Curved momentum spaces from quantum (anti–)de Sitter groups in (3 + 1) dimensions

A. Ballesteros,[1,*] G. Gubitosi,[2,3,†] I. Gutierrez-Sagredo,[1,‡] and F. J. Herranz[1,§]

[1]*Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain*
[2]*Radboud University, Institute for Mathematics, Astrophysics and Particle Physics,*
Heyendaalseweg 135, NL-6525 AJ Nijmegen, The Netherlands
[3]*Dipartimento di Fisica, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*

Curved momentum spaces associated to the κ-deformation of the $(3 + 1)$ de Sitter and anti–de Sitter algebras are constructed as orbits of suitable actions of the dual Poisson-Lie group associated to the κ-deformation with nonvanishing cosmological constant. The κ-de Sitter and κ-anti–de Sitter curved momentum spaces are separately analyzed, and they turn out to be, respectively, half of the $(6 + 1)$-dimensional de Sitter space and half of a space with $SO(4, 4)$ invariance. Such spaces are made of the momenta associated to spacetime translations and the "hyperbolic" momenta associated to boost transformations. The known κ-Poincaré curved momentum space is smoothly recovered as the vanishing cosmological constant limit from both of the constructions.

I. INTRODUCTION

It is now understood that in models of Planck-scale deformed special relativity (DSR), where the Planck energy plays the role of a second relativistic invariant besides the speed of light, momentum space has a nontrivial geometry [1–5]. In particular, its curvature can be nonvanishing and governed by the Planck energy [6,7].

This realization recently allowed for a description of the phenomenological features of DSR as effects induced by the geometrical properties of momentum space, dual to the effects we are familiar with in the context of general relativity. For example, curvature of momentum space induces an energy dependence in the distance covered by a free massless particle in a given time. This can be seen as the dual to the well-known redshift of energy in curved spacetime [8]. Another effect that can be ascribed to a nontrivial geometry of momentum space is the so-called dual-gravity lensing [9], such that the direction from which a particle emitted by a given source reaches a faraway detector is energy dependent.

This kind of phenomenology raised considerable interest in the last two decades as it became clear that it could induce observable effects on particles traveling over cosmological distances [10–14]. However, the missing link to make the connection between properties of the momentum space and cosmological observations more stringent was the definition of momentum space when spacetime is itself curved. DSR models are in fact modifications of special relativity, so that they need to be generalized to a curved-spacetime setting in order to be applicable to physical frameworks where spacetime curvature cannot be neglected. Recently several proposals to extend DSR to curved spacetime scenarios have been put forward [15–20], but it was generally believed that it would not be possible to look at the geometrical properties of the momentum space on its own, because the interplay between curvature of spacetime and of momentum space would make the phase space too much intertwined [21].

In recent work we demonstrated that in fact it is possible to define momentum space in the presence of a cosmological constant [22]. The nontrivial interplay between curvature of spacetime and of momentum space is resolved by generalizing the momentum space so that it contains the hyperbolic momenta associated to boosts besides the momenta associated to spacetime translations. Our construction focused on the mathematical framework of quantum deformed algebras, which is known to be very effective in studying features of DSR models in the flat spacetime case [23–25]. Specifically, we constructed the generalized momentum

[*]angelb@ubu.es
[†]g.gubitosi@science.ru.nl
[‡]igsagredo@ubu.es
[§]fjherranz@ubu.es

space associated to the κ-deformation of the de Sitter (dS) algebras in $(1+1)$ and $(2+1)$ dimensions. We obtained, respectively, (half of) a $(2+1)$-dimensional dS manifold and (half of) a $(4+1)$-dimensional dS manifold. The coordinates of these momentum spaces are the local group coordinates associated to spacetime translations and boosts and the two spaces are the orbits of the corresponding dual Poisson-Lie group. The construction crucially relied on the observation that spacetime translations and boosts play a similar role in the structure of the algebra and coalgebra of κ-dS in both $(1+1)$ and $(2+1)$ dimensions. Additionally, in $(2+1)$ dimensions the rotation generator is not relevant to the construction of the generalized momentum space, since its role is limited to generating the isotropy subgroup of the origin of the momentum space.

While this previous work provided a crucial proof of concept, of course the physically relevant scenario is that with $(3+1)$ spacetime dimensions. This is the case we focus on in this paper, where we construct the generalized momentum space of the $(3+1)$-dimensional κ-dS algebra. Moreover, we also present the construction of the curved momentum space associated to the κ-deformation of the $(3+1)$ anti–de Sitter (AdS) algebra, which follows the same lines and—as it was conjectured in [22]—leads to a different manifold. Also the nontrivial features that arise in the $(3+1)$ construction with respect to the $(2+1)$ case for both quantum dS and AdS symmetries are emphasized.

We find it useful to begin in Sec. II with a summary of the construction of the curved momentum space associated to the κ-deformed Poincaré algebra in $(3+1)$ dimensions. While this is a repetition of known results, it allows us to present the framework that is adopted in the following within a context that should be more familiar to the reader. Concretely, we construct the dual Poisson-Lie group associated to the κ-Poincaré algebra and show that the orbits of this group define a curved momentum space manifold, which is half of a $(3+1)$-dimensional dS space.

Section III is devoted to presenting the $(3+1)$-dimensional κ-(A)dS algebra and its dual Poisson-Lie group. The two cases with positive and negative cosmological constant can be treated in a unified framework, which also admits the κ-Poincaré case as the vanishing cosmological constant limit of either of κ-(A)dS. We emphasize that the spacetime translation sector and the boosts enter in a very similar way in the structure of the deformed coalgebra. Moreover, the fact that a momentum space relying only on the spacetime translations cannot be constructed is made apparent by the fact that the coalgebra of translations does not close on its own, but it contains generators of the Lorentz sector. These two features are analogous to what we

had observed for the lower-dimensional models studied in our previous paper [the mixing between the translation sector and the Lorentz sector being somewhat less invasive in the $(1+1)$ case, since it only concerns the algebra sector]. A new feature that arises in the $(3+1)$-dimensional scenario is the fact that the rotations sector is no longer undeformed at the coalgebraic level.

In Secs. IV and V we construct the generalized momentum spaces associated to, respectively, the κ-AdS and κ-dS algebra. Again, for each algebra the momentum space is generated by the orbits of the dual Poisson-Lie group, and its coordinates are given by the local group coordinates associated to spacetime translations and boosts. However, while, as we mentioned, the two algebras and their dual Poisson-Lie groups admit a unified description, this is not the case for the two momentum spaces. This is because one needs to adapt the construction in order to deal with the imaginary quantities that arise when dealing with the two different signs of the cosmological constant. As a consequence, the generalized momentum spaces that we find have different geometrical properties. The κ-AdS algebra has a momentum space that is half of a $SO(4,4)$ quadric. The κ-dS algebra has a momentum space that is half of a $(6+1)$-dimensional dS manifold. Similarly to what we had found in the lower-dimensional models, rotations do not enter in the construction of the momentum space, and only generate the isotropy subgroup of the origin.

The concluding Sec. VI summarizes our findings and additional insights on the nontrivial properties of rotations that emerge in $(3+1)$ dimensions are included in an appendix.

II. THE κ-POINCARÉ MOMENTUM SPACE

The aim of this section is to summarize the construction of the curved momentum space corresponding to the $(3+1)$ κ-Poincaré quantum algebra [26–29]. We follow the presentation of [7], and make use of the group-theoretical framework recently introduced in [22]. We show that the κ-Poincaré curved momentum space can be obtained as a specific orbit of the action of the dual κ-Poisson-Lie group on a $(4+1)$-dimensional ambient Minkowski space, and it turns out to be (half of) the $(3+1)$ dS space. The rest of this paper is devoted to the generalization of this construction to the κ-deformed spacetime symmetries with nonvanishing cosmological constant.

