Study of $B_c \rightarrow \psi(2S)K, \eta_c(2S)K, \psi(3770)K$ decays with perturbative QCD approach

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We study the $B_c \rightarrow \psi(2S) K, \eta_c(2S) K, \psi(3770) K$ decays with perturbative QCD approach based on k_T factorization. The new orbitally excited charmonium distribution amplitudes $\psi(1^3D_1)$ based on the Schrödinger wave function of the $n = 1, l = 2$ state for the harmonic-oscillator potential are employed. By using the corresponding distribution amplitudes, we calculate the branching ratio of $B_c \rightarrow \psi(2S) K, \eta_c(2S) K, \psi(3770) K$ decays and the form factors $A_{0,1,2}$ and V for the transition $B_c \rightarrow \psi(1^3D_1)$. We obtain the branching ratio of both $B_c \rightarrow \psi(2S) K$ and $B_c \rightarrow \eta_c(2S) K$ are at the order of 10^{-5} . The effects of two sets of the S-D mixing angle $\theta = -12^\circ$ and $\theta = 27^\circ$ for the decay $B_c \rightarrow \psi(3770) K$ are studied first in this paper. Our calculations show that the branching ratio of the decay $B_c \rightarrow \psi(3770) K$ can be raised from the order of 10^{-6} to the order of 10^{-5} at the mixing angle $\theta = -12^\circ$, which can be tested by the running LHC-b experiments.

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I. INTRODUCTION

The detailed study of B meson decay can provide a good chance for testing the standard model (SM), searching new physics signals beyond the SM [1]. The meson B_c , being the heaviest ground pseudoscalar meson, was observed for the first time via the decay $B_c \rightarrow J/\psi \ell \nu$ in 1.8 TeV $P\bar{P}$ collisions through the CDF detector at the Fermilab Tevatron in 1998 [2]. Because it is indifferent to the strong and electromagnetic interactions, it can decay only through the weak interaction. The meson B_c has rich decay channels [3], for either of its components, i.e., b or c quarks, can decay individually. Because of this innate advantage, it can also provide a very ideal platform to study weak decays of heavy quarks.

Recently, the decay $B_c \rightarrow \psi(2S)\pi$ was updated by the LHC-b Collaboration accompanied by the measured ratio of the branching fractions [4] since its first observation in 2013,

$$\frac{\text{BR}(B_c \rightarrow \psi(2S)\pi)}{\text{BR}(B_c \rightarrow J/\psi\pi)} = 0.268 \pm 0.032(\text{stat}) \pm 0.007(\text{syst}) \pm 0.006(\text{BF}). \quad (1)$$

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The first uncertainty is statistical, the second uncertainty is systematic, and the last term indicates the uncertainty from $\text{Br}(\psi(2S) \rightarrow \mu^+\mu^-)/\text{Br}(J/\psi \rightarrow \mu^+\mu^-)$. Here $\psi(2S)$ is the first radially excited charmonium meson. About the vector charmonium meson $\psi(2S)$ contained in a B_c decays, it has been studied in various approaches. For example, in Ref. [5], the authors calculated the branching ratios for the $B_c \rightarrow \psi(2S)X$ by means of the modified harmonic-oscillator wave function based on the light front quark model; in Ref. [6], the authors used ISGW2 quark model to research the production of radially excited charmonium mesons in B_c decays; the relativistic (constituent) quark model, the potential model, the QCD relativistic potential model, and the improved instantaneous Bethe-Salpeter equation and Mandelstam approach were adopted in Refs. [7–11], respectively. However, it is regrettable that all of these computations are based on a so-called naive factorization assumption, with various form factor inputs. There are also some uncontrolled large theoretical errors with quite different numerical results. Constraining by the unreliability of their models, most of them cannot give any theoretical error estimates. In the work [12], the authors successfully used the perturbative QCD (pQCD) approach [13] to study the S-wave ground state charmonium decays of B_c meson based on the harmonic-oscillator wave functions for the charmonium 1S states. In this paper, we also take harmonic-oscillator wave functions as the approximate wave function of both 2S and 1D charmonium states to study the $B_c \rightarrow \psi(2S) K, \eta_c(2S) K, \psi(3770) K$ decays. Here the 1D charmonium state is the component of $\psi(3770)$ resonance. $\psi(3770)$, the lowest-lying

charmonium state above the open-charm $D\bar{D}$ threshold, is of great interest in quarkonium physics. The rate of decay mode $B \rightarrow \psi(3770) K$, observed in the Belle Collaboration [14], is surprisingly large. It might seemingly indicate that the $\psi(3770)$ is mainly the vector charmonium state $\psi(1^3D_1)$ with a small admixture of vector charmonium state $\psi(2^3S_1)$. It is expected to be expressed as

$$\psi(3770) = \cos\theta |c\bar{c}(1^3D_1)\rangle - \sin\theta |c\bar{c}(2^3S_1)\rangle. \quad (2)$$

Here, the S-D mixing angle θ arises from the ratio of the leptonic decay widths of $\psi(3686)$ and $\psi(3770)$ [15]. Calculations from nonrelativistic potential model provide two sets of mixing scheme: $\theta = -(12 \pm 2)^\circ$ or $\theta = (27 \pm 2)^\circ$ [16–18]. In Ref. [19], the authors have used the light-cone QCD sum rules to calculate the form factors $A_{0,1,2}^{B_c \rightarrow \psi(1^3D_1)}$ and $V^{B_c \rightarrow \psi(1^3D_1)}$. In view of the simple analysis, we calculate the form factors $A_{0,1,2}^{B_c \rightarrow \psi(1^3D_1)}$ and $V^{B_c \rightarrow \psi(1^3D_1)}$ by using the pQCD approach in this work.

This paper is organized as follows. In Sec. II, we describe the theoretical framework and the wave function for the radially excited charmonium mesons $\psi(2S)$, $\eta_c(2S)$ and the orbitally excited charmonium state $\psi(1^3D_1)$. In Sec. III, we present the corresponding form factor expressions for the transition $B_c \rightarrow \psi(1^3D_1)$. The decay amplitudes for the two-body decays $B_c \rightarrow X_{c\bar{c}}K$ ($X = \psi(2S)$, $\eta_c(2S)$, $\psi(3770)$) are computed by employing the pQCD approach in Sec. IV. The numerical results and several points of discussions are presented in Sec. V. Finally we finish this paper with a brief summary.

