Universal terms of holographic entanglement entropy in theories with hyperscaling violation

M. Reza Tanhayi*

*Department of Physics, Faculty of Basic Science,
Islamic Azad University Central Tehran Branch (IAUCTB), P.O. Box 14676-86831, Tehran, Iran
and School of Physics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5531, Tehran, Iran*

(Received 12 December 2017; published 15 May 2018)

We examine the holographic entanglement entropy in hyperscaling violating backgrounds. Precisely in such theories, by semianalytic computation, we use holographic methods to derive the universal terms of entanglement entropy in various dimensions. We also find such terms when the bulk action includes higher-order curvature terms. We make further comments on the mutual information in hyperscaling violating backgrounds with higher curvature interactions. We explore the sign of tripartite information, which results in a monogamy property on mutual information. It is shown that for various hyperscaling parameters, the mutual and tripartite information are always positive and negative, respectively. This indicates that mutual information is monogamous in theories with hyperscaling violation.

DOI: [10.1103/PhysRevD.97.106008](https://doi.org/10.1103/PhysRevD.97.106008)

I. INTRODUCTION

The gauge/gravity correspondence [1] claims an equivalence between strongly coupled gauge theory and gravity theory, which lives in one higher-dimensional anti-de Sitter (AdS) spacetime. In this correspondence the objects in the gravitational theory are related to those in the dual field theory, and in this sense it is known as the duality. This duality could actually provide a framework to study quantum physics of strongly correlated many-body systems via the classical dynamics of gravity [2]. As a matter of fact, AdS geometries are dual to the field theory with conformal symmetry; however, studying the field theories that are scale invariant but not conformal invariant and finding the corresponding dual metrics that are not asymptotically AdS can be a challenging task. The main motivation for studying such generalization comes from the fact that many physical systems in their critical points exhibit a rather different scaling in space and time; thus, finding the holographic dual models for such systems could be a relevant question. For example, in addressing the Landau-Fermi liquids [2,3], one needs a Lifshitz-type metric in dual gravity theory where the spatial and time coordinates of the field theory have been

scaled differently [4]. The class of dual geometries with such scaling may be written as

$$ds_{d+2}^2 = \tilde{L}^2 \left(\frac{-dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{1}{r^2} \sum_{i=1}^d dx_i^2 \right), \quad (1.1)$$

where z is the dynamical critical exponent and $\tilde{L}$ stands for the curvature scale. The above metric reduces to AdS_{d+2} for $z = 1$ and enjoys the following scaling of coordinates:

$$t \rightarrow \zeta^z t, \quad \vec{x} \rightarrow \zeta \vec{x}, \quad r \rightarrow \zeta r. \quad (1.2)$$

Note that the metric (1.1) arises as the solution of simple gravity theories coupled with some sort of matter field [4,5]. One may also add a scalar dilaton field in the bulk theory to obtain a larger class of scaling metrics which has been studied in several works [6]. On the other hand, studying both the dilaton scalar field and an Abelian gauge field leads to a full class of scaling metrics called hyperscaling violating geometries [7]:

$$ds_{d+2}^2 = \tilde{L}^2 r^{2\theta} \left(\frac{-dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{1}{r^2} \sum_{i=1}^d dx_i^2 \right), \quad (1.3)$$

where θ is the hyperscaling violating exponent. Note that one recovers a nonrelativistic scaling symmetry, namely, the Lifshitz geometry, by setting $\theta = 0$ and $z \neq 1$. It is clear that the metric (1.3) cannot be an ordinary solution of the Einstein equation, and in fact, the isotropy should be broken

*mtanhayi@ipm.ir

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

by a gauge field or a massive vector field coupled to a scalar field [5–10]. This metric under the scale transformation (1.2) transforms as $ds \rightarrow \zeta^{\frac{\theta}{2}} ds$.

It was shown that in $d + 1$ -dimensional field theories dual to the hyperscaling violating metric (1.3), the corresponding thermodynamical parameters, such as entropy, scale with respect to temperature as $T^{(d-\theta)/z}$ [7,11]; on the other hand, in the Lifshitz metric, the entropy scales as $T^{d/z}$, which indicates that, by turning on the hyperscaling violating parameter, the degrees of freedom can effectively be described in $d - \theta$ dimensions [12]. Remarkably, by associating an effective dimension to such theories, one obtains the log behavior of the entanglement entropy. This could be utilized in a holographic description of the systems with a Fermi surface in condensed matter physics [13]; namely, in any dimension, one can engineer the parameters to provide a gravitational dual for a system with a Fermi surface.

The parameters in metric (1.3) are constrained by imposing some conditions; e.g., in Ref. [6] it was shown that holographic consideration of the null energy condition yields

$$(d - \theta) \left(z - 1 - \frac{\theta}{d} \right) \geq 0, \quad (z - 1)(d + z - \theta) \geq 0, \quad (1.4)$$

where we consider $0 \leq \theta < d$.

In this work we further investigate some specific features of hyperscaling violating theories by making use of the gravity dual which is described by Einstein gravity (EG) or Gauss-Bonnet (GB) gravity coupled to some specific matter fields. Precisely, we deal with the structure of entanglement in the vacuum of such theories by computing the holographic entanglement entropy for a strip, a sphere, and a cylinder as the entangling regions. In the case of a strip, an explicit form of the entanglement entropy is computed, whereas the logarithmically divergent terms are studied for a spherical or a cylindrical entangling surface.

It is known that entanglement entropy for local quantum field theories is infinite, and generally, for a given entangling region with area $\mathcal{A}$ and in a $d + 1$ -dimensional field theory, one obtains [14]

$$S = c_{d-1} \frac{\mathcal{A}}{\epsilon^{d-1}} + \cdots + s_{d+1} + \mathcal{O}\left(\frac{\epsilon}{\ell}\right),$$

$$s_{d+1} = \begin{cases} c_{d+1} \log \frac{\ell}{\epsilon} & \text{for even-dimensional CFTs} \\ c'_{d+1} & \text{for odd-dimension CFTs,} \end{cases} \quad (1.5)$$

where c_i 's are some constants, ϵ is the UV cutoff and ℓ stands for the characteristic length of the corresponding entangling region. The most divergent term is proportional to the area of the entangling surface, and this is known as the area law which is due to the infinite correlations between degrees of freedom near the boundary of the

entangling surface. Here, s_{d+1} is the universal part of the entanglement entropy, and for even-dimensional CFTs (odd d in our notation), there is a logarithmic term where its coefficient is physical and universal in the sense that it is not affected by cutoff redefinitions. It is argued that it could carry some universal information about the underlying quantum field theory and thus is of great interest. For example, the entanglement entropy of a strip of width ℓ in two dimensions becomes [15]

$$S = \frac{c}{3} \log \frac{\ell}{\epsilon}, \quad (1.6)$$

where its coefficient is indeed the central charge of the theory. On the other hand, for spherical and cylindrical entangling regions there are also logarithmic terms, and the corresponding coefficients could potentially address the anomalies of underlying quantum field theory. It is known that in four-dimensional conformal theory, the coefficients of these logarithmic terms are related to a and c central charges [16], and notably, for higher-dimensional field theories, not much is known about the universal logarithmic term of entanglement entropy [17].

In this paper, we investigate the logarithmically divergent terms of holographic entanglement entropy (HEE), and by a semianalytic consideration, we find the logarithmic terms for some values of the parameters in hyperscaling violating theories. We use a holographic description of entanglement entropy for both Einstein gravity and higher-order gravity theories for three entangling regions—the strip, sphere, and cylinder—in theories with hyperscaling violation.

