems, and asymptotic symmetries [7,8], an increasing interest in asymptotic flat spaces has emerged.

The striking aspect of asymptotic flat spaces is the enhancement of their symmetry group to the Bondi-Metzner-Sachs (BMS) group [9–13]. In three dimensions, the BMS₃ algebra can be obtained from two copies of the Virasoro algebra, with generators $(\mathcal{L}, \tilde{\mathcal{L}})$ and central charges $(c, \bar{c})$, respectively, when we define the operators

$$L_m := \mathcal{L}_m - \tilde{\mathcal{L}}_{-m}, \quad M_m := \frac{1}{r_0}(\mathcal{L}_m - \tilde{\mathcal{L}}_{-m}) \quad (1.1)$$

and we assume that the AdS₃ radius r_0 goes to infinity. The BMS₃ algebra is given by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0} \frac{m(m^2-1)}{12} c_1 \\ [L_m, M_n] &= (m-n)M_{m+n} + \delta_{m+n,0} \frac{m(m^2-1)}{12} c_M \\ [M_m, M_n] &= 0, \end{aligned} \quad (1.2)$$

where $c_1 = c - \bar{c}$ and $c_M = \frac{1}{r_0} c_L$.

*thiago.araujo@apctp.org

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BMS algebra appears in several physical contexts, integrable models, for example [14,15], but we are particularly interested in the study of some field theories that are invariant under this symmetry algebra. Theories of this type may arise through the so-called *flat space holography* [16–22]. Evidently, one would like to extend to asymptotic flat spaces the AdS/CFT duality of which the reason for being is to explain some remarkable connections between the symmetries of the AdS_{D+1} string theory solutions and D -dimensional field theories living in their conformal boundary [23,24].

As with the Virasoro case, the BMS₃ symmetry is infinite dimensional, and this fact is responsible for some surprising algebraic features. First of all, it is isomorphic to the two-dimensional (2D) *Galilean-conformal algebra*, which is an Inönü-Wigner contraction of the Virasoro algebra; see Refs. [25–31]. Therefore, we may regard these two algebras as the “same,” but the drawback of these constructions is that we do not have the holomorphicity properties usually available in meromorphic conformal field theories (CFTs) as defined initially by Belavin *et al.* [32].

We try to remedy this situation using a third realization of the BMS₃ algebra: it has been shown that it is also isomorphic to the $W(2, 2)$ algebra [31,33–37], and that is the viewpoint we adopt in the present work. Roughly speaking, the $W(2, 2)$ algebra is a $\mathcal{W}$ algebra defined by the stress-energy tensor and an additional two-spin field $W(z)$. Similar ideas have been proposed in Ref. [38], but with the strong assumptions of that work, the additional spin-2 field was restricted to be the stress-energy tensor. We apply the isomorphism $\text{BMS}_3 \simeq W(2, 2)$ to study 2D CFTs with this algebra as a chiral symmetry, and then we try to use the powerful holomorphicity properties of meromorphic CFTs.

For example, in a generic 2D CFT, we can solve the theory without the use of an action principle or even the equations of motion in a miraculous way known as the

bootstrap approach. This formalism is an outcome of what a quantum field theory is: the result of two pieces of data, *correlation functions* and *symmetries*. Correlation functions are generally defined from the dynamical details of a particular physical system, while the symmetries impose nontrivial relations among the correlation functions. Evidently, the main role in the bootstrap formalism is played by the infinite-dimensional algebra, which manifests itself as the Ward identities [32,39–47].

Before going any further, we should remark that, although the flat space holography is, probably, a rich source of BMS₃ invariant field theories, the CFTs we consider in this work are not expected to be dual to the asymptotically flat spaces that we have mentioned before. The main motivation in this text is toward the construction of CFTs with extended symmetries defined by another spin-2 field of which the modes satisfy the BMS₃ algebra. Under this perspective, we should consider that these (elusive) theories are similar to $\mathfrak{sl}(n)$ Toda CFTs, which are endowed with extended $\mathcal{W}_\ell$ algebras $\ell \geq 3$ as chiral symmetries; see, for example, Refs. [48,49] and references therein.

The paper is organized as follows. In Sec. II, we review the isomorphism $\text{BMS}_3 \simeq W(2, 2)$. Despite the fact that these results are not new, we try to write this discussion in a chiral CFT language. Additionally, we slightly change the hypothesis of Ref. [31] and assume that the zero-mode Lie algebra is not necessarily semisimple [50]. We show that outside the semisimple Lie algebras realm it is possible to generate the centrally extended algebra through a Sugawara construction. In the same section, we briefly review the detailed analysis of null states in BMS₃ field theories performed in Ref. [26]. We pay special attention to some kind of monster in the theory, which is necessary to the main proposal of this work in Sec. V. Namely, there are zero-norm states that are not null in the usual sense, since they are not, in principle, annihilated by all positive $W(2, 2)$ -algebra modes. In Sec. III, we study the Ward identities, and we will see that the constraints naively imposed by these identities to the correlation function also impose that the central charge c_M vanishes. As we will argue along this work, a null central charge c_M is the worst scenario that we can find; the resulting theory is equivalent to a CFT without extended chiral symmetry at all, and in this case, the action of the generators $\{W_n | n \in \mathbb{Z}\}$ is completely trivial. In Sec. IV, we avoid the case $c_M = 0$ in our theory by changing some of our hypotheses: we embrace the existence of the seminull states, and we accept the fate of nonunitarity. We will see how we can organize the descendant fields into multiplets, and we also propose a modification of the Ward identity that takes into account the Jordan structure of the theory. We conclude in Sec. V and discuss further research directions.

II. BMS(3) AS A CHIRAL W(2, 2) ALGEBRA

In this section, we review the isomorphism between the BMS₃ and the $W(2, 2)$ algebras [35–37,51]. We mainly

follow the notation of Ref. [43], but Refs. [32,39–47] may be useful. Let us assume that we have a CFT living in the Riemann sphere $\mathbb{CP}^1$. Given two states $|v\rangle$ and $|w\rangle$ with conformal weights h_v and h_w , respectively, the operator product expansion (OPE) between their corresponding vertex operators is

$$V_v(z)V_w(\zeta) = V(V(|v\rangle; z - \zeta)|w\rangle; \zeta) = \sum_{n \leq h_w} V(V_n^v|w\rangle; \zeta)(z - \zeta)^{-n-h_v}. \quad (2.1)$$

Observe that the sum has an upper bound, and it arises from the fact that the theory cannot have any state with negative conformal dimension in a unitary theory. The OPE between a quasiprimary field $W(z)$ of conformal dimension 2 and the stress-energy tensor reads

$$T(z)W(\zeta) \sim V(L_2|\phi_2\rangle; \zeta)(z - \zeta)^{-4} + V(L_1|\phi_2\rangle; \zeta)(z - \zeta)^{-3} + V(L_0|\phi_2\rangle; \zeta)(z - \zeta)^{-2} + V(L_{-1}|\phi_2\rangle; \zeta)(z - \zeta)^{-1}, \quad (2.2)$$

where $|\phi_2\rangle$ is the state associated to the vertex operator $W(z)$. The commutator of their components is

$$\begin{aligned} [L_m, W_n] &= \sum_{k=-1}^2 \binom{m+1}{m-k} V_{m+n}(L_k|\phi_2\rangle) \\ &= V_{n+m}(L_{-1}|\phi_2\rangle) + (m+1)V_{n+m}(L_0|\phi_2\rangle) \\ &\quad + \frac{m(m+1)}{2} V_{n+m}(L_1|\phi_2\rangle) \\ &\quad + \frac{m(m^2-1)}{6} V_{n+m}(L_2|\phi_2\rangle). \end{aligned} \quad (2.3)$$

Given that $W(z)$ is quasiprimary, we have some simplifications:

- (1) The state $|\phi_2\rangle$ has conformal weight 2. Then, $L_0|\phi_2\rangle = 2|\phi_2\rangle$ and $L_1|\phi_2\rangle = 0$.
- (2) It is easy to show that

$$\begin{aligned} V(L_{-1}|\phi_2\rangle; z) &= \sum_{n \in \mathbb{Z}} V_n^{L_{-1}|\phi_2\rangle} z^{-n-3} \\ &= \partial V(|\phi_2\rangle; z) \\ &= \sum_{n \in \mathbb{Z}} (-n-2) W_n z^{-n-3}; \end{aligned} \quad (2.4)$$

therefore,

$$V_n(L_{-1}|\phi_2\rangle) = -(n+2)W_n. \quad (2.5)$$

- (3) Finally, the state $L_2|\phi_2\rangle$ has conformal weight $h = 0$. In fact,

$$L_0(L_2|\phi_2\rangle) = ([L_0, L_2] + L_2 L_0)|\phi_2\rangle. \quad (2.6)$$

Using now that $[L_0, L_2] = -2L_2$ and $L_0|\phi_2\rangle = 2|\phi_2\rangle$, we have $L_0(L_2|\phi_2\rangle) = 0$. All in all, we conclude that $L_2|\phi_2\rangle$ is proportional to the vacuum $|0\rangle$, and it is usually taken to be zero. In the present case, we write

$$L_2|\phi_2\rangle := \frac{c_2}{2}|0\rangle. \quad (2.7)$$

Additionally, we know that $V_n(|0\rangle) = \delta_{n,0}$. Putting all these facts together, we have

$$[L_m, W_n] = (m-n)W_{m+n} + \frac{c_2}{12}m(m^2-1)\delta_{n+m,0}. \quad (2.8)$$

Finally, it is clear from (2.1) that the general form of the OPE $W(z)W(\zeta)$ is still undetermined, but we can make a consistent choice:

$$W(z)W(\zeta) = 0 \Leftrightarrow [W_m, W_n] = 0. \quad (2.9)$$

Relaxing this condition may be interesting from the physical viewpoint, but we keep this choice along this text.

Definition: The $W(2,2)$ algebra is defined by the generators $\{L_m, W_n | m, n \in \mathbb{Z}\}$ satisfying the following commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c_1}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, W_n] &= (m-n)W_{m+n} + \frac{c_2}{12}m(m^2-1)\delta_{m+n,0} \\ [W_m, W_n] &= 0. \end{aligned} \quad (2.10)$$

The Verma module for the $W(2,2)$ algebra has been constructed in Refs. [35–37]. Furthermore, the algebras (1.2) and (2.10) are clearly isomorphic; therefore, we make the following identifications: $c_M \equiv c_2$ and $W(z) \equiv M(z)$.

The symmetry algebra above can be equivalently defined in terms of the holomorphic fields

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-2} \quad (2.11)$$

satisfying the OPEs,

$$\begin{aligned} T(z)T(w) &\sim \frac{1}{2} \frac{c_1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \\ T(z)W(w) &\sim \frac{1}{2} \frac{c_2}{(z-w)^4} + \frac{2W(w)}{(z-w)^2} + \frac{\partial W(w)}{(z-w)} \\ W(z)W(w) &\sim 0. \end{aligned} \quad (2.12)$$

These expressions have been written down in Ref. [51], in which the authors studied a free-field realization of the BMS₃ algebras in terms of a $\beta\gamma$ -ghost system. In their

construction, the $\beta\gamma$ system generates a BMS₃ algebra with central charges $(c_1, c_2) = (26, 0)$, and the value of c_2 may be changed by a twist.

A. Sugawara construction

In Ref. [31], the authors, regarding the BMS(3) algebra as a Galilean-Virasoro algebra, obtained the operators $W(z)$, with central charge $c_2 = 0$, and from the same Lie algebra, they built the energy-momentum tensor $T(z)$ through a Sugawara construction. Now, we would like to show that, even when we assume that the underlying Lie algebras that generate the operators $T(z)$ and $W(z)$ are different, and we shall see that depending on the properties of the Lie algebra, it is possible to generate a nontrivial central charge c_2 .

Suppose that among the vertex operators there exists a complete set of currents

$$\mathfrak{g} = \{j^a(z) \equiv V(|v^a\rangle; z) | L_0 | v^a \rangle = | v^a \rangle, a = 1, \dots, q\}. \quad (2.13)$$

Using (2.1), we write their OPE as

$$j^a(z)j^b(\zeta) \sim \frac{KZ^{ab}}{(z-\zeta)^2} + \frac{C_c^{ab}j^c(\zeta)}{(z-\zeta)} + (j^a j^b)(\zeta), \quad (2.14)$$

where K , Z^{ab} , and C_c^{ab} are constants. The current algebra reads

$$[j_m^a, j_n^b] = C_c^{ab}j_{m+n}^c + mKZ^{ab}\delta_{m+n,0}. \quad (2.15)$$

We do not assume that the zero mode algebra $\mathfrak{g}_0$, $[j_0^a, j_0^b] = C_c^{ab}j_0^c$, is semisimple. Therefore, the structure constants $C_c^{ab} = -C_c^{ba}$ do not need to be completely antisymmetric, and we do not impose that Z^{ab} is proportional to the Cartan-Killing form

$$\kappa^{ab} = \frac{1}{\mathcal{N}_0} C^{ae} C^{bc}_e, \quad (2.16)$$

where $\mathcal{N}_0$ is the Dynkin index. Finally, the nondegeneracy of the Cartan-Killing form [52] is not necessary in this setting. In other words, we allow the current algebra to be as arbitrary as possible.

We try to build the operator $W(z)$ from a Sugawara construction defined by the currents $\mathfrak{g}$ as

$$W(z) = \gamma X^{ab}(j^a j^b)(z). \quad (2.17)$$

Evidently, $X^{ab} = X^{ba}$, and we allow the possibility $X^{ab} \propto \kappa^{ab}$, but we do not take it for granted.

The properties of the matrices X^{ab} and Z^{ab} are determined from the OPEs

$$W(z)W(\zeta) \quad \text{and} \quad T(z)W(\zeta), \quad (2.18)$$

and the constraints imposed by the BMS(3) algebra:

- (1) It is easy to see that the OPE (2.1) implies that the fields $j^a(z)$ are primaries. Then, their transformation is

$$T(z)j^a(\zeta) \sim \frac{j^a(\zeta)}{(z-\zeta)^2} + \frac{\partial j^a(\zeta)}{(z-\zeta)}. \quad (2.19)$$

Now, we are able to calculate $T(z)W(\zeta)$ as

$$\begin{aligned} T(z)W(\zeta) &= \frac{\gamma X^{ab}}{2\pi i} \oint_{\zeta} \frac{dx}{x-\zeta} \\ &\quad \times \{T(z)j^a(x)j^b(\zeta) + T(z)j^a(x)j^b(\zeta)\} \\ &\sim \frac{\gamma K X^{ab} Z^{ab}}{(z-\zeta)^4} + \frac{2W(\zeta)}{(z-\zeta)^2} + \frac{\partial W(\zeta)}{(z-\zeta)}, \end{aligned} \quad (2.20)$$

and we see that $c_2 = 2\gamma K X^{ab} Z^{ab}$.

- (2) Now, we calculate the following OPE:

$$\begin{aligned} W(z)j^c(\zeta) &= \frac{\gamma X^{ab}}{2\pi i} \oint_{\zeta} \frac{dx}{x-\zeta} \\ &\quad \times \{j^a(z)j^b(x)j^c(\zeta) + j^a(z)j^b(x)j^c(\zeta)\}. \end{aligned} \quad (2.21a)$$

Then,

$$\begin{aligned} W(z)j^c(\zeta) &\sim \frac{2\gamma K X^{ab} Z^{ad} C^{bc}_d}{(z-\zeta)^3} \\ &\quad + \frac{2\gamma (K Z^{ca} X^{ae} + X^{ab} C^{bc}_d C^{ad}_e) j^e(\zeta)}{(z-\zeta)^2} \\ &\quad + \frac{2\gamma (K X^{ab} Z^{bc} \partial j^a(\zeta) + X^{ab} C^{bc}_d (j^a j^d)(\zeta))}{(z-\zeta)}, \end{aligned} \quad (2.21b)$$

that is,

$$\begin{aligned} W(z)j^c(\zeta) &\equiv \frac{\lambda_{(1)}^c}{(z-\zeta)^3} + \frac{\lambda_{(2)}^{ca} j^a(\zeta)}{(z-\zeta)^2} \\ &\quad + \frac{\lambda_{(2)}^{ca} \partial j^a(\zeta) + \lambda_{(3)}^{bac} (j^a j^c)(\zeta)}{(z-\zeta)}. \end{aligned} \quad (2.21c)$$