II. THEORETICAL FRAME AND THE WAVE FUNCTION

The $B_c(X_{c\bar{c}})$ meson momentum $p_1(p_2)$ and the light quarks momentum k_i included in each meson are written in the light-cone coordinates as

$$\begin{aligned} p_1 &= \frac{M_{B_c}}{\sqrt{2}}(1, 1, 0_\perp), \quad k_1 = x_1 p_1 + (0, 0, k_{1\perp}), \\ p_2 &= \frac{M_{B_c}}{\sqrt{2}}(1, r_2^2, 0_\perp), \quad k_2 = x_2 p_2 + (0, 0, k_{2\perp}), \\ p_3 &= \frac{M_{B_c}}{\sqrt{2}}(0, 1 - r_2^2, 0_\perp), \quad k_3 = x_3 p_3 + (0, 0, k_{3\perp}), \end{aligned}$$

with $r_2 = \frac{M_{X_{c\bar{c}}}}{M_{B_c}}$. The K meson momentum $p_3 = p_1 - p_2$. The polarization vectors of the vector mesons $\psi(2S)$ and $\psi(3770)$ are given as

$$\epsilon_L = \frac{1}{\sqrt{2}}(\frac{1}{r_2}, -r_2, 0), \quad \epsilon_\perp = (0, 0, 1_\perp).$$

For the decay $B_c \rightarrow X_{c\bar{c}}K$, the relevant effective Hamiltonian is written as [20]

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{us} V_{cb}^* \{C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)\} + \text{H.c.}, \quad (3)$$

with the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements V_{us} and V_{cb}^* , and the local four-fermion operators

$$\begin{aligned} O_1 &= (\bar{b}_i c_j)_{V-A} (\bar{u}_j s_i)_{V-A}, \\ O_2 &= (\bar{b}_i c_i)_{V-A} (\bar{u}_j s_j)_{V-A}. \end{aligned} \quad (4)$$

Here i and j denote SU(3) color indices. $C(\mu)$ is the Wilson coefficient estimated at renormalization scale μ . Apparently, because there are not any two of the same quarks in the four-quark operators, penguin diagrams cannot contribute. Therefore there is no CP violation in the decays of $B_c \rightarrow X_{c\bar{c}}K$ within the standard model.

In the pQCD theoretical frame, the decay amplitude can be decomposed as the convolution [21–24]

$$\begin{aligned} \mathcal{A}(B_c \rightarrow M_2 M_3) &\sim \int d^4 k_1 d^4 k_2 d^4 k_3 \text{Tr}[\phi_{B_c}(k_1) \phi_X(k_2) \phi_K(k_3) \\ &\quad \times H(k_1, k_2, k_3, t)] e^{-S(t)}, \end{aligned} \quad (5)$$

where Tr refers to the trace over Dirac and color indices; the function $H(k_1, k_2, k_3, t)$ describes the so-called hard scattering kernels, which is scale dependent but can be perturbatively calculated fortunately. The functions $\phi_{B_c}(k_1)$, $\phi_X(k_2)$, and $\phi_K(k_3)$ denote hadron wave functions, which play the role of absorbing the infrared divergence. The Sudakov factors $S(t)$ arise from both $k_\perp$ and threshold resummation, aiming to avoid the end-point singularity.

In our calculations, the distribution amplitude of the realistic model for hadron B_c can be found in Refs. [25,26]

$$\phi_{B_c}(u) = 6u(1-u) \left[1 + \sum_{n=1}^{\infty} a_n(\mu) C_n^{3/2}(2u-1) \right], \quad (6)$$

where $C_1^{3/2}(x) = 3x$, $C_2^{3/2}(x) = \frac{3}{2}(5x^2 - 1)$, and a_n denotes the Gegenbauer moments. In Ref. [25], the authors have calculated the relativistic corrections of Gegenbauer moments and found that they are comparable with the next to leading order radiative corrections, and they have also given the total correction values for the first two Gegenbauer moments a_1 and a_2 , which contain leading order contribution, one-loop QCD radiative corrections, and relativistic corrections.

For the light pseudoscalar meson kaon, the wave function can generally be defined as [27]

$$\begin{aligned} \phi(p, x, \xi) &= \frac{i}{2N_c} \gamma_5 [\not{p} \phi_k^A(x) + m_0 \phi_k^P(x) \\ &\quad + \xi m_0 (\not{p} \psi - 1) \phi_k^T(x)]. \end{aligned} \quad (7)$$

We adopt the distribution amplitudes $\phi_k^{A,P,T}$ from Refs. [28,29],

$$\phi_k^A(x) = \frac{6f_K}{2\sqrt{2N_c}}x(1-x)[1 + 0.15t + 0.405(5t^2 - 1)], \quad (8)$$

$$\phi_k^P(x) = \frac{f_K}{2\sqrt{2N_c}}[1 + 0.106(3t^2 - 1) - 0.148(3 - 30t^2 + 35t^4)/8], \quad (9)$$

$$\phi_k^T(x) = \frac{f_K}{2\sqrt{2N_c}}t[1 + 0.1581(5t^2 - 3)], \quad (10)$$

with $t = 1 - 2x$. Here the wave function ϕ_K^A refers to the twist-2 distribution amplitude, and both ϕ_K^P and ϕ_K^T refer to the twist-3 distribution amplitudes.

In Refs. [30,31], the harmonic-oscillator wave functions have been applied to describe the charmonium ground state J/ψ , and the theoretical results agree well with the published experimental data. In Ref. [32], the authors also adopted the harmonic-oscillator wave functions for the mesons $\psi(2S)$ and $\eta_c(2S)$; the ratio $B_r(B_c \rightarrow \psi(2S)\pi)/B_r(B_c \rightarrow J/\psi\pi) = 0.29_{-0.11}^{+0.17}$ they got is very close to the experimental data $0.268 \pm 0.032 \pm 0.007$ [4]. It pushes us to try to predict the branching ratio for the $B_c \rightarrow \psi(3770)K$ decays, for which no one has yet made a theoretical prediction. As is mentioned before, the meson $\psi(3770)$ is a 2S-1D mixing state. The pure 2S state has been mentioned above. The pure 1D state indicates the principal quantum number $n = 1$ and the orbital angular momentum $l = 2$, which means that it is only an angular excitation state. We are going to describe it by using harmonic-oscillator wave functions for the $n = 1$, $l = 2$ Schrödinger state.