It is worth noting, however, that the entanglement entropy is a scheme-dependent quantity due to the appearance of the UV cutoff parameter. Moreover, the entanglement entropy cannot demonstrate all the physical content of a field theory; for example, as mentioned above, for a two-dimensional conformal field theory, entanglement entropy for a single interval is completely fixed by the central charge of the theory. Thus, to investigate the other aspects of conformal field theory and improve our knowledge of the Hilbert space of a quantum system, it seems some other entanglement measurements are needed. One of the most powerful measurements is indeed the mutual information, which is a UV cutoff-independent quantity [18]. We also study the mutual information and its sign in hyperscaling violating theories.

The organization of the paper is as follows. In the next section, we consider the holographic entanglement entropy for strip, spherical, and cylindrical entangling regions in hyperscaling violating theories. In Sec. III, we explore the curvature corrections to holographic entanglement entropy; the universal terms in higher-curvature hyperscaling gravity theories are also addressed in this section. We make further comments on mutual information and its monogamy property for a trip geometry in Sec. IV. Section V is devoted to discussions.

II. HOLOGRAPHIC ENTANGLEMENT ENTROPY IN HYPERSCALING VIOLATING BACKGROUNDS: EINSTEIN GRAVITY

Let us suppose a system with Hilbert space $\mathcal{H}$, which has been divided into two parts, A (mostly supposed as a spatial region) and its complement $\bar{A}$; then, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. In principle, entanglement entropy can be used as a measure of quantum correlation between subsystems A and $\bar{A}$, where the corresponding entanglement entropy for subsystem A is given by the von Neumann entropy as follows [19,20]:

$$S_A = -\text{Tr}(\rho_A \log \rho_A). \quad (2.1)$$

The reduced density matrix ρ_A can be obtained by tracing over the degrees of freedom living in the environment of A , namely, in $\bar{A}$. The entanglement entropy of A depends on the chosen region and also on the state of the field theory. Basically, computing the entanglement entropy is a difficult task, but, in a field theory with a holographic dual described by classical Einstein gravity, there is an elegant proposal to compute this quantity made by Ryu and Takayanagi known as the RT-proposal [15]. According to this proposal, in a given entangling region, the entanglement entropy is

$$S = \frac{\mathcal{A}}{4G_N}, \quad (2.2)$$

where G_N is the Newton constant and $\mathcal{A}$ is the minimal surface in the bulk whose boundary coincides with the boundary of the entangling region. This is indeed the holographic description of the entanglement entropy.

In this section, we study the holographic entanglement entropy for systems with hyperscaling violation. It was shown that an Einstein-Maxwell-dilaton model with a particular potential, which is given by the following action,

$$\mathcal{I} = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g} \times \left[R - \frac{1}{2}(\partial\phi)^2 + V_0 e^{\gamma\phi} - \frac{1}{4} e^{\alpha\phi} F^{\mu\nu} F_{\mu\nu} \right], \quad (2.3)$$

admits a solution of the form (1.3), where γ , V_0 , and α are some free parameters of the model. The gauge field produces an anisotropic scaling symmetry of the theory, and a hyperscaling violating factor comes from the above particular form of the potential. The scalar and gauge fields are given by

$$\phi = \phi_0 + \beta \ln r, \quad A_t = \sqrt{\frac{2(z-1)}{d-\theta+z}} e^{\frac{d-1+\theta/d}{\beta}\phi_0} r^{d-\theta+z}, \quad (2.4)$$

where $\beta = \sqrt{2(d-\theta)(z-1-\theta/d)}$. Note that the free parameters of the action are fixed by the hyperscaling violating and dynamical exponent parameters as follows [9]:

$$V_0 = \frac{(d-\theta+z-1)(d-\theta+z)}{e^{\gamma\phi_0}}, \quad \lambda = \frac{2\theta(d-1)-2d^2}{\beta d}, \quad \gamma = \frac{2\theta}{\beta d}. \quad (2.5)$$

In the hyperscaling violating background, we first review the holographic entanglement entropy for a strip and then generalize the consideration to spherical and cylindrical entangling surfaces by computing the logarithmically divergent terms.

A. Strip entangling region

First let us review the holographic entanglement entropy for a strip at a fixed time on the boundary, which is defined by

$$t = \text{fixed}, \quad -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \quad 0 \leq x_2, \dots, x_d \leq H, \quad (2.6)$$

where $(t, \vec{x})$ are the spacetime coordinates. Now the aim is to find the surface in the bulk with the boundary of the above strip and then minimize it. In the metric (1.3), the profile of the hypersurface in the bulk may be parametrized by $x(r)$; thus, the induced metric on this hypersurface is given by.¹

$$ds_{\text{ind}}^2 = r^{2\frac{d}{d-1}} \left[(1+x'^2) dr^2 + \sum_{a=2}^d dx_a^2 \right], \quad (2.7)$$

where here the prime stands for the derivative with respect to r . Thus, the area of the profile in the bulk is given by

$$\mathcal{A} = \int \sqrt{\sigma} d^d x = \frac{H^{d-1}}{2} \int dr (r^{\theta-d}) \sqrt{1+x_1'^2}, \quad (2.8)$$

where σ stands for the induced metric determinant. After minimizing the functional of (2.8) and using the condition of the hypersurface turning point r_t , in which $x_1'(r_t) \rightarrow \infty$, one gets the following conserved quantity along the radial profile:

$$-\frac{r^{\theta-d} x_1'}{\sqrt{x_1'^2 + 1}} = -r_t^{\theta-d}. \quad (2.9)$$

One can use the above relation to find the width of the strip as a function of the turning point, and for $\theta \neq d-1$, one obtains

$$\ell = 2 \int_0^{r_t} x_1'(r) dr = \frac{2\sqrt{\pi} \Gamma(\frac{1+d-\theta}{2(d-\theta)})}{\Gamma(\frac{1}{2(d-\theta)})} r_t. \quad (2.10)$$

¹In this section, we set $\tilde{L}$ to 1.

This can be inverted to find the turning point of the proposed hypersurface in the bulk, and finally, the HEE for EG reads as follows:

$$S_{EG} = \frac{H^{d-1}}{2G_N(d-\theta-1)} \times \left(\frac{1}{\epsilon^{(d-\theta-1)}} - \left(\frac{\sqrt{\pi}\Gamma(\frac{1+d-\theta}{2(d-\theta)})}{\Gamma(\frac{1}{2(d-\theta)})} \right)^{d-\theta} \left(\frac{\ell}{2} \right)^{\theta-d+1} \right), \quad (2.11)$$

where ϵ stands for the UV scale which has been defined by the radial profile. This was found in [6], and it is clear that the scaling behavior of HEE has been modified by an additional term as $(\text{length})^\theta$. Note that $\theta = 0$ leads to the HEE for the AdS_{d+2} background.