The starting point for the construction is the Poisson version of the $(3+1)$ κ-Poincaré algebra in the so-called bicrossproduct basis [30]. This is given by the Poisson brackets (the notation $\mathbf{P}^2 = P_1^2 + P_2^2 + P_3^2$ is used hereafter for vectors),

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \qquad \{J_a, P_b\} = \epsilon_{abc} P_c,$$
$$\{J_a, K_b\} = \epsilon_{abc} K_c, \qquad \{K_a, P_0\} = P_a,$$
$$\{K_a, K_b\} = -\epsilon_{abc} J_c, \qquad \{P_0, J_a\} = 0,$$
$$\{P_0, P_a\} = 0, \qquad \{P_a, P_b\} = 0,$$
$$\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}(1 - e^{-2zP_0}) + \frac{z}{2}\mathbf{P}^2\right) - zP_aP_b, \quad (1)$$

together with the compatible (as a Poisson algebra homomorphism) coproduct map,

$$\Delta_z(P_0) = P_0 \otimes 1 + 1 \otimes P_0,$$
$$\Delta_z(P_a) = P_a \otimes 1 + e^{-zP_0} \otimes P_a,$$
$$\Delta_z(J_a) = J_a \otimes 1 + 1 \otimes J_a,$$
$$\Delta_z(K_a) = K_a \otimes 1 + e^{-zP_0} \otimes K_a + z\epsilon_{abc}P_b \otimes J_c. \quad (2)$$

We define $z = 1/\kappa$ as the (Planck-scale) quantum deformation parameter, with $a, b, c = 1, 2, 3$ (sum over repeated indices is assumed), and ϵ_{abc} is the totally antisymmetric tensor, with $\epsilon_{123} = 1$. As usual, the generators P_a, J_a, and K_a denote respectively translations, rotations, and boosts.

This quantum deformation of the Poincaré algebra induces a deformed Casimir function $\mathcal{C}_z$ for the Poisson algebra (1), given by

$$\mathcal{C}_z = \frac{4}{z^2}\sinh^2(zP_0/2) - e^{zP_0}\mathbf{P}^2$$
$$= \frac{2}{z^2}[\cosh(zP_0) - 1] - e^{zP_0}\mathbf{P}^2. \quad (3)$$

This deformed Casimir constitutes the keystone for the interpretation of the κ-Poincaré algebra (in bicrossproduct basis) as the modified kinematical symmetry underlying deformed dispersion relations arising in the quantum gravity context [2,7].[1] It is also worth recalling that a second invariant $\mathcal{W}_z$ does exist for the Poisson structure (1), which is just a deformed analogue of the square of the modulus of the Pauli-Lubanski four-vector,

$$\mathcal{W}_z = \left(\cosh(zP_0) - \frac{z^2}{4}e^{zP_0}\mathbf{P}^2\right)W_{z,0}^2 - \mathbf{W}_z^2,$$

where the deformed components are

$$W_{z,0} = e^{\frac{z}{2}P_0}\mathbf{J} \cdot \mathbf{P},$$
$$W_{z,a} = -J_a\frac{\sinh(zP_0)}{z} + e^{zP_0}\epsilon_{abc}\left(K_b + \frac{z}{2}\epsilon_{bkl}J_kP_l\right)P_c.$$

---

[1] A different choice of basis for the κ-Poincaré algebra would result in a different Casimir and thus characterize a different dispersion relation and associated kinematical symmetries.

We stress that all the expressions included in this paper are analytic in the deformation parameter z, and the non-deformed limit $z \to 0$ always gives rise to the usual $(3 + 1)$ relativistic symmetries.

When dealing with Hopf algebra kinematical symmetries, the coproduct can be interpreted as the composition law for observables. In particular, the coproduct (2) is such that the κ-deformation induces a nonlinear composition rule for momenta in interaction vertices. As we are going to show, it is because of this deformed composition rule that curvature in the κ-Poincaré momentum space emerges. In more technical terms, the curvature of the momentum space arises as a consequence of the non-cocommutativity of the coproduct map for the translation generators, and such a curved momentum space can be explicitly constructed as follows.

First, the noncocommutativity of the κ-translations can be characterized by writing the skew-symmetric part of the first-order deformation (in z) of the coproduct. Namely, if we write the coproduct (2) for the translation generators as a power series expansion in terms of the deformation parameter z we obtain

$$\Delta_z(P_0) = P_0 \otimes 1 + 1 \otimes P_0,$$
$$\Delta_z(P_a) = P_a \otimes 1 + 1 \otimes P_a - zP_0 \otimes P_a + o[z^2],$$

and the skew symmetrization of the first-order term in z of the previous expressions gives rise to the map

$$\delta(P_0) = 0,$$
$$\delta(P_1) = z(P_1 \wedge P_0),$$
$$\delta(P_2) = z(P_2 \wedge P_0),$$
$$\delta(P_3) = z(P_3 \wedge P_0),$$

which is called the cocommutator map and it endows the Poincaré algebra g with a Lie bialgebra structure δ: $g \to g \otimes g$. Moreover, δ characterizes the Hopf algebra deformation through the first-order information it encodes (see [31–33] and references therein for details).

Secondly, the dual δ^*: $g^* \otimes g^* \to g^*$ of the cocommutator map defines the Lie algebra g^* of the so-called dual Poisson-Lie group G^*. In the κ-Poincaré case, if we denote by $\{X^0, X^1, X^2, X^3\}$ the generators in g^* dual to, respectively, $\{P_0, P_1, P_2, P_3\}$, their dual Lie brackets are given by

$$[X^0, X^i] = -zX^i, \qquad [X^i, X^j] = 0, \qquad i, j = 1, 2, 3. \quad (4)$$

Note that when the deformation parameter vanishes, $z \to 0$, all the coproducts are cocommutative $(\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y)$. As a consequence, δ vanishes and the dual Lie algebra (and group) is Abelian. It is also worth recalling that the dual Lie algebra of the translations sector given by (4) is just the so-called κ-Minkowski spacetime [34–37].

A. Dual Poisson-Lie group and curved momentum space

The dual Poisson-Lie group G^* can be explicitly constructed starting from the five-dimensional faithful representation ρ of the dual Lie algebra g^*, given by

$$\rho(X^0) = z \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho(X^1) = z \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(X^2) = z \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \rho(X^3) = z \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5}$$

If $\{p_0, p_1, p_2, p_3\}$ are the local dual group coordinates associated, respectively, to $\{X^0, X^1, X^2, X^3\}$, then the dual Lie group element can be constructed through the exponentiation,

$$G^*(p_0, p_1, p_2, p_3) = \exp(p_1 \rho(X^1)) \exp(p_2 \rho(X^2))$$
$$\times \exp(p_3 \rho(X^3)) \exp(p_0 \rho(X^0)),$$

which explicitly reads

$$G^*(p) = \begin{pmatrix} \cosh(zp_0) + \frac{1}{2} e^{zp_0} z^2 \bar{p}^2 & zp_1 & zp_2 & zp_3 & \sinh(zp_0) + \frac{1}{2} e^{zp_0} z^2 \bar{p}^2 \\ e^{zp_0} zp_1 & 1 & 0 & 0 & e^{zp_0} zp_1 \\ e^{zp_0} zp_2 & 0 & 1 & 0 & e^{zp_0} zp_2 \\ e^{zp_0} zp_3 & 0 & 0 & 1 & e^{zp_0} zp_3 \\ \sinh(zp_0) - \frac{1}{2} e^{zp_0} z^2 \bar{p}^2 & -zp_1 & -zp_2 & -zp_3 & \cosh(zp_0) - \frac{1}{2} e^{zp_0} z^2 \bar{p}^2 \end{pmatrix}, \tag{6}$$

where we have defined $\bar{p}^2 = p_1^2 + p_2^2 + p_3^2$.

The significance of the dual Poisson-Lie group relies on the fact that the coproduct (2) is just the group law for G^* (see [33] for details). In fact, if we multiply two matrices of the type (6) we get another group element

$$G^*(p'') = G^*(p) \cdot G^*(p').$$

It can be straightforwardly checked that the group law $p'' = f(p, p')$ reads

$$p_0'' = p_0 + p_0', \qquad p_i'' = p_i + e^{-zp_0} p_i', \tag{7}$$

which is consistent with (4) in the sense that X^0 generates a dilation and the X^i generators correspond to (dual) translations.

