Conformal blocks from Wilson lines with loop corrections

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We compute the conformal blocks of the Virasoro minimal model or its W_N extension with large central charge from Wilson line networks in a Chern-Simons theory including loop corrections. In our previous work, we offered a prescription to regularize divergences from loops attached to Wilson lines. In this paper, we generalize our method with the prescription by dealing with more general operators for $N = 3$ and apply it to the identity W_3 block. We further compute general light-light blocks and heavy-light correlators for $N = 2$ with the Wilson line method and compare the results with known ones obtained using a different prescription. We briefly discuss general W_3 blocks.

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I. INTRODUCTION

Two dimensional conformal field theory (CFT) with large central charge c is supposed to be dual to the semiclassical regime of three-dimensional gravity on anti-de Sitter (AdS) space. Since holography is expected to provide a formulation of quantum gravity in terms of dual CFT, we would obtain some insights on the quantum aspects of gravity from $1/c$ corrections in dual CFT. As a simple example, we examine the W_N minimal model, which has a coset description as

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \quad (1.1)$$

with the central charge

$$c = (N-1) \left(1 - \frac{N(N+1)}{(k+N)(k+N+1)} \right). \quad (1.2)$$

The dual bulk description was proposed to be the Chern-Simons gauge theory based on $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Lie algebra [1]. In particular, we can study the quantum corrections of AdS_3 gravity in terms of the Virasoro minimal model given by the coset (1.1) with $N = 2$. We consider the large c

regime of the minimal model, which corresponds to a perturbative regime of Chern-Simons theory. The expression of central charge (1.2) implies that $k = -1 - N + \mathcal{O}(c^{-1})$; thus, the large c regime of the model should be nonunitary. Nevertheless, the holography has been argued to work [2–4], and it should be useful at least for problems related to its symmetry algebra.

The basic quantities of CFT are the spectrum of operators and correlation functions or conformal blocks. In particular, the conformal block decomposition of the correlator is an essential tool for recent developments on conformal bootstrap programs (see, e.g., Ref. [5]). In terms of Chern-Simons theory, conformal blocks at the large c limit were proposed to be computed by the networks of open Wilson lines, and the proposal was confirmed for explicit examples [6,7]. In this paper, we develop techniques to evaluate conformal blocks beyond the large c limit by applying recent works on loop effects for Wilson lines in Refs. [8–10]; see also Ref. [11]. The proposal of the map between conformal blocks and Wilson line networks in Refs. [6,7] can be easily extended beyond the large c limit as in Ref. [9]. We evaluate the $1/c$ corrections of conformal blocks from the loop corrections of the Wilson line networks. The computations are rather subtle, and a main issue is how to regularize divergences associated with loop diagrams. In the language of the usual Lagrangian formulation, how to deal with this kind of divergences was established. Since we are dealing with open Wilson lines instead of Lagrangian, it is quite nontrivial to formulate a general procedure. In our previous paper [10], we offered a way to regularize divergences by renormalizing parameters in Wilson lines, which correspond to couplings between bulk scalars and a higher spin current, along with rescaling

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the overall factor of open Wilson lines. Moreover, we showed that ambiguities related to the renormalization procedure can be removed by requiring that the correlators of boundary theory are consistent with the symmetry. In this paper, we adopt the renormalization procedure in Ref. [10] for regularizing divergences.

The primary operators of the W_N minimal model (1.1) are labeled as $(\Lambda_+; \Lambda_-)$, where $\Lambda_{\pm}$ represent the highest weights of $SU(N)$ Lie algebra. The operator with $(f; 0)$ [where f denotes the fundamental representation of $SU(N)$] is dual to a bulk scalar field, and the operators with $(\Lambda; 0)$ correspond to their composite fields [1]. On the other hand, the operator with $(0; \Lambda)$ has its conformal weight of order c , and it is proposed to be dual to a conical defect geometry [2–4]. The symmetry of W_N algebra for the model (1.1) is generated by higher spin currents $J^{(s)}$ with $s = 2, 3, \dots, N$. In Ref. [10], we evaluated the correlators as

$$\begin{aligned} & \langle \bar{\mathcal{O}}_{(\Lambda;0)}(z_1) \mathcal{O}_{(\Lambda;0)}(z_2) \rangle, \\ & \langle \bar{\mathcal{O}}_{(\Lambda;0)}(z_1) \mathcal{O}_{(\Lambda;0)}(z_2) J^{(s)}(z_3) \rangle \end{aligned} \quad (1.3)$$

with general $\Lambda = 2j$ (where j represents the spin of representation) for $N = 2$ and with $\Lambda = f$ for $N = 3$. In particular, we reproduced the $1/c$ corrections of conformal weight for operators dual to bulk scalars up to $1/c^2$ order by adopting our regularization prescription.

In this paper, we extend the previous analysis and apply our Wilson line method to four-point conformal blocks with $N = 2$ and $N = 3$. For $N = 2$, we consider the correlator as

$$\langle \bar{\mathcal{O}}_{(\Lambda;0)}(\infty) \mathcal{O}_{(\Lambda;0)}(1) \bar{\mathcal{O}}_{(\Xi;0)}(z) \mathcal{O}_{(\Xi;0)}(0) \rangle \quad (1.4)$$

with $(\Lambda, \Xi) = (2q, 2j)$. The conformal blocks were already examined by utilizing open Wilson lines in Ref. [8]. Since we adopt a different regularization prescription, our formulation is not the same as theirs. We can easily relate the two methods for the identity Virasoro block, so we focus on general Virasoro blocks.² In Ref. [8], they obtained the all-order expression in z for the general Virasoro block up to $1/c$ order except for the j, q -independent part. For the part, they obtained the first few terms in z expansion. We compute the general block with our formulation and obtain the all-order expression in z up to $1/c$ order including the j, q -independent part. For $N = 3$, we examine the four-point conformal block (1.4) with setting $\Xi = f$ for simplicity. We explicitly compute the identity W_3 block using the product

of two open Wilson lines and reproduce the CFT result. For the purpose, we first extend the previous analysis for the two- and three-point functions (1.3) by dealing with general Λ for $N = 3$. The general W_3 blocks at the leading order in $1/c$ were obtained in Ref. [12] from the CFT viewpoint, and they were reproduced in Ref. [7] using the networks of the open Wilson line. We point out that our formulation combines these results in a quite nice way. We also analyze the $1/c$ correction of a simple nonidentity W_3 block with $(\Lambda, \Xi) = (f, f)$.

We further consider the four-point function with heavy operators $\mathcal{O}_{(0;\Lambda)}$ as

$$\langle \bar{\mathcal{O}}_{(0;\Lambda)}(\infty) \mathcal{O}_{(f;0)}(1) \bar{\mathcal{O}}_{(f;0)}(z) \mathcal{O}_{(0;\Lambda)}(0) \rangle \quad (1.5)$$

with $N = 2$. The heavy operator corresponds to a conical defect geometry as mentioned above, and we compute the correlator using an open Wilson line in the conical space. For quantum corrections to the open Wilson line, we adopt our formulation developed so far. For quantum corrections to the conical geometry, we make use of the results in Ref. [13] obtained by applying the coadjoint orbits of the Virasoro group in Refs. [14,15]. In this way, we reproduce the previous results from CFT computation in Refs. [16–18]. The Wilson line method was used in Refs. [7,8] at the leading order in $1/c$, where they used open Wilson lines for the heavy operators instead of conical spaces. For general N , the computation with our method reduces to those in Refs. [19,20] at the leading order in $1/c$.

The organization of this paper is as follows. In the next section, we introduce the W_N minimal model and write down the expressions of conformal blocks examined in terms of Wilson lines in this paper. In Sec. III, we explain our bulk method for conformal blocks, particularly focusing on our prescription for regularizing divergences. In Sec. III B, we review our previous work in Ref. [10] on the two- and three-point functions (1.3) for $N = 2$, and in Sec. III C, we extend the analysis by considering general light operators with $N = 3$. In Sec. IV, we apply the method to compute conformal blocks from the bulk viewpoint. In Sec. IV A, we obtain the identity W_3 block from the product of two Wilson lines and compare it to the CFT result. In Sec. IV B, we examine general Virasoro blocks with our formulation and see the relation to previous results in Ref. [8]. We further discuss general W_3 blocks. In Sec. V, we apply our formulation to correlators with two heavy operators along with two light operators for $N = 2$. We conclude this paper and discuss open problems in Sec. VI. In Appendix A, we summarize CFT computations of four-point conformal blocks in the W_N minimal model. In particular, we obtain four-point function of (1.4) with Λ corresponding a rectangular Young diagram and $\Xi = f$. As an application of Refs. [10,21], we compute the $1/c$ corrections of higher spin charges for the scalar operators using the method of conformal block decomposition. In

¹For $N = 2$, the conformal weight of operator was reproduced up to $1/c$ order in Ref. [9] by applying a Wilson line method, which is possible without specifying any regularization prescription.

²We refer to the conformal block with the identity operator or its descendants exchanged as an identity block and to other conformal blocks as general blocks.

Appendix B, we include some technical details of $\mathfrak{sl}(3)$ Lie algebras. In Appendix C, we compute the $1/c$ correction of a simple nonidentity W_3 block. In Appendix D, we examine how correlators obtained from Wilson lines change under local conformal transformations.