B. Sphere entangling region

The holographic entanglement entropy for a spherical entangling region has been studied in the literature [15]. Here, after briefly summarizing the results, we generalize the study to the hyperscaling violating backgrounds. In this case, let us modify the metric (1.3) as $\sum dx_i^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2$. On the boundary, the entangling region is a sphere with radius $r < R$; therefore, the corresponding

codimension-two hypersurface in the bulk is realized by $t = 0$ and $\rho = \rho(r)$. With this assumption, the induced metric becomes

$$ds_{\text{ind}}^2 = r^{\frac{2\theta}{d-2}} ((1 + \rho'^2) dr^2 + \rho^2 d\Omega_{d-1}^2), \quad (2.12)$$

and the corresponding entropy functional reads as

$$S = \frac{\mathcal{A}_{d-1}}{4G_N} \int dr \rho^{d-1} r^{\theta-d} \sqrt{\rho'^2 + 1}, \quad (2.13)$$

where $\mathcal{A}_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is indeed the area of the unit $d-1$ -dimensional sphere. The above entropy functional should be minimized, which leads to the following equation of motion:

$$(\rho'(r)^2 + 1)((\theta - d)\rho(r)\rho'(r) - (d-1)r) + r\rho(r)\rho''(r) = 0. \quad (2.14)$$

For $\theta = 0$, it has a simple solution as follows:

$$\rho(r) = \sqrt{R^2 - r^2}, \quad (2.15)$$

and the corresponding HEE is given by [15]

$$S = \frac{\pi^{d/2}}{2G_N\Gamma(\frac{d}{2})} \left[\mathcal{P}_1 \left(\frac{R}{\epsilon} \right)^{d-1} + \mathcal{P}_3 \left(\frac{R}{\epsilon} \right)^{d-3} + \dots + \begin{cases} \mathcal{P}_{d-1} \left(\frac{R}{\epsilon} \right) + \mathcal{P}_d & \text{for even } d \\ \mathcal{P}_{d-2} \left(\frac{R}{\epsilon} \right)^2 + \mathcal{Q} \log \left(\frac{R}{\epsilon} \right) & \text{for odd } d \end{cases} \right]$$

$$\mathcal{P}_1 = \frac{1}{d-1}, \quad \mathcal{P}_3 = -\frac{d-2}{2(d-3)}, \quad \mathcal{P}_d = \frac{\Gamma(\frac{d}{2})}{2\sqrt{\pi}} \Gamma\left(\frac{1-d}{2}\right), \quad \mathcal{Q} = (-1)^{\frac{d-1}{2}} \frac{(d-2)!!}{(d-1)!!}. \quad (2.16)$$

The coefficient of the leading UV-divergent term is proportional to the area of the boundary which is the area law. On the other hand, when d is odd, there is a logarithmic term, and its coefficient is universal in the sense that it is independent of the rescaling.

The next step is to investigate the effect of dynamical and hyperscaling violation parameters on the universal term of HEE. However, it is important to note that to identify the universal contribution, the near-boundary behavior of the minimal surface would be sufficient. Thus, one needs to consider the asymptotic behavior of the profile near the boundary which can be written as

$$\rho(r) = R + c_1 r + c_2 r^2 + c_3 r^3 + \dots \quad (2.17)$$

Taking into account the condition $\rho(0) = R$ and after minimizing the entropy functional, one finds

$$c_1 = 0, \quad c_2 = \frac{(d-1)}{2R(\theta-d+1)}, \quad c_3 = 0. \quad (2.18)$$

Thus, near the boundary, one obtains

$$S = \frac{1}{2G_N\Gamma(\frac{d}{2})} \int dr \left[r^{\theta-d} \left(R^{d-1} + \frac{(d-1)^2 R^{d-3} (\theta-d+2)}{2(\theta-d+1)^2} r^2 \right) + \mathcal{O}(r^3) \right]. \quad (2.19)$$

By semianalytic computation, we collect universal log terms in Table I. One observes that for odd d and $\theta = 0$, there is a universal term which is given by

$$S_{\text{uni}}^{\theta=0} = (-1)^{\frac{d-1}{2}} \frac{\pi^{d/2}}{2G_N\Gamma(\frac{d}{2})} \frac{(d-2)!!}{(d-1)!!} \log \left(\frac{R}{\epsilon} \right), \quad (2.20)$$

TABLE I. Coefficient of the universal term $[\log(R/\epsilon)]$ which appears when reading the HEE for the sphere in the $d+2$ -dimensional hyperscaling violating background. Note that they appear up to the overall factor of $\frac{\pi^{d/2}}{2G_N\Gamma(\frac{d}{2})}$.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\theta=0$	...	$-1/2$	...	$3/8$	...	$-5/16$
$\theta=1$	R	...	$-(9/8)R$	...	$(2375/2048)R$	...
$\theta=2$	...	R^2	...	$-2R^2$	...	$(351/128)R^2$
$\theta=3$	...	...	R^3	...	$-(25/8)R^3$	...
$\theta=4$	...	...	...	R^4	...	$-9/2R^4$
$\theta=5$	...	...	...	...	R^5	...
$\theta=6$	...	...	...	...	...	R^6

and it is in agreement with (2.17). For $\theta = d-1$, the log term indicates that the corresponding background could provide a gravitational dual for a theory with a Fermi surface, and the corresponding HEE is given by²

$$S_{\text{uni}}^{\theta=d-1} = \frac{\pi^{d/2}}{2G_N\Gamma(\frac{d}{2})} R^{d-1} \log\left(\frac{R}{\epsilon}\right). \quad (2.21)$$

The other universal log terms appear for $d \geq 3$ and $\theta = d-3$ and, moreover, for $d \geq 5$ and $\theta = d-5$, which we have computed in Table I. It is worth mentioning that, in the absence of logarithms, finite contributions might be universal as well.³ For even d , e.g., for a six-dimensional background, we find the finite part of the HEE as follows:

$$S_{\text{finite}} = \frac{\pi^2}{4G_N} \frac{R^\theta (2\theta^2 + \theta - 12)}{(\theta-3)^2(\theta-1)}, \quad (2.22)$$

where for $\theta = 0$ one recovers the finite part of the HEE given by Eq. (2.17) for $d = 4$.

C. Cylindrical entangling region

For a cylinder, we parametrize the metric (1.3) as

$$ds^2 = r^{\frac{2d}{d-2}} (-r^{2(z-1)} dt^2 + dr^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2), \quad (2.23)$$

where the entangling surface is a cylinder with $\rho = \ell$ on the $t = 0$ surface in this boundary geometry. We also introduce a regulator H along the cylinder length. By taking the profile as $\rho = \rho(r)$, the entropy functional becomes

$$S = \frac{H}{2G_N} \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int dr r^{\theta-d} \rho(r)^{d-2} \sqrt{\rho'(r)^2 + 1}. \quad (2.24)$$

One should minimize the above integrand to find the profile in the bulk. However, similar to the previous case, we are interested in the universal terms, so what we need is actually

²Note that such theories have a dimensionful scale that does not decouple in the infrared; hence, dimensional analysis is restored identically [6].

³We thank the referee for commenting on this point.

TABLE II. Coefficient of the universal term $[\log(\ell/\epsilon)]$ up to the overall factor $\frac{H}{2G_N} \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ which appears when reading the HEE for the cylinder in the $d+2$ -dimensional hyperscaling violating background.

	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\theta=0$	$-(1/8\ell)$	...	$(135/2048\ell)$	...	$\sim 0.025 \frac{1}{\ell}$
$\theta=1$	...	$-1/2$	...	$3/8$	...
$\theta=2$	ℓ	...	$-(9/8)\ell$	...	$\sim 1.15\ell$
$\theta=3$	...	ℓ^2	...	$-2\ell^2$	...
$\theta=4$	...	...	ℓ^3	...	$-(25/8)\ell^3$
$\theta=5$	...	...	...	ℓ^4	...
$\theta=6$	...	...	...	...	ℓ^5

to find the near-boundary behavior of the profile. By imposing the condition in which, at the boundary $r = 0$, one has $\rho(0) = \ell$, and after minimizing the entropy functional, near the boundary the profile takes the following form:

$$\rho(r) = \ell + \frac{(d-2)r^2}{2(\theta-d+1)\ell} + \dots \quad (2.25)$$

Correspondingly, the holographic entanglement entropy near the boundary becomes

$$S = \frac{H}{2G_N} \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int dr r^{\theta-d} \times \left(\ell^{d-2} + \frac{(d-2)^2 r^2 \ell^{d-4} (\theta-d+2)}{2(\theta-d+2)^2} + \mathcal{O}(r^3) \right). \quad (2.26)$$

We have computed the logarithmic terms of the HEE in Table II. As one observes, similar to the previous case, for the special values of the hyperscaling violating exponent, the holographic entanglement entropy shows a logarithmic violation of the area law. The finite part of the HEE for a six-dimensional background is given by

$$S_{\text{finite}} = \frac{\pi H R^{\theta-1}}{G_N} \frac{[(\theta-11)\theta+42]-86\theta+181\theta-319}{(\theta-5)^4(\theta-3)(\theta-1)}. \quad (2.27)$$

In the following section, we add the higher-order (Gauss-Bonnet) terms to the Lagrangian and investigate the logarithmic terms in a sphere and a cylinder as the entangling regions on the boundary in hyperscaling violating backgrounds.