From this expression, it is easy to see that the last term is a bit problematic, since the OPE $W(z)W(\zeta)$ will give recursively cubic terms in the currents. For semisimple Lie algebras, the complete antisymmetry

of the structure constant implies that this term vanishes trivially. In the present case, if we want to avoid the obvious antisymmetry choice, the most economical constraint is $\sum_b X^{ab} C^{bc}_d = 0$. Therefore,

$$\lambda_{(1)}^b = 0, \quad \lambda_{(3)}^{abc} = 0. \quad (2.22)$$

- (3) Using these results, it is easy to compute the OPE:

$$\begin{aligned} W(z)W(\zeta) &= \frac{\gamma X^{cd}}{2\pi i} \oint_{\zeta} \frac{dx}{x-\zeta} \\ &\quad \times \{W(z)j^c(x)j^d(\zeta) + W(z)j^c(x)j^d(\zeta)\}. \end{aligned} \quad (2.23a)$$

Therefore,

$$\begin{aligned} W(z)W(\zeta) &\sim \frac{\gamma K Z^{ab} X^{bc} \lambda_{(2)}^{ca}}{(z-\zeta)^4} + \frac{2\gamma X^{ac} \lambda_{(2)}^{cb}}{(z-\zeta)^2} (j^a j^b)(\zeta) \\ &\quad + \frac{\gamma X^{ac} \lambda_{(2)}^{cb}}{z-\zeta} \partial (j^a j^b)(\zeta), \end{aligned} \quad (2.23b)$$

where the condition $\sum_b X^{ab} C^{bc}_d = 0$ implies that the term with $(z-\zeta)^{-3}$ vanishes trivially. The condition $W(z)W(\zeta) \sim 0$ is satisfied if $K = 0$ regardless of the Lie algebra properties, and it gives vanishing central charge $c_2 = 0$, or if $\sum_{c,d} X^{ac} Z^{cd} X^{db} = 0$, that gives nontrivial constraints to the Lie algebras. In the latter case, depending on the underlying details of the Lie algebra, we can generate a nonzero central charge c_2 .

We have seen that the condition $W(z)W(\zeta) \sim 0$ was a bit arbitrary. We can change this OPE, taking the condition $X = Z^{-1}$. Therefore,

$$T(z)W(\zeta) \sim \frac{\gamma K \dim(\mathfrak{g})}{(z-\zeta)^4} + \frac{2W(\zeta)}{(z-\zeta)^2} + \frac{\partial W(\zeta)}{z-\zeta} \quad (2.24a)$$

$$\begin{aligned} W(z)W(\zeta) &\sim 2\gamma K \left(\frac{K \gamma \dim(\mathfrak{g})}{(z-\zeta)^4} + \frac{2\gamma K}{(z-\zeta)^2} W(\zeta) + \frac{\gamma K}{z-\zeta} \partial W(\zeta) \right) \\ &= 2\gamma K T(z)W(\zeta). \end{aligned} \quad (2.24b)$$

We conclude that $W(z) = 2\gamma K T(z)$.

In summary, given a stress-energy momentum tensor $T(z)$, possibly, built from a Sugawara approach, we can construct a second quasiprimary field $W(z)$ with vanishing central charge c_2 provided a Lie algebra $\{j^a\}$ is given. If we insist on the nontriviality of the central charge, we find that the W -algebra generator $W(z)$ is equivalent to the stress-energy tensor $T(z)$. The fact that we could not generate a nontrivial central charge c_2 is just a byproduct of a general result: we cannot consider the usual unitary CFT results to study a generic BMS₃ invariant field theory.

B. Null states

The complete analysis of generic singular states in the BMS₃ invariant field theories has been performed in Ref. [26], and the authors¹ were able to find null vector for special values of the CFT data $(h, h^{(2)}, c_1, c_2)$. In this subsection, we review their construction, paying special attention to zero-norm states that appear in the representation theory of BMS₃ invariant field theories.

We know that the highest-weight representation of a generic $\mathcal{W}_\ell$ algebra is defined by

$$\begin{aligned} L_0|h, \ell\rangle &= h|h, \ell\rangle, & W_0^{(\ell)}|h, h^{(\ell)}\rangle &= \ell|h, h^{(\ell)}\rangle \\ L_n|h, h^{(\ell)}\rangle &= 0, & W_n^{(\ell)}|h, h^{(\ell)}\rangle &= 0, \quad \forall n \geq 1, \end{aligned} \quad (2.25a)$$

with Verma module

$$\mathcal{V}_{h, h^{(\ell)}} := \text{span} \left\{ \prod_{n_i \in \mathbb{N}} L_{-n_i} \prod_{m_j \in \mathbb{N}} W_{-m_j} |h, h^{(\ell)}\rangle \mid n_i, m_j \in \mathbb{N}, n_{i+1} \geq n_i, m_{j+1} \geq m_j \right\}. \quad (2.25b)$$

As usual, there is a natural L_0 grading:

$$\mathcal{V}_{h, h^{(\ell)}}^{(N)} := \text{span} \left\{ \prod_{n_i \in \mathbb{N}} L_{-n_i} \prod_{m_j \in \mathbb{N}} W_{-m_j} |h, h^{(\ell)}\rangle \in \mathcal{V}_{h, h^{(\ell)}} \mid \sum_i n_i + \sum_j m_j = N \right\}. \quad (2.25c)$$

Then, the Verma module is written as

$$\mathcal{V}_{h, h^{(\ell)}} = \bigoplus_{N \in \mathbb{Z}} \mathcal{V}_{h, h^{(\ell)}}^{(N)}. \quad (2.25d)$$

In our case, the $\mathcal{W}$ -algebra generator has $\ell = 2$, and we write the highest-weight states as $|h\rangle \equiv |h, h^{(2)}\rangle$ with $L_0|h\rangle = h|h\rangle$ and $W_0|h\rangle = h^{(2)}|h\rangle$. Additionally, there may be states satisfying

$$\begin{aligned} L_0|\chi\rangle &= (h + A)|\chi\rangle, & W_0|\chi\rangle &= h^{(2)}|\chi\rangle, \\ L_n|\chi\rangle &= 0, & W_n|\chi\rangle &= 0 \quad \forall n \in \mathbb{N}^*, \end{aligned} \quad (2.26)$$

called *null* or *singular*, that are simultaneously primaries and descendants. Null states generate their own module $\mathcal{V}_0$, and also they are orthogonal to $\mathcal{V}_{h, h^{(2)}}$ and, in particular, to themselves, that is, $\langle \chi | \chi \rangle = 0$. The important point is that these vectors decouple from the set physical states, and the final space is a submodule generated by the quotient $\mathcal{V}_{h, h^{(2)}} / \mathcal{V}_0$ [41, 42].

Using these results, we should observe that in the BMS₃ algebra any state of the form $|\chi\rangle = W_{-n}|h\rangle$ is a zero-norm state, that is,

$$\|\chi\|^2 = \langle v | W_n W_{-n} | h \rangle = \langle v | W_{-n} W_n | v \rangle = 0. \quad (2.27)$$

On the other hand, these states are not, necessarily, annihilated by all the generators L_m , with $m \geq 1$. In other words, such states are not necessarily null. Acting with the Virasoro generators, we find

$$L_0|\chi\rangle = L_0(W_{-n}|h\rangle) = (n + h)|\chi\rangle, \quad (2.28a)$$

and

$$\begin{aligned} L_m|\chi\rangle &= L_m(W_{-n}|h\rangle) \\ &= \begin{cases} m \left(2h^{(2)} + \frac{c_2}{12}(m^2 - 1) \right) |h\rangle, & \text{if } m = n \\ 0, & \text{if } m > n > 0 \\ (m + n) W_{m-n}^{(2)} |h\rangle, & \text{if } n > m > 0. \end{cases} \end{aligned} \quad (2.28b)$$

Therefore, the condition

$$2h^{(2)} + \frac{c_2}{12}(m^2 - 1) = 0 \quad (2.29)$$

needs to be imposed. We need to be careful with the range $n > m > 0$ since it is not zero, in principle. We say that zero-norm states that are not annihilated by all positive modes are *seminull*. Therefore, we would like to see under which conditions the seminull states are singular.