For the wave function of the vector charmonium states $\psi(2S)$, $\psi_{c\bar{c}}(1D)$, we refer to the vector mesons ω , ρ , and D^* in Refs. [33,34], and the wave function of the pseudoscalar meson $\eta_c(2S)$ gets the same access to $\eta_c(1S)$ [30].

$$\begin{aligned} &\langle \psi(p_2, \epsilon_L) | \bar{c}_\alpha(0) c_\beta(z) | 0 \rangle \\ &= \frac{1}{\sqrt{2N_c}} \int d^4k_2 e^{+ik_2 \cdot z} \not{\epsilon}_L [m_{\psi_{c\bar{c}}(1D)} \phi_\psi^L(k_2) + \not{\epsilon}_2 \phi_\psi^T(k_2)]_{\beta\alpha} \end{aligned} \quad (11)$$

$$\begin{aligned} &\langle \psi(p_2, \epsilon_T) | \bar{c}_\alpha(0) c_\beta(z) | 0 \rangle \\ &= \frac{1}{\sqrt{2N_c}} \int d^4k_2 e^{+ik_2 \cdot z} \not{\epsilon}_T \phi_\psi^T(k_2) [m_{\psi_{c\bar{c}}(1D)} + \not{\epsilon}_2]_{\beta\alpha} \end{aligned} \quad (12)$$

$$\begin{aligned} &\langle \eta_c(2S)(p_2) | \bar{c}_\alpha(0) c_\beta(z) | 0 \rangle \\ &= \frac{-i}{\sqrt{2N_c}} \int d^4k_2 e^{+ik_2 \cdot z} [\gamma_5 \not{\epsilon}_2 \phi^p(k_2) + \gamma_5 m \phi^s(k_2)]_{\beta\alpha}, \end{aligned} \quad (13)$$

where p_2 plays the part of the momentum of the charmonium mesons $\psi(2S)$, $\psi(1^3D_1)$, or $\eta_c(2S)$ and m is their corresponding mass. The $\epsilon^{L(T)}$ means the longitudinal (transverse) polarization vector. Here the functions ϕ_ψ^L , ϕ_ψ^T , and ϕ^p pertain to twist-2 distribution amplitudes, and both the ϕ_ψ^L and ϕ^s pertain to the twist-3 distribution amplitudes. Their distribution amplitudes of the asymptotic models for the radially excited charmonium mesons $\psi(2S)$ and $\eta_c(2S)$ have been studied in Ref. [32]. We focus on the distribution amplitude of the asymptotic model for the $\psi(1^3D_1)$ state as follows.

First of all, we give the isochronous Schrödinger equation based on the harmonic-oscillator potential as

$$\psi_{1D}(r) \propto (\alpha r)^2 e^{-\frac{\alpha^2 r^2}{2}} Y_2 m(\theta, \varphi), \quad (14)$$

where $Y_2 m(\theta, \varphi)$ is the spherical harmonic function. $\alpha^2 = \frac{m_c \omega}{2}$ and ω is the harmonic vibration frequency.

In order to get its function in the momentum space, we apply the Fourier transform to it,

$$\psi_{1D}(\vec{k}) = \int d^3\vec{r} e^{-i\vec{r} \cdot \vec{k}} \psi_{1D}(r) \propto (\vec{k}^2 - 3k_z^2) e^{-\frac{\vec{k}^2}{2\alpha^2}}, \quad (15)$$

where k represents three-dimensional momentum.

Taking the substitution ansatz [35,36]

$$\vec{k}_\perp \rightarrow \vec{k}_\perp, \quad k_z \rightarrow (x - \bar{x}) \frac{m_0}{2}, \quad m_0^2 = \frac{m_c^2 + \vec{k}_\perp^2}{x\bar{x}}, \quad (16)$$

where $\bar{x} = 1 - x$, and x is the momentum fraction associated with one of the partons. Now we write down the relationship formula

$$k^2 \rightarrow \frac{k_\perp^2 + (x - \bar{x})^2 m_c^2}{4x\bar{x}}; \quad (17)$$

then the wave function (15) is replaced by

$$\begin{aligned} \psi_{1D}(\vec{k}) &\rightarrow \psi_{1D}(x, \vec{k}_\perp) \\ &\propto \left(\frac{k_\perp^2 + (x - \bar{x})^2 m_c^2}{4x\bar{x}} - 3 \frac{(k_\perp^2 + m_c^2)(x - \bar{x})^2}{4x\bar{x}} \right) \\ &\quad \times e^{-\frac{k_\perp^2 + (x - \bar{x})^2 m_c^2}{8x\bar{x}\alpha^2}}. \end{aligned} \quad (18)$$

Now we convert the transverse momentum $k_\perp$ to its conjugate variable b ; the oscillator wave function $\psi_{1D}(x, b)$ can be written as

$$\begin{aligned} \psi_{1D}(x, b) &\sim \int d^2k_\perp e^{-ib \cdot k_\perp} \psi_{1D}(x, k_\perp) \\ &\propto x\bar{x} \mathcal{I}(x) e^{-x\bar{x} \frac{m_c^2}{\omega} [\omega^2 b^2 + (\frac{x - \bar{x}}{2x\bar{x}})^2]}, \end{aligned} \quad (19)$$

with

$$\mathcal{I}(x) = \left(\frac{1}{x\bar{x}} - m\omega b^2 \right) (6x^4 - 12x^3 + 7x^2 - x) - \frac{m_c(1-2x)^2}{4\omega x\bar{x}}. \quad (20)$$

The modified wave functions can be given as

$$\psi_{1D}(x, b) \propto \Phi^{\text{asy}}(x) \mathcal{I}(x) e^{-x\bar{x} \frac{m_c}{\omega} [\omega^2 b^2 + (\frac{x-\bar{x}}{2x\bar{x}})^2]} \quad (21)$$

with the $\Phi^{\text{asy}}(x)$ being the asymptotic models [36]. We then obtain the distribution amplitudes for the orbitally excited charmonium state $\psi(1^3D_1)$,

$$\psi_{1D}^{L,T}(x, b) = \frac{f_{1D}}{2\sqrt{2N_c}} N^{L,T} x\bar{x} \mathcal{I}(x) e^{-x\bar{x} \frac{m_c}{\omega} [\omega^2 b^2 + (\frac{x-\bar{x}}{2x\bar{x}})^2]}, \quad (22)$$

$$\psi_{1D}^t(x, b) = \frac{f_{1D}}{2\sqrt{2N_c}} N^t (x - \bar{x})^2 \mathcal{I}(x) e^{-x\bar{x} \frac{m_c}{\omega} [\omega^2 b^2 + (\frac{x-\bar{x}}{2x\bar{x}})^2]}, \quad (23)$$

with the normalization conditions

$$\int_0^1 \psi_{1D}^i(x, 0) dx = \frac{f_{1D}}{2\sqrt{2N_c}} \quad (24)$$

where $N_c = 3$ is the color number, $N^i (i = L, T, t)$ are the normalization constants, and $f_{1D} = 47.8$ MeV [19] is the decay constant of the orbitally excited $\psi(1^3D_1)$ state. Both the wave functions Eqs. (22) and (23) are symmetric under $x \leftrightarrow \bar{x}$.