Now, by making use of the Poisson version of the quantum duality principle [38–40], the group multiplication law (7) can be immediately rewritten in algebraic terms as a comultiplication map Δ_z through the identification of the two copies of the dual group coordinates as

$$p \equiv p \otimes 1, \qquad p' \equiv 1 \otimes p. \tag{8}$$

In this algebraic language, the multiplication law for the group G^* can be written as a coproduct in the form

$$\Delta_z(p_0) \equiv p_0'' = p_0 \otimes 1 + 1 \otimes p_0,$$
$$\Delta_z(p_i) \equiv p_i'' = p_i \otimes 1 + e^{-zp_0} \otimes p_i,$$
$$i = 1, 2, 3. \tag{9}$$

This coproduct is just the one for the translation sector of the κ-Poincaré algebra in bicrossproduct basis once the following identification between the dual group coordinates and the generators of the κ-Poincaré algebra is performed:

$$p_0 \equiv P_0, \qquad p_1 \equiv P_1, \qquad p_2 \equiv P_2, \qquad p_3 \equiv P_3.$$

Note that if a different ordering is chosen for the exponentials, one would recover the coproducts corresponding to a different basis of the κ-Poincaré algebra. Moreover, the

unique Poisson-Lie structure on G^* that is compatible with the coproduct (9) and has the undeformed Poincaré Lie algebra as its linearization is given by the κ-Poincaré Poisson brackets for the translation sector.

Under this approach, the κ-Poincaré momentum space admits a straightforward geometric interpretation [7]. The entries of the fifth column in G^* can be rewritten as the following S_i functions:

$$S_0 = \sinh(zp_0) + \frac{1}{2}e^{zp_0}z^2\bar{p}^2,$$
$$S_1 = ze^{zp_0}p_1,$$
$$S_2 = ze^{zp_0}p_2,$$
$$S_3 = ze^{zp_0}p_3,$$
$$S_4 = \cosh(zp_0) - \frac{1}{2}e^{zp_0}z^2\bar{p}^2. \qquad (10)$$

Surprisingly enough, these satisfy the defining relation of the $(3+1)$-dimensional dS space,

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 = 1.$$

This means that the κ-Poincaré momentum space parametrized by the ambient coordinates $(S_0, S_1, S_2, S_3, S_4)$ can be obtained as the orbit arising from a linear action of the Lie group matrix $G^*(p)$ onto a $(4+1)$-dimensional ambient Minkowski space and passing through the point $(0,0,0,0,1)$. Namely,

$$G^* \cdot (0,0,0,0,1)^T = (S_0, S_1, S_2, S_3, S_4)^T. \qquad (11)$$

Moreover, the ambient coordinates fulfil the condition

$$S_0 + S_4 = e^{zp_0} > 0,$$

which means that only half of the $(3+1)$-dimensional dS space is generated through the action (11). We denote this manifold as M_{dS_4}. Note that in the limit $z \to 0$ the dual Lie group G^* generated by (4) is Abelian.

III. THE κ-(A)dS ALGEBRA AND ITS DUAL POISSON-LIE GROUP

The aim of this paper is the generalization of the previous construction to the case with nonvanishing cosmological constant, by following the approach presented in [22] for the $(1+1)$- and $(2+1)$-dimensional dS cases. In this way, a global picture of the interplay between the cosmological constant and the Planck-scale deformation parameter $z = 1/\kappa$ can be presented. In our approach the cosmological constant $\Lambda = -\omega$ is included as an explicit deformation parameter and all the expressions are analytic both in terms of Λ and z. The construction of the $(3+1)$ κ-(A)dS algebra was first exposed in [41]. It turns out to be much more complicated than its κ-Poincaré limit, which is recovered when $\Lambda \to 0$.

Before getting into the κ-deformed case, let us briefly revisit the so-called AdS_ω algebra as a unified way to

describe the AdS, dS and Poincaré symmetries (see [41]). In the kinematical basis the AdS_ω Lie algebra $\{P_0, P_a, K_a, J_a\}$ is defined by the brackets,

$$[J_a, J_b] = \epsilon_{abc}J_c, \quad [J_a, P_b] = \epsilon_{abc}P_c, \quad [J_a, K_b] = \epsilon_{abc}K_c,$$
$$[K_a, P_0] = P_a, \quad [K_a, P_b] = \delta_{ab}P_0, \quad [K_a, K_b] = -\epsilon_{abc}J_c,$$
$$[P_0, P_a] = \omega K_a, \quad [P_a, P_b] = -\omega\epsilon_{abc}J_c, \quad [P_0, J_a] = 0.$$
$$(12)$$

As we mentioned, the parameter ω is related to the cosmological constant via $\omega = -\Lambda$. Therefore, the one-parametric AdS_ω algebra reduces to the AdS Lie algebra $\mathfrak{so}(3,2)$ when $\omega > 0$, to the dS Lie algebra $\mathfrak{so}(4,1)$ when $\omega < 0$, and to the Poincaré Lie algebra $\mathfrak{iso}(3,1)$ when $\omega = 0$.

The algebra (12) has two Casimir invariants [42]. The first one is quadratic and comes from the Killing-Cartan form,

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 + \omega(\mathbf{J}^2 - \mathbf{K}^2).$$

The second one is a fourth-order invariant,

$$\mathcal{W} = W_0^2 - \mathbf{W}^2 + \omega(\mathbf{J} \cdot \mathbf{K})^2,$$

where $W_0 = \mathbf{J} \cdot \mathbf{P}$ and $W_a = -J_aP_0 + \epsilon_{abc}K_bP_c$ are the components of the (A)dS analogue of the Pauli-Lubanski four-vector. We recall that in the Poincaré case $\omega = 0$ the invariant $W_0^2 - \mathbf{W}^2$ provides the square of the spin/helicity operator, which in the rest frame is proportional to the square of the angular momentum operator. It is worth emphasizing that the presence of a nonvanishing ω implies that the quadratic invariant (the one linked to dispersion relations) has a new contribution coming from the Lorentz sector of the AdS_ω algebra.

By making use of this unified description, the three $(3+1)$ Lorentzian symmetric homogeneous spaces with constant sectional curvature ω are defined as the coset spaces $\mathbf{AdS}_\omega^{3+1} \equiv \mathrm{SO}_\omega(3,2)/\mathrm{SO}(3,1)$, namely, the following:

  (i) When $\omega > 0$ ($\Lambda < 0$) we have the AdS spacetime $\mathbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.

  (ii) When $\omega < 0$ ($\Lambda > 0$) we get the dS spacetime $\mathbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$.

  (iii) The Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$ arises when $\omega = \Lambda = 0$.

Explicit ambient space coordinates for these three maximally symmetric Lorentzian spacetimes can be obtained by making use of a suitable realization of the Lie groups obtained by exponentiation of a faithful representation of the AdS_ω Lie algebra [see [43–45] for details in the $(2+1)$-dimensional case].

In the following sections we summarize the main aspects of the κ-deformation of the AdS_ω Poisson algebra by making use of this unified description, and then we construct its associated dual Poisson-Lie group. The construction and analysis of the associated curved momentum spaces are performed separately for the AdS and dS cases, since their geometric properties turn out to be different.

A. The κ-deformation of the $(3+1)$ AdS$_\omega$ algebra

The generalization of the $(3+1)$ Poisson κ-Poincaré algebra to the nonvanishing ω case has been recently presented explicitly in [41]. In particular, the deformed coproduct for the AdS$_\omega$ algebra reads

$$\Delta_z(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \qquad \Delta_z(J_1) = J_1 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_1, \qquad \Delta_z(J_2) = J_2 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_2, \quad (13)$$

$$\Delta_z(P_0) = P_0 \otimes 1 + 1 \otimes P_0,$$
$$\Delta_z(P_1) = P_1 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes P_1 - \sqrt{\omega}K_2 \otimes \sinh(z\sqrt{\omega}J_3)$$
$$\qquad - z\sqrt{\omega}P_3 \otimes J_1 + z\omega K_3 \otimes J_2 + z^2\omega(\sqrt{\omega}K_1 - P_2) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3}$$
$$\qquad - \frac{1}{2}z^2\omega(\sqrt{\omega}K_2 + P_1) \otimes (J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3},$$
$$\Delta_z(P_2) = P_2 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes P_2 + \sqrt{\omega}K_1 \otimes \sinh(z\sqrt{\omega}J_3)$$
$$\qquad - z\sqrt{\omega}P_3 \otimes J_2 - z\omega K_3 \otimes J_1 - z^2\omega(\sqrt{\omega}K_2 + P_1) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3}$$
$$\qquad - \frac{1}{2}z^2\omega(\sqrt{\omega}K_1 - P_2) \otimes (J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3},$$
$$\Delta_z(P_3) = P_3 \otimes 1 + e^{-zP_0} \otimes P_3 + z(\omega K_2 + \sqrt{\omega}P_1) \otimes J_1 e^{-z\sqrt{\omega}J_3} - z(\omega K_1 - \sqrt{\omega}P_2) \otimes J_2 e^{-z\sqrt{\omega}J_3}, \quad (14)$$