II. W_N MINIMAL MODEL

The minimal model of W_N algebra can be described by the coset (1.1) with the central charge (1.2); see, e.g., Refs. [22–24] for some details. The primary states are labeled as $(\Lambda_+; \Lambda_-)$ with $\Lambda_{\pm}$ as the highest weights of $SU(N)$ Lie algebra. We denote $n_i^{\pm}$ ($n_N^{\pm} = 0$) as the numbers of boxes in the i th row of Young diagrams corresponding to $\Lambda^{\pm}$ and define power sums as

$$C_s(\Lambda_+; \Lambda_-) \equiv \frac{2}{s} \sum_{i=1}^N \left[\alpha_+ \left(n_i^+ - \frac{|\Lambda_+|}{N} + \rho_i \right) + \alpha_- \left(n_i^- - \frac{|\Lambda_-|}{N} + \rho_i \right) \right]^s \quad (2.1)$$

by following Ref. [4]. Here, $|\Lambda_{\pm}|$ denote the numbers of boxes in $\Lambda_{\pm}$, and $\alpha_{\pm}$ are given by

$$\alpha_+ = \sqrt{\frac{N+k+1}{2(N+k)}}, \quad \alpha_- = -\sqrt{\frac{N+k}{2(N+k+1)}}. \quad (2.2)$$

The components of the Weyl vector ρ are $\rho_i = (N+1)/2 - i$. The conformal weight of primary state $(\Lambda_+; \Lambda_-)$ is then written as

$$h(\Lambda_+; \Lambda_-) = C_2(\Lambda_+; \Lambda_-) - \frac{c - N + 1}{24}. \quad (2.3)$$

The two-point function of the operator $(\Lambda_+; \Lambda_-)$ is fixed by the symmetry as

$$\langle \bar{\mathcal{O}}_{(\Lambda_+; \Lambda_-)}(z_1) \mathcal{O}_{(\Lambda_+; \Lambda_-)}(z_2) \rangle = \frac{1}{z_{12}^{2h(\Lambda_+; \Lambda_-)}}. \quad (2.4)$$

The overall normalization can be set arbitrary by changing the definition of operator. Here and in the following, we only consider the holomorphic sector, but it is straightforward to include the antiholomorphic sector.

We are interested in a regime with the large central charge (1.2) but finite N , which implies that we should take

$$k = -1 - N + \frac{N(N^2 - 1)}{c} + \frac{N(1 - N^2)(1 - N^3)}{c^2} + \mathcal{O}(c^{-3}). \quad (2.5)$$

In the regime, the conformal weight behaves as $\mathcal{O}(c^0)$ for $\Lambda_- = 0$, while $\mathcal{O}(c)$ for $\Lambda_- \neq 0$. For a while, we focus on

the case only with light operators $(\Lambda_+; \Lambda_-) = (\Lambda; 0)$, and later, we examine correlators with $\Lambda_- \neq 0$ as well. For $N = 2$, we label the representation of $\mathfrak{su}(2)$ by spin quantum number $j = n_1/2$, where $n_i = n_i^+$ is defined above. The conformal weight of the operator with j is given by

$$h_j \equiv h(2j; 0) = -j - \frac{6j(j+1)}{c} - \frac{78j(j+1)}{c^2} + \mathcal{O}(c^{-3}) \quad (2.6)$$

in $1/c$ expansion. For general N , we expand the highest weight in the fundamental weights ω_i as $\Lambda = \sum_{i=1}^{N-1} \lambda_i \omega_i$, where λ_i are the Dynkin labels. As a simple but nontrivial example, we also study in some details the $N = 3$ case, where the Dynkin labels (λ_1, λ_2) are related as $n_1 = \lambda_1 + \lambda_2$ and $n_2 = \lambda_2$. The conformal weight of the operator with $(\lambda_1, \lambda_2; 0)$ is

$$h(\lambda_1, \lambda_2; 0) = -\lambda_1 - \lambda_2 - \frac{8(\lambda_1^2 + \lambda_1 \lambda_2 + 3\lambda_1 + \lambda_2^2 + 3\lambda_2)}{c} + \mathcal{O}(c^{-2}). \quad (2.7)$$

For general N , the computation of correlators is simplified when $\Lambda = n\omega_m$, which corresponds to the case with $n_1 = \dots = n_m = n$, $n_{m+1} = \dots = n_N = 0$. The conformal weight of the operator with $\Lambda = n\omega_m$ is

$$h(n\omega_m, 0) = \frac{1}{2} mn(m - N) + \frac{mn(N^2 - 1)(m - N)(n + N)}{2c} + \mathcal{O}(c^{-2}). \quad (2.8)$$

Expanding the conformal weight (2.3) in $1/c$ as

$$h(\Lambda; 0) = h_0(\Lambda; 0) + \frac{h_1(\Lambda; 0)}{c} + \mathcal{O}(c^{-2}), \quad (2.9)$$

the two-point function (2.4) becomes

$$\langle \bar{\mathcal{O}}_{(\Lambda; 0)}(z_1) \mathcal{O}_{(\Lambda; 0)}(z_2) \rangle = \frac{1}{z_{12}^{2h_0(\Lambda; 0)}} - \frac{2h_1(\Lambda; 0) \log(z_{12})}{c z_{12}^{2h_0(\Lambda; 0)}} + \mathcal{O}(c^{-2}). \quad (2.10)$$

Later, we shall see that the bulk computations reproduce the shifts of conformal weight from the terms proportional to $\log(z_{12})$.

The symmetry of W_N algebra is generated by higher spin currents $J^{(s)}(z)$ with $s = 2, 3, \dots, N$. In this paper, we adopt the convention for the higher spin currents in Refs. [10, 21]. In particular, the two-point functions are given by

$$\langle J^{(s)}(z_1) J^{(s)}(z_2) \rangle = \frac{B_s}{z_{12}^{2s}} \quad (2.11)$$

with

$$B_s = \frac{(1-2s)N_s c}{6},$$

$$N_s = \frac{3\sqrt{\pi}\Gamma(s)(1-N)_{s-1}(N+1)_{s-1}}{2^{2s-2}(N^2-1)\Gamma(s+\frac{1}{2})}. \quad (2.12)$$

For $s = 2, 3$, we have

$$B_2 = \frac{c}{2}, \quad B_3 = \frac{(4-N^2)c}{6},$$

$$N_2 = -1, \quad N_3 = \frac{N^2-4}{5}, \quad (2.13)$$

and, in particular, the normalization for the spin 2 current is the same as the usual one for the energy momentum tensor, i.e., $J^{(2)}(z) = T(z)$. The three-point function with a higher spin current $J^{(s)}$ is fixed by the Ward identity as

$$\langle \bar{\mathcal{O}}_{(\Lambda_+; \Lambda_-)}(z_1) \mathcal{O}_{(\Lambda_+; \Lambda_-)}(z_2) J^{(s)}(z_3) \rangle$$

$$= w^{(s)}(\Lambda_+; \Lambda_-) \left(\frac{z_{12}}{z_{13}z_{23}} \right)^s \langle \bar{\mathcal{O}}_{(\Lambda_+; \Lambda_-)}(z_1) \mathcal{O}_{(\Lambda_+; \Lambda_-)}(z_2) \rangle, \quad (2.14)$$

where $w^{(s)}(\Lambda_+; \Lambda_-)$ denotes the spin s charge of the operator $(\Lambda_+; \Lambda_-)$. The spin 2 charge is nothing but the conformal weight as $w^{(2)}(\Lambda_+; \Lambda_-) = h(\Lambda_+; \Lambda_-)$ in (2.3). The expression of spin 3 charge $w^{(3)}(\Lambda_+; \Lambda_-) \equiv w(\Lambda_+; \Lambda_-)$ may be found in Refs. [2,4,25] as

$$w(\Lambda_+; \Lambda_-) = \sqrt{\frac{2N(N+2)(1-N^2)}{((c-1)(2+N)+3N^2)}} C_3(\Lambda_+; \Lambda_-), \quad (2.15)$$

where the overall factor is due to our convention for the spin 3 current. For $N = 3$, the spin 3 charge of the operator $(\lambda_1, \lambda_2; 0)$ is given by

$$w(\lambda_1, \lambda_2; 0) = \frac{\lambda_1 - \lambda_2}{3} + \frac{8(15\lambda_1 + 15\lambda_2 + 13)(\lambda_1 - \lambda_2)}{15c}$$

$$+ \mathcal{O}(c^{-2}). \quad (2.16)$$

For general N , the spin 3 charge of the operator $(n\omega_m; 0)$ is

$$w(n\omega_m; 0) = \frac{1}{6}mn(2m^2 - 3mN + N^2) + \frac{mn(N^2 - 1)(2m^2 - 3mN + N^2)(3n(N+2) + N(N+2) - 2)}{6(N+2)c} + \mathcal{O}(c^{-2}) \quad (2.17)$$

in $1/c$ expansion.

A main aim of this paper is to examine conformal blocks using open Wilson lines in the bulk theory. For $N = 2$, we consider the four-point function of scalar operators in (1.4) with $(\Lambda, \Xi) = (2q, 2j)$, that is,

$$\langle \mathcal{O}_q(\infty) \mathcal{O}_q(1) \mathcal{O}_j(z) \mathcal{O}_j(0) \rangle, \quad \mathcal{O}_l \equiv \mathcal{O}_{(2l;0)}. \quad (2.18)$$

Here, we have used $\bar{\mathcal{O}}_l = \mathcal{O}_l$ since the finite representation of $\mathfrak{su}(2)$ is self-conjugate. The identity Virasoro block has been evaluated using Wilson lines up to order $1/c^2$ as [8]

$$z^{2h_j} \mathcal{V}_0(z) = 1 + \frac{2h_j h_q}{c} z^2 {}_2F_1(2, 2; 4; z)$$

$$+ \frac{1}{c^2} [h_j^2 h_q^2 k_a(z) + (h_j^2 h_q + h_j h_q^2) k_b(z)$$

$$+ h_j h_q k_c(z)] + \mathcal{O}(c^{-3}), \quad (2.19)$$

where the expressions of k_a, k_b, k_c may be used as in (A.11) of Ref. [21]. Here, $z^2 {}_2F_1(2, 2; 4; z)$ at the $1/c$ order corresponds to the global block³ with the exchange of the energy momentum tensor. We do not compute the identity block with the Wilson line method since there is no

³The global part of Virasoro algebra is given by $\mathfrak{sl}(2)$ algebra, and only the $\mathfrak{sl}(2)$ descendants are considered as intermediate states for the global block.

significant difference from the previous computation in Ref. [8]. The Virasoro block with the intermediate operator $\mathcal{O}_p$ was similarly obtained up to $1/c$ order as

$$z^{2h_j} \mathcal{V}_p(z) = z^{h_p} {}_2F_1(h_p, h_p; 2h_p; z)$$

$$+ \frac{1}{c} [h_j h_q f_a(h_p, z) + (h_j + h_q) f_b(h_p, z)$$

$$+ f_c(h_p, z)] + \mathcal{O}(c^{-2}). \quad (2.20)$$

All order expressions in z for f_a and f_b and the first few orders in z for f_c were obtained in Ref. [8]. In Sec. IV B, we examine the general block, applying our method, and, in particular, we obtain all order expressions in z even for f_c .