To calculate the decay branching ratio of the model $B_c \rightarrow \psi(1^3D_1) + K$, it is the frequency of oscillations ω_{1D} that cannot be determined easily. For light $q\bar{q}$ systems, the best value of oscillation frequency from spectroscopy and decays is 0.379 GeV, but for heavy $Q\bar{Q}$ states, it is a bit larger and also has a range $\omega = 0.4\text{--}0.6$ GeV in the literature (see Ref. [37] and references therein). According to the quark model theory, studies show that the effective oscillatory parameter ω for higher $c\bar{c}$ multiplets is smaller than the corresponding lower ones [37], which is based on the fact that the excited states have large spatial extensions. For example, in Ref. [30], the authors tried to take the ground charmonium state J/ψ frequency from 0.5 to 0.8 GeV; in Ref. [32], the authors tried the value $\omega = 0.2$ GeV for the first radially excited charmonium mesons

$\psi(2S)$ and $\eta_c(2S)$. Although it is still difficult to define precisely what the ω for $\psi(1^3D_1)$ state is, we try to adopt $\omega = 0.35\text{--}0.55$ GeV for the $\psi(1^3D_1)$ state in this work, which is reasonable based on the above theoretical analysis.

III. $B_c \rightarrow \psi(1^3D_1)$ FORM FACTORS

Based on the pQCD theoretical framework, the form factors of the transition $B_c \rightarrow \psi(1^3D_1)$ are similar to that of $B_c \rightarrow J/\psi$, which can be defined as [38,39]

$$\begin{aligned} & \langle \psi(1^3D_1)(p_2, \epsilon_T) | \bar{c} \gamma_\mu b | B_c(p_1) \rangle \\ &= \frac{2iV(q^2)}{M_{B_c} + m_{\psi(1D)}} \epsilon^{\mu\nu\rho\sigma} \epsilon_\nu^* p_{2\rho} p_{2\sigma} \end{aligned} \quad (25)$$

$$\begin{aligned} & \langle \psi(1^3D_1)(p_2, \epsilon_T) | \bar{c} \gamma_\mu \gamma_5 b | B_c(p_1) \rangle \\ &= \frac{\epsilon^* \cdot q}{q^2} 2m_{\psi(1D)} q^\mu A_0^{B_c \rightarrow \psi(1^3D_1)}(q^2) + (M_{B_c} + m_{\psi(1D)}) \\ & A_1^{B_c \rightarrow \psi(1^3D_1)}(q^2) \left[\epsilon^{*\mu} - \frac{\epsilon^* \cdot q}{q^2} q^\mu \right] - \frac{\epsilon^* \cdot q}{M_{B_c} + m_{\psi(1D)}} \\ & \left[(p_1 + p_2)^\mu - \frac{M_{B_c}^2 - m_{\psi(1D)}^2}{q^2} q^\mu \right] A_2^{B_c \rightarrow \psi(1^3D_1)}(q^2), \end{aligned} \quad (26)$$

where $q = p_1 - p_2$ is the momentum transfer and ϵ^* represents the polarization vector of the $\psi(1^3D_1)$ charmonium state. $V^{B_c \rightarrow \psi(1^3D_1)}$ and $A_{0,1,2}^{B_c \rightarrow \psi(1^3D_1)}$ are the transition form factors. Furthermore, in the large-recoil limit, i.e., $q^2 = 0$, we have

$$A_0(0) = \frac{1+r_2}{2r_2} A_1(0) - \frac{1-r_2}{2r_2} A_2(0), \quad (27)$$

where $r_2 = \frac{m_{\psi(1D)}}{M_{B_c}}$.

Based on the single gluon exchange, the lowest-order diagrams (a) and (b) in Fig. 1. contribute to the transition form factor for the $B_c \rightarrow \psi(1^3D_1)$ at the maximally recoiling point ($q^2 = 0$). Our predictions of the form factors are collected in Table I and are compared with the results from the light-cone QCD sum rule [19].

In the pQCD theory scheme, the expressions for the form factor $A_{0,1,2}^{B_c \rightarrow \psi(1^3D_1)}$ and $V^{B_c \rightarrow \psi(1^3D_1)}$ are written directly as

FIG. 1. Feynman diagrams contributing to the $B_c \rightarrow X_{c\bar{c}}$ form factors.

$$A_0 = -8\pi M_{B_c}^2 C_F \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) \{[(1-x_2-2r_b)\phi^L(x_2, b_2) + (2x_2-2+r_b)r_2\phi'(x_2, b_2)] \\ \times E_a(t_a)H_a(\alpha, \beta_a, b_1, b_2)S_t(x_2) + [r_2^2(x_1-1)-x_1]\phi^L(x_2, b_2)E_b(t_b)H_b(\alpha, \beta_b, b_1, b_2)S_t(x_1)\}, \quad (28)$$

$$A_1 = -\frac{r_2}{1+r_2} 8\pi M_{B_c}^2 C_F \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) \left\{ \left[(2-x_2-4r_b-x_2r_2^2)\phi^L(x_2, b_2) \right. \right. \\ \left. \left. + \left(r_b r_2 - 2r_2 + 4x_2 r_2 + \frac{r_b}{r_2} - \frac{2}{r_2} \right) \phi'(x_2, b_2) \right] E_a(t_a)H_a(\alpha, \beta_a, b_1, b_2)S_t(x_2) \right. \\ \left. - (1+2r_c-2x_1+r_2^2)\phi^L(x_2, b_2)E_b(t_b)H_b(\alpha, \beta_b, b_1, b_2)S_t(x_1) \right\}, \quad (29)$$