$$\Delta_z(K_1) = K_1 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes K_1 + P_2 \otimes \frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}}$$
$$\qquad - zP_3 \otimes J_2 - z\sqrt{\omega}K_3 \otimes J_1 - z^2(\omega K_2 + \sqrt{\omega}P_1) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3}$$
$$\qquad - \frac{1}{2}z^2(\omega K_1 - \sqrt{\omega}P_2) \otimes (J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3},$$
$$\Delta_z(K_2) = K_2 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes K_2 - P_1 \otimes \frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}}$$
$$\qquad + zP_3 \otimes J_1 - z\sqrt{\omega}K_3 \otimes J_2 - z^2(\omega K_1 - \sqrt{\omega}P_2) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3}$$
$$\qquad + \frac{1}{2}z^2(\omega K_2 + \sqrt{\omega}P_1) \otimes (J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3},$$
$$\Delta_z(K_3) = K_3 \otimes 1 + e^{-zP_0} \otimes K_3 + z(\sqrt{\omega}K_1 - P_2) \otimes J_1 e^{-z\sqrt{\omega}J_3} + z(\sqrt{\omega}K_2 + P_1) \otimes J_2 e^{-z\sqrt{\omega}J_3}. \quad (15)$$

Notice that this coproduct is written in a "bicrossproduct-type" basis that generalizes the one corresponding to the $(2+1)$ κ-AdS$_\omega$ algebra [44,45].

As it can be easily checked, the κ-Poincaré coproduct (2) is obtained from the above expressions in the limit $\omega \to 0$. A direct comparison between both sets of expressions makes it evident that the degree of complexity of the κ-deformation is greatly increased when the cosmological constant ω is turned on. In fact, the κ-AdS$_\omega$ algebra can be thought of as a two-parametric deformation, which is ruled by a quantum deformation parameter $z = 1/\kappa$ (the Planck scale) and a "classical" deformation parameter $\omega = -\Lambda$ (the cosmological constant), which has a well-defined geometrical meaning. As we show in the following, the roles of the two deformation parameters are interchanged when the dual Poisson-Lie group is considered, in the spirit of the semidualization approach to $(2+1)$ quantum gravity [46,47].

There are several differences between the coproducts (13)–(15) and (2) that have to be emphasized. First, $\Delta_z(K)$ and $\Delta_z(P)$ are structurally similar when $\omega \neq 0$, in contrast with (2). Second, translations in (14) do not close a Hopf subalgebra, since when $\omega \neq 0$ the coproducts $\Delta_z(P)$ contain boosts and rotations as well. Finally, in the non-vanishing cosmological constant case the rotation sector (13) is deformed, while in (2) all of the coproducts for J_a are primitive. In particular, the rotation generators have no longer all the same role in the coalgebra, signaling a departure from standard isotropy. Further comments on this are found below, as well as in the concluding section.[2] These three features are

---

[2] Some technical comments on the deformation (13) of the rotation subalgebra in the $(3+1)$-dimensional κ-AdS$_\omega$ algebra are presented in the appendix. In fact, this last feature is quite different from other kinds of deformations that have been studied in the context of Planck-scale deformed symmetries and—to the best of our knowledge—has not been appropriately emphasized in the literature.

induced by the interplay between the cosmological constant and the quantum deformation, and the first two are essential for the construction of the curved momentum space when $\omega \neq 0$.

We do not reproduce here the deformed brackets for the Poisson version of the κ-AdS$_\omega$ algebra that can be explicitly found in [41], since they are quite involved and strongly present nonlinear expressions when compared to the AdS$_\omega$ Lie algebra (12). It is worth stressing that, in contradistinction to (1), the full Lorentz sector has now deformed Poisson brackets like, for instance,

$$\{J_1, J_2\} = \frac{e^{2z\sqrt{\omega}J_3} - 1}{2z\sqrt{\omega}} - \frac{z\sqrt{\omega}}{2}(J_1^2 + J_2^2),$$

$$\{J_1, K_2\} = K_3 - z\sqrt{\omega}J_1 K_1,$$

$$\{K_2, K_3\} = -\frac{1}{2}J_1(1 + e^{-2z\sqrt{\omega}J_3}[1 + z^2\omega(J_1^2 + J_2^2)])$$
$$- z\sqrt{\omega}K_1 K_3,$$

which show evident differences with respect to its $\omega \to 0$ limit (1), and where the role of the J_3 generator is neatly distinguished with respect to the one played by J_1 and J_2.

The deformed quadratic Casimir for the κ-AdS$_\omega$ algebra reads

$$\mathcal{C}_z = \frac{2}{z^2}[\cosh(zP_0)\cosh(z\sqrt{\omega}J_3) - 1] + \omega\cosh(zP_0)(J_1^2 + J_2^2)e^{-z\sqrt{\omega}J_3}$$

$$- e^{zP_0}(\mathbf{P}^2 + \omega\mathbf{K}^2)\left[\cosh(z\sqrt{\omega}J_3) + \frac{z^2\omega}{2}(J_1^2 + J_2^2)e^{-z\sqrt{\omega}J_3}\right]$$

$$+ 2\omega e^{zP_0}\left[\frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}}T_3 + z\left(J_1 T_1 + J_2 T_2 + \frac{z\sqrt{\omega}}{2}(J_1^2 + J_2^2)T_3\right)e^{-z\sqrt{\omega}J_3}\right], \tag{16}$$

where $T_a = \epsilon_{abc}K_b P_c$. Again, the $\omega \to 0$ limit is just (3), and comparing (16) to its flat limit (3) gives a clear idea of the kind of deformation we are dealing with. This deformed invariant (16) is relevant in what follows, since it is connected to the deformed dispersion relation that can be deduced from the curved momentum space with cosmological constant that we are going to construct.

B. The dual Poisson-Lie group

Mimicking the procedure used in the previous section, the first step for the construction of the curved momentum spaces of the κ-(A)dS algebras is to obtain the cocommutator map δ associated to the κ-deformed coproduct map with nonvanishing cosmological constant. This can be found by extracting the first-order deformation in z of the coproduct (13)–(15), which has a skew-symmetric part given by [41]

$$\delta(P_0) = 0,$$

$$\delta(J_3) = 0,$$

$$\delta(J_1) = z\sqrt{\omega}J_1 \wedge J_3,$$

$$\delta(J_2) = z\sqrt{\omega}J_2 \wedge J_3,$$

$$\delta(P_1) = z(P_1 \wedge P_0 - \omega J_2 \wedge K_3 + \omega J_3 \wedge K_2 + \sqrt{\omega}J_1 \wedge P_3),$$

$$\delta(P_2) = z(P_2 \wedge P_0 - \omega J_3 \wedge K_1 + \omega J_1 \wedge K_3 + \sqrt{\omega}J_2 \wedge P_3),$$

$$\delta(P_3) = z(P_3 \wedge P_0 - \omega J_1 \wedge K_2 + \omega J_2 \wedge K_1 - \sqrt{\omega}J_1 \wedge P_1 - \sqrt{\omega}J_2 \wedge P_2),$$

$$\delta(K_1) = z(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2 + \sqrt{\omega}J_1 \wedge K_3),$$

$$\delta(K_2) = z(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3 + \sqrt{\omega}J_2 \wedge K_3),$$

$$\delta(K_3) = z(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1 - \sqrt{\omega}J_1 \wedge K_1 - \sqrt{\omega}J_2 \wedge K_2). \tag{17}$$

The differences between the (A)dS and Poicaré deformations that were mentioned in the previous subsection leave their traces in the δ map. In particular, we stress two main features of the cocommutator (17): First, that $\delta(P)$ does not close a sub-Lie bialgebra since it includes the full Lorentz sector in the definition of the cocommutator; secondly, that $\delta(P)$ and

$\delta(K)$ are structurally similar when $\omega \neq 0$. The latter statement is just the footprint of a general property of classical dS and AdS symmetries that disappears in the Poincaré limit, since the AdS$_\omega$ Lie algebra (12) with $\omega \neq 0$ can be straightforwardly endowed with the following automorphism that interchanges the P_a and K_a generators:

$$\tilde{P}_0 = P_0, \quad \tilde{P}_a = \sqrt{\omega}K_a, \quad \tilde{K}_a = -\frac{1}{\sqrt{\omega}}P_a, \quad \tilde{J}_a = J_a. \quad (18)$$

It can directly be checked that the transformed generators close the commutation rules (12), and that the cocommutator (17), coproduct (13)–(15), and deformed Poisson brackets given in [41] remain in the same form, provided that the deformation parameter z is unchanged. Therefore, when the cosmological constant does not vanish, translations and boosts have similar algebraic properties and play complementary geometric roles (see [45] for a more detailed discussion in the context of homogeneous spaces of worldlines).