Moreover, it has been shown that if we assume (2.29) with $c_2 \neq 0$ the null vector is unique, up to a multiplicative constant, and it is of the form $W_{-n_1}^{k_1} \cdots W_{-n_n}^{k_n} |h\rangle$, and that when $c_2 = 0$ we recover the Verma module of an ordinary CFT with a symmetry given by the Virasoro algebra only [26, 36]. In this last case, the Verma module is reducible, and we can build usual null states using the generators L_{-n} . But observe that, even in that case, we still have the seminull states, and in this section, we would like to pay attention to them.

I. $N = 1$

For the first level in the Verma module, we have two states,

¹Their analysis can be easily translated into our notation by the transformation $(\Delta, \xi, C_1, C_2)_{\text{theirs}} \equiv (h, h^{(2)}, c_1/12, c_2/12)_{\text{ours}}$.

$$|\chi_1\rangle = W_{-1}|h\rangle \quad \text{and} \quad |\chi_2\rangle = L_{-1}|h\rangle: \quad (2.30)$$

These conditions can be derived from the 2×2 Gram matrix²

- (i) The vector $|\chi_1\rangle = W_{-1}|h\rangle$ is a zero-norm state, that is, $\|\chi_1\|^2 = 0$. Moreover, it is easy to see that

$$L_0|\chi_1\rangle = (h+1)|\chi_1\rangle, \quad W_0|\chi_1\rangle = h^{(2)}|\chi_1\rangle \quad \text{and} \quad W_n|\chi_1\rangle = 0 \quad n \geq 1. \quad (2.31)$$

In addition,

$$\begin{aligned} L_n|\chi_1\rangle &= \left((n+1)W_{n-1} + \frac{c_2}{12}n(n^2-1)\delta_{n-1,0} \right)|h\rangle \\ &= 2\delta_{n,1}h^{(2)}|h\rangle, \quad \forall n \in \mathbb{N}^*. \end{aligned} \quad (2.32)$$

We conclude that this state is seminull, unless we impose $h^{(2)} = 0$. At this point, the condition (2.29) does not give $c_2 = 0$. Then, if $h^{(2)} = 0$, $W_{-1}|v\rangle$ is null.

- (ii) Consider now

$$|\chi_2\rangle = L_{-1}|h\rangle. \quad (2.33)$$

As usual, we have $\|\chi_2\|^2 = 2h$. In addition,

$$\begin{aligned} L_0|\chi_2\rangle &= (h+1)|\chi_2\rangle, \\ W_0|\chi_2\rangle &= |\chi_1\rangle + h^{(2)}|\chi_2\rangle, \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} L_n|\chi_2\rangle &= 2h\delta_{n,1}|h\rangle, \\ W_n|\chi_2\rangle &= 2h^{(2)}\delta_{n,1}|h\rangle. \end{aligned} \quad (2.35)$$

The state is null just in the case $h = h^{(2)} = 0$.

$$G^{(1)} = \begin{pmatrix} \langle h|L_1L_{-1}|h\rangle & \langle h|W_1L_{-1}|h\rangle \\ \langle h|L_1W_{-1}|h\rangle & \langle h|W_1W_{-1}|h\rangle \end{pmatrix}, \quad (2.36)$$

the determinant $\det G^{(2)} = -4h^{(2)}$ of which vanishes just when $h^{(2)} = 0$. Furthermore, this condition says that if $h^{(2)} > 0$ we have a nonunitary theory.

2. $N=2$

At level 2, we can construct a generic vector of the form

$$\begin{aligned} |\mu_1\rangle &= (a_1W_{-2} + a_2L_{-2} + a_3L_{-1}W_{-1} \\ &\quad + a_4W_{-1}^2 + a_5L_{-1}^2)|h\rangle, \end{aligned} \quad (2.37)$$

and imposing the null vectors conditions, we determine constants a_i , see Ref. [26]:

- (i) The textbook example of null vector at level $N = 2$,

$$|\mu_2\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2 \right)|h\rangle, \quad (2.38)$$

is seminull in our case, unless $h^{(2)} = 0$.

- (ii) We also have a seminull state of the form

$$|\mu_3\rangle = (W_{-2} + a_0W_{-1}W_{-1})|h\rangle, \quad (2.39)$$

where a_0 is a constant to be determined. Obviously, $\|\mu_3\|^2 = 0$, and $W_n|\mu_3\rangle = 0$. Additionally,

$$L_0|\mu_3\rangle = (h+2)|\mu_3\rangle \quad \text{and} \quad W_0|\mu_3\rangle = h_2^{(2)}|\mu_3\rangle. \quad (2.40)$$

Therefore, we need to calculate

$$\begin{aligned} L_n|\mu_3\rangle &= L_n(W_{-2} + a_0W_{-1}W_{-1})|h\rangle \\ &= \left[(n+2)W_{n-2} + \frac{c_2}{12}n(n^2-1)\delta_{n-2,0} + 2a_0 \left((n+1)W_{n-1} + \frac{c_2}{12}n(n^2-1)\delta_{n-1,0} \right) W_{-1} \right]|h\rangle. \end{aligned} \quad (2.41)$$

From this expression, we easily see that

$$L_m|h\rangle = 0, \quad \forall n \in \mathbb{N}^* \quad (2.42a)$$

is identically satisfied if $h^{(2)} = c_2 = 0$ or

$$h_2^{(2)} = -\frac{c_2}{8} \quad \text{and} \quad a_0 = \frac{6}{c_2}, \quad (2.42b)$$

which is consistent with (2.29).

²The order is twice the order of a theory with the Virasoro symmetry only.

Again, these conditions can be derived from the 4×4 Gram matrix

$$G^{(2)} = \begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}^2|h\rangle & \langle h|L_2L_{-1}W_{-1}|h\rangle & \langle h|L_2W_{-1}^2|h\rangle \\ \langle h|L_1^2L_{-2}|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle & \langle h|L_1^2L_{-1}W_{-1}|h\rangle & \langle h|L_1^2W_{-1}^2|h\rangle \\ \langle h|W_2L_{-2}|h\rangle & \langle h|W_2L_{-1}^2|h\rangle & \langle h|W_2L_{-1}W_{-1}|h\rangle & \langle h|W_2W_{-1}^2|h\rangle \\ \langle h|W_1^2L_{-2}|h\rangle & \langle h|W_1^2L_{-1}^2|h\rangle & \langle h|W_1^2L_{-1}W_{-1}|h\rangle & \langle h|W_1^2W_{-1}^2|h\rangle \end{pmatrix} \quad (2.43a)$$

so that

$$G^{(2)} = \begin{pmatrix} 4h + c/2 & 6h & 6h^{(2)} & 0 \\ 6h & 4h(1 + 2h) & 4h^{(2)}(1 + 2h) & 8(h^{(2)})^2 \\ 4h^{(2)} + c_2/2 & 6h^{(2)} & 0 & 0 \\ 0 & 8(h^{(2)})^2 & 0 & 0 \end{pmatrix}, \quad (2.43b)$$

the determinant $\det G^{(2)} = 192(h^{(2)})^5(c_2 + 8h^{(2)})$ of which vanishes when $h^{(2)} = 0$ or $h^{(2)} = -c_2/8$.

3. $N=3$

For completeness, at level 3, we have

$$|\nu\rangle = (W_{-3} + a_1W_{-2}W_{-1} + a_2(W_{-1})^3)|h\rangle. \quad (2.44)$$

It is easy to see that $\|\nu\|^2 = 0$, $W_n|\nu\rangle = 0 \ \forall n > 0$, and

$$L_0|\nu\rangle = (h + 3)|\nu\rangle, \quad W_0|\nu\rangle = h^{(2)}|\nu\rangle. \quad (2.45)$$

Furthermore,

$$\begin{aligned} L_n|\nu\rangle = & \left\{ (n+3)W_{n-3} + \frac{c_2}{12}n(n^2-1)\delta_{n-3,0} \right. \\ & + a_1 \left[(n+1)W_{n-1} + \frac{c_2}{12}n(n^2-1)\delta_{n-1,0} \right] W_{-2} \\ & + a_1 \left[(n+2)W_{n-2} + \frac{c_2}{12}n(n^2-1)\delta_{n-2,0} \right] W_{-1} \\ & \left. + 3a_2 \left((n+1)W_{n-1} + \frac{c_2}{12}n(n^2-1)\delta_{n-1,0} \right) W_{-1}^2 \right\} |h\rangle, \end{aligned} \quad (2.46)$$

and the expression

$$L_n|\nu\rangle = 0 \quad \forall n \in \mathbb{N}^* \quad (2.47a)$$

is, again, identically satisfied for $h^{(2)} = c_2 = 0$ or if the following conditions are satisfied:

$$h_3^{(2)} = -\frac{c_2}{3}, \quad a_1 = \frac{6}{c_2} \quad \text{and} \quad a_2 = 0. \quad (2.47b)$$

Obviously, these conditions are consistent with (2.29). Using similar techniques, we can determine null states at arbitrary levels.