$$A_2 = \frac{r_2}{1-r_2} 8\pi M_{B_c}^2 C_F \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) \left\{ \left[(x_2 r_2^2 - x_2)\phi^L(x_2, b_2) + \left(r_b r_2 - 2r_2 - \frac{r_b}{r_2} + \frac{2}{r_2} \right) \phi'(x_2, b_2) \right] \right. \\ \left. \times E_a(t_a)H_a(\alpha, \beta_a, b_1, b_2)S_t(x_2) + (1-2x_1)(1-r_2^2)\phi^L(x_2, b_2)E_b(t_b)H_b(\alpha, \beta_b, b_1, b_2)S_t(x_1) \right\}, \quad (30)$$

$$V = -(1+r_2)8\pi M_{B_c}^2 C_F \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) \phi^L(x_2, b_2) \{ (x_2 r_2 + r_b - 2) \times E_a(t_a)H_a(\alpha, \beta_a, b_1, b_2)S_t(x_2) \\ - r_2 \times E_b(t_b)H_b(\alpha, \beta_b, b_1, b_2)S_t(x_1) \}, \quad (31)$$

where $r_{b,c} = \frac{m_{b,c}}{M_{B_c}}$ and $C_F = 4/3$ is the group factor of the $SU(3)_c$ gauge group. The function $E_i(t_i)$, the hard-scattering kernel function H_i , and the parametrization factor $S_t(x)$ are displayed together in the Appendix.

IV. THE DECAY AMPLITUDES

The decays $B_c \rightarrow X_{c\bar{c}}K$ are dominated by tree diagrams based on the operator product expansion. Through the analysis in Sec. II, there is no pollution from penguins and annihilation diagrams. In the perturbative QCD approach, the Feynman diagrams are displayed in Fig. 2, where

(a) and (b) are of factorizable topology; (c) and (d) are of nonfactorizable topology. Furthermore, we can directly use Eq. (5) to give the decay amplitudes

$$\mathcal{A}(B_c \rightarrow X_{c\bar{c}}K) = \frac{G_F}{\sqrt{2}} V_{us} V_{cb}^* \sum_{i=a,b,c,d} \mathcal{A}_i. \quad (32)$$

The explicit expressions for the relative amplitudes $\mathcal{A}_i$ are displayed as follows.

The amplitudes for $B_c \rightarrow \psi(2S)K$ decay,

$$\mathcal{A}_a = -8\pi C_F f_k M_{B_c}^4 (1-r_2^2) \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) E_a(t_a) C_a(t_a) H_a(\alpha, \beta_a, b_1, b_2) S_t(x_2) \\ \times [(1-x_2-2r_b)\phi_\psi^L(2S)(x_2, b_2) + (2x_2-2+r_b)r_2\phi_\psi'(2S)(x_2, b_2)], \quad (33)$$

$$\mathcal{A}_b = -8\pi C_F f_k M_{B_c}^4 (1-r_2^2) \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) E_b(t_b) C_b(t_b) H_b(\alpha, \beta_b, b_1, b_2) S_t(x_1) \\ \times [r_2^2(x_1-1)-x_1]\phi_\psi^L(2S)(x_2, b_2), \quad (34)$$

$$\mathcal{A}_c = -\frac{32\pi C_F}{\sqrt{2N_c}} M_{B_c}^4 (1-r_2^2) \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_2 b_3 db_2 db_3 \phi_{B_c}(x_1) \phi_k^a(x_3) E_c(t_c) C_c(t_c) H_c(\alpha, \beta_c, b_2, b_3) \\ \times [(1-r_2^2)(1-x_1-x_3)\phi_\psi^L(2S)(x_2, b_2) + r_2(x_2-x_1)\phi_\psi'(2S)(x_2, b_2)], \quad (35)$$

$$\mathcal{A}_d = -\frac{32\pi C_F}{\sqrt{2N_c}} M_{B_c}^4 (1-r_2^2) \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_2 b_3 db_2 db_3 \phi_{B_c}(x_1) \phi_k^a(x_3) E_d(t_d) C_d(t_d) H_d(\alpha, \beta_d, b_2, b_3) \\ \times \{ [2x_1-x_2-x_3+(x_3-x_2)r_2^2]\phi_\psi^L(2S)(x_2, b_2) + r_2(x_2-x_1)\phi_\psi'(2S)(x_2, b_2) \}, \quad (36)$$

FIG. 2. The lowest-order Feynman diagrams for the $B_c \rightarrow X_{c\bar{c}}$ decays.

with $r_2 = \frac{m_{\psi(2S)}}{M_{B_c}}$, and the distribution amplitudes $\phi^{L,t}(2S)$ can be found in Ref. [32].

The amplitude for $B_c \rightarrow \eta_c(2S)K$ decay.

$$\mathcal{A}_a = -i8\pi C_F f_k M_{B_c}^4 (1 - r_2^2) \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) E_a(t_a) C_a(t_a) H_a(\alpha, \beta_a, b_1, b_2) S_t(x_2) \times \{[(1 - x_2 - 2r_b)\phi_{\eta_c}^v(2S)(x_2, b_2) + (2x_2 - 2 + r_b)r_2\phi_{\eta_c}^s(2S)(x_2, b_2)]\}, \quad (37)$$

$$\mathcal{A}_b = -i8\pi C_F f_k M_{B_c}^4 (1 - r_2^2) \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 db_1 db_2 \phi_{B_c}(x_1) E_b(t_b) C_b(t_b) H_b(\alpha, \beta_b, b_1, b_2) S_t(x_1) \times \{[r_2^2(1 - x_1) + r_c]\phi_{\eta_c}^v(2S)(x_2, b_2) - 2r_2[(1 - x_1) + r_c]\phi_{\eta_c}^s(2S)(x_2, b_2)\}, \quad (38)$$

$$\mathcal{A}_c = -\frac{i32\pi C_F}{\sqrt{2N_c}} M_{B_c}^4 (1 - r_2^2) \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_2 b_3 db_2 db_3 \phi_{B_c}(x_1) \phi_k^a(x_3) E_c(t_c) C_c(t_c) H_c(\alpha, \beta_c, b_2, b_3) \times \{[1 - x_1 - x_3 + r_2^2(2x_2 + x_3 - x_1 - 1)]\phi_{\eta_c}^v(2S)(x_2, b_2) - r_2(x_2 - x_1)\phi_{\eta_c}^s(2S)(x_2, b_2)\}, \quad (39)$$

$$\mathcal{A}_d = \frac{i32\pi C_F}{\sqrt{2N_c}} M_{B_c}^4 (1 - r_2^2) \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_2 b_3 db_2 db_3 \phi_{B_c}(x_1) \phi_k^a(x_3) E_d(t_d) C_d(t_d) H_d(\alpha, \beta_d, b_2, b_3) \times \{[x_2 + x_3 - 2x_1 + r_2^2(x_2 - x_3)]\phi_{\eta_c}^v(2S)(x_2, b_2) - r_2(x_2 - x_1)\phi_{\eta_c}^s(2S)(x_2, b_2)\}, \quad (40)$$

with $r_2 = \frac{m_{\eta_c(2S)}}{M_{B_c}}$, and the distribution amplitudes $\phi_{\eta_c}^{v,s}(2S)$ have been given in Ref. [32].