For correlators in the W_N minimal model with general N , we may compute correlators by applying the Coulomb gas method as explained in Appendix A; see, e.g., Refs. [19,26–29]. A conformal block is given by a choice of integration contour over the positions of screening operators inserted. Since it is nontrivial to identify proper integration contours,⁴ we consider the simple correlator

⁴We are particularly interested in the identity W_N block, which appears in the s -channel decomposition of the four-point function (1.4). However, we cannot find out any references giving the s -channel expressions. The expressions in the t channel or the u channel were obtained in Refs. [26,27] when Λ or Ξ corresponded to a degenerate representation; see also Ref. [29].

$$\langle \bar{\mathcal{O}}_{(n\omega_m;0)}(\infty) \mathcal{O}_{(n\omega_m;0)}(1) \bar{\mathcal{O}}_{(f;0)}(z) \mathcal{O}_{(f;0)}(0) \rangle. \quad (2.21)$$

In this case, the Coulomb integrals reduce to a hypergeometric function ${}_2F_1(a, b; c; z)$, and we can apply the analysis for the Virasoro minimal model; see, e.g., Ref. [30]. The final result obtained in Appendix A is

$$\begin{aligned} & |z|^{-4h(f;0)} |1-z|^{2nm \frac{k+N+1}{N(k+N)}} \left[\left| {}_2F_1 \left(n + \frac{n}{k+N}, -\frac{m}{k+N}; -\frac{N}{k+N}; z \right) \right|^2 \right. \\ & \left. + \mathcal{N} \left| z^{\frac{k+2N}{k+N}} (1-z)^{\frac{m-N-n(k+N+1)}{k+N}} {}_2F_1 \left(1 + \frac{m}{k+N}, -n+1 - \frac{n}{k+N}; 2 + \frac{N}{k+N}; z \right) \right|^2 \right] \end{aligned} \quad (2.22)$$

with

$$\mathcal{N} \equiv - \frac{\Gamma(1 + \frac{m}{k+N}) \Gamma(-n+1 - \frac{n}{k+N}) \Gamma(1 + \frac{N-m}{k+N}) \Gamma(n+1 + \frac{N+n}{k+N}) \Gamma(-\frac{N}{k+N})^2}{\Gamma(n + \frac{n}{k+N}) \Gamma(-\frac{m}{k+N}) \Gamma(-n - \frac{N+n}{k+N}) \Gamma(-\frac{N-m}{k+N}) \Gamma(2 + \frac{N}{k+N})^2}. \quad (2.23)$$

The first and second terms correspond to the W_N blocks of identity and adjoint operators, respectively. In Sec. IV A, we examine the identity W_3 block up to the $1/c^2$ order. In Sec. IV B 3, we give some comments on general W_3 blocks, and in Appendix C, we analyze the adjoint W_3 block with $n = m = 1$ up to the $1/c$ order.

The identity W_N block is a useful quantity even in the CFT perspective. For instance, we proposed a way to obtain the $1/c$ correction of spin s charge for the primary operator in the W_N minimal model (1.1) by utilizing the Virasoro block decomposition of the identity W_N block in Refs. [10,21]. Since the W_N minimal model is a solvable model, we know how to obtain the higher spin charges with finite N, k as in Refs. [22,31,32]. However, in practice, it is difficult to obtain their explicit expressions for larger s , and our method provides a simple way to obtain them including $1/c$ corrections. In Refs. [10,21], we have considered higher spin charges for operators only in the fundamental representation. In Appendix A, we extend the analysis by examining higher spin charges for the operator with $\Lambda = n\omega_m$ including $1/c$ corrections.⁵ Among others, we can reproduce the $1/c$ correction of spin 3 charge in (2.17).

We also apply our approach to heavy-light correlators. At the leading order in $1/c$, general CFT correlators of this type were reproduced from the analysis of the open Wilson line in a conical space in Refs. [19,20]. For the analysis of $1/c$ corrections, we may work on a simple form of the correlator as

$$\langle \bar{\mathcal{O}}_{(0;n\omega_m)}(\infty) \mathcal{O}_{(f;0)}(1) \bar{\mathcal{O}}_{(f;0)}(z) \mathcal{O}_{(0;n\omega_m)}(0) \rangle. \quad (2.24)$$

In this case, there is only one type of exchanged operator, and the Coulomb integrals can be carried out as (see, e.g., Ref. [19])

⁵See Ref. [33] for a recent work on higher spin charges for general operators in a supersymmetric model.

$$|1-z|^{-4h(f;0)} |z|^{-\frac{2nm}{N}} \left| {}_2F_1 \left(-n, -\frac{m}{N+k}; -\frac{N}{N+k}; 1-z \right) \right|^2. \quad (2.25)$$

We reproduce the CFT result for $N = 2$ by applying our method for open Wilson lines and the analysis for conical spaces in Ref. [13]. The bulk approach should be applicable also for general N , but we leave detailed analysis to future work.

III. METHOD FOR BULK COMPUTATION

In this section, we explain our method to compute correlators or conformal blocks from the networks of open Wilson lines. In the next subsection, we first introduce $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons gauge theory and the networks of Wilson lines corresponding to conformal blocks. We then explain our prescription developed in Ref. [10] to remove divergences associated with loop diagrams. In Sec. III B, we review our previous work in Ref. [10] on the two-point function of a scalar operator and the three-point function with two scalars and a spin 2 current for $N = 2$. The analysis can be extended straightforwardly for $N = 3$ if the scalar operator belongs to the (anti)fundamental representation of $\mathfrak{su}(3)$. In Sec. III C, we extend our method by analyzing correlators involving general scalar operators.

A. Chern-Simons theory and open Wilson lines

The bulk description of the W_N minimal model with large c was proposed to be given by $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons gauge theory [2], the action of which is

$$\begin{aligned} S &= S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}], \\ S_{\text{CS}}[A] &= \frac{\hat{k}}{4\pi} \int \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \end{aligned} \quad (3.1)$$

Here, $\hat{k}$ is the level of the Chern-Simons theory, and A takes values in $\mathfrak{sl}(N)$ Lie algebra.⁶ As a higher spin gravity, it is important to decompose the generators of $\mathfrak{sl}(N)$ Lie algebra in terms of embedded $\mathfrak{sl}(2)$. We consider the principal embedding as

$$\mathfrak{sl}(N) = \mathfrak{sl}(2) \oplus \left(\bigoplus_{s=3}^N g^{(s)} \right). \quad (3.2)$$

Here, $g^{(s)}$ denotes the $(2s-1)$ -dimensional representation of $\mathfrak{sl}(2)$, which is generated by V_n^s ($n = -s+1, -s+2, \dots, s-1$). A solution to the equations of motion can be written in a form as

$$A = e^{-\rho V_0^2} a(z) e^{\rho V_0^2} dz + V_0^2 dp \quad (3.3)$$

by fixing a gauge. Here, ρ denotes the radial coordinate, and the boundary is located at $\rho = \infty$. Moreover, $z, \bar{z}$ are coordinates parallel to the boundary. With the coordinates, the AdS metric is given by $ds^2 = d\rho^2 + e^{2\rho} dz d\bar{z}$, and the corresponding configuration of the gauge field is with $a(z) = V_1^2$. As for the application to AdS gravity, we assign the asymptotic AdS condition, which restricts the form of the gauge field as [34–37]

$$a(z) = V_1^2 - \frac{1}{\hat{k}} \sum_{s=2}^N \frac{1}{N_s} J^{(s)}(z) V_{-s+1}^s \quad (3.4)$$

with N_s in (2.12). The asymptotic symmetry near the AdS boundary is identified as the W_N algebra generated by $J^{(s)}$. The central charge of the algebra is

$$c = 6\hat{k} \quad (3.5)$$

at the classical level, which is same as for the pure gravity in Ref. [38]. With the coefficients of $J^{(s)}$ in (3.4), the normalization of currents is the same as that in (2.11) at the leading order in $1/c$.

We would like to compute the correlators or conformal blocks of the W_N minimal model in terms of Wilson line operator

$$W(z_f; z_i) = P \exp \left(\int_{z_i}^{z_f} dz a(z) \right) \quad (3.6)$$

in the $\mathfrak{sl}(N)$ Chern-Simons theory. Here, we remove the ρ -dependence by a gauge transformation as $A = a(z)$. At the leading order in $1/c$, it has been argued that n -point conformal blocks can be evaluated as [6,7,9]

⁶We neglect $\tilde{A}$ throughout this paper, but $\tilde{A}$ can be analyzed similarly.

$$G_n(\Lambda_j | z_j) \equiv \langle S | \prod_{j=1}^n W_{\Lambda_j}(z_0; z_j) | \text{hw} \rangle_j, \quad (3.7)$$

where $|\text{hw}\rangle_j$ denotes the highest-weight state in the representation Λ_j of $\mathfrak{sl}(N)$. Moreover, $\langle S |$ belongs to a singlet representation in $\bigotimes_{j=1}^n \Lambda_j$. There are several ways to construct a singlet state, and a choice of singlet leads to one of conformal blocks; see Ref. [7] for more explanations and explicit examples. The n open Wilson lines are connected at a point z_0 , but the expression should be independent of z_0 . For $n = 2$, we may set $z_0 = z_2$;⁷ then, the expression of (3.7) becomes

$$G_2(\Lambda, \bar{\Lambda} | z_1, z_2) = \langle \text{lw} | W_{\Lambda}(z_2; z_1) | \text{hw} \rangle. \quad (3.8)$$

Here, $\langle \text{lw} |$ represents the lowest-weight state in the representation $\bar{\Lambda}$, which is the conjugate of Λ .

