Axionic black branes with conformal coupling

Adolfo Cisterna,[1,2,*] Cristián Erices,[3,4,†] Xiao-Mei Kuang,[5,‡] and Massimiliano Rinaldi[2,6,§]

[1]*Universidad Central de Chile, Vicerrectoría académica, Toesca 1783, Santiago, Chile*
[2]*Dipartimento di Fisica, Università di Trento, Via Sommarive 14, 38123 Povo (TN), Italy*
[3]*Department of Physics, National Technical University of Athens, Zografou Campus, GR 15773, Athens, Greece*
[4]*Universidad Católica del Maule, Avenida San Miguel 3605, Talca, Chile*
[5]*Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China*
[6]*TIFPA—INFN, Via Sommarive 14, 38123 Povo (TN), Italy*

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We find neutral and charged black brane solutions with axion fields in the context of a conformally coupled gravitational theory in four dimensions. These solutions describe anti–de Sitter (AdS) black branes supported by axion fields that break translational invariance at the boundary, which provides momentum dissipation. The conformally coupled scalar field is regular inside and outside the event horizon and there is no need for any self-interaction, obtaining in this way solutions without fine-tuned parameters. We analyze the thermodynamics of our solutions considering the effects of the axionic charges. In asymptotically AdS configurations, the axionic and electric charges are related, implying vanishing scalar field contributions to the mass. The rotating solution is obtained by means of a Lorentz boost having angular momentum sustained by the axion parameter. We compute the holographic DC conductivity and we show how it is affected by the inclusion of the conformal scalar field, which provides a temperature-independent behavior. Finally, we include a k-essence term that modifies the DC conductivity and provides more general behaviors.

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I. INTRODUCTION

For any well-posed gravitational theory such as Einstein's general relativity (GR) or any of its extensions, the existence of black hole solutions is a matter of primary interest. Black holes offer the perfect arena to study the theory in the strong gravity regime and provide information about the causal structure of the spacetime and astrophysically relevant predictions. Moreover, their semiclassical descriptions through the study of their thermodynamical properties provide the perfect setup to study quantum gravity effects that would lead to the characterization of a fundamental theory of quantum gravity [1]. However, in standard GR, to find black hole solutions dressed with matter fields is typically a nontrivial task due to the existence of the topological censorship theorem [2] as well as the no-hair conjecture [3].

On the one hand, topological obstructions in four dimensions can be evaded by relaxing the asymptotic behavior through the inclusion of a cosmological constant, obtaining static solutions with planar and hyperbolic horizons [4] along with several solutions with nontrivial topology at infinity [5].[1] In contrast, topological obstructions are weaker in dimension $D > 4$, allowing asymptotically flat solutions with nonspherical topology such as black rings [6] and diverse black object solutions [7]. Moreover, for higher-dimensional GR, the unicity theorems [8–10] are no longer valid, permitting black hole solutions such as the Schwarzschild-Tangherlini black hole [11] and the black p-brane.

On the other hand, the no-hair conjecture states that black holes cannot be described by any quantity apart from its mass, electric charge and angular momentum [12]. This implies that, after gravitational collapse, black holes can only be characterized by quantities that follows

*adolfo.cisterna@ucentral.cl
†crerices@central.ntua.gr
‡xmeikuang@gmail.com
§massimiliano.rinaldi@unitn.it

[1]Solutions with nontrivial topology at spatial infinity are obtained by compactifying the base manifold directions. They are usually dubbed topological black holes.

a Gauss-type law, namely charges that can be measured at infinity. Any other characteristic of the matter that falls into the black hole is lost. This is because the assumptions forbid black holes with nontrivial matter fields apart from the electromagnetic one, which is the well known Kerr-Newman solution [13]. As shown in Ref. [14], there are no nontrivial regular solutions in GR when minimally coupled scalar fields are considered. The no-hair conjecture renders the scalar field trivial and the solution is nothing else than the Schwarzschild black hole. Nevertheless, this conjecture can be circumvented by allowing the introduction of suitable potentials or nonminimal couplings to the matter field. The nonminimal coupling of conformal type with electromagnetic interaction and in the absence of a cosmological constant, was first considered by Bronnikov, Melnikov and Bocharova [15] and Bekenstein [16,17].[2] This was the first counterexample to the no-hair conjecture using scalar fields and it represents a black hole only in four dimensions [19]. Due to the fact that the scalar field does not introduce any new integration constant in the back-reaction these solutions are dubbed "solutions with secondary hair." However, although the metric of this so-called "BMBB" black hole turns out to be the extreme Reissner-Nordström solution, the scalar field diverges at the horizon, making its physical properties difficult to interpret. As it was shown in Refs. [20,21], this physical pathology can be fixed by introducing a cosmological constant which pushes the scalar field singularity behind the event horizon. This solution, dubbed the "MTZ" black hole, possesses a spherical or hyperbolic horizon depending on the sign of the cosmological constant and exists only for a particular combination of the cosmological and the quartic self-interacting coupling constants.[3] No planar solution is allowed to exist. In particular, planar solutions are typically affected by singular behaviors, which is a symptom of a shortage of a curvature scale on the horizon. In Ref. [23] the authors, based on a family of metrics that accomplish a weaker version of Birkhoff's theorem, constructed four-dimensional asymptotically anti–de Sitter (AdS) black holes with planar horizons supported by matter represented by p-forms.[4] Then, regular solutions can be found by charging the horizon with homogeneously distributed axionic charges along planar directions. These axion fields endow spacetime with an effective intrinsic curvature scale, making it possible to regularize black hole solutions. In Ref. [24] the authors constructed a nontrivial planar version of the MTZ black hole including, apart from the MTZ solution ingredients, two axion fields given by three forms originated from two Kalb-Ramond potentials. In this situation, both the scalar fields and the axion fields are nonminimally coupled to gravity. These solutions were also generalized to the case in which the nonminimal coupling is arbitrary and to the case of higher dimensions [25].

It is known that the flat geometry of the horizon opens the possibility to study holographic applications based on the gauge/gravity duality. The AdS/CFT correspondence [28] establishes a duality between gravitational theories in D dimensions and conformal field theories in the $(D-1)$-dimensional boundary. In this scenario, black holes became suitable laboratories to study strongly coupled systems, opening a new range of applicability of gravitational solutions at the service of condensed matter systems [29,30]. In this respect, planar/toroidal black holes with scalar fields possess special relevance due, in particular, to their applications in the dual description of superconductor systems [31,32]. Hairy black holes can undergo spontaneous undressing in a phase transition process reminiscent of nonzero condensate behavior in unconventional superconductors. Some examples of holographic applications, such as interesting AC conductivity behavior of hairy black branes, can be found by introducing a dilaton-dependent gauge coupling between the dilaton field and the Maxwell field, as well as phase transitions in the dual field theory [33,34]. To successfully describe real materials using the holographic tools [29,30], it is very important to include a mechanism of momentum dissipation. Several methods are employed in order to accomplish this, such as the so-called scalar lattice technique implemented by a periodic scalar source [35,36], the framework of massive gravity where the diffeomorphism invariance of the theory breaks down in the bulk [37–40], or the Q-lattice model in which the phase of a complex scalar field breaks the translational invariance of the theory [41,42]. A very simple way to describe systems with momentum dissipation is the case of massless scalar fields that depend linearly on the base manifold coordinates. This technique was introduced in Ref. [43] as an effective way to break translational invariance on the dual field theory and obtain several novel properties for the dual condense matter sector. Several solutions regarding these ideas were reported in Refs. [44–50].[5] This paper is devoted to the construction of AdS black brane solutions with a conformally coupled scalar field where the translational invariance at the boundary is broken by means of axion fields that depend linearly on the base manifold directions. This solution represents the generalization to the conformal

[2] An alternative way to circumvent the conjecture is to consider a complex scalar field with different symmetries than those exhibited by the spacetime. This is constructed in such a way that its stress-energy tensor shares the same symmetries of spacetime. Using this approach Kerr black holes with scalar hair have been numerically constructed in Ref. [18].

[3] A number of solutions were found including more general self-interactions in four and in higher dimensions [22].

[4] Applications of these ideas have provided several new regular black brane solutions [24–27].

[5] A very interesting application is the construction of homogeneous black strings and black p-branes with a negative cosmological constant, with no ingredients other than minimally coupled scalar fields [51].

coupling case of Ref. [43] in four dimensions and can be viewed as an economic way to obtain the planar version of the BMBB solution. The paper is organized as follows. Section II presents the theory under consideration. In Sec. III the solutions are exposed and their principal features are discussed. Section IV provides a general description of the black brane thermodynamic analysis while in Sec. V we describe the holographic DC conductivity and Hall angle for these solutions. We include a nonlinear k-essence term in order to modify the classical expected behavior of conductivities in these types of models. Finally we conclude in Sec. VI.