As for the decay amplitude $B_c \rightarrow \psi(1^3D_1)K$, it is similar to the decay amplitude for $B_c \rightarrow \psi(2S)K$, but with the replacement $\phi_{\psi(2S)}^{L,t} \rightarrow \phi_{\psi(1D)}^{L,t}$.

The branching ratios for the decay $B_c \rightarrow X_{c\bar{c}}K$ in the B_c meson rest frame can be written as

$$\text{BR}(B_c \rightarrow X_{c\bar{c}}K) = \frac{\tau_{B_c}}{8\pi M_{B_c}^2} |\mathcal{A}(B_c \rightarrow X_{c\bar{c}}K)|^2, \quad (41)$$

where the $X_{c\bar{c}}$ represents the meson $\psi(2S)$, $\eta_c(2S)$, and the charmonium state $\psi(1^3D_1)$, respectively, and the common momentum $p = (M_{B_c}^2 - m_{X_{c\bar{c}}}^2)/2M_{B_c}$. As for the decay mode $B_c \rightarrow \psi(3770)K$, we give the expression based on the idea of S-D mixing scheme,

$$\text{BR}(B_c \rightarrow \psi(3770)K) = \frac{\tau_{B_c}}{8\pi} \frac{p}{M_{B_c}^2} |\mathcal{A}(B_c \rightarrow \psi(3770)K)|^2, \quad (42)$$

with

$$\mathcal{A}(B_c \rightarrow \psi(3770)K) = \cos \theta \mathcal{A}(B_c \rightarrow \psi(1^3D_1)K) - \sin \theta \mathcal{A}(B_c \rightarrow \psi(2S)K). \quad (43)$$

V. NUMERICAL EVALUATION AND DISCUSSIONS

In our numerical calculation, we adopt the following input parameters [12,25,32,40,41]:

$$\begin{aligned} m_{\psi_{2S}} &= 3.686 \text{ GeV}, & m_{\eta_c(2S)} &= 3.639 \text{ GeV}, & M_{B_c} &= 6.277 \text{ GeV}, \\ f_{\psi_{2S}} &= 296_{-2}^{+3} \text{ MeV}, & f_{B_c} &= 489 \pm 4 \text{ MeV}, & f_{\eta_c(2S)} &= 243_{-111}^{+79} \text{ MeV}, \\ m_c &= 1.28 \text{ GeV}, & m_b &= 4.18 \text{ GeV}, & m_{\psi_{1D}} &= 3.77 \text{ GeV}, & a_1 &= 0.32, & a_2 &= -0.34 \\ \tau_{B_c} &= (0.453 \pm 0.041) \text{ ps}, & |V_{us}| &= 0.2252 \pm 0.0009, & |V_{cb}| &= 0.0409 \pm 0.0011. \end{aligned} \quad (44)$$

If not specified, we adopt their central values as the default input.

Our numerical results for the form factors $A_{0,1,2}^{B_c \rightarrow \psi(1^3D_1)}$ and $V^{B_c \rightarrow \psi(1^3D_1)}$ are listed in Table I, and in Table II we show the result of the branching fractions for $B_c \rightarrow X_{c\bar{c}}K$ decays. The branching fractions for $B_c \rightarrow \psi(3770)K$ decay are displayed in Table V, which contains two sets of S-D mixing angles. The relative theoretical uncertainties are listed in both Tables III and IV.

From Table I, we can find that the form factors of $A_{1,2}$ decrease as the frequency of oscillations parameter ω_{1D}

gets bigger, while A_0 and V increase first and then decrease. In addition, the form factors are not sensitive to the shape parameter ω_{1D} except for the value of the form factor V at $\omega_{1D} = 0.35 \text{ GeV}$. We also notice that our calculations of form factors are close to the calculations in Ref. [19] from the light-cone QCD sum rule approach except for the form factor V , which is almost four times the result of ours. We expect that it could be compared with more results calculated by other theoretical methods.

From Table II, we find that our predictions of branching fraction for the decay $B_c \rightarrow \psi(2S)K$ and $B_c \rightarrow \eta_c(2S)K$ are close to the predictions in Refs. [5,7,9,42]; this indicates that the harmonic-oscillator wave functions for radially excited 2S charmonium states are reasonable and applicable. It is clear that the branching fraction of the decay mode $B_c \rightarrow \psi(1^3D_1)K$ decreases with the increasing parameter ω_{1D} , and its change trend becomes slowly as $\omega_{1D} \geq 0.5 \text{ GeV}$, so we take the branching fraction result at $\omega_{1D} = 0.5 \text{ GeV}$ as the preferred value. In view of the success about the B meson exclusive decay $B \rightarrow \psi(3770)K$ [43], we calculate the branching fraction of $B_c \rightarrow \psi(3770)K$ based on the S-D mixing scheme, whose computations are listed in Table V; here the selection basis

TABLE I. The form factors $A_{0,1,2}$ and V of the transition $B_c \rightarrow \psi(1^3D_1)$ are calculated in the pQCD approach.

	A_0	A_1	A_2	V
$\omega_{1D} = 0.35$	4.86×10^{-2}	10.6×10^{-2}	0.38	0.42×10^{-2}
$\omega_{1D} = 0.40$	4.97×10^{-2}	10.0×10^{-2}	0.33	2.14×10^{-2}
$\omega_{1D} = 0.45$	5.01×10^{-2}	9.18×10^{-2}	0.30	2.50×10^{-2}
$\omega_{1D} = 0.50$	4.91×10^{-2}	8.59×10^{-2}	0.27	2.73×10^{-2}
$\omega_{1D} = 0.55$	4.46×10^{-2}	7.98×10^{-2}	0.25	2.66×10^{-2}
[19]	2.86×10^{-2}	4.9×10^{-2}	0.11	0.11

TABLE II. Branching fractions of the decays $B_c \rightarrow X_{c\bar{c}}K$ in the pQCD approach.