For evaluating quantum effects in the Wilson line network (3.7), we should integrate over gauge fields in the path integral with the insertions of Wilson line operators. Following Refs. [8–10], we evaluate the expectation values by using the correlators of $J^{(s)}(z)$ in (3.6) with (3.4) in terms of the theory with W_N symmetry as in (2.11). Here, the normalization (2.12) is set in terms of the central charge c , so it is regarded as the full propagator after including loop corrections; see Ref. [9] for the arguments. We also need to integrate over z along the Wilson line in (3.6). However, the integration would diverge when at least two of $J^{(s)}(z)$ collide at the same point. As in Ref. [10], we introduce a regular ϵ and replace (2.11) by

$$\langle J^{(s)}(z_1) J^{(s)}(z_2) \rangle = \frac{B_s}{z_{12}^{2s-2\epsilon}}. \quad (3.9)$$

A merit of this choice is that the regulator does not break the scale invariance. The integration over z_i would diverge when $\epsilon \rightarrow 0$, and we remove the divergence by renormalizing the overall normalization of Wilson lines and parameters corresponding to interactions as in Ref. [10]. We denote the parameters as

$$c_s = 1 + \frac{1}{c} c_s^{(1)} + \mathcal{O}(c^{-2}) \quad (3.10)$$

and insert them into the Wilson line operator as

$$W(z_f; z_i) = P \exp \left[\int_{z_i}^{z_f} dz \left(V_1^2 - \frac{6}{c} \sum_{s=2}^N \frac{c_s}{N_s} J^{(s)}(z) V_{-s+1}^s \right) \right]. \quad (3.11)$$

⁷If we instead set $z_0 = z_1$, then we obtain $G_2(\Lambda, \bar{\Lambda} | z_1, z_2) = \langle \text{lw} | W_{\bar{\Lambda}}(z_1; z_2) | \text{hw} \rangle$ with $|\text{hw}\rangle$ and $\langle \text{lw} |$ in the representations $\bar{\Lambda}$ and Λ , respectively.

We can remove some divergences at the $1/c$ order by using $c_s^{(1)}$ and similarly for higher orders. There are still ambiguities for the terms of order ϵ^0 , and we offer to fix them such that the boundary correlators are consistent with the W_N symmetry.

B. Two- and three-point functions for $N=2$

In order to illustrate our method and fix notations, we review our analysis in Ref. [10] for the two- and three-point functions in (1.3). For $N=2$, it is convenient to work with the x -basis, which can be applied to the general N case but with operators in the (anti)fundamental representation. As in Refs. [8,10,39], we consider the x -basis expression of (3.8) as

$$\begin{aligned} \langle 1w | W_{h_0}(z_f; z_i) | hw \rangle \\ = \int dx \langle 1w | x \rangle W_{h_0}(z_f; z_i) \langle x | hw \rangle. \end{aligned} \quad (3.12)$$

For brevity, we call the left-hand side $W_{h_0}(z_f; z_i)$. With the basis, we can use

$$\langle 1w | x \rangle = \delta(x), \quad \langle x | hw \rangle = \frac{1}{x^{2h_0}}, \quad (3.13)$$

where $h_0 = -j$ for $N=2$ and $h_0 = (1-N)/2$ for general N . The generators are [40]

$$\begin{aligned} V_n^s = \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{(n-s+1)_{s-1-i} (-2h_0-s+2)_{s-1-i}}{(s+i)_{s-1-i}} \\ \times x^{-n+i} \partial_x^i, \end{aligned} \quad (3.14)$$

and, in particular, the $\mathfrak{sl}(2)$ subalgebra is generated by

$$\begin{aligned} V_1^2 = \partial_x, \quad V_0^2 = x\partial_x + h_0, \\ V_{-1}^2 = x^2\partial_x + 2h_0x. \end{aligned} \quad (3.15)$$

As in Refs. [8–10], we express (3.12) as

$$\begin{aligned} W_{h_0}(z_f; z_i) = \sum_{n=0}^{\infty} \left(-\frac{6}{c} \right)^n \int_{z_i}^{z_f} dz_n \cdots \int_{z_i}^{z_2} dz_1 \\ \times \sum_{s_j=2}^N \left[\prod_{j=1}^n \frac{c_{s_j}}{N_{s_j}} J^{(s_j)}(z_j) \right] \\ \times f_{h_0}^{(s_n, \dots, s_1)}(z_f; z_i; z_n, \dots, z_1) \end{aligned} \quad (3.16)$$

with $(X = x)$

$$\begin{aligned} f_{h_0}^{(s_n, \dots, s_1)}(z_f; z_i; z_n, \dots, z_1) \\ = [e^{(z_f - z_n)V_1^2} V_{-s_n+1}^{s_n} e^{(z_n - z_f)V_1^2}] \cdots \\ \times [e^{(z_f - z_1)V_1^2} V_{-s_1+1}^{s_1} e^{(z_1 - z_f)V_1^2}] e^{(z_f - z_i)V_1^2} \langle X | hw \rangle \Big|_{X=0} \end{aligned} \quad (3.17)$$

in $1/c$ expansion.

Let us first evaluate the two-point function of operator $\mathcal{O}_j$ for $N=2$ as

$$\langle \mathcal{O}_j(z) \mathcal{O}_j(0) \rangle = \langle W_{h_0}(z; 0) \rangle \quad (3.18)$$

with $h_0 = -j$ by applying our Wilson line method. At the leading order in $1/c$, the two-point function becomes

$$\langle W_{h_0}(z; 0) \rangle|_{\mathcal{O}(c^0)} = \frac{1}{z^{2h_0}} \quad (3.19)$$

as expected. The contribution at the next order is

$$\begin{aligned} \langle W_{h_0}(z; 0) \rangle|_{\mathcal{O}(c^{-1})} \\ = \left(\frac{6}{c} \right)^2 \int_0^z dz_2 \int_0^{z_2} dz_1 f_{h_0}^{(2,2)}(z; 0; z_2, z_1) \langle J^{(2)}(z_2) J^{(2)}(z_1) \rangle \\ = \frac{1}{c z^{2h_0}} \left[\frac{6(h_0-1)h_0}{\epsilon} + 12h_0(h_0-1)\log(z) + 2h_0(5h_0-2) \right], \end{aligned} \quad (3.20)$$

where we have used (3.9). We can remove the term proportional to $1/\epsilon$ by changing the overall normalization of Wilson line operator. The shift of conformal weight can be read off from the term proportional to $\log(z)$ as in (2.10), and the $1/c$ order term in (2.6) is successfully reproduced.

We also examine a three-point function using Wilson line operator as

$$\langle \mathcal{O}_j(z) \mathcal{O}_j(0) J^{(2)}(y) \rangle = \langle W_{h_0}(z; 0) J^{(2)}(y) \rangle \quad (3.21)$$

with the extra insertion of the spin 2 current. At the leading order in $1/c$, we have

$$\begin{aligned} \lim_{y \rightarrow -\infty} |y|^4 \langle W_{h_0}(z; 0) J(y) \rangle|_{\mathcal{O}(c^0)} \\ = \lim_{y \rightarrow -\infty} |y|^4 \frac{6}{c} \int_0^z dz_1 f_{h_0}^{(2)}(z; 0; z_1) \langle J^{(2)}(z_1) J^{(2)}(y) \rangle \\ = h_0 z^2 \langle W_{h_0}(z; 0) \rangle|_{\mathcal{O}(c^0)} \end{aligned} \quad (3.22)$$

as in (2.14). At the next leading order, there are two types of contributions as

$$\begin{aligned}
& \left(\frac{6}{c}\right)^2 \int_0^z dz_2 \int_0^{z_2} dz_1 f_{h_0}^{(2,2)}(z; 0; z_2, z_1) \\
& \times \langle J^{(2)}(z_2) J^{(2)}(z_1) J^{(2)}(y) \rangle, \\
& \left(\frac{6}{c}\right)^3 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 f_{h_0}^{(2,2,2)}(z; 0; z_3, z_2, z_1) \\
& \times \langle J^{(2)}(z_3) J^{(2)}(z_2) J^{(2)}(z_1) J^{(2)}(y) \rangle.
\end{aligned} \quad (3.23)$$

We use the correlators

$$\langle J^{(2)}(z_3) J^{(2)}(z_2) J^{(2)}(z_1) \rangle = \frac{c}{(z_{21} z_{32} z_{31})^{2-\epsilon}} \quad (3.24)$$

for the three-point function and

$$\begin{aligned}
& \langle J^{(s)}(z_4) J^{(s)}(z_3) J^{(s)}(z_2) J^{(s)}(z_1) \rangle \\
& = \frac{B_s^2}{(z_{43} z_{21})^{2s-2\epsilon}} + \frac{B_s^2}{(z_{42} z_{31})^{2s-2\epsilon}} + \frac{B_s^2}{(z_{41} z_{32})^{2s-2\epsilon}}
\end{aligned} \quad (3.25)$$

with $s = 2$ up to order c^2 for the four-point function. The sum of the integrals leads to

$$\begin{aligned}
& \lim_{y \rightarrow -\infty} |y|^{4-2\epsilon} \langle W_{h_0}(z; 0) J^{(2)}(y) \rangle \\
& = \left[h_0 - \frac{6h_0}{c} \left(\frac{1}{\epsilon} + h_0 - \frac{1}{2} \right) \right] z^2 \langle W_{h_0}(z; 0) \rangle + \dots
\end{aligned} \quad (3.26)$$

We remove the divergence by renormalizing the coupling c_2 in (3.11) as (see (4.21) of Ref. [10])

$$c_2 = 1 + \frac{1}{c} \left(\frac{6}{\epsilon} + 3 \right) + \mathcal{O}(c^{-2}). \quad (3.27)$$

Here, the constant term is chosen such that the three-point function becomes

$$\lim_{y \rightarrow -\infty} |y|^4 \langle W_{h_0}(z; 0) J^{(2)}(y) \rangle = h_j z^2 \langle W_{h_0}(z; 0) \rangle \quad (3.28)$$

with h_j in (2.6) up to order $1/c$. In other words, we chose the renormalization scheme to be consistent with the conformal Ward identity in the boundary theory. Here, we remark that the renormalized coupling c_2 in (3.27) is independent of h_0 ; thus, the same Wilson line operator (3.11) can be used to evaluate the network in (3.7).