III. HOLOGRAPHIC ENTANGLEMENT ENTROPY IN HYPERSCALING VIOLATING BACKGROUNDS: CURVATURE CORRECTIONS

In this section, we consider the HEE for systems with hyperscaling violation when the gravity action contains higher-curvature terms. We add the higher-order (Gauss-Bonnet) terms into the Lagrangian and investigate the

logarithmic terms of the HEE in a sphere and a cylinder as the entangling regions on the boundary in hyperscaling violating backgrounds.

As already mentioned in the previous section, one can use the RT-proposal to compute the HEE; however, this

elegant picture does not work for actions with higher-derivative terms. There are some proposals to compute the HEE in higher-curvature theories [21–23]. We use the proposal of Ref. [24], where for the following action in the bulk with curvature squared terms [25],

$$\mathcal{I} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+2}x \sqrt{-g} \left[R + \frac{d(d+1)}{L^2} + L^2 (a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) \right], \quad (3.1)$$

the HEE is given by

$$S = \frac{1}{4G_N} \int_{\Sigma} d^d x \sqrt{\sigma} \left\{ 1 + L^2 \left[2a_1 R + a_2 \left(R_{\mu\nu} n_i^\mu n_i^\nu - \frac{1}{2} \sum_i (\text{Tr} \mathcal{K}^{(i)})^2 \right) + 2a_3 \left(R_{\mu\nu\alpha\beta} n_i^\mu n_i^\alpha n_j^\nu n_j^\beta - \sum_i \mathcal{K}_{\mu\nu}^{(i)} \mathcal{K}_{(i)}^{\mu\nu} \right) \right] \right\}. \quad (3.2)$$

In the above relations, L is a length scale which for Einstein gravity reduces to an AdS radius, and Σ is a codimension-two hypersurface in the bulk spacetime $\mathcal{M}$ which extremizes the entropy functional. Note that introducing the curvature scale L makes the coupling constants dimensionless. The orthogonal normal vectors of Σ are denoted by $n_i (i = 1, 2)$, whereas the extrinsic curvature tensors are defined by

$$\mathcal{K}_{\mu\nu}^{(i)} = h_\mu^\lambda h_\nu^\rho (n_i)_{\lambda;\rho}, \quad h_\mu^\lambda = \delta_\mu^\lambda + \xi \sum_i (n_i)_\mu (n_i)^\lambda, \quad (3.3)$$

where ξ is $+1$ for timelike and -1 for spacelike vectors. It is noted that by setting the higher-curvature coupling to zero, Eq. (3.2) reduces to the RT-proposal.

We consider GB gravity, which can be achieved by setting $a_1 = a_3 = -\frac{1}{4}a_2 = \frac{1}{(d-1)(d-2)}\lambda$ in (3.1), where λ is a dimensionless coupling constant that controls the strength of the GB term. The HEE for GB gravity has been studied in detail in [22]. Also, in the hyperscaling violating background, Ref. [26] studied HEE for the strip. In the rest of this section, we first review the effect of curvature correction (GB term) on the HEE of a strip and then investigate the logarithmically divergent terms of the HEE for spherical and cylindrical entangling surfaces.

A. Strip

Similar to Hilbert-Einstein gravity, in the case of the squared curvature gravity, by adding a dilatonic scalar field

with a nontrivial potential and a gauge field, one can extract a hyperscaling violating solution such as (1.3) [27]. In this metric, the following are the Ricci scalar and nonzero component of the Ricci tensor:

$$\begin{aligned} R &= -\frac{2r^{-2\theta/d}}{\tilde{L}^2 d} \left(z^2 d + z(d^2 - (d+1)\theta) + \frac{d+1}{2}(\theta-d)^2 \right), \\ R_{tt} &= \frac{r^{-2z}}{d} (z^2 d + z(d^2 - (d+1)\theta) + (\theta-d)\theta), \\ R_{rr} &= \frac{-z^2 + (\theta-d) + \frac{z\theta}{d}}{r^2}, \\ R_{ii} &= \frac{(d+z-\theta)(\theta-d)}{r^2 d}, \quad i = 1, \dots, d. \end{aligned} \quad (3.4)$$

In order to compute the HEE for the strip, similar to Einstein gravity, the profile can be parametrized as $x_1 = x_1(r)$, so the induced metric reads

$$ds_{\text{ind}}^2 = \tilde{L}^2 r^{\frac{2(\theta-d)}{d}} \left[(1 + x_1'^2) dr^2 + \sum_{i=2}^d dx_i^2 \right]. \quad (3.5)$$

After doing some calculations and making use of the relation (3.2), the expression of HEE in the hyperscaling violating background when the gravity action contains higher-order curvature terms is given by

$$\begin{aligned} S &= \frac{1}{4G_N} \int_{\Sigma} r^{(\theta-d)} \sqrt{1 + x_1'^2} d^d x \left\{ 1 - \frac{L^2}{\tilde{L}^2} \left[\frac{a_1 r^{-2\theta/d}}{d(1 + x_1'^2)^3} (4z^2 d + z(4d^2 - (d+1)^2\theta) + 2(d+1)(\theta-d)) \right. \right. \\ &\quad + a_2 (1 + x_1'^2)^2 ((d + zd - 2\theta)(d + z - \theta) + (2z^2 d + z(d^2 + z(d^2 - (d+2)\theta)) + (\theta-d)^2) x_1'^2 \\ &\quad - \frac{d}{2} ((\theta-d)(x_1' + x_1'^3) + r x_1'')) + a_3 \left(\frac{4}{d} (\theta - zd)(1 + x_1'^2)(\theta - d - z d x_1'^2) \right. \\ &\quad \left. \left. - r^2 x_1'^2 d + 2(\theta-d)^2 (x_1' + x_1'^3)^2 + 4r(\theta-d)(x_1' + x_1'^3) x_1'' \right) \right] \right\}. \end{aligned} \quad (3.6)$$

Now one should find the profile of the hypersurface $x_1(r)$, which is supposed to be a smooth differentiable function with the condition $x_1(0) = \ell/2$. This is a complicated expression; nevertheless, it can be done by minimizing the entropy functional (3.6). One may consider the entropy functional as a one-dimensional action by considering it as $S = \int dr \mathcal{L}$, in which the Lagrangian is independent of $x_1(r)$. Therefore, the corresponding equation of motion becomes

$$\frac{\partial}{\partial r} \left(\frac{\partial \mathcal{L}}{\partial x_1'} \right) - \left(\frac{\partial \mathcal{L}}{\partial x_1} \right) = C, \quad \text{with} \quad \frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad (3.7)$$

where C is a constant fixed by imposing the turning point of the hypersurface in the bulk, where one has $x_1'(r_t) \rightarrow \infty$. Therefore, for the GB gravity one gets

$$\begin{aligned} & \frac{\tilde{L}^{-2} r^{\frac{(d-2)\theta}{d}} x_1' (2(\theta-d)^2 L^2 \lambda - \tilde{L}^2 d^2 r^{2\theta/d} ((x_1')^2 + 1))}{d^2 ((x_1')^2 + 1)^{3/2}} \\ &= -\frac{1}{r_t^{d-\theta}}. \end{aligned} \quad (3.8)$$