Remark: Compatibility of all states (and single valuedness of $h^{(2)}$) imposes that $h^{(2)} = c_2 = 0$. In other words, if one sets the seminull states to zero, the central charge c_2 and the weights $h^{(2)}$ associated to the BMS₃ symmetry are both zero. It is easy to see that the quotient $\mathcal{V}_{h,h^{(2)}}/\mathcal{V}_0$ is, equivalently, defined as a space where the generators W_m act trivially, that is,

$$W_n|v\rangle \mapsto 0 \quad \forall n \in \mathbb{Z}. \quad (2.48)$$

Therefore, $\mathcal{V}_{h,h^{(2)}}/\mathcal{V}_0 \simeq \mathcal{V}_{h,0} \simeq \mathcal{V}_h$; that is, the final module is the Verma module solely generated by $\{L_{-n}, n \geq 1\}$.

In summary, we have seen that for $24h^{(2)} = -c_2(n^2 - 1)$, $n \in \mathbb{Z}^*$, the Verma module is reducible and that if we want to avoid seminull states we need to take $c_2 = 0$. In that case, the resulting module is isomorphic to the states of a Virasoro algebra [26,36].

However, we have two strong hypotheses in our construction³: unitarity and the absence of zero-norm states in the CFT spectrum. Therefore, in order to avoid trivial central charges and W weights, we need to take seriously the existence of seminull states and leave aside the positivity of the Hilbert space.

Additionally, it is obvious that given the level-1 states $|\chi_1\rangle = W_{-1}|h\rangle$ and $|\chi_2\rangle = L_{-1}|h\rangle$ we have the Jordan structure

$$W_0|\chi_1\rangle = h^{(2)}|\chi_1\rangle, \quad W_0|\chi_2\rangle = h^{(2)}|\chi_2\rangle + |\chi_1\rangle, \quad (2.49)$$

where $|\chi_1\rangle$ is evidently a seminull state for generic values of $h^{(2)}$. It is easy to see that this structure also appears for higher-level states; therefore, one might try to consider each

³There is another important hypothesis: the factorization of the CFT into holomorphic and antiholomorphic sectors.

pair of states, e.g., $(|\chi_1\rangle, |\chi_2\rangle)$, as “conjugate” vectors. We return to all these points in Sec. IV and propose some solutions to the problems mentioned above.

III. CORRELATION FUNCTIONS

We would like to study the constraints imposed by the BMS₃ extended symmetry to the correlation functions of a unitary CFT. We will see that the Ward identities for two- and three-point functions trivialize the action of the W_n generators. In other words, the results of this section show that it is not possible to define a unitary field theory with extended symmetry given by $W(2, 2)$ and nontrivial actions of the generators W_n .

Let us rewrite the OPE between the algebra generators and primary fields as

$$W^{(\ell)}(z)V_v(\zeta) = \frac{V(W_0^{(\ell)}|v;\zeta)}{(z-\zeta)^\ell} + \frac{V(W_{-1}^{(\ell)}|v;\zeta)}{(z-\zeta)^{-1+\ell}} + \dots + \frac{V(W_{1-\ell}^{(\ell)}|v;\zeta)}{(z-\zeta)}, \quad (3.1a)$$

where

$$V(W_{-n}^{(\ell)}|v;\zeta) \equiv W_{-n}^{(\ell)}V_v(\zeta) = \frac{1}{2\pi i} \oint_w dz \frac{1}{(z-\zeta)^{n+1-\ell}} W^{(\ell)}(z)V_v(\zeta), \quad (3.1b)$$

and $W^{(1)}(z) \equiv T(z)$ and $W^{(2)}(z) \equiv W(z)$. It is easy to see that the Ward identities are

$$\langle T(z)V_1(z_1)\dots V_m(z_m) \rangle = \sum_{k=1}^n \left(\frac{h_k}{(z-z_k)^2} + \frac{\partial_k}{z-z_k} \right) \langle V_1(z_1)\dots V_m(z_m) \rangle \quad (3.2a)$$

$$\langle W(z)V_1(z_1)\dots V_m(z_m) \rangle = \sum_{k=1}^n \left(\frac{h_k^{(2)}}{(z-z_k)^2} + \frac{W_{-1}}{z-z_k} \right) \langle V_1(z_1)\dots V_m(z_m) \rangle. \quad (3.2b)$$

Regularity imposes the conditions

$$\lim_{z \rightarrow \infty} T(z) = \frac{1}{z^4} \quad \text{and} \quad \lim_{z \rightarrow \infty} W^{(2)}(z) = \frac{1}{z^4}. \quad (3.3)$$

Using the Ward identities, we can write the correlators for descendants as

$$\begin{aligned} \langle L_{-n}V(w)V_1(z_1)\dots V_m(z_m) \rangle \\ = \hat{L}_{-n}\langle V(w)V_1(z_1)\dots V_m(z_m) \rangle \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \langle W_{-n}V(w)V_1(z_1)\dots V_m(z_m) \rangle \\ = \hat{W}_{-n}\langle V(w)V_1(z_1)\dots V_m(z_m) \rangle, \end{aligned} \quad (3.4b)$$

where⁴

$$\hat{L}_{-n} = (-1)^{n-1} \sum_{k=1}^n \left(\frac{(1-n)h_k}{(w-z_k)^n} - \frac{1}{(w-z_k)^{n-1}} \partial_k \right) \quad (3.4c)$$

$$\hat{W}_{-n} = (-1)^{n-1} \sum_{k=1}^n \left(\frac{(1-n)h_k^{(2)}}{(w-z_k)^n} - \frac{1}{(w-z_k)^{n-1}} {}^k W_{-1} \right). \quad (3.4d)$$

The modes associated to $W(z)$ are not (at the quantum level) differential operators. Therefore, given a null vector $|\chi_{h^{(2)}, c_2}\rangle$ associated to highest-weight state $|h, h^{(2)}\rangle$, the constraints that this state imposes to the correlation functions are not, contrary to the usual Virasoro null states, differential equations. For example, using the results of the previous section, the null states at levels 2 and 3 give, respectively, the two conditions

$$\begin{aligned} \langle \chi(z)\phi_1(w_1)\dots \rangle &= \left(W_{-2} + \frac{6}{c_2} W_{-1}^2 \right) \langle V_{h_1}(z)\phi_1(w_1)\dots \rangle = 0 \\ \langle \mu(z)\phi_1(w_1)\dots \rangle &= \left(W_{-3} + \frac{6}{c_2} W_{-2}W_{-1} \right) \langle V_{h_1}(z)\phi_1(w_1)\dots \rangle \\ &= 0, \end{aligned} \quad (3.5)$$

which in the present state seem to be completely algebraic.

Let us assume the existence of an $SL(2, \mathbb{C}) \cup \{W_{-1}, W_0, W_{+1}\}$ invariant vacuum $|0\rangle$. Using $V(w) = V(|0\rangle; w) = \mathbb{1}$ and taking $w = 0$, we have the equations

$$\begin{aligned} \sum_{k=1}^n \partial_k G_m = 0, \quad \sum_{k=1}^n (h_k + z_k \partial_k) G_m = 0, \\ \sum_{k=1}^n (2z_k h_k + z_k^2 \partial_k) G_m = 0 \end{aligned} \quad (3.6a)$$

$$\begin{aligned} \sum_{k=1}^n {}^k W_{-1} G_m = 0, \\ \sum_{k=1}^n (h_k^{(2)} + {}^k W_{-1} z_k) G_m = 0, \\ \sum_{k=1}^n (2z_k h_k^{(2)} + {}^k W_{-1} z_k^2) G_m = 0, \end{aligned} \quad (3.6b)$$

where $G_m = \langle V_1(z_1)\dots V_m(z_m) \rangle$.