In Ref. [10], we also examined the $1/c^2$ order corrections of the two-point function, and we reproduced the shift of conformal weight at the $1/c^2$ order as in (2.6). This is possible only after properly regularizing divergences, and this fact gives a support for our renormalization prescription; see Ref. [10] for more details. The analysis here can be directly applied to the case with $N = 3$, but involving scalar operators only in the (anti)fundamental representation. Similarly for $N = 2$, we showed in Ref. [10] that the divergences at the $1/c$ order in the two- and three-point

functions of the form (1.3) can be removed by renormalizing the overall factor of Wilson line operator and couplings c_2 and c_3 introduced as in (3.11). With the renormalization at the $1/c$ order, we reproduced the shift of conformal weight of scalar operator up to the $1/c^2$ order.

C. Two- and three-point functions for $N = 3$

In Ref. [10], we restrict ourselves to operators only in the (anti)fundamental representation. However, the fusion of this type of operators would yield another operator belonging to a representation other than the (anti)fundamental one. Therefore, it is necessary to extend the previous analysis and deal with correlators involving general operators. It is known that $\mathfrak{sl}(N)$ generators acting on general states can be represented by using parameters x_a ($a = 1, 2, \dots, (N-1)N/2$) with the number of the positive root. For $N = 3$, we introduce three parameters $X = (x, y, w)$, which makes the expression (3.16) with (3.17) applicable to the general situation. The explicit expressions of states and generators are summarized in Appendix B.

Just like the $N = 2$ case, we start from computing the two-point function of operator $\mathcal{O}_{(\lambda_1, \lambda_2; 0)}$ using the Wilson line as

$$\langle \bar{\mathcal{O}}_{(\lambda_1, \lambda_2; 0)}(z) \mathcal{O}_{(\lambda_1, \lambda_2; 0)}(0) \rangle = \langle W_{(\lambda_1, \lambda_2)}(z; 0) \rangle. \quad (3.29)$$

The expression of the Wilson line in $1/c$ expansions in (3.16) with (3.17) leads to

$$\langle W_{(\lambda_1, \lambda_2)}(z) \rangle|_{\mathcal{O}(c^0)} = z^{2\lambda_1 + 2\lambda_2} \quad (3.30)$$

with the normalization of (B3). This is consistent with the leading order expression of conformal weight in (2.7). At the next leading order in $1/c$, we have the contributions ($s = 2, 3$)

$$\left(\frac{6}{c}\right)^2 \int_0^z dz_2 \int_0^{z_2} dz_1 f_{(\lambda_1, \lambda_2)}^{(s,s)}(z; 0; z_2, z_1) \langle J^{(s)}(z_2) J^{(s)}(z_1) \rangle, \quad (3.31)$$

where $f_{(\lambda_1, \lambda_2)}^{(s,s)}(z; 0; z_2, z_1)$ are given by (3.17) with (B7), (B9), and (B10). Moreover, we use the two-point functions of currents in (3.9) with

$$B_2 = \frac{c}{2}, \quad B_3 = -\frac{5c}{6}; \quad (3.32)$$

see (2.13) with $N = 3$. The sum of the both contributions is

$$\begin{aligned}
\langle W_{(\lambda_1, \lambda_2)}(z) \rangle &= z^{2\lambda_1 + 2\lambda_2} \\
&+ \frac{1}{c} z^{2\lambda_1 + 2\lambda_2} \left[\frac{8(\lambda_1^2 + \lambda_1(\lambda_2 + 3) + \lambda_2(\lambda_2 + 3))}{\epsilon} \right. \\
&+ \frac{1}{15} (240(\lambda_1^2 + \lambda_1(\lambda_2 + 3) + \lambda_2(\lambda_2 + 3)) \log(z) \\
&+ 197\lambda_1^2 + \lambda_1(206\lambda_2 + 213) \\
&\left. + \lambda_2(197\lambda_2 + 213)) \right] \quad (3.33)
\end{aligned}$$

up to the orders of ϵ^0 and $1/c$. The z -independent overall factor can be set as 1 by changing the normalization of the Wilson line operator. From the coefficient of $\log(z)$, the shift of conformal dimension can be read off as

$$-\frac{1}{2} \cdot \frac{1}{c} \cdot 16(\lambda_1^2 + \lambda_1(\lambda_2 + 3) + \lambda_2(\lambda_2 + 3)), \quad (3.34)$$

which reproduces the $1/c$ order term in (2.7).

Next, we move to the evaluation of Wilson line operator with the insertion of a higher spin current $J^{(s)}$ with $s = 2$ or $s = 3$. We would like to show that the three-point functions can be reproduced as

$$\langle \bar{\mathcal{O}}_{(\lambda_1, \lambda_2, 0)}(z) \mathcal{O}_{(\lambda_1, \lambda_2, 0)}(0) J^{(s)}(y) \rangle = \langle W_{(\lambda_1, \lambda_2)}(z; 0) J^{(s)}(y) \rangle. \quad (3.35)$$

Let us define

$$G_3^{(s)}(z) = \lim_{y \rightarrow -\infty} |y|^{2s} \langle W_{(\lambda_1, \lambda_2)}(z) J^{(s)}(y) \rangle \quad (3.36)$$

with $s = 2, 3$. We first examine the leading order in $1/c$. From the expression of the open Wilson line in (3.16) with (3.17), we find

$$\begin{aligned}
G_3^{(2)}(z)|_{\mathcal{O}(c^0)} &= \frac{6}{c} B_2 \int_0^z dz_1 f_1^{(2)}(z_1) = -(\lambda_1 + \lambda_2) z^{2+2\lambda_1+2\lambda_2}, \\
G_3^{(3)}(z)|_{\mathcal{O}(c^0)} &= -\frac{6}{c} B_3 \int_0^z dz_1 f_1^{(3)}(z_1) = \frac{1}{3} (\lambda_1 - \lambda_2) z^{3+2\lambda_1+2\lambda_2}
\end{aligned} \quad (3.37)$$

using (B7), (B9), and (B10). There are no divergences from these tree level computations, which reproduce (2.14) with (2.7) and (2.16) at this order.

Divergences arise from the next order in $1/c$. For $G_3^{(2)}(z)$ in (3.36) at this order, a type of contributions comes from the integrals⁸

⁸We take the limit $G_3^{(s)}(z)|_{\mathcal{O}(c^{-1})} = \lim_{y \rightarrow -\infty} |y|^{2s-2\epsilon} \times \langle W_{(\lambda_1, \lambda_2)}(z) J^{(s)}(y) \rangle|_{\mathcal{O}(c^{-1})}$ for these computations.

$$\begin{aligned}
&\left(\frac{6}{c}\right)^2 c \int_0^z dz_2 \int_0^{z_2} dz_1 f_{(\lambda_1, \lambda_2)}^{(2,2)}(z; 0; z_2, z_1) \frac{1}{z_{21}^{2-\epsilon}}, \\
&- \left(\frac{6}{c}\right)^2 \frac{5c}{2} \int_0^z dz_2 \int_0^{z_2} dz_1 f_{(\lambda_1, \lambda_2)}^{(3,3)}(z; 0; z_2, z_1) \frac{1}{z_{21}^{4-\epsilon}}.
\end{aligned} \quad (3.38)$$

Here, we have used the three-point functions (3.24) and

$$\langle J^{(3)}(z_3) J^{(3)}(z_2) J^{(2)}(z_1) \rangle = \frac{-5c/2}{z_{32}^{4-\epsilon} z_{21}^{2-\epsilon} z_{31}^{2-\epsilon}}, \quad (3.39)$$

where the powers are obtained by shifting the conformal weights of $J^{(2)}$ and $J^{(3)}$ as $2 \rightarrow 2 - \epsilon$ and $3 \rightarrow 3 - \epsilon$, respectively. Another type is

$$\begin{aligned}
&\left(\frac{6}{c}\right)^3 B_s B_2 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\
&\quad \times f_{(\lambda_1, \lambda_2)}^{(s,s,2)}(z; 0; z_3, z_2, z_1) \frac{1}{z_{32}^{2s-2\epsilon}}, \\
&\left(\frac{6}{c}\right)^3 B_s B_2 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\
&\quad \times f_{(\lambda_1, \lambda_2)}^{(2,s,s)}(z; 0; z_3, z_2, z_1) \frac{1}{z_{21}^{2s-2\epsilon}}, \\
&\left(\frac{6}{c}\right)^3 B_s B_2 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\
&\quad \times f_{(\lambda_1, \lambda_2)}^{(s,2,s)}(z; 0; z_3, z_2, z_1) \frac{1}{z_{31}^{2s-2\epsilon}}
\end{aligned} \quad (3.40)$$

with $s = 2, 3$. Here, the large c factorization of four-point functions is used as (3.25), and

$$\langle J^{(3)}(z_4) J^{(3)}(z_3) J^{(2)}(z_2) J^{(2)}(z_1) \rangle = \frac{B_2 B_3}{z_{43}^{6-2\epsilon} z_{21}^{4-2\epsilon}} + \mathcal{O}(c). \quad (3.41)$$

Totally, we have

$$\begin{aligned}
z^{-2} G_3^{(2)}(z) &= \left[-(\lambda_1 + \lambda_2) + \frac{1}{c} \left(\frac{36(\lambda_1 + \lambda_2)}{\epsilon} \right. \right. \\
&\quad \left. \left. - (8\lambda_1^2 + \lambda_1(8\lambda_2 + 11) + \lambda_2(8\lambda_2 + 11)) \right) \right] \\
&\quad \times \langle W_{(\lambda_1, \lambda_2)}(z; 0) \rangle
\end{aligned} \quad (3.42)$$

up to orders ϵ^0 and $1/c$. There is a divergent term proportional to $1/\epsilon$, which is removed by shifting the coupling c_2 introduced in (3.11) from its classical value $c_2 = 1$. We choose the ϵ -independent part of c_2 as

$$c_2 = 1 + \frac{1}{c} \left(\frac{36}{\epsilon} + 13 \right) + \mathcal{O}(c^{-2}); \quad (3.43)$$

then, we reproduce the CFT result in (2.14) with (2.7) as

$$z^{-2}G_3^{(2)}(z) = h(\lambda_1, \lambda_2; 0) \langle W_{(\lambda_1, \lambda_2)}(z) \rangle \quad (3.44)$$

up to the order $1/c$. Note that the shift of coupling c_2 in (3.43) is the same as (5.17) of Ref. [10], and it is independent of the representation (λ_1, λ_2) .