II. THE THEORY

We consider four-dimensional gravity with a cosmological constant interacting with a matter source given by a conformally coupled real scalar field and two free axions, described by the action

$$S[g,\phi,\psi_I] = \int d^4x\sqrt{-g}\left[\kappa(R-2\Lambda) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}\phi^2 R - \frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2\right], \tag{1}$$

where $\kappa \equiv \frac{1}{16\pi G}$ and G is the four-dimensional Newton's constant. The field equations are

$$\kappa(G_{\mu\nu} + \Lambda g_{\mu\nu}) = \frac{1}{2}T^{\phi}_{\mu\nu} + \frac{1}{2}T^{\psi}_{\mu\nu}, \tag{2}$$

$$\left(\Box - \frac{1}{6}R\right)\phi = 0, \tag{3}$$

$$\Box\psi_I = 0, \tag{4}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. The energy-momentum tensor is given by contributions from the scalar and axion fields which are respectively

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 + \frac{1}{6}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2, \tag{5}$$

$$T^{\psi}_{\mu\nu} = \sum_{I=1}^{2}\left(\partial_\mu\psi_I\partial_\nu\psi_I - \frac{1}{2}g_{\mu\nu}(\partial\psi_I)^2\right). \tag{6}$$

We look for static and planar four-dimensional metrics given by

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(dx^2 + dy^2), \tag{7}$$

where $0 \le r < \infty$, $0 \le x \le \beta_x$ and $0 \le y \le \beta_y$. The base manifold is assumed to be compact, without boundary and of vanishing curvature, i.e., it is locally isometric to flat space $\mathbb{R}^2$. Imposing the axion fields to depend only on the boundary directions the Klein-Gordon equation for each axion field is trivially integrated, yielding

$$\psi_I = \zeta_{Ii}x^i + \alpha_I, \tag{8}$$

where $x^1 \equiv x$, $x^2 \equiv y$ and ζ_{Ii}, α_I are integration constants. Due to the form of the axion kinetic term in the action, the axion field enjoys global ISO(2) symmetry. As it was pointed out in Ref. [27], this means that for planar solutions of the field equations, the global ISO(2) symmetry is isomorphic to the spatial isometries of the conformal boundary. Nevertheless, Eq. (8) completely breaks the global ISO(2) symmetry. As a matter of choice, we are interested in solutions that break translation symmetry in the conformal boundary, for which $\alpha_I = 0$. This is because, from a holographic point of view, it is more interesting, since it induces momentum dissipation in the dual field theory. We have preserved only the $SO(2)$ symmetry of the conformal boundary, allowing to rearrange the expression for the axion field in terms of only two integration constants λ_I, which translate into the constraint $\sum_{I=1}^{2}(\zeta_{Ii}\zeta_{Ij} - \lambda_I^2\delta_{ij}) = 0$. In this way, we may write the solution as $\psi_I = \lambda x_I$.

III. FOUR DIMENSIONAL BLACK BRANE SOLUTION

The field equations (2), (3) and (4) admit an exact solution where the metric, scalar and axion fields are given by

$$ds^2 = -\frac{(r-3\lambda l)(r+\lambda l)^3}{r^2 l^2}dt^2 + \frac{r^2 l^2}{(r-3\lambda l)(r+\lambda l)^3}dr^2 + r^2(dx^2+dy^2), \tag{9}$$

$$\phi = 2\sqrt{3}\frac{\lambda l}{r+\lambda l}, \tag{10}$$

$$\psi_I = 2\sqrt{3}\lambda x_I, \tag{11}$$

where we have redefined the axion parameter $\lambda \to 2\sqrt{3}\lambda$, set $\kappa = 1$ for simplicity and defined the AdS radius $l^{-2} := -\frac{\Lambda}{3}$. The metric (9) describes a planar black hole, provided the cosmological constant is negative, with an asymptotically AdS behavior as it can be seen by the large-r behavior of the components $g_{tt} \sim -\frac{r^2}{l^2} + \mathcal{O}(r^0)$, $g_{rr} \sim \frac{l^2}{r^2} + \mathcal{O}(r^0)$. There is a curvature singularity at the origin as it can be checked by evaluating the Ricci scalar

$$R = -\frac{12}{l^2} + \frac{12\lambda^2}{r^2}, \tag{12}$$

which is dressed by a single event horizon located at $r_+ = 3\lambda l$, when λ is positive, and at $r_+ = -\lambda l$ when λ is negative. In the latter case the scalar field diverges at the event horizon resembling what occurs with the BMBB configuration, making its physical interpretation not clear as the entropy is not well defined. However, unlike the BMBB and MTZ configurations, the scalar field has no poles when λ is positive, being regular everywhere and hereafter, we analyze this well-behaved solution. In contrast with the case of the MTZ black hole configuration, where the cosmological constant is fine-tuned by the quartic self-interacting coupling constant, here the cosmological constant is completely arbitrary. This means that this solution naturally emerges as the most economic way to obtain a toroidal black hole with a conformally coupled scalar field which is free of singularities in four dimensions. The inclusion of both free axion fields allows us to obtain planar solutions and regularizes everywhere the conformally coupled scalar field in a simpler way than in the case with a conformally coupled scalar field with quartic self-interaction.

An electrically and magnetically charged black hole is obtained by adding the Maxwell term,

$$-\frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}, \tag{13}$$

to the action (1), which gives the same metric (9) and axion field (11), but with a scalar field and gauge potential of the form

$$\begin{aligned} \phi &= \frac{\sqrt{Q_E^2 + Q_M^2 + 12\lambda^4 l^2}}{\lambda(r+\lambda l)}, \\ A &= -\frac{Q_E}{r}dt + \frac{Q_M}{2}(xdy - ydx). \end{aligned} \tag{14}$$

At large r, the scalar field is approximated by

$$\phi = \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + \mathcal{O}(r^{-3}), \tag{15}$$

where

$$\begin{aligned} \phi_1 &\equiv \lambda^{-1}\sqrt{Q_E^2 + Q_M^2 + 12l^2\lambda^4}, \\ \phi_2 &\equiv -l\sqrt{Q_E^2 + Q_M^2 + 12l^2\lambda^4}. \end{aligned} \tag{16}$$

It is worth pointing out that, as it was shown in Refs. [52,53], the boundary conditions that ensure the AdS symmetry of the scalar field's asymptotic behavior are $\{\phi_1 = 0, \phi_2 \neq 0\}$, $\{\phi_1 \neq 0, \phi_2 = 0\}$ and $\phi_1^2 = \alpha\phi_2$, where α is a constant without variation. However, only the third boundary condition allows a nonvanishing scalar field. By means of Eq. (16), and in order to satisfy this boundary condition, a relation between the parameters of the solution must be fulfilled as follows:

$$\lambda^4 = \frac{Q_E^2 + Q_M^2}{(\alpha^2 - 12)l^2}, \qquad |\alpha| > 2\sqrt{3}. \tag{17}$$

It can be checked that the dominant energy condition is satisfied for the charged and neutral black hole. In fact, by computing the stress-energy tensor in the orthonormal frame,

$$\begin{aligned} T^{ab} = \mathrm{diag}\bigg(&\frac{12r^2\lambda^2 - 6l^2\lambda^4}{r^4}, -\frac{12r^2\lambda^2 - 6l^2\lambda^4}{r^4}, \\ &-\frac{6l^2\lambda^4}{r^4}, -\frac{6l^2\lambda^4}{r^4}\bigg), \end{aligned} \tag{18}$$

we note that its canonical form is of type I according to the classification in Ref. [54]. We can identify the energy density ρ and the principal pressures p_a ($a = 1, 2, 3$), as

$$\rho = -p_1 = \frac{12r^2\lambda^2 - 6l^2\lambda^4}{r^4}, \quad p_2 = p_3 = -\frac{6l^2\lambda^4}{r^4}, \tag{19}$$

verifying directly that $\rho \geq 0$ and $-\rho \leq p_a \leq \rho$ for $r \geq \lambda l$. Note that the event horizon $r_+ > 3\lambda l$ covers this region, implying that the dominant energy condition holds in the causally connected region of the spacetime. Our solutions are easily extended to the case in which they possess angular momentum. Due to our planar base manifold this can be done by applying an improper gauge transformation on the time coordinate and on one of the transverse manifold directions upon identification of this last coordinate. We explicitly show this construction and its thermodynamic analysis in Appendix A.

IV. BLACK HOLE THERMODYNAMICS

For a more complete understanding of the black hole solutions above we perform a complete thermodynamical analysis using the Euclidean approach. In this context, the partition function for a thermodynamical ensemble is identified with the Euclidean path integral in the saddle-point approximation around the classical Euclidean solution [55]. We consider spacetimes with a manifold of topology $\mathbb{R}^2 \times \mathbb{R}^2$ where the first plane $\mathbb{R}^2$ is parametrized by the radial coordinate r and the Euclidean time τ, while the second refers to the base manifold which is assumed to be compact with volume σ and, as we mentioned, spanned by coordinates x and y. Therefore the Euclidean continuation of the black hole metric (7) reads,

$$ds_E^2 = N^2(r)F(r)d\tau^2 + \frac{dr^2}{F(r)} + r^2(dx^2 + dy^2), \tag{20}$$

and the scalar, axionic and gauge fields are

$$\phi = \phi(r), \qquad \psi_I = \psi_I(x^i),$$
$$A = A_\tau(r)d\tau + A_x(y)dx + A_y(x)dy, \tag{21}$$

where $x^1 = x$ and $x^2 = y$. These plane coordinates range as $0 \le \tau \le \beta$, $r_+ \le r < \infty$, $0 \le x \le \beta_x$ and $0 \le y \le \beta_y$ where the period β is identified with the inverse temperature and the Euclidean action is related to the Gibbs $\mathcal{G}$ free energy by $I_E = \beta\mathcal{G}$. These considerations cause the reduced Euclidean action I_E to take the Hamiltonian form,