As a consequence, the main idea introduced in [22] for the construction of curved momentum spaces for the κ-(A)dS algebras in $(2+1)$ dimensions becomes fully applicable: when $\omega \neq 0$ the momentum space has to be enlarged by including the angular momenta associated to the rotation symmetries and the hyperbolic momenta associated to boosts. This means that the momentum space arises as the orbit of an appropriate action of the dual Poisson-Lie group G_ω^*, whose Lie algebra g_ω^* is obtained by dualizing δ. Namely,

$$[R^1,R^2]=0, \quad [R^1,R^3]=z\sqrt{\omega}R^1, \quad [R^2,R^3]=z\sqrt{\omega}R^2,$$
$$[R^1,X^1]=-z\sqrt{\omega}X^3, \quad [R^1,X^2]=zL^3,$$
$$[R^1,X^3]=-z(L^2-\sqrt{\omega}X^1), \quad [R^2,X^1]=-zL^3,$$
$$[R^2,X^2]=-z\sqrt{\omega}X^3, \quad [R^2,X^3]=z(L^1+\sqrt{\omega}X^2),$$
$$[R^3,X^1]=zL^2, \quad [R^3,X^2]=-zL^1, \quad [R^3,X^3]=0,$$
$$[R^1,L^1]=-z\sqrt{\omega}L^3, \quad [R^1,L^2]=-z\omega X^3,$$
$$[R^1,L^3]=z(\sqrt{\omega}L^1+\omega X^2), \quad [R^2,L^1]=z\omega X^3,$$
$$[R^2,L^2]=-z\sqrt{\omega}L^3, \quad [R^2,L^3]=z(\sqrt{\omega}L^2-\omega X^1),$$
$$[R^3,L^1]=-z\omega X^2, \quad [R^3,L^2]=z\omega X^1, \quad [R^3,L^3]=0,$$
$$[L^a,X^0]=zL^a, \quad [L^a,L^b]=0, \quad [L^a,X^b]=0,$$
$$[X^a,X^0]=zX^a, \quad [X^a,X^b]=0, \quad [X^0,R^a]=0, \quad (19)$$

where $\{X^0,X^1,X^2,X^3,L^1,L^2,L^3,R^1,R^2,R^3\}$ are dual to $\{P_0,P_1,P_2,P_3,K_1,K_2,K_3,J_1,J_2,J_3\}$, respectively. Notice that (18) induces, through duality and provided that $\omega \neq 0$, the

following automorphism for the generators of the dual Lie algebra g_ω^*,

$$\tilde{X}^0=X^0, \quad \tilde{X}^a=\frac{1}{\sqrt{\omega}}L^a, \quad \tilde{L}^a=-\sqrt{\omega}X^a, \quad \tilde{R}^a=R^a,$$

which leaves the commutation relations (19) invariant and shows that the X^a and L^a generators can be also interchanged at the dual Lie algebra level.

Next from the expressions (19) we deduce that in g_ω^* there exists a seven-dimensional solvable Lie subalgebra generated by

$$[X^0,X^i]=-zX^i, \quad [X^0,L^i]=-zL^i, \quad [X^i,L^j]=0,$$
$$[X^i,X^j]=0, \quad [L^i,L^j]=0, \quad i=1,2,3. \quad (20)$$

This subalgebra is ω independent, and the dual of the rotation sector generates a three-dimensional solvable subalgebra,

$$[R^1,R^2]=0, \quad [R^1,R^3]=z\sqrt{\omega}R^1, \quad [R^2,R^3]=z\sqrt{\omega}R^2.$$

In the limit $\omega \to 0$ this turns out to be Abelian, which is the dual counterpart of the fact that $\Delta_z(J_a)=\Delta_0(J_a)$ when the quantum deformation disappears.

We stress that the first-order noncommutative κ-AdS$_\omega$ spacetime is given by the dual of the translations sector, namely,

$$[X^0,X^i]=-zX^i, \quad [X^i,X^j]=0, \quad i=1,2,3.$$

This is indeed ω independent but, as it was shown in [44,48], when the all-orders quantum group is computed, the quantum spacetime with nonvanishing ω is a nonlinear algebra whose higher-order contributions explicitly depend on the cosmological constant.

IV. THE κ-AdS CURVED MOMENTUM SPACE

In this section and the following we separately analyze the κ-AdS and κ-dS dual Poisson-Lie groups and construct the associated momentum spaces, since their geometric properties are different. The matrix representation for the Lie algebra (19) when $\omega > 0$ can be found to be

$$D(X^0)=z\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D(X^1)=z\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D(X^2) = z \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D(X^3) = z \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D(L^1) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D(L^2) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$D(L^3) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \qquad D(R^1) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D(R^2) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D(R^3) = z\sqrt{\omega} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This faithful representation has been found by imposing that its $\omega \to 0$ limit should lead to a reducible representation in which the matrices (5) can be obtained by suppressing appropriate rows and columns. After that, the rest of entries were found through direct computation by imposing the commutation rules (19) to hold.

If we denote as $\{p_0, p_1, p_2, p_3, \chi_1, \chi_2, \chi_3, \theta_1, \theta_2, \theta_3\}$ the local dual group coordinates that correspond, respectively, to $\{X^0, X^1, X^2, X^3, L^1, L^2, L^3, R^1, R^2, R^3\}$, a representation of the Lie group G_ω^* can be explicitly obtained as

$$G_\omega^*(\theta, p, \chi) = e^{\theta_3 D(R^3)} e^{\theta_2 D(R^2)} e^{\theta_1 D(R^1)} e^{p_1 D(X^1)} e^{p_2 D(X^2)} e^{p_3 D(X^3)} e^{\chi_1 D(L^1)} e^{\chi_2 D(L^2)} e^{\chi_3 D(L^3)} e^{p_0 D(X^0)}. \tag{21}$$

Moreover, a long but straightforward computation shows that if we multiply two G_ω^* elements,

$$G^*_\omega(\theta'', p'', \chi'') = G^*_\omega(\theta, p, \chi) \cdot G^*_\omega(\theta', p', \chi'),$$

the group law

$$\theta'' = f(\theta, \theta', p, p', \chi, \chi'),$$
$$p'' = g(\theta, \theta', p, p', \chi, \chi'),$$
$$\chi'' = h(\theta, \theta', p, p', \chi, \chi'),$$

can be explicitly obtained and it can be exactly written as the coproduct (13)–(15) for $\omega > 0$, provided the identification

$$\theta_a \equiv J_a, \qquad p_0 \equiv P_0, \qquad p_a \equiv P_a, \qquad \chi_a \equiv K_a \quad (22)$$

is assumed and by following the convention (8). Similarly to what we observed for the $\omega = 0$ case, the fact that we are able to recover the coproducts of the algebra in the bicrosscroduct basis is due to the specific choice of ordering of the exponential in (21). A different ordering choice would reflect into a group law compatible with the coproduct of the algebra in a different basis. We recall that the ordering (21) was just the one introduced in [41] in order to obtain the κ-deformation of the $(3+1)$ AdS algebra presented in Sec. III, and this choice guarantees the self-consistency of all the new results here presented.