We can similarly analyze $G_3^{(3)}(z)$ in (3.36). At the $1/c$ order, contributions are

$$\begin{aligned} & \left(\frac{6}{c}\right)^2 \frac{5c}{2} \int_0^z dz_2 \int_0^{z_2} dz_1 \\ & \times [f_{(\lambda_1, \lambda_2)}^{(2,3)}(z; 0, z_2, z_1) + f_{(\lambda_1, \lambda_2)}^{(3,2)}(z; 0, z_2, z_1)] \frac{1}{z_{21}^{2-\epsilon}}, \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} & - \left(\frac{6}{c}\right)^3 B_s B_3 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\ & \times f_{(\lambda_1, \lambda_2)}^{(s,3)}(z; 0, z_3, z_2, z_1) \frac{1}{z_{32}^{2s-2\epsilon}}, \\ & - \left(\frac{6}{c}\right)^3 B_s B_3 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\ & \times f_{(\lambda_1, \lambda_2)}^{(3,s)}(z; 0, z_3, z_2, z_1) \frac{1}{z_{21}^{2s-2\epsilon}}, \\ & - \left(\frac{6}{c}\right)^3 B_s B_3 \int_0^z dz_3 \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \\ & \times f_{(\lambda_1, \lambda_2)}^{(s,3,s)}(z; 0, z_3, z_2, z_1) \frac{1}{z_{31}^{2s-2\epsilon}} \end{aligned} \quad (3.46)$$

with $s = 2, 3$. Here, we have used (3.39), (3.25), and (3.41). The sum of them leads to

$$\begin{aligned} z^{-3}G_3^{(3)}(z) &= \left[\frac{\lambda_1 - \lambda_2}{3} + \frac{1}{c} \left(-\frac{12(\lambda_1 - \lambda_2)}{\epsilon} \right. \right. \\ & \quad \left. \left. + \frac{1}{5}(\lambda_1 - \lambda_2)(40\lambda_1 + 40\lambda_2 + 33) \right) \right] \\ & \times \langle W_{(\lambda_1, \lambda_2)}(z) \rangle \end{aligned} \quad (3.47)$$

up to orders ϵ^0 and $1/c$. Changing the coupling c_3 in (3.11) as (see (5.30) of Ref. [10])

$$c_3 = 1 + \frac{1}{c} \left(\frac{36}{\epsilon} + 1 \right) + \mathcal{O}(c^{-2}), \quad (3.48)$$

the above expression becomes

$$z^{-3}G_3^{(3)}(z) = w(\lambda_1, \lambda_2; 0) \langle W_{(\lambda_1, \lambda_2)}(z; 0) \rangle \quad (3.49)$$

up to the $1/c$ order as in (2.14) with (2.16). Here, we remark that the shift of c_3 is independent of λ_1, λ_2 as for c_2 .

Adopting the regularization, we should be able to reproduce the shift of conformal weight up to $1/c^2$ order just like the case with $(\lambda_1, \lambda_2) = (1, 0)$ in Ref. [10]. Here, we do not repeat the computation since it is not necessary for the following analysis.

IV. CONFORMAL BLOCKS FROM OPEN WILSON LINES

In this section, we evaluate four-point conformal blocks of the form (1.4) by including quantum effects on the networks of open Wilson lines. For $N = 2$, similar computations were done in Ref. [8], but their formulation is different from ours. In the next subsection, we start from the identity blocks, which are computable by the products of two open Wilson lines. Since computations turn out to be quite similar to those in Ref. [8] for $N = 2$, we focus on the $N = 3$ case by applying the method developed in the previous section. In Sec. IV B, we examine general conformal blocks, where we have to really deal with the networks of open Wilson lines. We concentrate on the simpler case with $N = 2$ and briefly discuss the $N = 3$ extensions; see also Appendix C.

A. Identity W_3 blocks

In order to express a conformal block for (1.4) in terms of the Wilson line network as in (3.7), we have to choose a proper singlet S from the product of representations $\bar{\Lambda} \otimes \Lambda \otimes \bar{\Xi} \otimes \Xi$ as explained in Sec. III A. For the identity block, we set S as the direct product of two singlets given by $s_{12} \in \bar{\Lambda} \otimes \Lambda$ and $s_{34} \in \bar{\Xi} \otimes \Xi$. With the choice, the Wilson line network factorizes as

$$\begin{aligned} G_4(\Lambda_j | z_j) &= \left(\langle s_{12} | \prod_{j=1,2} W_{\Lambda_j}(z_0; z_j) | \text{hw} \rangle_j \right) \\ & \times \left(\langle s_{34} | \prod_{l=3,4} W_{\Lambda_l}(z'_0; z_l) | \text{hw} \rangle_l \right) \end{aligned} \quad (4.1)$$

with $z_0 = z'_0$. Setting $z_0 = z_1 = \infty$ and $z'_0 = z_3 = z$ with $z_2 = 1$ and $z_4 = 0$, the Wilson line network becomes the product of two open Wilson lines. Therefore, the identity block should be computed as

$$\langle G_4(\Lambda_j | z_j) \rangle = \langle W_{\Lambda}(\infty; 1) W_{\Xi}(z; 0) \rangle, \quad (4.2)$$

where the products of currents are evaluated by the correlators of W_N theory as before. In this subsection, we explicitly compute (4.2) for $N = 3$ with $\Lambda = (\lambda_1, \lambda_2)$ and $\Xi = \mathbf{f}$, that is,

$$\begin{aligned} G_4^0(z) &= \langle W_{(\lambda_1, \lambda_2)}(\infty; 1) W_{\mathbf{f}}(z; 0) \rangle \\ &= \lim_{w \rightarrow \infty} w^{2h(\lambda_1, \lambda_2; 0)} \langle W_{(\lambda_1, \lambda_2)}(w; 1) W_{\mathbf{f}}(z; 0) \rangle, \end{aligned} \quad (4.3)$$

by applying the method developed in the previous section. We compare the result with the identity block from the CFT computation in (2.22); see Appendix A for some details. The $1/c$ expansion is given as (A28) with (A29), but now we set $N = 3$ and $m = 1$.

The leading order expression in $1/c$ is simply given by the product of two expectation values as

$$G_4^0(z)|_{\mathcal{O}(c^0)} = \langle W_{(\lambda_1, \lambda_2)}(\infty; 1) \rangle|_{\mathcal{O}(c^0)} \langle W_f(z; 0) \rangle|_{\mathcal{O}(c^0)} = z^2. \quad (4.4)$$

This can be expressed in diagram (a) of Fig. 1. Among higher order corrections in $1/c$, there are contributions of the self-energy type as in other diagrams. For contributions corresponding to diagrams (b)–(f), we replace $\langle W_f(z; 0) \rangle|_{\mathcal{O}(c^0)}$ in (4.4) the integrals of the forms

$$\int_0^z dz_n \cdots \int_0^{z_2} dz_1 f_f^{(s_n, \dots, s_1)}(z; 0; z_n, \dots, z_1) \times \langle J^{(s_n)}(z_n) \cdots J^{(s_1)}(z_1) \rangle \quad (4.5)$$

with $n = 2, 3, 4$ and $s_j = 2, 3$. Here, the integrals with $n = 2$ and $n = 3$ correspond to diagrams (b) and (c). The integrals with $n = 4$ include the four-point functions of currents with several different terms as in (3.25) and (3.41), which are expressed by diagrams (d), (e), and (f). The $1/c$ corrections to $W_f(z; 0)$ up to order $1/c^2$ have been evaluated in Ref. [10], and they can be summarized as the shift of conformal dimension from (4.4) to [see (2.7) for a general operator]

$$\frac{1}{z^{2h(f;0)}}, \quad h(f;0) = -1 - \frac{32}{c} - \frac{1600}{c^2} + \mathcal{O}(c^{-3}) \quad (4.6)$$

after renormalizing the overall factor of the open Wilson line and the coupling constants c_s in (3.11). The other diagrams correspond to the similar corrections of another Wilson line $W_{(\lambda_1, \lambda_2)}(\infty; 1)$, but they do not appear in the four-point block $G_4^0(z)$ (4.3) since an operator is put at the infinity.

Nontrivial contributions start to enter from the next order in $1/c$. They correspond to diagram (a) of Fig. 2, and the sum of them is

$$H_1(z) = H_1^{(2)}(z) + H_1^{(3)}(z) \quad (4.7)$$

with

$$H_1^{(s)}(z) = \left(\frac{6}{c}\right)^2 \int_1^\infty dz_2 \int_0^z dz_1 f_{(\lambda_1, \lambda_2)}^{(s)}(\infty; 1; z_2) \times f_f^{(s)}(z; 0; z_1) \langle J^{(s)}(z_2) J^{(s)}(z_1) \rangle. \quad (4.8)$$

With the two-point functions of currents in (2.11) with (3.32), we evaluate the integrals as

$$H_1^{(2)}(z) = \frac{2(\lambda_1 + \lambda_2)}{c} z^4 {}_2F_1(2, 2; 4; z),$$

$$H_1^{(3)}(z) = -\frac{2(\lambda_1 - \lambda_2)}{15c} z^5 {}_2F_1(3, 3; 6; z). \quad (4.9)$$

They correspond to the global blocks with spin 2 and 3 currents as the intermediate operators as they should be. The coefficients are also given by the products of three-point functions divided by a current-current two-point function as

FIG. 1. Contribution to the identity W_3 block from the product of open Wilson lines at the leading order in $1/c$ and its corrections of self-energy type up to the order $1/c^2$. The straight and wavy lines represent open Wilson lines and propagators of current with spin $s = 2$ or $s = 3$. (a) Leading order contribution. (b)–(f) Corrections associated with the open Wilson line from 0 to z . (g) Correction associated with both the open Wilson lines. (h)–(l) Corrections associated with the open Wilson line from 1 to ∞ .