One can solve the equations of motion for small λ and find the following expression for the profile:

$$\begin{aligned} x_1'(r) &= r^d r_t^\theta \sqrt{\frac{1}{r_t^{2d} r^{2\theta} - r^{2d} r_t^{2\theta}}} \\ &+ \frac{2}{d^2} (\theta-d)^2 r^{d-\frac{2\theta}{d}} r_t^\theta \sqrt{\frac{1}{r_t^{2d} r^{2\theta} - r^{2d} r_t^{2\theta}}} \lambda + \mathcal{O}(\lambda^2). \end{aligned} \quad (3.9)$$

After making use of the relation $\ell = 2 \int_0^{r_t} x_1'(r) dr$, the following expression between the turning point r_t and the width of the entangling region can be obtained:

$$\begin{aligned} \ell &= \frac{2\sqrt{\pi} \Gamma(\frac{-d+\theta-1}{2(\theta-d)}) r_t}{\Gamma(\frac{1}{2d-2\theta})} - \frac{2\sqrt{\pi} (\theta-d) \Gamma(\frac{d(d+1)-(d+2)\theta}{2d(d-\theta)}) r_t^{1-\frac{2\theta}{d}}}{d^2 \Gamma(\frac{d(2d+1)-2(d+1)\theta}{2d(d-\theta)})} \lambda \\ &+ \mathcal{O}(\lambda^2). \end{aligned} \quad (3.10)$$

Equation (3.10) can be inverted to find the turning point, and by plugging the profile (3.9) into the entropy function and after doing some calculation, one obtains the expression of the HEE in terms of the turning point. Finally, this leads to HEE, which is as follows:

$$\begin{aligned} S &= \frac{\tilde{L}^d}{2G_N} \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \int r^{\theta-d} \rho^{d-1} \sqrt{\rho'^2 + 1} dr \left[1 + \frac{2L^2 \lambda}{\tilde{L}^2 d^2 (d-1)(d-2) \rho^2 (1 + \rho'^2)} r^{-2\theta/d} \right. \\ &\quad \times ((1-d)(1 + \rho'^2)((\theta-d)((d-2)\theta - d^2) \rho^2 + d(d+4)r(\theta-d)\rho\rho' + d^2 r^2 \rho'^2) + d(d+4)r\rho((\theta-d)\rho\rho' - rd)\rho'') \left. \right]. \end{aligned} \quad (3.12)$$

TABLE III. The numerical constant terms appear in the computation HEE of a strip in five- and six-dimensional hyper-scaling violating backgrounds.

$d+2=5$		$d+2=6$	
$\theta=0$	$\theta=1$	$\theta=0$	$\theta=1$
$\mathfrak{A} \quad -3$	$-\frac{32}{15}$	-4	$-\frac{63}{20}$
$\mathfrak{C} \quad -\frac{12\pi^{3/2}\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{6})^3}$	$-\frac{2^{8/3}\pi^{4/3}\Gamma(-\frac{5}{12})\Gamma(\frac{1}{3})^{5/3}}{9\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})^{5/3}}$	$-\frac{64\pi^2\Gamma(\frac{5}{8})^4}{\Gamma(\frac{1}{8})^4}$	$-\frac{9\pi^{11/4}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})^{3/2}}{10\Gamma(\frac{1}{6})^{5/2}\Gamma(\frac{13}{12})^2}$

$$S_{GB} = S_{EG} + \frac{\tilde{L}^d H^{d-1}}{2G_N} \left(\frac{\mathfrak{A}}{\epsilon^{\frac{(d-2)\theta}{d} + d-1}} + \frac{\mathfrak{C}}{\ell^{\frac{(d-2)\theta}{d} + d-1}} \right) \lambda + \mathcal{O}(\lambda^2), \quad (3.11)$$

where S_{EG} stands for the HEE in Einstein gravity, which is given by (2.11). Note that $\mathfrak{A}$ and $\mathfrak{C}$ are two numerical constants, and for $\theta=0$ and $\theta=1$, we have listed them in Table III. It is worth mentioning that a similar statement has been made in Ref. [26], where the authors used the proposal of Ref. [22] to compute the HEE of a strip in GB gravity. Noting that GB gravity is a special form of the curvature squared action, the final result of HEE deduced from the proposal of [22] (see also [23]) would be the same as [24]; namely, the universal parts of the two computations coincide. However, taking into account the boundary terms would modify the coefficient of the leading UV-divergent term. Regarding this fact, our result of HEE for a strip coincides with [26].

It is clear that in computing the HEE in the static background of GB gravity, the dynamical exponent z does not play any role. Moreover, it can be seen that, in any case, at the zero order of the GB parameter, ϵ and ℓ scale as a power of $d-\theta-1$, whereas at the first order of λ , they scale as $d-\theta+\frac{2\theta}{d}-1$.

B. Spherical and cylindrical entangling regions

In the case of the spherical entangling region, similar to Einstein gravity, the entangling region on the boundary is given by $r < R$ where $t=0$. Therefore, the codimension-two surface in the bulk is parametrized by $\rho = \rho(r)$, and in this case, the functional of the holographic entanglement entropy becomes

TABLE IV. Coefficient of the universal term $[\log(R/\epsilon)]$ which appears in the HEE for the sphere in the $d+2$ -dimensional hyperscaling violating background. The coefficients appear up to the overall factor of $\frac{\pi^{d/2} \tilde{L}^d}{2G_N \Gamma(\frac{d}{2})}$.

	$d=3$	$d=4$	$d=5$	$d=6$
$\theta=0$	$-\frac{1}{2} + \frac{15}{4}\lambda$	$\dots$	$\frac{1}{16}(6-35\lambda)$	$\dots$
$\theta=1$	$\dots$	$\frac{9}{8}R(-1+2\lambda)$	$\dots$	$\sim 1.15R(1-3\lambda)$
$\theta=\frac{3}{2}$	$\dots$	$\dots$	$\dots$	$\sim -0.94R\lambda$
$\theta=2$	$R^2(1-\frac{3}{2}\lambda)$	$\frac{3}{2}R\lambda$	$R^2(-2+5\lambda)$	$\dots$
$\theta=3$	$\dots$	$R^3(1-2\lambda)$	$\dots$	$\frac{25}{8}R^3(-1+3\lambda)$
$\theta=\frac{10}{3}$	$\dots$	$\dots$	$\frac{4}{3}R^2\lambda$	$\dots$
$\theta=4$	$\dots$	$\dots$	$R^4(1-\frac{5}{2}\lambda)$	$\dots$
$\theta=\frac{9}{2}$	$\dots$	$\dots$	$\dots$	$\frac{5}{8}R^3\lambda$
$\theta=5$	$\dots$	$\dots$	$\dots$	$R^5(1-3\lambda)$
$\theta=6$	$\dots$	$\dots$	$\dots$	$\dots$

Similar to the previous sections, one can consider the above entropy functional as an action; thus, one just needs to solve the equations of motion coming from the variation of the action to find the profile. However, as long as we want to address the universal part of the HEE, we only need to write the near-boundary behavior of the hypersurface in the bulk. We find the expansion of the profile near the boundary ($r \approx 0$) as follows:

$$\rho(r) = R + \frac{d-1}{2R(\theta-d+1)}r^2 - \frac{2\theta^2}{Rd(\theta-d+1)(d-2)\theta-d(d-1)}r^{2-2\theta/d}\lambda + \mathcal{O}(\lambda^2). \quad (3.13)$$

After doing some calculations, we summarize the logarithmic terms of HEE up to first order of the coupling constant for the sphere entangling region in Table IV.