Now, the κ-AdS momentum space can be constructed by considering the left action of the group element $G^*_\omega(\theta, p, \chi)$ on an eight-dimensional ambient space. The points that can be reached from the origin $\mathcal{O} \equiv (0,0,0,0,0,0,0,1)$ under such action are those with coordinates $(S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7)$ given by

$$G^*_\omega \cdot (0,0,0,0,0,0,0,1)^T = (S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7)^T.$$

These can explicitly be written as

$$S_0 = \sinh(zp_0) + \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2),$$
$$S_1 = A(p_1 B_{21}^+ + \sqrt{\omega}(C + \chi_2 B_{21}^-)),$$
$$S_2 = A(p_2 B_{12}^+ + \sqrt{\omega}(D - \chi_1 B_{12}^-)),$$
$$S_3 = ze^{zp_0}(p_3 - z\sqrt{\omega}(\theta_1 p_1 + \theta_2 p_2 + \sqrt{\omega}(\theta_1\chi_2 - \theta_2\chi_1))),$$
$$S_4 = A(-p_2 B_{21}^- + \sqrt{\omega}(D + \chi_1 B_{21}^+)),$$
$$S_5 = A(-p_1 B_{12}^- + \sqrt{\omega}(C - \chi_2 B_{12}^+)),$$
$$S_6 = -z\sqrt{\omega}e^{zp_0}(\chi_3 - z(\theta_2 p_1 - \theta_1 p_2 + \sqrt{\omega}(\theta_1\chi_1 + \theta_2\chi_2))),$$
$$S_7 = \cosh(zp_0) - \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2), \quad (23)$$

where we have defined

$$\bar{p}^2 = p_1^2 + p_2^2 + p_3^2,$$
$$\bar{\chi}^2 = \chi_1^2 + \chi_2^2 + \chi_3^2,$$
$$A = \frac{1}{2}ze^{z(p_0 - \theta_3\sqrt{\omega})},$$
$$B_{ij}^\pm = \omega z^2(\theta_i^2 - \theta_j^2) + e^{2z\sqrt{\omega}\theta_3} \pm 1, i \in \{1,2\},$$
$$C = 2z(\theta_2\sqrt{\omega}(z\theta_1(-p_2 + \sqrt{\omega}\chi_1) - \chi_3) + \theta_1 p_3),$$
$$D = 2z(\theta_1\sqrt{\omega}(z\theta_2(-p_1 - \sqrt{\omega}\chi_2) + \chi_3) + \theta_2 p_3). \quad (24)$$

Note that, when evaluated at $(\theta_1, \theta_2, \theta_3) = (0,0,0)$, the last four functions give $A \to \frac{1}{2}ze^{zp_0}$, $B_{ij}^+ \to 2$, $B_{ij}^- \to 0$, $C \to 0$, and $D \to 0$.

We stress that the $\omega \to 0$ limit of these expressions makes S_4, S_5, and S_6 vanish, and for the remaining ambient coordinates we get exactly the κ-Poincaré curved momentum space (10). In other words, this means that for $\omega = 0$ the matrix (21) is a reducible representation of the dual κ-Poincaré group, which is consistent with the fact that the ambient space has been enlarged when the cosmological constant has been introduced.

From (23) we can deduce the geometrical properties of the κ-AdS momentum space. In fact, it is straightforward to check that the following relations hold:

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 - S_4^2 - S_5^2 - S_6^2 + S_7^2 = 1,$$
$$S_0 + S_7 = e^{zp_0} > 0. \quad (25)$$

This means that, if we consider an $\mathbb{R}^{4,4}$ ambient space, the κ-AdS momentum space is (half of) a SO(4,4) quadric. From the expressions (23) it is also straightforward to check that the subgroup of dual rotations,

$$G_0 = e^{\theta_3 D(R^3)}e^{\theta_2 D(R^2)}e^{\theta_1 D(R^1)}, \quad (26)$$

leaves the point $\mathcal{O}$ invariant. The action of the remaining seven-parameter subgroup [generated by the Lie subalgebra (20)] is obtained by evaluating (23) at $(\theta_1, \theta_2, \theta_3) = (0,0,0)$,

$$S_0 = \sinh(zp_0) + \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2),$$
$$S_1 = ze^{zp_0}p_1, \quad S_2 = ze^{zp_0}p_2, \quad S_3 = ze^{zp_0}p_3,$$
$$S_4 = ze^{zp_0}\sqrt{\omega}\chi_1, \quad S_5 = -ze^{zp_0}\sqrt{\omega}\chi_2,$$
$$S_6 = -ze^{zp_0}\sqrt{\omega}\chi_3, \quad S_7 = \cosh(zp_0) - \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2). \quad (27)$$

These expressions encode the essential information concerning the nonvanishing cosmological constant generalization of

(10), since dual rotations leave the point $\mathcal{O}$ invariant. Therefore, we can think of the κ-AdS momentum space (25) as the seven-dimensional orbit in $\mathbb{R}^{4,4}$ that can be parametrized through (27) in terms of the dual translation and boost coordinates, while the dual rotation coordinates θ do not play any role in the description of the curved momentum space.

We recall that the deformed Poisson brackets for the κ-AdS algebra would be the ones in [41] for $\omega > 0$, and (22) allows them to be interpreted as a Poisson-Lie structure on the dual Lie group G_ω^* for which the multiplication on G_ω^* [i.e., the coproduct (13)–(15) for the κ-AdS algebra] is a Poisson map. If we now apply the identification (22) onto the deformed Casimir function (16) and afterwards we project it onto the curved momentum space parametrized by the p and χ coordinates by setting $\theta_i \to 0$, we obtain

$$C_z = \frac{2}{z^2}(\cosh(zp_0) - 1) - e^{zp_0}(\bar{p}^2 + \omega\bar{\chi}^2), \quad (28)$$

which could be considered as the deformed dispersion relation that corresponds to the $(3 + 1)$ κ-AdS momentum space.[3] The physical interpretation of such a dispersion relation requires giving a physical meaning to the hyperbolic momenta χ_a. This can be done thanks to the identification (22), which states that the local group coordinates have the same Poisson brackets as the generators of the κ-(A)dS algebra, as discussed in detail in [22]. It is then possible to represent the local coordinates of the dual group in terms of the usual phase space coordinates π_μ and x^ν with $\{\pi_\mu, x^\nu\} = \delta_\mu^\nu$, following a procedure similar to that found in [8,49]. In particular, the χ_a are expected to be given by a combination of both the components π_μ and x^ν of the phase

space. An explicit example of this is found in [49], where the $(1 + 1)$ κ-de Sitter algebra is represented on phase space.[4] In that case it is found that the dispersion relation depends on both momenta and spatial coordinates in a way that encodes a deformed gravitational redshift. We would expect something similar to show in this higher-dimensional model; however we are not yet at the stage where the explicit dependence of the dispersion relation on the phase space coordinates can be exposed. In fact, finding the appropriate phase space realization of the $(3 + 1)$ κ-(A)dS algebra is a nontrivial task, which is a work in progress and will be presented in a forthcoming publication [50], along with a thorough analysis of the physical implications of the findings presented here.

V. THE κ-dS CURVED MOMENTUM SPACE

As it could be expected, if we apply the construction presented in the previous section to the case $\omega < 0$ we obtain the same kind of geometric construction for the κ-dS momentum space, which should generalize the $(2 + 1)$ results presented in [22]. The only aspect we have to be careful about is the appearance of complex quantities when $\omega < 0$, due to the presence of $\sqrt{\omega}$ in some of the expressions [for instance, see (17)]. This is not a major obstacle to the construction of the momentum space, since, as we are going to show, all the complex contributions are linked to the dual of the rotation subgroup, which is again the isotropy subgroup of the origin of the momentum space. So they disappear when we consider the projection to the submanifold parametrized by momenta and boost coordinates.

The matrix representation of the algebra (19) when $\omega < 0$ can be found to be

$$D(X^0) = z \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D(X^1) = z \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

---

[3]As we show in the following section, for $\omega < 0$ this same expression describes the deformed dispersion relation corresponding to the $(3 + 1)$ κ-dS momentum space.