⁴We use the notation ${}^k W_{-1} \langle V_1(z_1)\dots V_k(z_k)\dots V_n(z_n) \rangle \equiv \langle V_1(z_1)\dots W_{-1} V_k(z_k)\dots V_n(z_n) \rangle$.

The T -Ward identities impose the usual conditions for the one-, two-, and three-point functions, and the W -Ward identities impose further restrictions that we shall verify now. The one-point functions are zero, so let us start with the two-point functions.

Equations (3.6b) can be organized as the linear system

$$R \cdot \vec{V} = \begin{pmatrix} 0 & 1 & 1 \\ h_1^{(2)} + h_2^{(2)} & z_1 & z_2 \\ 2(h_1^{(2)} z_1 + h_2^{(2)} z_2) & z_1^2 & z_2^2 \end{pmatrix} \times \begin{pmatrix} \langle V_1(z_1) V_2(z_2) \rangle \\ \langle (W_{-1} V_1(z_1)) V_2(z_2) \rangle \\ \langle V_1(z_1) (W_{-1} V_2(z_2)) \rangle \end{pmatrix} = 0, \quad (3.7)$$

and nontrivial solutions exist iff $\det R = -(h_1^{(2)} - h_2^{(2)})(z_1 - z_2)^2 = 0$. Therefore, $h_1^{(2)} = h_2^{(2)}$, and we conclude that the two-point functions are

$$\langle V_1(z_1) V_2(z_2) \rangle = \frac{c_{12} \delta_{h_1 h_2} \delta_{h_1^{(2)} h_2^{(2)}}}{z_{12}^{2h_1}}. \quad (3.8)$$

This condition also appears in Refs. [48,49] for the $\mathcal{W}_3$ algebra, except that in the present case the W_0 eigenvalues $h_k^{(2)}$ must be equal.

Given that $h_1^{(2)} = h_2^{(2)} \equiv h^{(2)}$ must be satisfied in order to get a nontrivial correlation function, the first condition in the Ward identity (3.7) is

$$\langle W_{-1} V_1(z_1) V_2(z_2) \rangle = -\langle V_1(z_1) W_{-1} V_2(z_2) \rangle, \quad (3.9a)$$

and the remaining equations form the system

$$\begin{aligned} 2h^{(2)} \langle V_1(z_1) V_2(z_2) \rangle + z_1 \langle W_{-1} V_1(z_1) V_2(z_2) \rangle \\ + z_2 \langle W_{-1} V_1(z_1) V_2(z_2) \rangle = 0 \\ 2h^{(2)} (z_1 + z_2) \langle V_1(z_1) V_2(z_2) \rangle + z_1^2 \langle W_{-1} V_1(z_1) V_2(z_2) \rangle \\ + z_2^2 \langle W_{-1} V_1(z_1) V_2(z_2) \rangle = 0. \end{aligned} \quad (3.9b)$$

Putting all these facts together, we find

$$\langle V_1(z_1) W_{-1} V_2(z_2) \rangle = \frac{h^{(2)} c_{12} \delta_{h_1 h_2} \delta_{h_1^{(2)} h_2^{(2)}}}{z_{12}^{2h_1+1}}. \quad (3.9c)$$

Let us turn our attention to the three-point functions. From (3.6b), one gets the linear system

$$\sum_{k=1}^3 {}^k W_{-1} \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle = 0 \quad (3.10a)$$

$$\sum_{k=1}^3 (h_k^{(2)} + {}^k W_{-1} z_k) \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle = 0 \quad (3.10b)$$

$$\sum_{k=1}^3 (2h_k^{(2)} z_k + {}^k W_{-1} z_k^2) \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle = 0. \quad (3.10c)$$

It is an underdetermined system: three equations and four unknowns, that is, $\langle V_1 V_2 V_3 \rangle$, $\langle W_{-1} V_1 V_2 V_3 \rangle$, $\langle V_1 W_{-1} V_2 V_3 \rangle$ and $\langle V_1 V_2 W_{-1} V_3 \rangle$. The first condition obviously gives

$$\begin{aligned} \langle V_1(z_1) V_2(z_2) W_{-1} V_3(z_3) \rangle \\ = -\langle W_{-1} V_1(z_1) V_2(z_2) V_3(z_3) \rangle \\ - \langle V_1(z_1) W_{-1} V_2(z_2) V_3(z_3) \rangle. \end{aligned} \quad (3.11)$$

Inserting the last expression into the second and third equations of the linear system gives

$$\begin{aligned} [(z_1 - z_3)({}^1 W_{-1}) + (z_2 - z_3)({}^2 W_{-1})] \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle \\ = -(h_1^{(2)} + h_2^{(2)} + h_3^{(2)}) \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle \end{aligned} \quad (3.12a)$$

and

$$\begin{aligned} [(z_1^2 - z_3^2)({}^1 W_{-1}) + (z_2^2 - z_3^2)({}^2 W_{-1})] \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle \\ = -2(h_1^{(2)} z_1 + h_2^{(2)} z_2 + h_3^{(2)} z_3) \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle, \end{aligned} \quad (3.12b)$$

respectively.

From these last conditions, we find

$$\begin{aligned} \langle (W_{-1} V_1(z_1)) V_2(z_2) V_3(z_3) \rangle &\equiv \Delta(z_k; h_k^{(2)}) \langle (V_1(z_1)) V_2(z_2) V_3(z_3) \rangle \\ &= \frac{C_{123} (2(h_1^{(2)} z_1 + h_2^{(2)} z_2 + h_3^{(2)} z_3) + (z_2 + z_3)(h_1^{(2)} + h_2^{(2)} + h_3^{(2)}))}{z_{12}^{h_{12}+1} z_{13}^{h_{13}+1} z_{23}^{h_{23}}}, \end{aligned} \quad (3.13)$$

where we have used the textbook result for the three-point function [41,42,45].

Using these three expressions, we can write the Eqs. (3.6b) as the homogeneous system

$$S \cdot \vec{U} = \begin{pmatrix} \Delta & 1 & 1 \\ h_1^{(2)} + h_2^{(2)} + h_3^{(2)} + z_1 \Delta & z_2 & z_3 \\ 2(h_1^{(2)} z_1 + h_2^{(2)} z_2 + h_3^{(2)} z_3) + z_1^2 \Delta & z_2^2 & z_3^2 \end{pmatrix} \times \begin{pmatrix} \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle \\ \langle V_1(z_1) (W_{-1} V_2(z_2)) V_3(z_3) \rangle \\ \langle V_1(z_1) V_2(z_2) (W_{-1} V_3(z_3)) \rangle \end{pmatrix} = 0, \quad (3.14)$$

and again, taking the determinant of S and setting it to zero, we have

$$-4(z_2 - z_3)(h_1^{(2)} z_1 + h_2^{(2)} z_2 + h_3^{(2)} z_3) = 0, \quad (3.15)$$

which must be satisfied for any value of z_1 , z_2 , and z_3 ; then,

$$h_1^{(2)} = h_2^{(2)} = h_3^{(2)} = 0. \quad (3.16)$$

Therefore, the action of the W_n modes in a unitary CFT algebra is trivialized.

IV. NONUNITARY CFTs

As we have seen in the previous sections, if assume that the BMS₃ invariant CFTs are unitary, we find very unpleasant results. First of all, we cannot generate the $W(z)$ operators with nontrivial central charges c_2 following a modification of the Sugawara construction based on semisimple Lie algebras. Additionally, the null states and the correlation function analysis indicate that the action of this symmetry algebra is trivial.

One important aspect of our study is the presence of zero-norm vectors, called seminull, which are not orthogonal to the whole Verma module. We know that in a unitary CFT null states generate their own Verma module and that the set of “physical” vectors is drastically reduced. In other words, we take the quotient of the whole Verma module by the space of null states, that is, $\mathcal{V}_{(h,c)}/\mathcal{V}_0$.

Unitarity is, evidently, an important property in fundamental theories, but we must be careful with its role and not overestimate its importance in general CFTs. First of all, nonunitary theories have their own place in the realm of condensed matter systems [53]. Additionally, unitarity of the vector space of physical states is much more important than the unitarity of the theory itself. For example, the gauge fixed plus ghost action is generally nonunitary (evidently, this action is not obtained from first principles); on the other hand, the Becchi-Rouet-Stora-Tyutin (BRST) cohomology must, evidently, be positive definite.