FIG. 2. (a) Contribution with the exchange of a current with $s = 2$ or $s = 3$. (b)–(i) Its $1/c$ corrections from loop diagrams.

$$H_1^{(2)}(z) = \frac{h_{(f;0)} h_{(\lambda_1, \lambda_2; 0)}}{B_2} z^4 {}_2F_1(2, 2; 4; z),$$

$$H_1^{(3)}(z) = \frac{w_{(f;0)} w_{(\lambda_1, \lambda_2; 0)}}{B_3} z^5 {}_2F_1(3, 3; 6; z) \quad (4.10)$$

at the leading order in $1/c$. Here, the spin 2 and 3 charges were given by (2.7) and (2.16).

The other diagrams in Fig. 2 yield loop corrections to interactions among an open Wilson line and an exchanged current. These diagrams can be evaluated precisely as for (3.36) but with y replaced by z_1 or z_2 . Therefore, after renormalizing the open Wilson lines and the coupling constants c_s , the net effects are summarized as the $1/c$ order terms in (2.7) and (2.16) as

$$h_{(f;0)} h_{(\lambda_1, \lambda_2; 0)} = \lambda_1 + \lambda_2 + \frac{8(7\lambda_1 + \lambda_1^2 + 7\lambda_2 + \lambda_2^2 + \lambda_1\lambda_2)}{c} + \mathcal{O}(c^{-2}),$$

$$w_{(f;0)} w_{(\lambda_1, \lambda_2; 0)} = \frac{\lambda_1 - \lambda_2}{9} + \frac{8(15\lambda_1^2 + 41\lambda_1 - 15\lambda_2^2 - 41\lambda_2)}{45c} + \mathcal{O}(c^{-2}). \quad (4.11)$$

Thus, we have a type of $1/c^2$ corrections as

$$H_2^{(2)}(z) = \frac{16(7\lambda_1 + \lambda_1^2 + 7\lambda_2 + \lambda_2^2 + \lambda_1\lambda_2)}{c^2} z^4 F(2, 2, 4; z),$$

$$H_2^{(3)}(z) = -\frac{16(15\lambda_1^2 + 41\lambda_1 - 15\lambda_2^2 - 41\lambda_2)}{75c^2} z^5 F(3, 3, 6; z) \quad (4.12)$$

coming from diagrams (b)–(i) of Fig. 2.

At the order of $1/c^2$, there are more interesting contributions with the exchange of two currents as in diagrams (a) and (b) of Fig. 3 in addition to those already given.

When the two currents have the same spin, they come from the integrals

$$H_2^{(s,s)}(z) = \left(\frac{6}{c}\right)^4 B_s^2 \int_1^\infty dz_4 \int_1^{z_4} dz_3 \int_0^z dz_2 \int_0^{z_2} dz_1$$

$$\times f_{(\lambda_1, \lambda_2)}^{(s,s)}(\infty; 1; z_4, z_3)$$

$$\times f_f^{(s,s)}(z; 0; z_2, z_1) \left(\frac{1}{z_{31}^{2s} z_{42}^{2s}} + \frac{1}{z_{41}^{2s} z_{32}^{2s}} \right) \quad (4.13)$$

with $s = 2, 3$. The first and second terms are expressed by diagrams (a) and (b), respectively. When the two currents have the different spins, the contributions corresponding to diagrams (a) and (b) are given by the integrals

$$H_{2,1}^{(s_1, s_2)}(z) = \left(\frac{6}{c}\right)^4 B_2 B_3 \int_1^\infty dz_4 \int_1^{z_4} dz_3 \int_0^z dz_2 \int_0^{z_2} dz_1$$

$$\times f_{(\lambda_1, \lambda_2)}^{(s_2, s_1)}(\infty; 1; z_4, z_3) f_f^{(s_2, s_1)}(z; 0; z_2, z_1) \frac{1}{z_{31}^{2s_1} z_{42}^{2s_2}},$$

$$H_{2,2}^{(s_1, s_2)}(z) = \left(\frac{6}{c}\right)^4 B_2 B_3 \int_1^\infty dz_4 \int_1^{z_4} dz_3 \int_0^z dz_2 \int_0^{z_2} dz_1$$

$$\times f_{(\lambda_1, \lambda_2)}^{(s_1, s_2)}(\infty; 1; z_4, z_3) f_f^{(s_2, s_1)}(z; 0; z_2, z_1) \frac{1}{z_{41}^{2s_1} z_{32}^{2s_2}} \quad (4.14)$$

FIG. 3. (a),(b) Contributions with the exchange of two currents with $s = 2$ or $s = 3$.

with $(s_1, s_2) = (2, 3)$ or $(3, 2)$. Summing over this type of contributions, we find

$$\begin{aligned} H_2^{\text{ex}}(z) = & \frac{8}{5} [20 \log^2(1-z)(3(\lambda_1^2 + \lambda_1(4\lambda_2 + 6) + (\lambda_2 - 3)\lambda_2) + z^2(2\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2 + 9) \\ & - 6\lambda_1 z(\lambda_1 + 2\lambda_2 + 6)) + 3z(10\lambda_1^2(31z - 30) + \lambda_1(160\lambda_2 z + 519z - 318) \\ & + \lambda_2(10\lambda_2(z + 30) + 201z + 318)) - 6 \log(1-z)(10\lambda_1^2(8(z-3)z + 15) \\ & + \lambda_1(2z(30\lambda_2(z-2) + 7z + 3) - 21) + \lambda_2(10\lambda_2(z(z+6) - 15) + (z-36)z + 21)) \\ & + 360\text{Li}_2(z)(\lambda_1(2(z-3)z + 3) + \lambda_2(z^2 - 3))] \end{aligned} \quad (4.15)$$

with $\text{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$. The total contribution at the $1/c^2$ order other than the self-energy type is thus

$$H_2(z) = H_2^{(2)}(z) + H_2^{(3)}(z) + H_2^{\text{ex}}(z) \quad (4.16)$$

with (4.12) and (4.15).

We confirm the bulk computation by showing that the above result reproduces the CFT one. The identity block in (2.22) is expanded in $1/c$ as (A28) with (A29), and we set $N = 3$, $m = 1$ and identify $(\lambda_1, \lambda_2) = (n, 0)$ for the comparison. We can see that the $1/c$ order term in (A28) is the same as (4.7) with (4.9). Furthermore, we can check that the $1/c^2$ order expression of (A28) with (A29) is reproduced by (4.16) at each order in z expansion.

Before ending this subsection, we would like to mention the relation to the analysis in Ref. [8] for $N = 2$ with a different regularization. The regularization prescription in Ref. [8] is to adopt the normal ordered product for currents coming from the same Wilson line operator. Moreover, they used the exact conformal weight h_j from the beginning instead of the quantum number j for the finite representation of $\text{sl}(2)$. Using the normal ordered prescription is equivalent to neglecting contributions coming from loop diagrams in Figs. 1 and 2; see also Ref. [9]. As we explained above, these contributions only shift the conformal weight for $N = 2$; therefore, the effect is already included in their treatment. The contributions represented by diagrams in Fig. 3 are the same in the both prescriptions; see Fig. 5 of Ref. [8]. Therefore, we can see that the both ways of computation should give the same result for $N = 2$.

B. General conformal blocks

In this subsection, we study the conformal blocks with the exchange of the general operator from the networks of open Wilson lines (3.7) by including quantum effects. For the general blocks, it turns out to be convenient to use the X -basis as

$$G_n(\Lambda_j|z_j) = \left[\prod_{j=1}^n \int dX_{jj} \langle 1w|X_j \rangle W_{\Lambda_j}(z_j; z_0; X_j) \langle X_j \rangle \right] |S\rangle \quad (4.17)$$

in its conjugated form. Here, the generators $V_m^s(X_j)$ are now written in terms of X_j as explicitly indicated in the arguments of the Wilson line operator. The condition for the singlet $|S\rangle$ becomes

$$\sum_{j=1}^n V_m^s(X_j) \left[\prod_{l=1}^n \langle X_l \rangle \right] |S\rangle = 0 \quad (4.18)$$

for all s and m . In general, there are several solutions, and a solution corresponds to a conformal block. Applying ${}_j\langle 1w|X_j \rangle = \delta(X_j)$ as in (3.13) or (B2), the expression is simplified as

$$G_n(\Lambda_j|z_j) = \left[\prod_{j=1}^n W_{\Lambda_j}(z_j; z_0; X_j) \langle X_j \rangle \right] |S\rangle \Big|_{X_j=0}. \quad (4.19)$$

The quantum effects are evaluated by using the correlators of higher spin currents inserted as before.