[4]Note that the algebra appearing in [49] is written in a different basis than the one used here. While both algebras are of bicrossproduct type, for $z \to 0$ the algebra in [49] reduces to the de Sitter algebra in the comoving basis, while the one used here reduces to the de Sitter algebra in the kinematical basis. It is worth stressing that the comoving momenta are defined as a linear superposition of kinematical translations and boosts, which is again consistent with the mixing of both types of kinematical transformations within (28).

$$
D(X^2) = z \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\qquad
D(X^3) = z \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
D(L^1) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix},
\qquad
D(L^2) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix},
$$

$$
D(L^3) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix},
\qquad
D(R^1) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -i & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
D(R^2) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & -1 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\qquad
D(R^3) = z\sqrt{-\omega} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Again, the left linear action of (21) onto the point $\mathcal{O} = (0,0,0,0,0,0,0,1)$ gives rise to an orbit whose points have ambient coordinates in $R^{1,7}$ given by

$$S_0 = \sinh(zp_0) + \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2),$$

$$S_1 = A(p_1 B_{21}^+ + i\sqrt{-\omega}(C + \chi_2 B_{21}^-)),$$

$$S_2 = A(p_2 B_{12}^+ + i\sqrt{-\omega}(D - \chi_1 B_{12}^-)),$$

$$S_3 = ze^{zp_0}(p_3 - iz\sqrt{-\omega}(\theta_1 p_1 + \theta_2 p_2$$
$$+ i\sqrt{-\omega}(\theta_1\chi_2 - \theta_2\chi_1))),$$

$$S_4 = A(ip_2 B_{21}^- + \sqrt{-\omega}(D + \chi_1 B_{21}^+)),$$

$$S_5 = A(-ip_1 B_{12}^- - \sqrt{-\omega}(C - \chi_2 B_{12}^+)),$$

$$S_6 = \sqrt{-\omega}ze^{zp_0}(\chi_3 - z(\theta_2 p_1 - \theta_1 p_2$$
$$+ i\sqrt{-\omega}(\theta_1\chi_1 + \theta_2\chi_2))),$$

$$S_7 = \cosh(zp_0) - \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2), \qquad (29)$$

where A, $B_{ij}^\pm$, C, D are the same functions appearing in (24). It is straightforward to check that such coordinates obey the constraints,

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 + S_6^2 + S_7^2 = 1,$$
$$S_0 + S_7 = e^{zp_0} > 0,$$

so that we obtain (half of) the $(6 + 1)$ dS space as the curved momentum space for the κ-dS quantum algebra.

Again, the isotropy subgroup for $\mathcal{O}$ is generated by the subgroup of dual rotations (26), and each point of the curved momentum space can be characterized by the seven momenta and rapidities by evaluating (29) at $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$,

$$S_0 = \sinh(zp_0) + \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2),$$

$$S_1 = ze^{zp_0}p_1,$$

$$S_2 = ze^{zp_0}p_2,$$

$$S_3 = ze^{zp_0}p_3,$$

$$S_4 = ze^{zp_0}\sqrt{-\omega}\chi_1,$$

$$S_5 = ze^{zp_0}\sqrt{-\omega}\chi_2,$$

$$S_6 = ze^{zp_0}\sqrt{-\omega}\chi_3,$$

$$S_7 = \cosh(zp_0) - \frac{1}{2}e^{zp_0}z^2(\bar{p}^2 - \omega\bar{\chi}^2). \qquad (30)$$

Note that these ambient space coordinates (30) are all real, since all the complex contributions in (29) are linked to the action of the dual rotation subgroup.

Finally, the projection of the deformed Casimir onto the curved momentum space leads to (28), which can again be interpreted as the dispersion relation for the $\omega < 0$ case. Also, the $\omega \to 0$ Poincaré limit of all of these expressions is straightforward, and leads to the results presented in Sec. II.

VI. DISCUSSION AND CONCLUDING REMARKS

In [22] we constructed the generalized momentum space associated to quantum deformed spacetime symmetries in the presence of a cosmological constant. We focused on the κ-dS algebra, which can be seen as a deformation of the standard Poincaré algebra governed by two deformation parameters: the cosmological constant $\Lambda = -\omega$ is a classical deformation parameter, controlling the effects of spacetime curvature, while $z = 1/\kappa$ is the quantum deformation parameter, controlling the Planck-scale effects (which in turn induce curvature in momentum space). The interplay between these two deformations is nontrivial and intertwines the coordinates of the whole phase space. So, while it was known for a while already that in the $\omega \to 0$ limit (where the symmetries are described by the κ-Poincaré algebra) the geometrical properties of the momentum space are those of half of a dS manifold, it was generally thought that it would not be possible to make an analogous analysis once $\omega \neq 0$.

Our recent result was an important breakthrough in this respect, since it demonstrated that the joint effects of spacetime curvature and of the quantum deformation can be taken into account if one constructs the momentum space by considering not just the momenta linked to spacetime translations, but also the hyperbolic momenta associated to boost transformations. Still, [22] focused only on low-dimensional cases, namely the κ-dS algebras in $(1 + 1)$ and $(2 + 1)$ dimensions. Of course the physically relevant model is that with $(3 + 1)$ dimensions, which, as explained in the introduction, allows one to connect to the phenomenology of particles propagating over cosmological distances. Describing the generalized momentum space of both quantum deformed dS and AdS algebras with a cosmological constant in $(3 + 1)$ dimensions was the goal of the work presented here.

Going from the $(2 + 1)$-dimensional case to the one with $(3 + 1)$ dimensions entails dealing with a deformed rotation sector, which is still classical in lower-dimensional models. Specifically, the coalgebra of the rotations is modified in $(3 + 1)$ dimensions, in such a way that one of the rotation generators takes a special role compared to the others (see the appendix). This might raise worries that the model breaks spatial isotropy. However, just as the deformed boost transformations do not break relativistic invariance, but simply deform the laws of transformation between inertial frames, the deformed rotations could imply that the concept of isotropy has to be adapted to fit within the new transformation rules. What the observational consequences of this deformed isotropy could be is still a matter of investigation.

Despite these novel features, the analysis of the generalized momentum space of the κ-dS algebra in $(3 + 1)$ dimensions led to a higher-dimensional version of the results found in [22]: the momentum space is half of a $(6 + 1)$-dimensional dS manifold and the rotations are the group of isotropy of its origin. The lower-dimensional results are recovered via canonical projection.

We expanded our construction to the κ-AdS algebra, which can be defined starting from the κ-dS algebra and changing sign to the cosmological constant parameter. While the difference between the two models is minimal at the level of the algebra and coalgebra, we found that the momentum space is characterized by a qualitatively different geometry. This is because the change of sign of the cosmological constant produces the appearance of complex quantities due to the presence of $\sqrt{\omega}$ factors. This analysis provides the first example where quantum effects do not produce a momentum space with dS geometry, but something different—we found that the κ-AdS algebra has a momentum manifold with $SO(4, 4)$ invariance.

Finally, we comment that the nonrelativistic limit of the results here presented is indeed worth studying, since it would give rise to the momentum spaces associated to quantum deformations of the $(3 + 1)$ Newton-Hooke symmetries when $\omega \neq 0$, and to quantum Galilei symmetries in the case $\omega = 0$. In this respect, we recall that the Galilean limit of $(2 + 1)$ quantum gravity based on quantum deformations of the Galilei and Newton-Hooke algebras was formerly presented in [51,52], and further studied in [53]. Work on this line is in progress.

APPENDIX: ROTATIONS IN THE $\kappa - AdS_\omega$ ALGEBRA

Let us recall that since δ in Eq. (17) is a coboundary Lie bialgebra, there exists a classical r-matrix such that $\delta(X) = [X \otimes 1 + 1 \otimes X, r]$. For coboundary deformations classical r-matrices provide the simplest way to condense the information of a given quantum algebra. In this case (see [41,54]) such a classical r-matrix reads

$$r_\omega = z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega}J_1 \wedge J_2),$$
(A1)

and its κ-Poincaré limit $\omega \to 0$ is the well-known classical r-matrix

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3). \quad (A2)$$

Note that rotations appear in the r-matrix only in $(3 + 1)$ dimensions. In fact, the $(2 + 1)$ κ-AdS$_\omega$ deformations are generated by the classical r-matrix given by (see [43,44])

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2), \quad\quad (A3)$$

which does not depend on ω and is just the projection of (A1) to the $(2 + 1)$-dimensional case, obtained by suppressing the J_1, J_2, P_3 and K_3 generators. The $J_1 \wedge J_2$ term, which does not appear in (A3), is the one responsible for the nonvanishing $\delta(J)$ in the $(3 + 1)$ case and, therefore, is the term that induces the deformation (13) of the rotation subalgebra.

At this point it is natural to wonder whether there could exist another quantum AdS$_\omega$ algebra in $(3 + 1)$ dimensions that generalizes the κ-Poincaré algebra and has a nondeformed rotation subalgebra, $\delta(J_1) = \delta(J_2) = \delta(J_3) = 0$. This question can be addressed by recalling that in [43,55] it was proven that all AdS$_\omega$ deformations in $(3 + 1)$ dimensions with primitive coproducts for P_0 and J_3 [i.e., with $\delta(P_0) = \delta(J_3) = 0$] are generated by one of the two (disjoint) families of two-parametric classical r-matrices,

$$r_{z_1,z_3} = z_1(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 \\ \pm \sqrt{\omega}J_1 \wedge J_2) + z_3 P_0 \wedge J_3, \quad (A4)$$

$$r_{z_2,z_3} = z_2(P_1 \wedge P_2 + \omega K_1 \wedge K_2 - \omega J_1 \wedge J_2 \\ \pm \sqrt{\omega}P_3 \wedge K_3) + z_3 P_0 \wedge J_3. \quad (A5)$$

Therefore, if we impose that the κ-Poincaré algebra has to be obtained in the limit $\omega \to 0$ [which means that the classical r-matrix that generates the deformation should still have (A2) as its $\omega \to 0$ limit], this implies that the only viable solution is (A4), that is, (A1) plus an additional twist generated by $z_3 P_0 \wedge J_3$. We point out that both classical r-matrices (A4) and (A5) are invariant under the automorphism (18) provided that the deformation parameters z_1, z_2, and z_3 remain unchanged.