$$I_E = \beta \int_{r_+}^{\infty} dr \int_0^{\beta_x} dx \int_0^{\beta_y} dy (N\mathcal{H} - A_\tau \mathcal{G}) + B_E \tag{22}$$

where B_E is a surface term. The reduced constraints are given by

$$\mathcal{H} = \frac{r^2}{8\pi G}\left[\left(1 - \frac{4\pi G}{3}\phi^2\right)\left(\frac{\partial_r F}{r} + \frac{F}{r^2}\right) + \Lambda\right]$$
$$+ \frac{r^2}{6}\left[F(\partial_r\phi)^2 - \left(\partial_r F + \frac{4F}{r}\right)\phi\partial_r\phi - 2F\phi\partial_r^2\phi\right]$$
$$+ \frac{1}{2}\sum_{i=1}^{2}(\partial_{x^i}\psi_I)^2 + \frac{1}{2r^2}(\partial_x A_y - \partial_y A_x)^2 + \frac{1}{2r^2}(\pi^r)^2,$$
$$\mathcal{G} = \partial_r \pi^r, \tag{23}$$

where π^r stands for the electromagnetic field momentum defined by,

$$\pi^r = -\frac{r^2 A'_\tau}{N}. \tag{24}$$

This is the only nonvanishing momenta conjugated to the fields. In order to get a well-defined variational problem, the Euclidean action must be a differential functional of the canonical variables $\{F, A_x, A_y, \phi, \pi^r, \psi_I\}$, i.e., $\delta I_E = 0$ on shell. This means that the boundary term B_E must cancel all the boundary terms induced by the variations of the bulk term when the total variation of the action is performed. It is easy to check that the equations of motion obtained by varying the reduced action with respect to $\{N, F, A_\tau, A_x, A_y, \phi, \pi^r\}$ are consistent with the original Einstein equations. It turns out that N is a constant which, without loss of generality, can be taken to be $N = 1$. It is certainly evident, from Eq. (22), that N and A_τ are Lagrange multipliers whose associated constraints are the Hamiltonian one $\mathcal{H} = 0$ and Gauss' law $\mathcal{G} = 0$ for the electromagnetic field's conjugate momentum. The former, in addition to variations with respect to F and ϕ, provides a closed set of equations whose solution is given by the expressions for F and ϕ found in Eq. (7). The latter determines a Coulomb form for the electric potential. Maybe the less direct equations of motion are given by variations with respect to ψ_I and A_{x^i} which turn out to be respectively,

$$\partial^2_{x^i}\psi_I = 0, \qquad \partial_{x^i}F_{xy} = 0, \tag{25}$$

whose solution, after a suitable redefinition of the constants, provides a linear dependence on the coordinates for the fields in agreement with the solution found previously [Eqs. (8) and (14)]. Finally, the variation with respect to the momenta π^r determines an equation trivially satisfied by its definition.

The variational principle on the boundary term gives

$$\delta B_E = \beta\sigma\left[A_\tau\delta\pi^r - \frac{rN}{8\pi G}\left(1 - \frac{4\pi G}{3}\phi^2 - \frac{4\pi G}{3}r\phi\phi'\right)\delta F - \frac{r^2 N}{6}(4F\phi' + F'\phi)\delta\phi + \frac{r^2 N}{3}F\phi\delta\phi'\right]_{r_+}^{\infty}$$
$$- \int_{r_+}^{\infty} dr \frac{N}{r^2}\left\{\left[\int_0^{\beta_x} dx(\partial_y A_x - \partial_x A_y)\delta A_x\right]_{y=0}^{y=\beta_y} - \left[\int_0^{\beta_y} dy(\partial_y A_x - \partial_x A_y)\delta A_y\right]_{x=0}^{x=\beta_x}\right\}$$
$$- \int_{r_+}^{\infty} dr\left\{\left[\int_0^{\beta_y} dy\partial_x\psi_1\delta\psi_1\right]_{x=0}^{x=\beta_x} + \left[\int_0^{\beta_x} dx\partial_y\psi_2\delta\psi_2\right]_{y=0}^{y=\beta_y}\right\}, \tag{26}$$

where we have used the fact that the volume of the base manifold is $\sigma = \beta_x\beta_y$. Hereafter N is considered as a constant and is chosen to be $N = 1$. From the last two boundary terms the contribution of the topological and axionic charge of the system can be identified. These terms lead to the variation of the magnetic charge multiplied by the magnetic potential as well as the variation of the axionic charge multiplied by its associated chemical potential.

Requiring regularity of the metric at the horizon yields $\beta F'(r_+) = 4\pi$ giving the value for the temperature,

$$T = \frac{16}{9\pi}\frac{\lambda}{l}. \tag{27}$$

The variation of the fields on the event horizon is given by

$$\delta F|_{r_+} = -\frac{4\pi}{\beta}\delta r_+, \qquad \delta\phi|_{r_+} = \delta\phi(r_+) - \phi'|_{r_+}\delta r_+, \tag{28}$$

$$\delta\psi_I|_{r_+} = 2\sqrt{3}x^i\delta\lambda, \qquad \delta\pi^r|_{r_+} = \delta Q_E,$$
$$\delta A_x|_{r_+} = \frac{\delta Q_M}{2}y, \qquad \delta A_y|_{r_+} = -\frac{\delta Q_M}{2}x. \tag{29}$$

It is convenient to define the effective Newton's constant at the horizon $\tilde{G}_+$ as,[6]

$$\tilde{G}_+ = \frac{G}{(1 - \frac{4\pi G}{3}\phi(r_+)^2)}. \tag{30}$$

With this definition and Eqs. (26), (28) and (29), the variation of the boundary term at the horizon is

$$\delta B_E(r_+) = \delta\left(\frac{A_+}{4\tilde{G}_+}\right) + \beta\Phi_e\delta(\sigma Q_E) + \beta\Phi_M\delta(\sigma Q_M) + \beta\Phi_{\psi_1}\delta(-2\sqrt{3}\sigma\lambda) + \beta\Phi_{\psi_2}\delta(-2\sqrt{3}\sigma\lambda), \tag{31}$$

where $A_+ = \sigma r_+^2$ is the horizon area. Additionally we have conveniently defined

$$\Phi_E \equiv \frac{Q_E}{r_+}, \qquad \Phi_M \equiv \frac{Q_M}{r_+},$$
$$\Phi_{\psi_1} \equiv 2\sqrt{3}\lambda r_+, \qquad \Phi_{\psi_2} \equiv 2\sqrt{3}\lambda r_+, \tag{32}$$

identifying the chemical potentials for the electric, magnetic and axion fields as $\Phi_E, \Phi_M, \Phi_{\psi_I}$, respectively. We reordered the variations in Eq. (31) using the fact that σ is fixed to make clear contact with the conserved charges later. Hereafter, we work in the grand canonical ensemble, where the temperature $T = \beta^{-1}$ and the chemical potentials at the horizon are fixed. Then, by virtue of these boundary conditions at the horizon, the boundary term can be integrated there, giving

$$B_E(r_+) = \frac{A_+}{4\tilde{G}_+} + \beta\Phi_e(r_+)(\sigma Q_E) + \beta\Phi_M(\sigma Q_M) + \beta\Phi_{\psi_1}(-2\sqrt{3}\sigma\lambda) + \beta\Phi_{\psi_2}(-2\sqrt{3}\sigma\lambda). \tag{33}$$

The variations of the fields at infinity are

$$\delta F|_\infty = -12\lambda\delta\lambda - \frac{24l\lambda^2\delta\lambda}{r} - \frac{12l^2\lambda^3\delta\lambda}{r^2},$$
$$\delta\phi|_\infty = \frac{\delta\phi_1}{r} + \frac{\delta\phi_2}{r^2} + \mathcal{O}(r^{-3}), \tag{34}$$

$$\delta\psi_I|_\infty = 2\sqrt{3}x^i\delta\lambda, \qquad \delta\pi^r|_\infty = \delta Q_E,$$
$$\delta A_x|_\infty = \frac{\delta Q_M}{2}y, \qquad \delta A_y|_\infty = -\frac{\delta Q_M}{2}x. \tag{35}$$

Then, from Eq. (26) and Eqs. (34)–(35) we obtain an expression for the variation of the boundary term at infinity as follows:

$$\delta B_E(\infty) = \beta\sigma 48l\lambda^2\delta\lambda + \frac{\beta\sigma}{3l^2}(2\phi_2\delta\phi_1 - \phi_1\delta\phi_2). \tag{36}$$

The integrability condition $\delta^2 B_E(\infty) = 0$ can be imposed, implying that $\phi_2 = \phi_2(\phi_1)$. Additionally, since β is fixed the boundary term at infinity generically takes the form

$$B_E(\infty) = \beta\sigma 16l\lambda^3 + \frac{\beta\sigma}{3l^2}\int\left(2\phi_2 - \phi_1\frac{d\phi_2}{d\phi_1}\right)d\phi_1. \tag{37}$$