Decay modes	ω	This work	Reference [5]	Reference [7]	Reference [9]	Reference [10]	Reference [42]
$B_c \rightarrow \psi(2S)K$	0.20	3.33×10^{-5}	$(2.30 \pm 0.32) \times 10^{-5}$	1.0×10^{-5}	2.0×10^{-5}	9.3×10^{-6}	5.7×10^{-5}
$B_c \rightarrow \eta_c(2S)K$	0.20	1.74×10^{-5}		1.0×10^{-5}	2.2×10^{-5}	5.0×10^{-6}	
$B_c \rightarrow \psi(1^3D_1)K$	0.35	9.55×10^{-6}					
	0.40	8.06×10^{-6}					
	0.45	6.94×10^{-6}					
	0.50	5.88×10^{-6}					
	0.55	5.08×10^{-6}					

TABLE III. The theoretical uncertainties derived from the decay constant f_{B_c} and the hard scale $t \pm 0.15t$, we compute the form factors $A_{0,1,2}$ and V at $q^2 = 0$ for the transition $B_c \rightarrow \psi(1^3D_1)$ in the pQCD approach based on the preferred shape parameter $\omega_{1D} = 0.50$ GeV.

ω_{1D}	A_0	A_1
0.5 GeV	$4.91^{+0.04+0.46}_{-0.04-0.02} \times 10^{-2}$	$8.59^{+0.07+0.52}_{-0.07-0.01} \times 10^{-2}$
	2.86×10^{-2} [19]	4.9×10^{-2} [19]
	A_2	V
0.5 GeV	$0.27^{+0.00+0.01}_{-0.00-0.00}$	$2.73^{+0.02+0.58}_{-0.02-0.10} \times 10^{-2}$
	0.11 [19]	0.11 [19]

of the S-D mixing angle has been introduced in Sec. I. By comparing Tables II and V, we find that, for the decay $B_c \rightarrow \psi(3770) K$ near our preferred shape parameter $\omega_{1D} = 0.5$ GeV, we get a branching fraction in order of magnitude 10^{-6} if we treat the $\psi(3770)$ as a pure 1D charmonium state. But the branching fraction can be raised from 5.88×10^{-6} to 1.264×10^{-5} obviously when we adopt the mixing angle $\theta = -12^\circ$, which is consistent with the arguments about mixing angle θ in Refs. [18,19,43–45]. We attribute this remarkable improvement to a very small decay constant of charmonium state $\psi(1^3D_1)$ (0.0478 GeV) compared with the decay constant of the meson $\psi(2S)$ (0.296 GeV). This decay mode has not been measured yet, but around $\mathcal{O}(10^9)B_c$ mesons can be anticipated with $1 fb^{-1}$ of data at the LHC [46], which means it could be soon tested at the LHC-b experiment; that will help us to understand the structure of $\psi(3770)$ and the constituent quark model.

Moreover, we give the relative theoretical uncertainties in Tables III and IV. In Table III, we analyzed the uncertainties of $B_c \rightarrow \psi(1^3D_1)$ transition form factors based on the preferred value $\omega_{1D} = 0.5$ GeV. The two theoretical uncertainties come from the decay constant f_{B_c} and the hard scale $t \pm 0.15t$ in Eq. (5), which characterizes

TABLE IV. The branching fractions of $B_c \rightarrow (\psi(2S), \eta_c(2S))K$ decays with the different theoretical uncertainties arisen from the shape parameters, the decay constants f_{B_c} , $f_{\psi(2S)}$, or $f_{\eta_c(2S)}$, and the hard scale $t \pm 0.15t$. For the $\psi(1^3D_1)$ charmonium state, we adopt the preferred parameter $\omega_{1D} = 0.5$ GeV. Because there is no uncertainty about the decay constant $f_{\psi_{1D}}$, we only give the branching fraction of the decay $B_c \rightarrow \psi(1^3D_1)K$ with the theoretical uncertainty induced by the decay constant f_{B_c} and the hard scale $t \pm 0.15t$.

Decay modes	
$B_c \rightarrow \psi(2S)K$	$3.33^{+0.211+0.054+0.067+0.443}_{-0.232-0.054-0.044-0.127} \times 10^{-5}$
$B_c \rightarrow \eta_c(2S)K$	$1.74^{+0.144+0.028+1.315+0.173}_{-0.114-0.028-1.227-0.054} \times 10^{-5}$
$B_c \rightarrow \psi(1^3D_1)K$	$5.88^{+0.096+0.238}_{-0.095-0.054} \times 10^{-6}$

TABLE V. Branching fractions of the decay $B_c \rightarrow \psi(3770)K$ in the pQCD approach based on two sets of S-D mixing angles.

ω_{1D}	$\theta = -12^\circ$	$\theta = 27^\circ$
0.35 GeV	1.770×10^{-5}	3.012×10^{-8}
0.40 GeV	1.570×10^{-5}	1.540×10^{-8}
0.45 GeV	1.414×10^{-5}	8.495×10^{-8}
0.50 GeV	1.264×10^{-5}	2.113×10^{-7}
0.55 GeV	1.148×10^{-5}	3.705×10^{-7}

the size of the next to leading order contribution. From Tables III and I, it is easy to see that the main error comes from the nonperturbative shape parameter, which needs more theoretical and experimental efforts to understand. In Table IV, we display the branching fractions of $B_c \rightarrow (\psi(2S), \eta_c(2S), \psi(1^3D_1))K$ decays with different theoretical uncertainties. For the decay mode $B_c \rightarrow \psi(2S)K$, the main theoretical uncertainties come from the shape parameter error $\omega_c = 0.2 \pm 0.01$ for $\psi(2S)$ meson and hard scale $t \pm 0.15t$, which produce an uncertainty in the range of -6.9% to 6.3% and -3.8% to 13% , respectively. For the decay mode $B_c \rightarrow \eta_c(2S)K$, the shape parameter error produces an uncertainty in the range of -6.5% to 8.3% and the hard scale t leads to an uncertainty from -3.1% to 9.9% . For the decay $B_c \rightarrow \psi(1^3D_1)K$, the larger uncertainty coming from the hard scale t , which can bring an uncertainty of -0.9% to 4.0% to the branching ratio. These small uncertainties show that the harmonic-oscillator wave function is an excellent candidate for describing charmonium states and the uncertainties from the next to leading order contributions are very limited and can be neglected safely for these decay modes. The largest uncertainty of the decay $B_c \rightarrow \eta_c(2S)K$ appears in the decay constant $f_{\eta_c(2S)}$; this point is easy to understand for the great uncertainty $f_{\eta_c(2S)} = 243^{+79}_{-111}$ MeV, whose origin has been studied in Ref. [32], and whose value is expected to be improved by future precise experimental measurements at LHC-b or Super-B factories. The other uncertain factors such as CKM matrix elements and B_c meson life are too small and can be neglected safely.