It is not clear at the present stage whether the presence of the seminull states necessarily gives a nonunitary CFT, but let us assume that it is the case. Therefore, we would like to discuss the resulting aspects of nonunitarity in our analysis;

that is, we would like to see the results of the condition $h^{(2)} \neq 0$. Consider again the states $|\lambda_1^{(N)}\rangle = W_{-N}|h\rangle$ and $|\lambda_2^{(N)}\rangle = L_{-N}|h\rangle$. The Jordan structure (2.49) is generalized to

$$\begin{aligned} L_0|\lambda_1^{(N)}\rangle &= (h+N)|\lambda_1^{(N)}\rangle, & L_0|\lambda_2^{(N)}\rangle &= (h+N)|\lambda_2^{(N)}\rangle \\ W_0|\lambda_1^{(N)}\rangle &= h^{(2)}|\lambda_1^{(N)}\rangle, & W_0|\lambda_2^{(N)}\rangle &= h^{(2)}|\lambda_2^{(N)}\rangle + N|\lambda_1^{(N)}\rangle. \end{aligned} \quad (4.1)$$

Given an infinite set of nilpotent variables $\{\theta_n|\theta_n^2 = 0, \forall n \in \mathbb{Z}\}$, we define the state

$$\begin{aligned} |\Phi_N\rangle &:= |h_N; h^{(2)} + \theta_N\rangle \\ &= |h_N, h^{(2)}\rangle_0 + \theta_N |h_N, h^{(2)}\rangle_1 \\ &\equiv |\lambda_1^{(N)}\rangle + \theta_N |\lambda_2^{(N)}\rangle, \end{aligned} \quad (4.2a)$$

where in the second line we have written the Taylor expansion in θ_N . Let us say that $|\lambda_2^{(N)}\rangle$ is the Virasoro state and $|\lambda_1^{(N)}\rangle$ is its seminull partner. Hence, we rewrite the Jordan structure (4.1) as

$$\begin{aligned} L_0|\Phi_N\rangle &= (h+N)|\Phi_N\rangle, \\ W_0|\Phi_N\rangle &= (h^{(2)} + N\theta_N)|\Phi_N\rangle. \end{aligned} \quad (4.2b)$$

Evidently, we still have states of the form $L_{-\vec{n}}W_{-\vec{m}}|h\rangle$, where $N = |\vec{n}| + |\vec{m}|$, but using the commutation relations, we can write all these cross terms as descendants of $|\lambda_1^{(N')}\rangle$ and $|\lambda_2^{(N')}\rangle$ with $N' < N$ in the following way: at each level N , let us define the operators $\mathbb{W}_n^{(N)}$ as

$$\mathbb{W}_n^{(N)} := W_n + N\theta_N L_n, \quad (4.2c)$$

where the role of the index $n = 1, \dots, N$ will be explained soon. Observe that the state $|\Phi_N\rangle$ is written as

$$|\Phi_N\rangle := \mathbb{W}_{-N}^{(N)}|h\rangle. \quad (4.2d)$$

Let us consider the first states. It easy to see that $|\Phi_0\rangle = |h\rangle$. At level 1, we have $n = 1$; therefore,

$$|\Phi_1\rangle = \mathbb{W}_{-1}^{(1)}|\Phi_0\rangle = |\lambda_1\rangle + \theta_1|\lambda_2\rangle, \quad (4.3)$$

where $|\lambda_1\rangle = W_{-1}|\Phi_0\rangle$ and $|\lambda_2\rangle = L_{-1}|\Phi_0\rangle$ have conformal weights $h_1 = h + 1$. Therefore,

$$\begin{aligned} L_0|\lambda_1\rangle &= h_1|\lambda_1\rangle & L_0|\lambda_2\rangle &= h_1|\lambda_2\rangle \\ W_0|\lambda_1\rangle &= h^{(2)}|\lambda_1\rangle & W_0|\lambda_2\rangle &= h^{(2)}|\lambda_2\rangle + |\lambda_1\rangle. \end{aligned} \quad (4.4)$$

We see that the operator W_0 cannot be diagonalized, and we can say that $|\lambda_2\rangle$ is the Jordan partner of $|\lambda_1\rangle$. Finally, we already know that $L_1|\lambda_1\rangle = 2h^{(2)}|h\rangle$.

At level 2, we have the operators $\mathbb{W}_{-1}^{(2)}$ and $\mathbb{W}_{-2}^{(2)}$. From (4.2d), one has

$$|\Phi_2\rangle = \mathbb{W}_{-2}^{(2)}|\Phi_0\rangle = |\lambda_1^{(2)}\rangle + 2\theta_2|\lambda_2^{(2)}\rangle, \quad (4.5)$$

where $|\lambda_1^{(2)}\rangle = W_{-2}|\Phi_0\rangle$ and $|\lambda_2^{(2)}\rangle = L_{-2}|\Phi_0\rangle$. Evidently, there are other states at level 2 in the Verma module. The remaining fields can be built by applying the operator $\mathbb{W}_{-1}^{(2)}$ in the state $|\Phi_1\rangle = \mathbb{W}_{-1}^{(1)}|\Phi_0\rangle$, that is,

$$\begin{aligned} \mathbb{W}_{-1}^{(2)}|\Phi_1\rangle &= (W_{-1}^2 + \theta_1 W_{-1} L_{-1} + 2\theta_2 L_{-1} W_{-1} + 2\theta_2 \theta_1 L_{-1}^2) \\ &\times |\Phi_0\rangle. \end{aligned} \quad (4.6)$$

Therefore, the level-2 Verma module is organized in the multiplet $\{|\Phi_2\rangle, \mathbb{W}_{-1}^{(2)}|\Phi_1\rangle\}$. Similarly, level 3 can be organized in $\{|\Phi_3\rangle, W_{-1}^{(3)}|\Phi_2\rangle, W_{-1}^{(3)}W_{-1}^{(2)}|\Phi_1\rangle\}$. All in all, we see that the Virasoro states and their seminull partners can be seen as the building blocks of this construction and that we can organize all Verma module in terms of the set $\{W_n^{(N)}, |\lambda_1^{(N)}\rangle, |\lambda_2^{(N)}\rangle\}$. Evidently, we have some redundancies to be removed, for example, $L_{-n}W_{-m}|\Phi_0\rangle$ and $W_{-m}L_{-n}|\Phi_0\rangle$ for $N = n + m$ that are equivalent modulo the state $W_{-N}|\Phi_0\rangle$.

Finally, using the operator-state map, we define

$$\begin{aligned} \Phi_N(z) &= \varphi_N(z) + \theta_N \psi_N(z) \\ \Rightarrow |\Phi_N\rangle &= \lim_{z \rightarrow 0} \Phi_N(z)|h\rangle \end{aligned} \quad (4.7a)$$

so that

$$|\lambda_1^{(N)}\rangle = \lim_{z \rightarrow 0} \varphi_N(z)|h\rangle, \quad |\lambda_2^{(N)}\rangle = \lim_{z \rightarrow 0} \psi_N(z)|h\rangle. \quad (4.7b)$$

The similarity between this theory and the structures of the LogCFTs [54–56] has been pointed out in Refs. [26,34,37,57,58], but the relation between them is not straightforward, and as the results of this section show, the Jordan structure in the present case is associated to the seminull descendant fields. The BMS₃ invariant CFTs are much more similar to the theories considered in Refs. [59,60]. In that case, the authors argued that, contrary to the unitarity CFTs, the existence of zero-norm states does not mean the reduction of the Verma module and that its degeneracy, that is, the presence of seminull states, should be compensated by the some *extension* (or *alien*) states $|\tilde{\chi}\rangle$ such that

$$\langle \tilde{\chi} | \chi \rangle \neq 0, \quad \langle \tilde{\chi} | \tilde{\chi} \rangle \neq 0, \quad (4.8)$$

where $|\tilde{\chi}\rangle$ is a “partner” of the zero-norm state $|\chi\rangle$.