Having computed the boundary terms and since the value of the reduced action on shell is the boundary term B_E, we obtain,

$$\begin{aligned} I_E &= B_E(\infty) - B_E(r_+) \\ &= \beta\sigma 16l\lambda^3 + \frac{\beta\sigma}{3l^2}\int\left(2\phi_2 - \phi_1\frac{d\phi_2}{d\phi_1}\right)d\phi_1 - \frac{A_+}{4\tilde{G}_+} \\ &\quad - \beta\Phi_e(r_+)(\sigma Q_E) - \beta\Phi_M(\sigma Q_M) \\ &\quad - \beta\Phi_{\psi_1}(-2\sqrt{3}\sigma\lambda) - \beta\Phi_{\psi_2}(-2\sqrt{3}\sigma\lambda), \end{aligned} \tag{38}$$

up to an arbitrary additive constant without variation. Since the Gibbs free energy is related to the Euclidean action as $I_E = \beta\mathcal{G} = \beta\mathcal{M} - \mathcal{S} - \beta\Phi_E\mathcal{Q}_E - \beta\Phi_M\mathcal{Q}_M - \beta\Phi_{\psi_1}\mathcal{Q}_1 - \beta\Phi_{\psi_2}\mathcal{Q}_2$ in the grand canonical ensemble, the mass $\mathcal{M}$, the electric charge $\mathcal{Q}_E$, the magnetic charge $\mathcal{Q}_M$, the axionic charges $\mathcal{Q}_i$ and entropy $\mathcal{S}$ are computed by means of the standard thermodynamical relations. From Eq. (38) it is straightforward that, the mass is generically given by

$$\begin{aligned} \mathcal{M} &= \left(\frac{\partial}{\partial\beta} - \beta^{-1}\Phi_E\frac{\partial}{\partial\Phi_E} - \beta^{-1}\Phi_M\frac{\partial}{\partial\Phi_M} - \beta^{-1}\Phi_{\psi_I}\frac{\partial}{\partial\Phi_{\psi_I}}\right)I_E \\ &= 16\sigma l\lambda^3 + \frac{\sigma}{3l^2}\int\left(2\phi_2 - \phi_1\frac{d\phi_2}{d\phi_1}\right)d\phi_1, \end{aligned} \tag{39}$$

whereas the entropy $\mathcal{S}$, axionic charges $\mathcal{Q}_i$, electric charge $\mathcal{Q}_E$ and magnetic charge $\mathcal{Q}_M$ read

$$\mathcal{S} = \left(\beta\frac{\partial}{\partial\beta} - 1\right)I_E = \frac{A_+}{4\tilde{G}_+}, \qquad \mathcal{Q}_i = -\frac{1}{\beta}\frac{\partial I_E}{\partial\Phi_{\psi_I}} = -2\sqrt{3}\lambda\sigma, \tag{40}$$

$$\mathcal{Q}_E = -\frac{1}{\beta}\frac{\partial I_E}{\partial\Phi_E} = \sigma Q_E, \quad \mathcal{Q}_M = -\frac{1}{\beta}\frac{\partial I_E}{\partial\Phi_M} = \sigma Q_M. \tag{41}$$

Thus, it turns out from these results, that the first law of black hole thermodynamics is satisfied,

$$d\mathcal{M} = TdS + \Phi_E d\mathcal{Q}_E + \Phi_M d\mathcal{Q}_M + \Phi_{\psi_1}d\mathcal{Q}_1 + \Phi_{\psi_2}d\mathcal{Q}_2, \tag{42}$$

which was expected since this is a consequence of the fact that the Euclidean reduced action attains an extremum.

[6]Here we have reestablished Newton's constant G for clearness. Note that we set $1/16\pi G = 1$ when evaluating our solution.

The positivity of the entropy requires $\tilde{G}_+ > 0$ which provides a lower bound for the axion parameter,

$$\lambda^4 > \frac{Q_E^2 + Q_M^2}{180 l^2}, \tag{43}$$

which implies an upper bound for the constant α through Eq. (17) such that $|\alpha| < 8\sqrt{3}$. Therefore, we see that, unlike the uncharged case, the axion parameter is restricted by the electric and magnetic charge; otherwise there is an unphysical negative entropy. The precise functional relation between the leading and subleading terms of the scalar field in the asymptotic region, can be determined by demanding that the scalar field respect the asymptotic AdS invariance. As we are interested in holographic applications, we will consider Eq. (17), and the boundary condition causes the integral term in Eq. (39) to vanish giving,

$$\mathcal{M} = 16\sigma l \lambda^3. \tag{44}$$

Therefore, there is no scalar field contribution to the black hole mass. As it was pointed out in Ref. [56], this is because to preserve the AdS symmetry of the scalar field in the asymptotic region the contribution of the scalar field to the mass must vanish.

The rotating solution can be found in Appendix A, in which for simplicity we analyze the neutral case. The main property of the rotating black brane is that the finiteness condition on the conserved charges imposes a relation between the angular momentum and the axion parameter. This means that the axion field can provide angular momentum to physically acceptable solutions whose mass $\mathcal{M}$ and angular momentum $\mathcal{J}$ are respectively given by

$$\mathcal{M} = 8\sigma l \lambda (3\omega_0^2 + 2\lambda^2), \qquad \mathcal{J} = 24\sigma l^2 \lambda \omega_0 \sqrt{\omega_0^2 + \lambda^2}, \tag{45}$$

where

$$\omega = \frac{\omega_0}{\sqrt{\omega_0^2 + \lambda^2}}, \tag{46}$$

where ω_0 is a constant without variation.

The local stability can be analyzed by computing the specific heat at fixed chemical potentials giving,

$$C_{\Omega_+,\Phi_{\psi_I}} = \frac{3\pi\sigma (r_+^2 + 9l^2\omega_0^2)^{3/2}(2r_+^2 + 9l^2\omega_0^2)}{2r_+(r_+^2 + 18 l^2 \omega_0^2)}, \tag{47}$$

which is positive always. In consequence, the black brane can always reach thermal equilibrium with a heat bath.

In general, the global stability can be analyzed through the comparison of the free energies between the hairy black brane and its counterpart in the absence of the scalar field. This is because, it is necessary to compare them by setting both configurations at the same fixed temperature T and chemical potentials in a given ensemble. Note that, because of Eq. (44), it is not possible to switch off the scalar field and keep a fixed mass and a nonvanishing temperature. This means that the only possible candidate to compare with the black brane is the black brane solution obtained in the absence of the conformally coupled scalar field found in Ref. [43]. In this case, and choosing the grand canonical ensemble we found that for the same boundary conditions and at fixed temperature, the free energy of our black brane is always positive while the other black brane possesses a nonpositive free energy for any temperature. In other words, the solution without a conformal scalar field is the thermodynamically preferred configuration.

V. HOLOGRAPHIC DC CONDUCTIVITIES AND HALL ANGLE

Now we move on to study the DC conductivities σ_{DC} and the Hall angle θ_H of the holographic theory dual to the charged hairy black hole. These transport properties have recently been widely studied in various theories because the conductivities do not evolve in the radial direction and hence can be analytically obtained by calculating the values at the horizon. It was discussed in Refs. [42,57] that σ_{DC} usually contains two terms, i.e., $\sigma_{\mathrm{DC}} = \sigma_{ccs} + \sigma_{\mathrm{diss}}$, where σ_{ccs} is the "charge-conjugation-symmetric" part [58] while σ_{diss} is related to the charge Q_E of the black hole and it is divergent in a translationally invariant theory. The relation between the Hall angle and σ_{diss} and the corresponding scaling were carefully studied in Refs. [57,59]. Thus, we will use the techniques of Ref. [42] to study the features of σ_{DC} and θ_H of our charged hairy black hole. To proceed, we turn on only the following relevant perturbations because the remaining perturbations are decoupled and have no relevance to our study:

$$\begin{aligned} &\delta A_x = -E_x t + a_x, \qquad \delta A_y = -E_y t + a_y, \\ &\delta g_{tx} = r^2 h_{tx}, \qquad \delta g_{rx} = r^2 h_{rx}, \\ &\delta g_{ty} = r^2 h_{ty}, \qquad \delta g_{ry} = r^2 h_{ry}, \\ &\delta\psi_1 = \Psi_1, \qquad \delta\psi_2 = \Psi_2, \end{aligned} \tag{48}$$

where E_x and E_y are constant, while a_x, a_y, $h_{tx}, h_{ty}, h_{rx}, h_{ry}, \Psi_1, \Psi_2$ are all functions of the radial coordinate r. Then, the two perturbed Maxwell equations are

$$F' a_x' + F a_x'' + Q_E h_{\mathrm{tx}}' + Q_M (F' h_{\mathrm{ry}} + F h_{\mathrm{ry}}') = 0, \tag{49}$$

$$F' a_y' + F a_y'' + Q_E h_{\mathrm{ty}}' - Q_M (F' h_{\mathrm{rx}} + F h_{\mathrm{rx}}') = 0, \tag{50}$$

where a prime denotes a derivative with respect to r. From the above equations, we define two conserved currents[7]