We would like to mention that though we mainly consider a theory with hyperscaling violation, when it

TABLE V. Coefficient of the universal term $[\log(\ell/\epsilon)]$ up to the overall factor $\frac{\tilde{L}^{d-1}H}{2G_N}\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$, which appears in the HEE for the cylinder in the $d+2$ -dimensional hyperscaling violating background.

	$d=3$	$d=4$	$d=5$	$d=6$
$\theta=0$	$\frac{1}{8\ell}(-1+2\lambda)$	$\dots$	$\frac{45}{2048\ell}(3-2\lambda)$	$\dots$
$\theta=1$	$\dots$	$-1/2$	$\dots$	$3/8$
$\theta=\frac{3}{2}$	$\dots$	$\dots$	$\dots$	$\sim -0.147\lambda$
$\theta=2$	ℓ	$-\frac{2}{3}\lambda$	$-(9/8)\ell$	$\dots$
$\theta=3$	$\dots$	ℓ^2	$\dots$	$-2\ell^2$
$\theta=\frac{10}{3}$	$\dots$	$\dots$	$\frac{3}{4}\ell\lambda$	$\dots$
$\theta=4$	$\dots$	$\dots$	ℓ^3	$\dots$
$\theta=\frac{9}{2}$	$\dots$	$\dots$	$\dots$	$\frac{2}{5}\ell^2\lambda$
$\theta=5$	$\dots$	$\dots$	$\dots$	ℓ^4
$\theta=6$	$\dots$	$\dots$	$\dots$	$\dots$

comes to the comparison, we restrict ourselves to $\theta=0$. In this way, in Ref. [17], the universal term of the entanglement entropy in even-dimensional CFTs was obtained as follows:

$$S_{\text{uni}} = (-1)^{\frac{d-1}{2}} \frac{\pi^{\frac{d-1}{2}} \tilde{L}^d}{2G_N \Gamma(\frac{d+1}{2})} \left(1 - \frac{2d}{d-2} f_\infty \lambda\right) \log\left(\frac{R}{\epsilon}\right), \quad (3.14)$$

where f_∞ is given by

$$\tilde{L}^2 = \frac{L^2}{f_\infty}, \quad f_\infty = \frac{1 - \sqrt{1-4\lambda}}{2\lambda}. \quad (3.15)$$

We note that at the linear level of the coupling constant λ , our results match (3.14) for $\theta=0$.

For a cylinder with $\rho = \ell$ at the $t=0$ surface, which is defined on the geometry (2.23), we find the following expression for the holographic entropy functional:

$$S = \frac{H \tilde{L}^{d-1} \pi^{(d-1)/2}}{2G_N \Gamma(\frac{d-1}{2})} \int r^{\theta-d} \rho^{d-2} \sqrt{\rho'^2 + 1} dr \left\{ 1 - \frac{2L^2 \lambda}{\tilde{L}^2 d^2 (d-1)(d-2)\rho^2 (1+\rho'^2)^2} r^{-2\theta/d} \right. \\ \times [(1+\rho'^2)((d-1)(\theta-d)((d-2)\theta-d^2)\rho^2 + 2d(d-1)(d-2)(\theta-d)r\rho\rho' + d^2(d-2)(d-3)r^2\rho'^2) \\ \left. + 2rd(d(d-2)r - (d-1)(\theta-d)\rho\rho')\rho\rho'' \right\}. \quad (3.16)$$

Note that in the bulk, the profile is defined by $\rho = \rho(r)$, and the prime stands for the derivative with respect to r . The entropy functional (3.16) should be minimized, and after making use of similar arguments of the sphere, we only need to write the near-boundary behavior of the hypersurface in the bulk. We summarize the coefficients of the logarithmic terms in Table V.

IV. MUTUAL AND TRIPARTITE INFORMATION IN HYPERSCALING VIOLATING BACKGROUNDS

We already mentioned that entanglement entropy could potentially address some information about the underlying conformal field theory. However, the appearance of the UV cutoff in the expression of entanglement entropy makes it a scheme-dependent quantity, so it is natural to explore other

measurements of entanglement to probe the field content of the theory. For example, by introducing an appropriate linear combination of entanglement entropies, one can define an important quantity called mutual information, where for two disjoint regions, say, A and B , it is defined by

$$I(A:B) = S_A + S_B - S_{A \cup B}, \quad (4.1)$$

where S_i 's stand for the entanglement entropies of the corresponding region i for the rest of system. It was shown that mutual information is a finite and positive valued quantity which measures the amount of entanglement shared between two systems A and B . In other words, mutual information quantifies the amount of information that we can obtain from one of the subsystems by looking at the other one, and for two uncorrelated systems, the mutual information becomes identically zero.⁴ In a two-dimensional conformal field theory, mutual information depends on the full operator content of the theory, which indicates that this entanglement measure could carry more information about the underlying conformal field theories [34].

In the context of gauge-gravity duality, by making use of the RT-proposal for computing HEE, we investigate some particular properties of mutual information between two disjoint rectangular systems in hyperscaling violating backgrounds. In order to study the mutual information, we compute this quantity for two disjoint strips, where their lengths and separation are given by ℓ_1 , ℓ_2 and h , respectively. In this case, as mentioned, the HEE is proportional to the minimal area in the bulk, and in the computation of the entanglement entropy for the union of two entangling strips, namely, $S_{A \cup B}$, there are always two candidates for minimal area surfaces in the bulk; depending on the separation of A and B , one of them is indeed the minimum one [35,36]. Therefore, one can write the entanglement entropy for the union of two entangling strips as follows:

$$S_{A \cup B} = \begin{cases} S(\ell_1 + \ell_2 + h) + S(h) & h < \ell_1, \ell_2 \\ S(\ell_1) + S(\ell_2) & h > \ell_1, \ell_2, \end{cases} \quad (4.2)$$

where $S(l)$ stands for the entanglement entropy of a strip with width l . Thus, the mutual information becomes

$$I(A:B) = \begin{cases} S(\ell_1) + S(\ell_2) - S(\ell_1 + \ell_2 + h) - S(h) & h < \ell_1, \ell_2 \\ 0 & h > \ell_1, \ell_2. \end{cases} \quad (4.3)$$

For two strips with the same width and after making use of the above relation and also (4.1), it is obvious that the mutual information becomes zero in the case of $h > \ell$; on the other hand, for $h < \ell$, one finds

$$I(A:B) = 2S(\ell) - S(2\ell + h) - S(h) \equiv I. \quad (4.4)$$

In what follows, we also consider $h < \ell$. This indicates that finding the mutual information of two parallel strips is identical to the entanglement entropy of three strips with widths h , ℓ and $2\ell + h$. We consider the case in which the widths of two parallel strips are the same; otherwise, one has to compute four entanglement entropies corresponding to ℓ_1 , ℓ_2 , h and $\ell_1 + \ell_2 + h$.

The entanglement entropies for a strip in a $d+1$ -dimensional CFT whose gravity dual is provided by the hyperscaling violating geometry, for Einstein and GB gravities, are given by (2.11) and (3.11), respectively. In the case of GB gravity in the AdS_5 background, one obtains

$$I_{GB} = \frac{H^2}{4G_N(2-\theta)} \left[\mathfrak{A} \left(\frac{-2}{\ell^{2-\theta}} + \frac{1}{(2\ell+h)^{2-\theta}} + \frac{1}{h^{2-\theta}} \right) + \mathfrak{C} \lambda \left(\frac{-2}{\ell^{2-\theta/3}} + \frac{1}{(2\ell+h)^{2-\theta/3}} + \frac{1}{h^{2-\theta/3}} \right) \right], \quad (4.5)$$

⁴It is worth mentioning that in Ref. [28] it was shown that the holographic mutual information is divergent for singular entangling regions. The time evolution of such measurements can be found, for example, in [29–33].

where $\mathfrak{A}$ and $\mathfrak{C}$ are two numerical constants given in Table III.