It can also be proven [43] that a linear change of basis $X \to \hat{X}$ in the κ-AdS$_\omega$ algebra transforms the r-matrix (A4) into

$$r_{z_1,\hat{z}_3} = z_1 \left(\hat{K}_1 \wedge \hat{P}_1 + \hat{K}_2 \wedge \hat{P}_2 + \hat{K}_3 \wedge \hat{P}_3 \\ \pm \frac{\sqrt{\omega}}{\sqrt{3}} (\hat{J}_1 \wedge \hat{J}_2 + \hat{J}_2 \wedge \hat{J}_3 + \hat{J}_3 \wedge \hat{J}_1) \right) \\ + \hat{z}_3 \hat{P}_0 \wedge (\hat{J}_1 + \hat{J}_2 + \hat{J}_3),$$

where $\hat{z}_3 = z_3/\sqrt{3}$, in such a way that the three rotation generators seem to play the same role, although now it can be checked that the primitive generator for the rotation subalgebra turns out to be $\hat{J}_1 + \hat{J}_2 + \hat{J}_3$. As a consequence, we can state as a general result that the generalization of the κ-Poincaré algebra to the nonvanishing cosmological constant case implies a nontrivial Planck-scale deformation of the rotation sector, unless we are willing to deform the time-translation sector.

[1] J. Kowalski-Glikman and S. Nowak, Classical Quantum Gravity **20**, 4799 (2003).
[2] J. Kowalski-Glikman, Phys. Lett. B **547**, 291 (2002).
[3] D. Raetzel, S. Rivera, and F. P. Schuller, Phys. Rev. D **83**, 044047 (2011).
[4] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, Phys. Rev. D **84**, 084010 (2011).
[5] S. Mignemi and A. Samsarov, Phys. Lett. A **381**, 1655 (2017).
[6] M. Arzano, G. Gubitosi, J. Magueijo, and G. Amelino-Camelia, Phys. Rev. D **92**, 024028 (2015).
[7] J. Kowalski-Glikman, Int. J. Mod. Phys. A **28**, 1330014 (2013).
[8] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, and N. Loret, Classical Quantum Gravity **30**, 235002 (2013).
[9] G. Amelino-Camelia, L. Barcaroli, and N. Loret, Int. J. Theor. Phys. **51**, 3359 (2012).
[10] M. Ackermann et al. (Fermi GBM/LAT Collaboration), Nature (London) **462**, 331 (2009).
[11] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature (London) **393**, 763 (1998).
[12] G. Amelino-Camelia, G. D'Amico, G. Rosati, and N. Loret, Nat. Astron. **1**, 0139 (2017).
[13] G. Amelino-Camelia and L. Smolin, Phys. Rev. D **80**, 084017 (2009).
[14] G. Amelino-Camelia, L. Barcaroli, G. D'Amico, N. Loret, and G. Rosati, Phys. Lett. B **761**, 318 (2016).
[15] G. Amelino-Camelia, A. Marciano, M. Matassa, and G. Rosati, Phys. Rev. D **86**, 124035 (2012).
[16] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer, Phys. Rev. D **92**, 084053 (2015).
[17] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer, Phys. Rev. D **95**, 024036 (2017).
[18] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer, Phys. Rev. D **96**, 084010 (2017).
[19] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, S. Liberati, and N. Loret, Phys. Rev. D **90**, 125030 (2014).
[20] F. Cianfrani, J. Kowalski-Glikman, and G. Rosati, Phys. Rev. D **89**, 044039 (2014).
[21] A. Marciano, G. Amelino-Camelia, N. R. Bruno, G. Gubitosi, G. Mandanici, and A. Melchiorri, J. Cosmol. Astropart. Phys. 06 (2010) 030.
[22] A. Ballesteros, G. Gubitosi, I. Gutiérrez-Sagredo, and F. J. Herranz, Phys. Lett. B **773**, 47 (2017).
[23] G. Gubitosi and F. Mercati, Classical Quantum Gravity **30**, 145002 (2013).
[24] J. Kowalski-Glikman and S. Nowak, arXiv:hep-th/0411154.
[25] G. Amelino-Camelia, M. Arzano, J. Kowalski-Glikman, G. Rosati, and G. Trevisan, Classical Quantum Gravity **29**, 075007 (2012).
[26] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoy, Phys. Lett. B **264**, 331 (1991).
[27] J. Lukierski, A. Nowicky, and H. Ruegg, Phys. Lett. B **271**, 321 (1991).
[28] S. Giller, P. Kosinski, J. Kunz, M. Majewski, and P. Maslanka, Phys. Lett. B **286**, 57 (1992).
[29] J. Lukierski, A. Nowicky, and H. Ruegg, Phys. Lett. B **293**, 344 (1992).
[30] S. Majid and H. Ruegg, Phys. Lett. B **334**, 348 (1994).
[31] V. Chari and A. Pressley, A Guide to Quantum Groups (Cambridge University Press, Cambridge, 1994).
[32] S. Majid, Foundations of Quantum Group Theory (Cambridge University Press, Cambridge, 1995).
[33] A. Ballesteros and F. Musso, J. Phys. A **46**, 195203 (2013).
[34] P. Maslanka, J. Phys. A **26**, L1251 (1993).
[35] S. Majid and H. Ruegg, Phys. Lett. B **334**, 348 (1994).
[36] S. Zakrzewski, J. Phys. A **27**, 2075 (1994).
[37] J. Lukierski and H. Ruegg, Phys. Lett. B **329**, 189 (1994).
[38] V. G. Drinfel'd, in Quantum Groups, Proc. Int. Cong. Math. Berkeley 1986, edited by A. V. Gleason (AMS, Providence, 1987), p. 798.
[39] M. A. Semenov-Tyan-Shanskii, Theor. Math. Phys. **93**, 1292 (1992).
[40] N. Ciccoli and F. Gavarini, Adv. Math. **199**, 104 (2006).
[41] A. Ballesteros, F. J. Herranz, F. Musso, and P. Naranjo, Phys. Lett. B **766**, 205 (2017).
[42] F. J. Herranz and M. Santander, J. Phys. A **30**, 5411 (1997).
[43] A. Ballesteros, F. J. Herranz, and F. Musso, J. Phys. Conf. Ser. **532**, 012002 (2014).
[44] A. Ballesteros, F. J. Herranz, C. Meusburger, and P. Naranjo, SIGMA **10**, 052 (2014).
[45] A. Ballesteros, F. J. Herranz, and N. R. Bruno, Adv. High Energy Phys. **2017**, 7876942 (2017).
[46] S. Majid and B. Schroers, J. Phys. A **42**, 425402 (2009).
[47] P. K. Osei and B. Schroers, J. Math. Phys. (N.Y.) **53**, 073510 (2012).
[48] A. Ballesteros, F. J. Herranz, and N. R. Bruno, Phys. Lett. B **574**, 276 (2003).
[49] L. Barcaroli and G. Gubitosi, Phys. Rev. D **93**, 124063 (2016).
[50] A. Ballesteros, G. Gubitosi, I. Gutiérrez-Sagredo, and F. J. Herranz (to be published).
[51] G. Papageorgiou and B. J. Schroers, J. High Energy Phys. 09 (2009) 009.
[52] G. Papageorgiou and B. J. Schroers, J. High Energy Phys. 10 (2010) 020.
[53] A. Ballesteros, F. J. Herranz, and P. Naranjo, Classical Quantum Gravity **31**, 245013 (2014).
[54] A. Ballesteros, N. A. Gromov, F. J. Herranz, M. A. del Olmo, and M. Santander, J. Math. Phys. **36**, 5916 (1995).
[55] A. Ballesteros, N. R. Bruno, and F. J. Herranz, Czech. J. Phys. **11**, 1321 (2004).