$$J_x = -r^2 F^{rx} = -F a_x' - Q_E h_{tx} - Q_M F h_{ry}, \quad (51)$$

$$J_y = -r^2 F^{ry} = -F a_y' - Q_E h_{ty} + Q_M F h_{rx}, \quad (52)$$

which satisfy $\frac{dJ_x(r)}{dr} = \frac{dJ_y(r)}{dr} = 0$ due to the Maxwell equations (49) and (50). This implies that J_x and J_y do not depend on r. In addition, according to the AdS/CFT dictionary, the holographic DC conductivities are determined by the conserved currents in the asymptotic boundary. As mentioned above, since J_x and J_y are independent of r, we shall evaluate them at the horizon instead of on the boundary. To impose the regularity conditions at the horizon, it is convenient to work in Eddington-Finkelstein coordinates (v, r) with $v = t - \int \frac{dr}{F}$. Thus, the regular conditions at the event horizon require the gauge field to take the form [60]

$$a_x = -E_x \int \frac{dr}{F}, \qquad a_y = -E_y \int \frac{dr}{F} \quad (53)$$

while the perturbed metric reads

$$h_{rx} = \frac{h_{tx}}{F}, \qquad h_{ry} = \frac{h_{ty}}{F}. \quad (54)$$

Moreover, we have $F(r_+) \sim 4\pi T(r - r_+)$ and we set $\Psi_{1,2}$ to be constant near the horizon [42]. Then, substituting the conditions (53) and (54) into the rx and ry components of the Einstein equation, we can solve for $h_{tx}(r_+)$ and $h_{ty}(r_+)$. Thus, the conductivities can be obtained

$$\sigma_{xx} = \frac{\partial J_x(r_+)}{\partial E_x} = \frac{3(12\lambda^4 + Q_M^2 + Q_E^2)(36\lambda^4 + 3Q_M^2 + Q_E^2)}{2Q_E^2(36\lambda^4 + 5Q_M^2) + 9(12\lambda^4 + Q_M^2)^2 + Q_E^4}, \quad (55)$$

$$\sigma_{yy} = \frac{\partial J_y(r_+)}{\partial E_y} = \sigma_{xx}, \quad (56)$$

$$\sigma_{xy} = \frac{\partial J_x(r_+)}{\partial E_y} = \frac{6Q_E Q_M(12\lambda^4 + Q_M^2 + Q_E^2)}{2Q_E^2(36\lambda^4 + 5Q_M^2) + 9(12\lambda^4 + Q_M^2)^2 + Q_E^4}, \quad (57)$$

$$\sigma_{yx} = \frac{\partial J_y(r_+)}{\partial E_x} = -\frac{6Q_E Q_M(12\lambda^4 + Q_M^2 + Q_E^2)}{2Q_E^2(36\lambda^4 + 5Q_M^2) + 9(12\lambda^4 + Q_M^2)^2 + Q_E^4} = -\sigma_{xy}, \quad (58)$$

where we have considered the event horizon $r_+ = 3\lambda$ and the temperature $T = 16\lambda/9\pi$ in Eq. (27) by setting $l = 1$. The DC conductivity and the Hall angle are

$$\sigma_{\mathrm{DC}} \coloneqq \sigma_{xx}(Q_M = 0) = \frac{3(12\lambda^4 + Q_E^2)}{36\lambda^4 + Q_E^2}, \quad (59)$$

$$\theta_H \coloneqq \frac{\sigma_{xy}}{\sigma_{xx}} = \frac{2Q_E Q_M}{36\lambda^4 + 3Q_M^2 + Q_E^2}, \quad (60)$$

and we see that σ_{DC} and θ_H are finite.

We rewrite the DC conductivity as

$$\sigma_{\mathrm{DC}} = 1 + \frac{2Q_E^2}{36\lambda^4 + Q_E^2} = 1 + \frac{2(\alpha^2 - 12)}{24 + \alpha^2} = \frac{3\alpha^2}{24 + \alpha^2}, \quad (61)$$

where we have used Eq. (17) in the second equality. When $Q_E = 0$, we have $\sigma_{\mathrm{DC}} = 1$ which is consistent with the result for a neutral black hole found in Ref. [42]. However, the term $\frac{2Q_E^2}{36\lambda^4 + Q_E^2}$ is very different from σ_{diss} in the case without a scalar field [43], which is divergent as $\lambda \to 0$. Theories in which other scalar fields appear, other than the minimally coupled axions, have been considered for example in Ref. [61]. We observe that, even in very exotic cases, in which other fields with nonminimal couplings are included, the conductivity still behaves as if those fields were not present, reproducing the same result found in Ref. [43]. Of course, once the theory under consideration modifies the location of the horizon in terms of these new contributions to the system, the dependence on the temperature will change. Nevertheless the asymptotic behavior of the conductivity remains unchanged. In our case, we observe that the conformally coupled scalar field $\phi(r)$ modifies the backreaction of the black brane solution such that σ_{DC} is temperature independent. Actually, after considering Eq. (17), we observe that it does not depend on λ at all, and then by means of Eq. (27) there is no dependence on the temperature. Thus, the holographic transport features in our model are more like those in a neutral black hole whose DC conductivity is a finite constant in the dual theory. We note that the DC conductivity in this model satisfies $\sigma_{\mathrm{DC}} > 1$ by inserting Eq. (17) into Eq. (61). The only dependence that σ_{DC} possesses is the dependence on the constant α. From Eq. (17) we know that the absolute value of α must be strictly greater than $2\sqrt{3}$. On the other hand we observe that in order to reproduce solutions with positive entropy, λ is bounded from below according to

[7] In principle, we can also define the current $r^2 F^{rx}$ because of the Maxwell equation $(r^2 F^{rx})' = 0$; however, we have defined $J_t = Q_E = -r^2 F^{rt} = r^2 (A_t)'$ in the black hole solution (14). In order to be consistent, we chose $J_i = -r^2 F^{ri} (i = x, y)$.

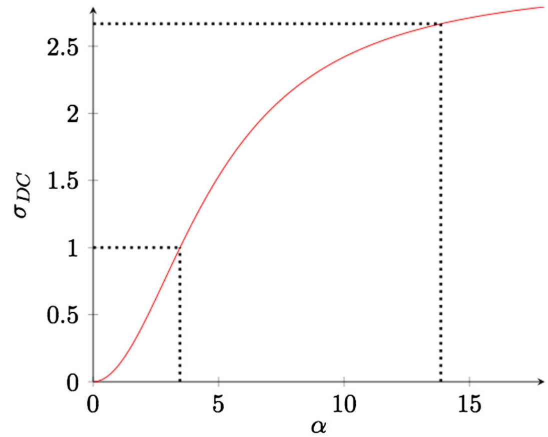

FIG. 1. DC conductivity with respect to α. Between the dashed lines, this curve represents conductivities for physically acceptable configurations.

Eq. (43). This implies an upper bound on α given by $8\sqrt{3}$. In Fig. 1 we observe that the conductivity approaches a constant value as α increases. Its minimum value is obtained by asymptotically approaching $\alpha = 2\sqrt{3}$ from the right. In addition, through Eqs. (17) and (27), it can be seen that the Hall angle decays as $\theta_H \sim 1/T^2$ in the high-temperature limit, resembling the quadratic-T inverse behavior found in cuprates holographically studied in Refs. [57,61,62].

To obtain a richer phenomenology, we have to modify the axionic part of the action by means of nonminimal couplings. Because our axion depends linearly on the flat coordinates, we need to modify the action in a way that maintains the original shift symmetry; otherwise the transverse manifold coordinates would appear in the equations of motion. The most simple way to do this is to consider a k-essence term as we did in Ref. [50]

$$\bar{S}[g,\phi,\psi_I] = \int d^4x\sqrt{-g}\Bigg[R - 2\Lambda - \frac{(\partial\phi)^2}{2} - \frac{\phi^2 R}{12} - \sum_{I=1}^{2}\left(\frac{X_I}{2} + \beta\left(\frac{X_I}{2}\right)^k\right) - \frac{F^2}{4}\Bigg] \tag{62}$$

where we have defined $X_I \coloneqq \partial_\mu\psi_I\partial^\mu\psi_I$. This goes back to the action (1) plus Eq. (13) when β, with mass dimension $4-4k$, goes to zero. The equations of motion for the above action are shown in Appendix B. We know that when $\phi(r) = 0$, the case $k = 2$ modifies the black brane solution found in Ref. [43] including a term which goes as $1/r^2$, like the electric charge. Then we provide an exact solution for Eq. (62) for this particular case

$$ds^2 = -\frac{(r-3\lambda l)(r+\lambda l)^3}{r^2 l^2}dt^2 + \frac{r^2 l^2}{(r-3\lambda l)(r+\lambda l)^3}dr^2 + r^2(dx^2+dy^2), \tag{63}$$

$$\psi_I = 2\sqrt{3}\lambda x^I, \tag{64}$$

and the scalar field and the Maxwell gauge field read

$$\phi = \frac{\sqrt{Q_E^2 + 12\lambda^4 l^2 + 576\lambda^4\beta}}{\lambda(r+\lambda l)}, \qquad A = -\frac{Q_E}{r}dt. \tag{65}$$