The above consideration can be easily generalized to systems consisting of n disjoint subsystems. If one is interested in the entanglement entropy for a system that consists of n disjoint regions A_i , $i = 1, \dots, n$, then a new quantity should be defined, which is known as the n -partite information. As a matter of fact, this new quantity examines the role of the measurement of the amount of information or correlations (both classical and quantum) between those subsystems. In terms of entanglement entropy, one may define the n -partite information as follows [37]⁵:

$$I^{[n]}(A_{\{i\}}) = \sum_{i=1}^n S(A_i) - \sum_{i < j}^n S(A_i \cup A_j) + \sum_{i < j < k}^n S(A_i \cup A_j \cup A_k) - \dots - (-1)^n S(A_1 \cup A_2 \cup \dots \cup A_n), \quad (4.6)$$

where, similar to the mutual information, $S(A_i \cup A_j \dots)$ is the entanglement entropy of the region $A_i \cup A_j \dots$ in the rest of the system. Note that for $n = 1$ and $n = 2$, this formula gives the entanglement entropy and mutual information.

⁵It is worth mentioning that this definition is not unique. For other generalizations, see [20].

FIG. 1. Holographic mutual information (left) and tripartite information (right) in five-dimensional hyperscaling violating background for $h = 0.5$, $\lambda = 0$, and different values of entangling width $\ell = 1.1, 1.2, 1.3, 1.4$, and 1.5 .

FIG. 2. Contour plot for mutual information (left) and tripartite information (right) in five-dimensional hyperscaling violating background for $\ell = 1$, $\lambda = 0$, and for different values of θ and h .

The case of $n = 3$, which is known as the three-partite (tripartite) information, is given by

$$I^{[3]}(A_1:A_2:A_3) = S(A_1) + S(A_2) + S(A_3) - S(A_1 \cup A_2) - S(A_1 \cup A_3) - S(A_2 \cup A_3) + S(A_1 \cup A_2 \cup A_3). \quad (4.7)$$

It was argued that holographic tripartite information has a fixed sign that is always negative [37,38]; this is indeed the negativity of tripartite information, which results in an important inequality relation on the mutual information. To explicitly see this constraint, let us write (4.7) in terms of the mutual information as

$$I^{[3]}(A_1:A_2:A_3) = I(A_1:A_2) + I(A_1:A_3) - I(A_1:A_2 \cup A_3), \quad (4.8)$$

where the negativity of tripartite information leads one to write

$$I(A_1:A_2 \cup A_3) \geq I(A_1:A_2) + I(A_1:A_3). \quad (4.9)$$

In the context of quantum information theory, this kind of inequality is known as the monogamy relation of mutual information. It is an important characteristic of measurements of quantum entanglement which states that correlations in holographic theories arise primarily from entanglement rather than classical correlations. The above constraint suggests that if two systems, say, A_1 and A_2 , with the same finite Hilbert space dimension, are maximally entangled, then neither of them can share any correlation with a third system A_3 [39].⁶

⁶To investigate other inequalities on holographic entanglement measurements, see [30].

FIG. 3. Holographic mutual information (left) and tripartite information (right) in five-dimensional hyperscaling violating background with the GB term. The graphs show different values of GB coupling for $\theta = 0$ (dashed line) and $\theta = 1$ (solid line), where in both cases we set $\ell = 1$ and $h = 0.5$.

Now the main aim is to investigate the sign of these measures in hyperscaling violating theories. For two intervals of the same length ℓ separated by distance h , by semianalytic computation we show that, in all ranges of parameters that we have considered, the mutual information and tripartite information are always positive and negative, respectively, which means the inequality (4.9) still holds. Therefore, the mutual information is monogamous in hyperscaling violation theories. Figures 1–3 demonstrate this fact for hyperscaling violating background in a five-dimensional background.

V. CONCLUSION

In this paper, by making use of AdS/CFT correspondence, we obtained the universal terms of entanglement entropy for strongly coupled theories with hyperscaling violation. Previously, it was shown that for $\theta = d - 1$ and any dynamical exponent z , the entanglement entropy receives a logarithmic term that indicates a violation of the area law [13]. In this work, we observed that due to the hyperscaling parameter, the divergence structure of the entropy gets new terms. Explicitly, in a $d + 1$ -dimensional field theory, the logarithmically divergent terms appear for $\theta = d - n$ in which $d \geq n$, $n = 1, 3, 5, \dots$. Keeping in mind the dimensional analysis of such theories, the coefficient of the log term in the holographic entanglement entropy scales as R^{d-n} for a sphere and ℓ^{d-n-1} for a cylinder. On the other hand, when the theory includes higher-curvature terms, some other new logarithmic terms appear in HEE. In addition, the universal terms of the entanglement entropy get corrected nontrivially, where in the case of GB terms, we obtained these corrections.

We also examined the mutual information in this setup for two parallel strips with width ℓ separated by distance h , assuming that $h < \ell$. Actually, for two separated systems, the mutual information is a measure of quantifying the amount of entanglement (or information) that these two systems can share. Mutual information is monogamous in the sense that it obeys the following inequality:

$$I(A_1 : A_2) + I(A_1 : A_3) \leq I(A_1 : A_2 \cup A_3).$$

The above inequality states that entangled correlations between A_1 and A_2 cannot be shared with a third system A_3 without spoiling the original entanglement [37]. In general, it is a difficult task to identify the monogamy relation; however, in principle, the holography methods provide a framework to study the above inequality by looking at the tripartite information. The expression of the tripartite information is given by (4.8), and its negativity leads to the above-mentioned monogamy of mutual information. Using the holographic methods, in hyperscaling violating theories, we showed that the tripartite information is always negative, and hence, mutual information is monogamous.

It is worth mentioning that investigating the entanglement inequalities which have been imposed by holographic prescriptions may be important in the sense that this may help us to distinguish between two quantum theories: those that can have a holographic dual and those that cannot. This might be thought of as an intrinsic property of a field theory with a gravity dual. One of the most important inequalities is the strong subadditivity, which states for any three subsystems A_1 , A_2 , and A_3 that do not intersect each other, one has [40]

$$S(A_1 \cup A_2) + S(A_2 \cup A_3) - S(A_1 \cup A_2 \cup A_3) - S(A_2) \geq 0.$$

By making A_2 empty, one obtains the subadditivity inequality $S(A_1) + S(A_2) - S(A_1 \cup A_2) \geq 0$, which leads to the definition of mutual information. It is known that this inequality is also satisfied by using the holographic RT-proposal of finding HEE. Intuitively, it would be interesting to generalize the above inequality to n disjoint subsystems. If we are interested in the measurement of the amount of information or correlations (both classical and quantum) between n disjoint regions A_i , $i = 1, \dots, n$, the n -partite information may provide a good measure for studying this process. Making use of the relation (4.6) for n intervals,

previously we showed that for the special case where all intervals have the same length and distance from their neighboring systems, one has [38,41]

$$I^{[4]}(A_1:A_2:A_3:A_4) \geq 0 \quad \text{and} \quad I^{[5]}(A_1:A_2:A_3:A_4:A_5) \leq 0.$$

Though these inequalities do not hold in general holographic setups [30,31], we observed that for the above-mentioned special case of the same width and separation in hyperscaling violating backgrounds, they are still satisfied.

As a future work, one can consider the time-dependent hyperscaling violating backgrounds and check how the sign of these quantities changes during the thermalization process.

Moreover, for an even-dimensional CFT, the coefficient of the logarithmic term in the entanglement entropy gives us the central charges, and they can be determined by the trace anomaly. This geometric approach was established for

two and higher (even-) dimensional CFT's ([17] and references therein). Thus, it would be interesting to investigate the geometric root of the new logarithmically divergent terms found in this paper.