VI. SUMMARY

In this paper, we calculated the form factors of $B_c \rightarrow \psi(1^3D_1)$ and gave the predictions for the branching fractions of two-body decays $B_c \rightarrow \psi(2S) K$, $\eta_c(2S)K$, $\psi(3770)K$ in the perturbative QCD approach. The new orbitally excited charmonium distribution amplitudes of $\psi(1^3D_1)$ based on the Schrödinger wave function of the $n = 1, l = 2$ state for the harmonic-oscillator potential are employed. We also discussed the theoretical uncertainties in this paper. In view of the mixing mechanism of the S-D wave, we gave a theoretical calculation of the branching ratio for the decay $B_c \rightarrow \psi(3770) K$ first in the literature. Our calculations show that the branching ratio of the decay

$B_c \rightarrow \psi(3770) K$ can reach the order of 10^{-5} , which can be tested by the running LHC-b experiments.

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APPENDIX: FORMULAS FOR THE CALCULATION USED IN THE TEXT

The function $E_i(t_i)$ in the decay amplitudes is defined by

$$E_i(t_i) = \alpha_s(t_i) e^{-S_i}, \quad (\text{A1})$$

in which the strong running coupling constant $\alpha_s(t_i)$ [47] based on the standard one-loop calculations is adopted in our calculations and

$$\begin{aligned} S_{a(b)} &= S_{B_c}(t_{a(b)}) + S_{X_{c\bar{c}}}(t_{a(b)}) = s(x_1 p_1^+, b_1) + s(x_2 p_2^+, b_2) + s(\bar{x}_2 p_2^+, b_2) + 2 \int_{1/b_1}^t \frac{d\mu}{\mu} \gamma_q + 2 \int_{1/b_2}^t \frac{d\mu}{\mu} \gamma_q \\ S_{c(d)} &= S_{B_c}(t_{a(b)}) + S_{X_{c\bar{c}}}(t_{c(d)}) + S_K(t_{c(d)}) = s(x_1 p_1^+, b_2) + s(x_2 p_2^+, b_2) + s(\bar{x}_2 p_2^+, b_2) + s(x_3 p_3^-, b_3) \\ &\quad + s(\bar{x}_3 p_3^-, b_3) + 2 \int_{1/b_2}^t \frac{d\mu}{\mu} \gamma_q + 2 \int_{1/b_2}^t \frac{d\mu}{\mu} \gamma_q + 2 \int_{1/b_3}^t \frac{d\mu}{\mu} \gamma_q, \end{aligned} \quad (\text{A2})$$

where the functions $s(Q, b)$ are called the Sudakov factor resulting from the resummation of double logarithms and can be found in [48]. $\gamma_q = -\alpha_s/\pi$ is the anomalous dimension of the quark. For killing the large logarithmic radiative corrections, the hard scale t_i in the amplitudes is chosen as the maximum in H_i ,

$$\begin{aligned} t_{a(b)} &= \max\left(\sqrt{|\alpha|}, \sqrt{|\beta_{a(b)}|}, 1/b_1, 1/b_2\right), \\ t_{c(d)} &= \max\left(\sqrt{|\alpha|}, \sqrt{|\beta_{c(d)}|}, 1/b_2, 1/b_3\right). \end{aligned} \quad (\text{A3})$$

The hard-scattering kernels function H arise from the Fourier transform of virtual quark and gluon propagators and are written as follows,

$$\begin{aligned} H_i(\alpha, \beta_i, b_1, b_2) &= H_1(\alpha, b_1) \times H_2(\beta_i, b_1, b_2) \\ H_1(\alpha, b_1) &= \begin{cases} K_0(b_1 \sqrt{\alpha}) & \alpha > 0 \\ K_0(ib_1 \sqrt{-\alpha}) & \alpha < 0 \end{cases} \\ H_2(\beta_i, b_1, b_2) &= \begin{cases} \theta(b_1 - b_2) K_0(b_1 \sqrt{\beta}) I_0(b_2 \sqrt{\beta}) + (b_1 \leftrightarrow b_2) & \beta > 0 \\ \theta(b_1 - b_2) K_0(ib_1 \sqrt{-\beta}) J_0(b_2 \sqrt{-\beta}) + (b_1 \leftrightarrow b_2) & \beta < 0 \end{cases} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \alpha &= -M_{B_c}^2 (x_1 - x_2)(x_1 - r_2^2 x_2), \\ \beta_a &= -M_{B_c}^2 [(1 - x_2)(1 - r_2^2 x_2) - r_b^2], \\ \beta_b &= -M_{B_c}^2 [(1 - x_1)(r_2^2 - x_1) - r_c^2], \\ \beta_c &= -M_{B_c}^2 (x_2 - x_1)[(\bar{x}_3 - x_1) + (x_2 - \bar{x}_3)r_2^2], \\ \beta_d &= -M_{B_c}^2 (x_2 - x_1)[(x_3 - x_1) + (x_2 - x_3)r_2^2], \end{aligned} \quad (\text{A5})$$

$$C_{a(b)} = C_1 + C_2/N_c, C_{c(d)} = C_2, \quad (\text{A6})$$

where J_0 is the Bessel function and K_0, I_0 are modified Bessel functions with $K_0(ix) = \frac{\pi}{2}(-N_0(x) + iJ_0(x))$. The $C_{1,2}$ are the Wilson coefficients.

The jet function $S_t(x)$ coming from the threshold resummation [33] contributes to the factorizable diagrams (a) and (b) in Fig. 2,

$$S_t(x) = \frac{2^{1+2c} \Gamma(\frac{3}{2} + c)}{\sqrt{\pi} \Gamma(1 + c)} [x(1 - x)]^c, \quad c = 0.3. \quad (\text{A7})$$

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