Here we are interested in the DC conductivity, so we turn off the magnetic field. For this solution, the AdS criterion is given by

$$\lambda^4 = \frac{Q_E^2}{(\alpha^2-12)l^2 - 576\beta}, \tag{66}$$

requiring

$$(\alpha^2-12)l^2 > 576\beta. \tag{67}$$

With the same method and algebra computation as in the previous study, we obtain the DC conductivity

$$\bar{\sigma}_{\rm DC} = \frac{3[12(48\beta+1)\lambda^4 + Q_E^2]}{36(48\beta+1)\lambda^4 + Q_E^2} = 1 + \frac{2Q_E^2}{36(48\beta+1)\lambda^4 + Q_E^2} = 1 + \frac{2(\alpha^2-12-576\beta)}{\alpha^2+24+1152\beta} = \frac{3\alpha^2}{\alpha^2+24+1152\beta} \tag{68}$$

where we have considered Eq. (66) with $l = 1$ in the second line. The above result is also independent of λ and it reduces to the DC conductivity (61) when $\beta = 0$ as we expected. We will consider only positive values of β in order to avoid phantom contributions to the axionic sector. It is observed that the conductivity depends on two parameters, α and β which satisfy Eq. (67), constraining β to take values in the interval $[0, \frac{\alpha^2-12}{576}[$; otherwise the axion parameter may become complex. In this case, α is still bounded from above by requiring configurations of positive entropy (43) such that $\alpha < 8\sqrt{3}$. For these considerations we note that in the region of parameters of physically acceptable configurations, the conductivity again reaches its maximum value $\sigma_{\rm DC} \to 8/3$ for $\alpha \to 8\sqrt{3}$ and $\beta = 0$. Now, the inclusion of the β parameter by means of the k-essence contribution, allows access to a wider range of conductivities. Namely, all the configurations for which β approaches its upper bound, which is $\beta \to \frac{\alpha^2-12}{576}$, possess $\sigma_{\rm DC} \to 1$. This means that the conductivity now lies on a two-dimensional surface described by α and β, whose extremal values give $\sigma_{\rm DC} \in]1, 8/3[$. We have to note that Eq. (67) prevents our solution from behaving as an insulator, as it imposes a lower bound for β that forbids access to the insulator state for physically acceptable configurations. Figure 2 shows this particular behavior.

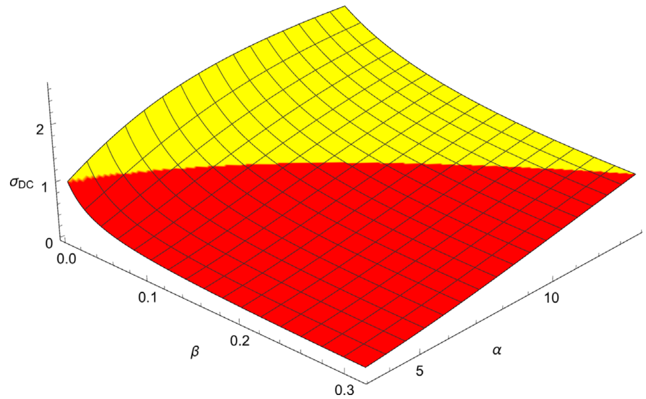

FIG. 2. DC conductivity with respect to α and β. The yellow region represents conductivities for physically acceptable configurations.

VI. FURTHER REMARKS

In this paper we have constructed black brane solutions in a conformally coupled scalar theory. These solutions are supported by two axion fields homogeneously distributed along the horizon that depend linearly on the transverse manifold coordinates. No self-interaction for the scalar field is needed. Solutions of this kind have been shown to produce momentum dissipation, making them excellent candidates to study holographic conductivities [43]. We observed that, for the neutral case, all parameters appearing in the solution are free from fine-tuning. The electric and magnetically charged case was also obtained. It was shown in this case, that demanding that the whole configuration respects invariance under the group AdS SO(3,2) at the boundary, implies that the axionic and electromagnetic charges must be related by means of Eq. (17). This implies that there is no contribution to the mass from the scalar fields. We then performed in detail the thermodynamic analysis of our charged solutions considering the nontrivial effect of the axionic charges. Unlike the BMBB black hole, the black brane interacts with a scalar field which is regular everywhere, allowing us to get a well-defined thermodynamics; in particular, the entropy is a finite quantity. The rotating black brane can be obtained from the static one, by performing a boost in the plane spanned by the temporal coordinate and one of the planar coordinates. It has been shown that the physically acceptable configurations with finite conserved charges, possess mass and angular momentum tuned by the axion parameter. This configuration satisfies the dominant energy condition and is thermodynamically locally stable, which means that it always attains equilibrium with a heat bath as well as its static counterpart. Once the whole configuration respects the asymptotic AdS symmetry, we focused on some holographic applications. Following the procedure stated in Ref. [42], and using the momentum relaxation techniques of Ref. [43], we obtained the DC conductivity and Hall angle of our charged solutions. We demonstrated that due to our AdS criterion (17) our solution mostly behaves as the neutral case originally studied in Ref. [42]. Nevertheless, in this case we have $\sigma_{\rm DC} > 1$ and, contrary to the charged black branes originally constructed in Ref. [23] and holographically studied in Ref. [43], its behavior is totally independent of the temperature. Despite this, our DC conductivity depends on the arbitrary constant α that ensures a nontrivial scalar field $\phi(r)$, whose asymptotic behavior is invariant under the group AdS SO(3,2). It was found that in the range of values for this constant that ensure physically acceptable configurations, the conductivity profile is a monotonically increasing function of α, taking values from $\sigma_{\rm DC} = 1$ to $\sigma_{\rm DC} = 8/3$. In order to look for different behaviors of our holographic conductor we have included a new contribution in the axionic sector of our system. Motivated by the result obtained in Ref. [50] for k-essence theories, we studied the k-essence case with $k = 2$. This case is integrated due to the fact that the k-essence contribution to the equations of motion possesses the same behavior as the electric charge. We observed that our AdS criterion is modified but the DC conductivity is still independent of the temperature. This behavior was expected since the $k = 2$ term behaves similarly to the electric charge. Despite this, the new coupling β modifies the conductivity in such a way that, while respecting the condition (67), $\sigma_{\rm DC} \sim 1$ for a wide range of values for β given by $\beta \to \frac{\alpha^2-12}{576}$. We observed that, for small values of α and large values of β, $\sigma_{\rm DC} \sim 0$ behaves as an insulator. Nevertheless, this region belongs to unphysical configurations since they violate the reality condition for the axion fields. As a final remark, we can conclude that, either in the case of pure minimally coupled axion fields or in more exotic theories such as k-essence, conformal scalar fields that respect the asymptotic AdS symmetry lead to a constant DC conductivity that depends on the coupling parameters and not on the temperature as it is usually expected. As further development of this work, we will disclose the thermal conductivity and the thermoelectric conductivity of the dual boundary theory elsewhere. Moreover, we expect to study more general conformally coupled scalar theories, such as the one studied in Ref. [63], that exist in dimensions higher than four. These would provide not only black brane solutions for such models but also a new setup to study holographic conductivities and their phenomenology. Additionally, other holographic applications of interest come from the idea that black branes are known to be dual to perfect fluids. In this context it would be interesting to explore when the solutions constructed here satisfy the shear viscosity entropy bound [64]

$$\frac{\eta}{S} = \frac{1}{4\pi} \tag{69}$$

which has been demonstrated to hold in models without higher curvature terms [65]. Nevertheless in Ref. [66] it was demonstrated that by including a nonminimal kinetic coupled scalar field the bound may be violated. We are aiming to investigate whether or not this is the case for black branes exhibiting a conformal nonminimal coupling with the curvature in the presence of scalar fields that induce momentum dissipation. We expect to report on this in our future related works.

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APPENDIX A: ROTATING BLACK BRANES

When the static solution possesses a planar base manifold, a rotating solution can be constructed from the static one by applying an improper local transformation [67]. Additionally, as it was noted in Ref. [68], by performing a topological identification along one of the planar coordinates, it is possible to obtain a stationary solution with angular momentum. For simplicity, we will consider the uncharged solution. To achieve this, the x coordinate now dubbed as φ and with the range $\varphi \in (-\infty, \infty)$, along with the t coordinate are transformed as follows:

$$t \to \frac{1}{\sqrt{1-\omega^2}}(t - l\omega\varphi), \qquad \varphi \to \frac{1}{\sqrt{1-\omega^2}}\left(\varphi - \frac{\omega}{l}t\right). \tag{A1}$$

This transformation is a boost in the t-φ plane parametrized by $\omega^2 < 1$. By means of a topological identification, the initial topology of the base manifold $\mathbb{R}\times\mathbb{R}$ can be transformed to $S^1\times\mathbb{R}$, by fixing the period of the angular coordinate φ to 2π. This implies that the local geometry of the static solution is preserved but not the global one. Hence, the transformation (A1) provides a resulting manifold which is globally stationary but locally static. Applying Eq. (A1) to our original static solution (9)–(11) we obtain

$$ds^2 = -N^2(r)F(r)dt^2 + \frac{dr^2}{F(r)} + H(r)(d\varphi + N^\varphi(r)dt)^2 + r^2dy^2 \tag{A2}$$

where $F(r)$ is given by Eq. (9) and

$$N^2(r) = \frac{r^2(1-\omega^2)}{r^2 - l^2\omega^2F(r)}, \qquad N^\varphi(r) = -\frac{r^2 - l^2F(r)}{r^2 - l^2\omega^2F(r)}\frac{\omega}{l},$$
$$H(r) = \frac{r^2 - l^2\omega^2F(r)}{1-\omega^2}. \tag{A3}$$