ACKNOWLEDGMENTS

I would like to thank Mohsen Alishahiha for related comments. Without discussions with M. Reza Mohammadi Mozaffar and Ali Mollabashi, this paper would probably not exist; hereby, I also thank them. I would also like to acknowledge A. Naseh, F. Omid, R. Vazirian, A. Akhavan, M. Vahidinia, A. Shirzad, and M. Hasanloo for some related discussions. I thank a very conscientious referee for offering several useful remarks. This work has been supported in part by the Islamic Azad University Central Tehran Branch.

-
- [1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, *Int. J. Theor. Phys.* **38**, 1113 (1999) [*Adv. Theor. Math. Phys.* **2**, 231 (1998)].
 - [2] S. A. Hartnoll, Lectures on holographic methods for condensed matter physics, *Classical Quantum Gravity* **26**, 224002 (2009).
 - [3] N. Ogawa, T. Takayanagi, and T. Ugajin, Holographic Fermi surfaces and entanglement entropy, *J. High Energy Phys.* **01** (2012) 125.
 - [4] M. Taylor, Non-relativistic holography, [arXiv:0812.0530](#).
 - [5] S. Kachru, X. Liu, and M. Mulligan, Gravity duals of Lifshitz-like fixed points, *Phys. Rev. D* **78**, 106005 (2008).
 - [6] X. Dong, S. Harrison, S. Kachru, G. Torroba, and H. Wang, Aspects of holography for theories with hyperscaling violation, *J. High Energy Phys.* **06** (2012) 041.
 - [7] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, Effective holographic theories for low-temperature condensed matter systems, *J. High Energy Phys.* **11** (2010) 151.
 - [8] K. Balasubramanian and J. McGreevy, Gravity Duals for Non-relativistic CFTs, *Phys. Rev. Lett.* **101**, 061601 (2008).
 - [9] M. Alishahiha, E. O. Colgain, and H. Yavartanoo, Charged black branes with hyperscaling violating factor, *J. High Energy Phys.* **11** (2012) 137.
 - [10] D. Roychowdhury, Holography for anisotropic branes with hyperscaling violation, *J. High Energy Phys.* **01** (2016) 105.
 - [11] B. Gouteraux and E. Kiritsis, Generalized holographic quantum criticality at finite density, *J. High Energy Phys.* **12** (2011) 036.
 - [12] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, England, 2010).
 - [13] M. Alishahiha and H. Yavartanoo, On holography with hyperscaling violation, *J. High Energy Phys.* **11** (2012) 034.
 - [14] M. Srednicki, Entropy and Area, *Phys. Rev. Lett.* **71**, 666 (1993); H. Casini and M. Huerta, Universal terms for the entanglement entropy in $2+1$ dimensions, *Nucl. Phys.* **B764**, 183 (2007).
 - [15] T. Nishioka, S. Ryu, and T. Takayanagi, Holographic entanglement entropy: An overview, *J. Phys. A* **42**, 504008 (2009).
 - [16] S. N. Solodukhin, Entanglement entropy, conformal invariance and extrinsic geometry, *Phys. Lett. B* **665**, 305 (2008).
 - [17] R. C. Myers and A. Singh, Comments on holographic entanglement entropy and RG flows, *J. High Energy Phys.* **04** (2012) 122; B. R. Safdi, Exact and numerical results on entanglement entropy in $(5+1)$ -dimensional CFT, *J. High Energy Phys.* **12** (2012) 005.
 - [18] H. Casini and M. Huerta, Remarks on the entanglement entropy for disconnected regions, *J. High Energy Phys.* **03** (2009) 048.
 - [19] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech.* (2004) P06002.
 - [20] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [21] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha, and M. Smolkin, Holographic GB gravity in arbitrary dimensions, *J. High Energy Phys.* **03** (2010) 111.
 - [22] L. Y. Hung, R. C. Myers, and M. Smolkin, On holographic entanglement entropy and higher curvature gravity, *J. High Energy Phys.* **04** (2011) 025.
 - [23] J. de Boer, M. Kulaxizi, and A. Parnachev, Holographic entanglement entropy in Lovelock gravities, *J. High Energy Phys.* **07** (2011) 109.
 - [24] D. V. Fursaev, A. Patrusev, and S. N. Solodukhin, Distributional geometry of squashed cones, *Phys. Rev. D* **88**, 044054 (2013).
 - [25] X. Dong, Holographic entanglement entropy for general higher derivative gravity, *J. High Energy Phys.* **01** (2014) 044.

- [26] P. Bueno and P.F. Ramirez, Higher-curvature corrections to holographic entanglement entropy in geometries with hyperscaling violation, *J. High Energy Phys.* **12** (2014) 078.
- [27] G. Knodel and J.T. Liu, Higher derivative corrections to Lifshitz backgrounds, *J. High Energy Phys.* **10** (2013) 002; M. Ghodrati, Hyperscaling violating solution in coupled dilaton-squared curvature gravity, *Phys. Rev. D* **90**, 044055 (2014).
- [28] M.R. Mohammadi Mozaffar, A. Mollabashi, and F. Omid, Holographic mutual information for singular surfaces, *J. High Energy Phys.* **12** (2015) 082.
- [29] M.R. Tanhayi, Thermalization of mutual information in hyperscaling violating backgrounds, *J. High Energy Phys.* **03** (2016) 202.
- [30] J. Erdmenger, D. Fernandez, M. Flory, E. Megias, A.K. Straub, and P. Witkowski, Time evolution of entanglement for holographic steady state formation, *J. High Energy Phys.* **10** (2017) 034.
- [31] M. Flory, J. Erdmenger, D. Fernandez, E. Megias, A.K. Straub, and P. Witkowski, Time dependence of entanglement for steady state formation in $\text{AdS}_3/\text{CFT}_2$, *J. Phys. Conf. Ser.* **942**, 012010 (2018).
- [32] S. Kundu and J.F. Pedraza, Spread of entanglement for small subsystems in holographic CFTs, *Phys. Rev. D* **95**, 086008 (2017).
- [33] A. Hajilou, M. Ali-Akbari, and F. Charmchi, A classical string in Lifshitz-Vaidya geometry, [arXiv:1707.00967](https://arxiv.org/abs/1707.00967).
- [34] P. Calabrese, J. Cardy, and E. Tonni, Entanglement entropy of two disjoint intervals in conformal field theory, *J. Stat. Mech.* (2009) P11001.
- [35] W. Fischler and S. Kundu, Strongly coupled gauge theories: High and low temperature behavior of non-local observables, *J. High Energy Phys.* **05** (2013) 098.
- [36] W. Fischler, A. Kundu, and S. Kundu, Holographic mutual information at finite temperature, *Phys. Rev. D* **87**, 126012 (2013).
- [37] P. Hayden, M. Headrick, and A. Maloney, Holographic mutual information is monogamous, *Phys. Rev. D* **87**, 046003 (2013).
- [38] M. Alishahiha, M.R. Mohammadi Mozaffar, and M.R. Tanhayi, On the time evolution of holographic n -partite information, *J. High Energy Phys.* **09** (2015) 165.
- [39] C. Lancien, S. Di Martino, M. Huber, M. Piani, G. Adesso, and A. Winter, Should Entanglement Measures Be Monogamous or Faithful?, *Phys. Rev. Lett.* **117**, 060501 (2016).
- [40] E.H. Lieb and M.B. Ruskai, A Fundamental Property of Quantum-Mechanical Entropy, *Phys. Rev. Lett.* **30**, 434 (1973).
- [41] S. Mirabi, M.R. Tanhayi, and R. Vazirian, On the monogamy of holographic n -partite information, *Phys. Rev. D* **93**, 104049 (2016).