Then, for each seminull state $|\chi\rangle$, we have an extended vector $|\tilde{\chi}\rangle$ that generates its own module and needs to be appended to the initial space of states. But this is exactly what we have found in our computations, and the only noticeable difference between the present case and Refs. [59,60] is that our zero-norm states are associated with descendant fields, while their construction is associated with the seminull states, extension operators that are neither descendant nor primaries. Despite this small difference, the seminull partners (extension operators) are natural features of our constructions and not mere *ad hoc* objects.

In particular, it is obvious that the stress-energy tensor $T(z)$ and the BMS₃ generator $W(z)$ are descendants. Given the states $|\mathbf{T}\rangle = L_{-2}|0\rangle$ and $|\mathbf{W}\rangle = L_{-2}|0\rangle$, their vertex operators are

$$T(z) := V(|\mathbf{T}\rangle; z) \quad \text{and} \quad W(z) := V(|\mathbf{W}\rangle; z). \quad (4.9)$$

Therefore, we have the pair

$$\mathcal{T}(z) := W(z) + \theta T(z), \quad (4.10)$$

and we see that the fields $(T(z), W(z))$ belong to the same “multiplet.”

We conclude that the symmetry transformations generated by the BMS₃ algebra must be considered simultaneously. As we have seen in the previous section, the W -ward identities impose severe constraints to our theory, namely, $h^{(2)} = 0$, and now we understand this fact as a consequence of our naive attempt to consider the symmetries associated with $T(z)$ and $W(z)$ as independent transformations. Consequently, we need to modify the Ward identities.

Let us start modifying the OPE $W(z)V_h(\zeta)$, that is,

$$W(z)V_h(\zeta) = \frac{h^{(2)}V_h(\zeta)}{(z-\zeta)^2} + \frac{(W_{-1}V_h)(\zeta) + (\partial V_h)(\zeta)}{(z-\zeta)}; \quad (4.11)$$

hence, the operator (3.4d) becomes

$$\widehat{W}_{-n} = (-1)^{n-1} \sum_{k=1}^n \left(\frac{(1-n)h_k^{(2)}}{(w-z_k)^n} - \frac{1}{(w-z_k)^{n-1}} ({}^k W_{-1} + \partial_k) \right). \quad (4.12)$$

Again, this is similar to the Logarithmic CFT (LogCFT), where we modify the Ward identity by a nilpotent operator [55,56].

Let us reconsider the expressions (3.4b) and (3.6b). Using (4.12); imposing the invariance of the vacuum under

$\{L_n, W_n | n = -1, 0, +1\}$, that is, $\widehat{W}_{-n}G_m = 0$; and finally using Eqs. (3.6a), we have

$$\sum_{k=1}^n {}^k W_{-1} G_m = 0 \quad (4.13a)$$

$$\sum_{k=1}^n (h_k^{(2)} - h_k + {}^k W_{-1} z_k) G_m = 0 \quad (4.13b)$$

$$\sum_{k=1}^n (2z_k(h_k^{(2)} - h_k) + {}^k W_{-1} z_k^2) G_m = 0. \quad (4.13c)$$

With this new identity, we can repeat the computations of the previous section for the two-point functions to see that they are not modified. For the three-point functions, we find the condition $h_k^{(2)} = h_k$.

The nice aspect now is that we can compute four-point functions using the usual technology we have in any CFT with Virasoro symmetry. The four-point function is written in terms of conformal blocks

$$G_4(x) = \sum_p C_{12}^p C_{34}^p \mathcal{F}(p|x), \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad (4.14)$$

and the crossing symmetry can be imposed via the bootstrap approach [41,42]. Evidently, the conditions imposed by the W-Ward identities are “essentially” algebraic, given that we do not have a clear understanding of the nature of the operators W_n in a general $\mathcal{W}$ algebra.

We have dealt with the existence of the seminull states by a simple modification of the W-Ward identity. But evidently, we need to get rid of the proper null states, since these fields must be absent in the final Verma module. In practice, we are very familiar with these null states, and we know from many standard examples, see Refs. [41,42,45], that these singular vectors give further conditions to the correlation functions.

For example, using the state (2.37) and supposing that it is indeed null, its associated vertex operator $\chi(z)$ can be inserted in a generic correlation function. Then,

$$(a_1 \widehat{W}_{-2} + a_2 \widehat{L}_{-2} + a_3 \widehat{L}_{-1} \widehat{W}_{-1} + a_4 \widehat{W}_{-1}^2 + a_5 \widehat{L}_{-1}^2) \times \langle V(w) V_1(z_1) \cdots V_m(z_m) \rangle = 0. \quad (4.15)$$

Consider now that the W weight is given by $24h^{(2)} = -c_2(n^2 - 1)$ for some $n \in \mathbb{Z}$; then, null vector is unique [36]. For a particular example, we assume that $n = 2$; then, the condition

$$\left(\widehat{W}_{-2} + \frac{6}{c_2} \widehat{W}_{-1}^2 \right) \langle V(w) V_1(z_1) \cdots V_m(z_m) \rangle = 0 \quad (4.16)$$

is defined by the state (2.39).

V. CONCLUSIONS AND FURTHER DIRECTIONS

In this paper, we explored the equivalence $\text{BMS}_3 \simeq W(2, 2)$ to remark on some aspects of meromorphic CFTs with extended chiral symmetries given by the BMS_3 algebra. Initially, we tried to study the structure of these generic theories using the results and insights of unitary theories, for example, the Sugawara construction. The drawback of this approach is that the algebraic aspects of the nonsemisimple Lie algebra are, at the present stage, very elusive.

To study these theories, we have seen that there are certain zero-norm vectors that are not necessarily annihilated by the modes $\{L_n, W_m\}$, with $n, m > 0$. The removal of these states provided us with a relation between the W weights $h^{(2)}$ and the central charge c_2 . In the particular case of vanishing central charge, $c_2 = 0$, it is well known that the Verma modules for these theories are equivalent to the Verma modules of ordinary CFTs [36].

We turned our attention to the analysis of Ward identities, which are, in a very precise way, the quantum manifestation of the classical symmetries of the system, and they are the cornerstone of any consistent quantum field theory (QFT) with classical local or global symmetries. Furthermore, the Ward identities for the W symmetry imply that the weights of all states must be zero; therefore, we cannot have a unitary field theory with a nontrivial W -symmetry action.

Next, we considered nonunitary CFTs, and we assumed that seminull states are important in our setting. It is not known whether the presence of seminull states necessarily gives a nonunitary CFT, and it would be interesting to check this condition carefully. In any case, the presence of these states gives a rich structure to the theory, and we organized the Verma module through a nilpotent operator that mixes the BMS_3 generators. Moreover, we have shown that in the nonunitary setting the W-Ward identities do not trivialize the action of the symmetry operators.

There are many directions for future research. One can try to perform a mathematical analysis of the conditions we considered in Sec. IV; in particular, one can try to understand the similarities between the BMS_3 theories and LogCFTs. Additionally, since we do not have an interpretation for the operators W_n , there is no good reason to believe that these operators respect the holomorphic and antiholomorphic factorization of the theory (for example, these operators could act as a multiplication by an antiholomorphic function); therefore, one could try to relax this condition.

One can try to change the OPE $W(z)W(\zeta)$ and see if one can find a consistent CFT with extended algebra or add higher-spin operators in this construction and see how the Verma module is modified. Remember that in the usual $\mathcal{W}_\ell$ -algebra approach one considers just one additional primary operator $W^{(\ell)}(z)$ for each integer $\ell \geq 3$. Then, the study we considered in this paper is part of a bigger

problem: the study of CFTs with extended symmetries associated with several quasiprimary operators of the same conformal dimension.

It is known that the $\mathcal{W}_\ell$ algebras are the symmetry of the $A_{\ell-1}$ Toda systems, and, in particular, the Virasoro algebra $\mathcal{W}_2 \equiv \text{Vir}$ is the symmetry algebra of the simplest Toda system, the Liouville CFT; see Refs. [48,49,61] and references therein. In Ref. [15], the relation between some “flat space limit” of the Liouville theory and BMS₃ symmetry has been analyzed; therefore, it would be interesting to see if we can connect the present construction to some deformation of the Liouville CFT. Even more speculatively, it would be interesting to see if conformal blocks of these elusive theories are connected to the partition functions of four-dimensional theories, *à la*

Alday-Gaiotto-Tachikawa (AGT) correspondence [62,63]. We hope to address some of these points in a future work.

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