On the other hand the conformally coupled scalar field $\phi(r)$ is not affected by the transformation while the axion field originally distributed along the x direction becomes time dependent

$$\psi_1 = 2\sqrt{3}\frac{\lambda}{\sqrt{1-\omega^2}}\left(\varphi - \frac{\omega}{l}t\right). \tag{A4}$$

From Eq. (A2) we see that the rotating solutions still possess a single event horizon located at $r_+ = 3\lambda l$ covering the curvature singularity in $r = 0$. The functions $N^2(r)$ and $H(r)$ are both positive, which ensures the proper signature for the black hole and a well-defined local area for the base manifold. All of these functions are monotonically increasing functions outside the horizon. The asymptotic behavior corresponds to a boosted AdS spacetime. It is possible to show that this solution indeed describes a rotating black hole. For this, we follow the Regge-Teitelboim approach [69] to determine the mass and angular momentum for this rotating solution. The Hamiltonian generator of the asymptotic symmetries $\xi^\mu = (\xi^\perp, \xi^i)$ for the Lagrangian in Eq. (1), is given by a linear combination of the Hamiltonian constraints $\mathcal{H}_\perp, \mathcal{H}_i$ supplemented with a surface term $Q[\xi^\mu]$ which ensures well-defined functional derivatives for the Hamiltonian generator. In the Arnowitt-Deser-Misner decomposition, γ_{ij} is the metric of the spacelike surfaces of constant time. Along with the scalar field and the axion fields ψ_I, they constitute the dynamical variables of the system with conjugate momenta π_{ij}, π_ϕ and π_{ψ_I}, respectively. The generator reads,

$$H[\xi^\mu] = \int d^3x(\xi^\perp\mathcal{H}_\perp + \xi^i\mathcal{H}_i) + Q[\xi^\mu]. \tag{A5}$$

For the kind of configurations considered in Eq. (A2), the expressions for the constraints are explicitly given by [70]

$$\mathcal{H}_\perp = -\sqrt{\frac{H}{F}}r\left[\left(\kappa - \frac{\xi}{2}\phi^2\right){}^{(3)}R - \frac{1}{2}F(\partial_r\phi)^2 - \kappa\Lambda\right] + \sqrt{\frac{f}{H}}\left(\frac{4}{2\kappa - \xi\phi^2}\right)\pi_{r\varphi}\pi^{r\varphi} + \sqrt{\frac{f}{H}}\frac{\pi_{\psi_1}^2}{2r} - \xi\partial_r(\sqrt{fH}r\partial_r\phi^2), \tag{A6}$$

$$\mathcal{H}_\varphi = -2\pi_\varphi{}^r{}_{|r} + \pi_{\psi_1}\partial_\varphi\pi_{\psi_1} \tag{A7}$$

where ${}^{(3)}R$ is the Ricci scalar of γij. The nonvanishing components of the momenta are given by,

$$\pi_\varphi{}^r = -\frac{rH^{3/2}}{2N}\left(\kappa - \frac{\xi}{2}\phi^2\right)\partial_rN^\varphi,$$
$$\pi_{\psi_1} = \frac{r\sqrt{H}}{NF}(\partial_r\psi_1 - N^\varphi\partial_\varphi\psi_1). \tag{A8}$$

Demanding $\delta H = 0$ on shell, we obtain the variation of the charge

$$\delta Q[\xi^\mu] = \lim_{r\to\infty}\Bigg\{\sigma\sqrt{\frac{H}{F}}\Bigg[\left(-\frac{F}{H}\left(\partial_r\delta H - \frac{\partial_r H}{2H}\delta H\right) - \left(\frac{\partial_r H}{2H} + \frac{1}{r}\right)\delta F\right)\left(\frac{1-\kappa\xi\Phi^2}{2\kappa}\right)\xi^\perp + \frac{F}{H}\partial_r\left[\left(\kappa - \frac{\xi}{2}\phi^2\right)\xi^\perp\right]\delta H + F[\xi^\perp(\xi\partial_r(\delta\phi^2) - \delta\phi\partial_r\phi) - \xi\partial_r\xi^\perp\delta\phi^2]\Bigg] + 2\sigma\xi^\varphi\delta\pi_\varphi{}^r - \int dS_i\left[\frac{r\sqrt{H}}{F}\xi^\perp\partial^i(\phi\delta\phi + \psi_1\delta\psi_1 + \psi_2\delta\psi_2) + \xi^i\pi_{\psi_1}\delta\psi_1\right]\Bigg\}, \tag{A9}$$

where σ stands for the volume of the base manifold. The deformation vectors ξ^μ in terms of the Killing vectors ∂_t and ∂_φ are given by,

$$\xi^\perp = N\sqrt{F}\partial_t, \tag{A10}$$

$$\xi^\varphi = \partial_\varphi + N^\varphi\partial_t. \tag{A11}$$

The mass M is the conserved charge associated with the time translation symmetry, while the angular momentum J is the conserved charge associated with the rotational translation symmetry. In fact, in this approach, both of them are obtained by evaluating $\delta\mathcal{M} = \delta Q[\partial_t]$ and $\delta\mathcal{J} = -\delta Q[\partial_\varphi]$, respectively. Namely,

$$\delta\mathcal{M} = \lim_{r\to\infty}\left\{(t + \beta_\varphi l\omega)\delta\Omega\frac{r}{l} + \frac{24l\sigma\lambda^2}{(1-\omega^2)^2}[(\omega^4 + \omega^2 - 2)\delta\lambda - 2\lambda\omega\delta\omega]\right\}, \tag{A12}$$

$$\delta\mathcal{J} = \lim_{r\to\infty}\left\{-(\omega t + \beta_\varphi l)\delta\Omega r + \frac{24l^2\sigma\lambda^2}{(1-\omega^2)^2}[3(1-\omega^2)\omega\delta\lambda + (1+\omega^2)\lambda\delta\omega]\right\} \tag{A13}$$

where $\delta\Omega$ is given by,

$$\delta\Omega = -\frac{12\lambda\beta_y}{(1-\omega^2)^2}[\omega(1-\omega^2)\delta\lambda + \lambda\delta\omega]. \tag{A14}$$

As it can be seen in Eqs. (A12) and (A13), the boost performed on the static solution, has introduced terms proportional to $\delta\Omega$ which are divergent. However, we can obtain finite conserved charges by demanding $\delta\Omega = 0$, which imposes a relation between the boost parameter ω and the axion parameter λ. This is determined by a differential equation of the form

$$\frac{d\omega}{d\lambda} + \frac{\omega}{\lambda}(1-\omega^2) = 0, \tag{A15}$$

whose solution gives

$$\omega = \frac{\omega_0}{\sqrt{\omega_0^2 + \lambda^2}}, \tag{A16}$$

where ω_0 is a constant without variation. This relation renders the variation of the charges finite and the remaining terms can be directly integrated. Therefore, the mass and angular momentum have the following expressions:

$$\mathcal{M} = 8\sigma l\lambda(3\omega_0^2 + 2\lambda^2), \qquad \mathcal{J} = 24\sigma l^2\lambda\omega_0\sqrt{\omega_0^2 + \lambda^2}, \tag{A17}$$

up to additive fixed constants. These constants have been fixed in order to have a static and massless background. In other words, switching off the axion field through $\lambda = 0$, the configuration

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2(d\varphi^2 + dy^2), \tag{A18}$$

possesses $\mathcal{M} = 0$ and $\mathcal{J} = 0$. Note also that, naturally the angular momentum vanishes when the boost parameter vanishes, recovering the mass for the static solution obtained in Sec. IV in the neutral case. Also, it is worth noting that the angular momentum is bounded from above by the mass $|\mathcal{J}| < \mathcal{M}l$. The temperature of the black hole can be computed by means of the surface gravity $k^2 = -\frac{1}{2}\nabla_\mu\chi_\nu\nabla^\mu\chi^\nu$ given in terms of the null Killing vector at the event horizon $\chi = \partial_t + \Omega_+\partial_\varphi$. Here, $\Omega_+ = \omega/l$ is the angular velocity at the horizon. This is

$$T = \frac{k}{2\pi} = \frac{16}{9\pi}\frac{\lambda^2}{l\sqrt{\omega_0^2 + \lambda^2}}, \tag{A19}$$

which, as it was expected, reduces to Eq. (27) when $\omega_0 = 0$. The conformal coupling modifies the standard Bekenstein-Hawking entropy, giving [71,72]

$$S = \frac{A_+}{4\tilde{G}_+}, \tag{A20}$$

where $A_+ = \sigma r_+\sqrt{H(r_+)}$ is the area of the horizon. As it was expected, the first law of thermodynamics is satisfied,

$$d\mathcal{M} = Td\mathcal{S} + \Omega_+ d\mathcal{J} + \Phi_{\psi_1}d\mathcal{Q}_1 + \Phi_{\psi_2}d\mathcal{Q}_2, \tag{A21}$$

with the chemical potentials for axion fields and axionic charges determined by

$$\Phi_{\psi_1} = 2\sqrt{3(1-\omega^2)}\lambda r_+, \quad \mathcal{Q}_{\psi_1} = -\frac{2\sqrt{3}\lambda\sigma}{\sqrt{1-\omega^2}}, \tag{A22}$$

$$\Phi_{\psi_2} = 2\sqrt{3}\lambda r_+, \qquad \mathcal{Q}_{\psi_2} = -2\sqrt{3}\lambda\sigma. \tag{A23}$$

The local thermal stability of the black hole can be analyzed by computing the specific heat at fixed angular velocity and chemical potentials,

$$C_{\Omega_+,\Phi_{\psi_I}} = \left(\frac{\partial M}{\partial T}\right)_{\omega,\Phi_{\psi_I}} = \left[\left(\frac{\partial M}{\partial r_+}\right)\left(\frac{\partial T}{\partial r_+}\right)^{-1}\right]_{\omega,\Phi_{\psi_I}}, \tag{A24}$$

which gives

$$C_{\Omega_+,\Phi_{\psi_I}} = \frac{3\pi\sigma(r_+^2+9l^2\omega_0^2)^{3/2}(2r_+^2+9l^2\omega_0^2)}{2r_+(r_+^2+18l^2\omega_0^2)}. \tag{A25}$$

The specific heat is always positive, and in consequence, the rotating black hole, as well as the static one ($\omega_0 = 0$), always attains equilibrium with a heat bath.

APPENDIX B: EQUATIONS OF MOTION FOR THE *k*-ESSENCE EXTENSION

The Einstein equations for Eq. (62) are given by

$$\kappa(G_{\mu\nu} + \Lambda g_{\mu\nu}) = \frac{1}{2}T^{\phi}_{\mu\nu} + \frac{1}{2}T^{\psi}_{\mu\nu} + \frac{1}{2}T^{em}_{\mu\nu} \tag{B1}$$

where we have defined

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 + \frac{1}{6}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2, \tag{B2}$$

$$T^{\psi}_{\mu\nu} = \sum_{I=1}^{2}\left[\partial_\mu\psi_I\partial_\nu\psi_I - \frac{1}{2}g_{\mu\nu}X_I + 2\beta\left(kX_I^{k-1}\partial_\mu\psi_I\partial_\nu\psi_I - \frac{1}{2}g_{\mu\nu}X_I^k\right)\right], \tag{B3}$$

$$T_{em} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{B4}$$

On the other hand, for our conformally coupled scalar and axion fields we have

$$\left(\Box - \frac{1}{6}R\right)\phi = 0, \tag{B5}$$

$$[(1+\beta kX_I^{k-1})g^{\mu\nu} + \beta k(k-1)X_I^{k-2}\nabla^\mu\psi_I\nabla^\nu\psi_I]\nabla_\mu\nabla_\nu\psi_I = 0. \tag{B6}$$

These equations provide our solution (63)–(65), for the $k = 2$ case.

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975); **46**, 206(E) (1976).
[2] J. L. Friedman, K. Schleich, and D. M. Witt, Phys. Rev. Lett. **71**, 1486 (1993); **75**, 1872(E) (1995).
[3] J. D. Bekenstein, arXiv:gr-qc/9808028.
[4] D. Birmingham, Classical Quantum Gravity **16**, 1197 (1999).
[5] L. Vanzo, Phys. Rev. D **56**, 6475 (1997).
[6] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002).
[7] *Black Holes in Higher Dimensions*, edited by G. T. Horowitz (Cambridge University Press, Cambridge, England, 2012).
[8] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).
[9] W. Israel, Phys. Rev. **164**, 1776 (1967); Commun. Math. Phys. **8**, 245 (1968).
[10] R. M. Wald, Phys. Rev. Lett. **26**, 1653 (1971).
[11] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).
[12] R. Ruffini and J. A. Wheeler, Phys. Today **24**, No. 1, 30 (1971).
[13] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[14] A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Lett. **20**, 878 (1968).
[15] N. M. Bocharova, K. A. Bronnikov, and V. N. Melnikov, Vestn. Mosk. Univ. Ser. III Fiz. Astron. **6**, 706 (1970).
[16] J. D. Bekenstein, Ann. Phys. (N.Y.) **82**, 535 (1974).
[17] J. D. Bekenstein, Ann. Phys. (N.Y.) **91**, 75 (1975).
[18] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112**, 221101 (2014).
[19] B. C. Xanthopoulos and T. E. Dialynas, J. Math. Phys. **33**, 1463 (1992).
[20] C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 024008 (2003).
[21] C. Martinez, J. P. Staforelli, and R. Troncoso, Phys. Rev. D **74**, 044028 (2006).
[22] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D **24**, 1542014 (2015).
[23] Y. Bardoux, M. M. Caldarelli, and C. Charmousis, J. High Energy Phys. 05 (2012) 054.
[24] Y. Bardoux, M. M. Caldarelli, and C. Charmousis, J. High Energy Phys. 09 (2012) 008.
[25] M. M. Caldarelli, C. Charmousis, and M. Hassaine, J. High Energy Phys. 10 (2013) 015.
[26] Y. Bardoux, M. M. Caldarelli, and C. Charmousis, J. High Energy Phys. 05 (2014) 039.

[27] M. M. Caldarelli, A. Christodoulou, I. Papadimitriou, and K. Skenderis, J. High Energy Phys. 04 (2017) 001.
[28] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).
[29] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[30] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[31] S. A. Hartnoll, Classical Quantum Gravity **26**, 224002 (2009).
[32] G. T. Horowitz, Lect. Notes Phys. **828**, 313 (2011).
[33] K. Goldstein, S. Kachru, S. Prakash, and S. P. Trivedi, J. High Energy Phys. 08 (**2010**) 078.
[34] M. Cadoni, G. D'Appollonio, and P. Pani, J. High Energy Phys. 03 (2010) 100.
[35] G. T. Horowitz, J. E. Santos, and D. Tong, J. High Energy Phys. 07 (2012) 168.
[36] A. Donos and S. A. Hartnoll, Nat. Phys. **9**, 649 (2013).
[37] R. A. Davison, Phys. Rev. D **88**, 086003 (2013).
[38] M. Blake and D. Tong, Phys. Rev. D **88**, 106004 (2013).
[39] A. Amoretti, A. Braggio, N. Maggiore, N. Magnoli, and D. Musso, Phys. Rev. D **91**, 025002 (2015).
[40] X. M. Kuang, E. Papantonopoulos, J. P. Wu, and Z. Zhou, Phys. Rev. D **97**, 066006 (2018).
[41] A. Donos and J. P. Gauntlett, J. High Energy Phys. 04 (2014) 040.
[42] A. Donos and J. P. Gauntlett, J. High Energy Phys. 06 (2014) 007.
[43] T. Andrade and B. Withers, J. High Energy Phys. 05 (2014) 101.
[44] R. A. Davison and B. Goutraux, J. High Energy Phys. 01 (2015) 039.
[45] B. Goutraux, J. High Energy Phys. 04 (2014) 181.
[46] M. Baggioli and M. Goykhman, J. High Energy Phys. 07 (2015) 035.
[47] L. Alberte, M. Baggioli, and O. Pujolas, J. High Energy Phys. 07 (2016) 074.
[48] M. Baggioli and O. Pujolas, Phys. Rev. Lett. **114**, 251602 (2015).
[49] X. M. Kuang and J. P. Wu, Phys. Lett. B **770**, 117 (2017).
[50] A. Cisterna, M. Hassaine, J. Oliva, and M. Rinaldi, Phys. Rev. D **96**, 124033 (2017).
[51] A. Cisterna and J. Oliva, Classical Quantum Gravity **35**, 035012 (2018).
[52] M. Henneaux, C. Martínez, R. Troncoso, and J. Zanelli, Ann. Phys. (Amsterdam) **322**, 824 (2007).
[53] T. Hertog and K. Maeda, J. High Energy Phys. 07 (2004) 051.
[54] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[55] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[56] A. Anabalón, D. Astefanesei, and C. Martínez, Phys. Rev. D **91**, 041501 (2015).
[57] M. Blake and A. Donos, Phys. Rev. Lett. **114**, 021601 (2015).
[58] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, J. High Energy Phys. 11 (2010) 151.
[59] Z. Zhou, J. P. Wu, and Y. Ling, J. High Energy Phys. 08 (2015) 067.
[60] A. Donos and J. P. Gauntlett, J. High Energy Phys. 11 (2014) 081.
[61] W. J. Jiang, H. S. Liu, H. Lu, and C. N. Pope, J. High Energy Phys. 07 (2017) 084.
[62] X. H. Ge, Y. Tian, S. Y. Wu, and S. F. Wu, Phys. Rev. D **96**, 046015 (2017).
[63] J. Oliva and S. Ray, Classical Quantum Gravity **29**, 205008 (2012).
[64] S. A. Hartnoll, D. M. Ramirez, and J. E. Santos, J. High Energy Phys. 03 (2016) 170.
[65] M. Brigante, H. Liu, R. C. Myers, S. Shenker, and S. Yaida, Phys. Rev. D **77**, 126006 (2008).
[66] X. H. Feng, H. S. Liu, H. L, and C. N. Pope, J. High Energy Phys. 11 (2015) 176.
[67] C. Erices and C. Martínez, Phys. Rev. D **92**, 044051 (2015).
[68] J. Stachel, Phys. Rev. D **26**, 1281 (1982).
[69] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
[70] C. Erices and C. Martínez, Phys. Rev. D **97**, 024034 (2018).
[71] M. Visser, Phys. Rev. D **48**, 5697 (1993).
[72] A. Ashtekar, A. Corichi, and D. Sudarsky, Classical Quantum Gravity **20**, 3413 (2003).