In the following, we mainly focus on the $N = 2$ case and make some comments on the $N = 3$ case at the end of this subsection. Before going into the details on three- and four-point Virasoro blocks, let us mention how the formalism explained above reproduces the previous analysis on the two-point function for $N = 2$. We examine $G_2(j, j|z, 0)$ in (4.19) with $X_i = x_i$. The vertex is given by the unique solution to the singlet condition (4.18) as (see, e.g., Ref. [12])

$$\langle x_1 | \langle x_2 | |S\rangle = v_2(j, j|x_1, x_2) \equiv x_{12}^{2j} \quad (4.20)$$

up to the overall factor. At the leading order in $1/c$, we have

$$\begin{aligned} \langle G_2(j, j|z, 0) \rangle |_{\mathcal{O}(c^0)} &= e^{(z-z_0)V_1^2(x_1)} e^{-z_0 V_1^2(x_2)} v_2(j, j|x_1, x_2) \Big|_{x_i=0} \\ &= z^{2j} \end{aligned} \quad (4.21)$$

as in (3.19). For the $1/c$ corrections, it is convenient to set $z_0 = 0$ (or $z_0 = z$), which reduces the number of diagrams; see Fig. 4. With $z_0 = 0$, the expression in (4.19) reduces to the expectation value of (3.12) with (3.13) and $(z_f, z_i) = (z, 0)$. We have also evaluated the $1/c$

$$\sum_{\text{one-loop diagrams}} z \text{---} z_0 \text{---} 0 = z \text{---} z_0 \text{---} 0$$

FIG. 4. Contribution to the two-point function from the network of open Wilson lines at the $1/c$ order.

corrections to the two-point function with keeping z_0 generic and checked that the final result is the same.

1. Three-point Virasoro block

As a simple but nontrivial example, we study three-point conformal block for $N = 2$. The three-point block is fixed by the conformal symmetry as

$$\langle \mathcal{O}_{j_1}(z_1) \mathcal{O}_{j_2}(z_2) \mathcal{O}_{j_3}(z_3) \rangle \propto \frac{1}{z_{12}^{h_{j_1}+h_{j_2}-h_{j_3}} z_{13}^{h_{j_1}+h_{j_3}-h_{j_2}} z_{32}^{h_{j_3}+h_{j_2}-h_{j_1}}} \quad (4.22)$$

with [see (2.6)]

$$\begin{aligned} h_{j_i} &= -j_i + \frac{1}{c} h_i^{(1)} + \mathcal{O}(c^{-2}), \\ h_i^{(1)} &= -6j_i(j_i + 1). \end{aligned} \quad (4.23)$$

We evaluate $G_3(j_i|z_i)$ in (4.19) with $X_i = x_i$. The vertex is uniquely fixed by the singlet condition (4.18) as (see, e.g., Refs. [8,12])

$$\begin{aligned} \left[\prod_{i=1}^3 \langle x_i | \right] |S\rangle &= v_3(j_i|x_i) \\ &\equiv x_{12}^{j_1+j_2-j_3} x_{13}^{j_1+j_3-j_2} x_{32}^{j_3+j_2-j_1} \end{aligned} \quad (4.24)$$

up to the overall factor. At the leading order in $1/c$, we find

$$\begin{aligned} \langle G_3(j_i|z_i) \rangle|_{\mathcal{O}(c^0)} &= \left[\prod_{j=1}^3 e^{z_{j0} V_1^2(x_j)} \right] v_3(j_i|x_i) \Big|_{x_i=0} \\ &= z_{12}^{j_1+j_2-j_3} z_{13}^{j_1+j_3-j_2} z_{32}^{j_3+j_2-j_1}. \end{aligned} \quad (4.25)$$

The expression reproduces the three-point function at the leading order in $1/c$. Here, we simply replace x_i by z_i ,⁹ and this is actually as the same as the prescription to obtain the large c limit of the three-point Virasoro block in Ref. [12].

We move to the next leading order in $1/c$. When evaluating the expectation value of $G_3(j_i|z_i)$, the correlators of $J^{(2)}(z)$ for Virasoro algebra are used, and divergences are removed as before. In order to make the computation simpler, we set $(z_1, z_2, z_3) = (z, 0, 1)$. Moreover, using $z_0 = z_3 = 1$,¹⁰ we consider the product of two Wilson lines as in Fig. 5. With this setup, we evaluate the integrals corresponding to the three diagrams in the right-hand side. Defining

$$\begin{aligned} t_1(z; x_1, x_2, x_3) &= \left(\frac{6}{c}\right)^2 \int_1^z dz_2 \int_1^{z_2} dz_1 \prod_{j=1}^2 [e^{(z-z_j)V_1^2(x_1)} V_{-1}^2(x_1) e^{(z_j-z)V_1^2(x_1)}] e^{(z-1)V_1^2(x_1)} e^{-V_1^2(x_2)} v_3(j_i|x_i) \langle J^{(2)}(z_2) J^{(2)}(z_1) \rangle, \\ t_2(z; x_1, x_2, x_3) &= \left(\frac{6}{c}\right)^2 \int_1^0 dz_2 \int_1^{z_2} dz_1 \prod_{j=1}^2 [e^{-z_j V_1^2(x_2)} V_{-1}^2(x_2) e^{z_j V_1^2(x_2)}] e^{(z-1)V_1^2(x_1)} e^{-V_1^2(x_2)} v_3(j_i|x_i) \langle J^{(2)}(z_2) J^{(2)}(z_1) \rangle, \\ t_3(z; x_1, x_2, x_3) &= \left(\frac{6}{c}\right)^2 \int_1^0 dz_2 \int_1^z dz_1 [e^{-z_2 V_1^{(2)}(x_2)} V_{-1}^{(s)}(x_2) e^{z_2 V_1^{(2)}(x_2)}] [e^{(z-z_1)V_1^2(x_1)} V_{-1}^2(x_1) e^{(z_1-z)V_1^2(x_1)}] \\ &\quad \times e^{(z-1)V_1^2(x_1)} e^{-V_1^2(x_2)} v_3(j_i|x_i) \langle J^{(2)}(z_2) J^{(2)}(z_1) \rangle, \end{aligned} \quad (4.26)$$

the integrals corresponding to diagrams (a), (b), and (c) are $t_a(z; 0, 0, 0)$ with $a = 1, 2, 3$, respectively. The $1/c$ order correction to the three-point conformal block is given by the sum of them as

$$\begin{aligned} \langle G_3(j_i|z_i) \rangle|_{\mathcal{O}(c^{-1})} &= \sum_{a=1}^3 t_a(z; 0, 0, 0) \\ &= -\frac{1}{c} z^{j_1+j_2-j_3} (z-1)^{j_1+j_3-j_2} [(h_1^{(1)} + h_2^{(1)} - h_3^{(1)}) \log(z) + (h_1^{(1)} + h_3^{(1)} - h_2^{(1)}) \log(z-1)] \end{aligned} \quad (4.27)$$

⁹Precisely speaking, x_i are replaced by $z_{i0} (\equiv z_i - z_0)$ with z_0 as the position of vertex for the Wilson line network. Since the results do not depend on z_0 , we can set $z_0 = 0$.

¹⁰As for the two-point function, we have checked that the final result in (4.27) does not depend on z_0 even for generic z_0 . For more complicated correlators, the computations would be quite messy without setting z_0 as a specific value.

FIG. 5. Contributions to the three-point function from the network of open Wilson lines at the $1/c$ order. (a),(b) Corrections of self-energy type. (c) Correction from the exchange of a current with $s = 2$.

with $h_i^{(1)}$ in (4.23) up to the order ϵ^0 term. Here, we have ignored the term proportional to the tree level expression, which is interpreted as a correction to the overall factor of the three-point function. The result reproduces the $1/c$ order correction of the three-point block in (4.22) with (4.23) by setting $(z_1, z_2, z_3) = (z, 0, 1)$.

2. Four-point Virasoro block

We move to the Virasoro block of operator $\mathcal{O}_p$ for the four-point function (2.18). For this correlator, we may set $(z_1, z_2, z_3, z_4) = (z, 0, \infty, 1)$ and $(j_1, j_2, j_3, j_4) = (j, j, q, q)$. We compute $G_4(j_i|z_i)$ in (4.19) with $X_i = x_i$. A solution to the singlet condition (4.18) is given by [12]

$$\left[\prod_{i=1}^4 \langle x_i | \right] |S\rangle = v_p(j_i|x_i), \quad (4.28)$$

where we have defined

$$v_p(j_i|x_i) \equiv \mathcal{C} \int dx v_3(j, j, p|x_1, x_2, x) \times v_3(q, q, -1-p|x_3, x_4, x). \quad (4.29)$$

With the integral contour as $x \in (0, \infty)$ and the overall factor as

FIG. 6. Contributions to the four-point Virasoro block of operator $\mathcal{O}_p$ from the network of open Wilson lines at the $1/c$ order. (a)–(c) Corrections of self-energy type. (d)–(f) Corrections from the exchange of a current with $s = 2$.

$$\mathcal{C} = (-1)^p \frac{\Gamma(-2p)}{(\Gamma(-p))^2}, \quad (4.30)$$

the leading order expression in $1/c$ becomes

$$\begin{aligned} \langle G_4(j_i|z_i) \rangle|_{\mathcal{O}(c^0)} &= \left[\prod_{j=1}^4 e^{z_{j0} V_1^2(x_j)} \right] v_p(j_i|x_i) \Big|_{x_i=0} \\ &= z^{2j-p} g_p(z), \end{aligned} \quad (4.31)$$

where $z^{2j-p} g_p(z)$ is the global block of operator $\mathcal{O}_p$ as

$$z^{2j-p} g_p(z) \equiv z^{2j-p} {}_2F_1(-p, -p; -2p; z). \quad (4.32)$$

As for the three-point block, this is essentially the way to obtain the large c limit of the four-point block in Ref. [12].

For the next leading order in $1/c$, we evaluate the network of open Wilson lines with $z_0 = z_4 = 1$. With this setting, the number of diagrams we have to consider is reduced as in Fig. 6. We choose the normalization of conformal block as

$$\langle G_4(j_i|z_i) \rangle = z^{2j-p} (1 + \mathcal{O}(z)), \quad (4.33)$$

and hence we subtract the terms proportional to the global block (4.32) such that there is no z^{2j-p} term at the $1/c$

order. With $t_a(z; x_1, x_2, x_3)$ defined in (4.26), the integrals corresponding to diagrams (a), (b), and (e) are written as

$$C \int_0^\infty dx t_a(z; 0, 0, x) x^{-1-p} \quad (4.34)$$

with $a = 1, 2, 3$, respectively. For other diagrams, we define the functions as

$$\begin{aligned} u_1(z; z'; x_1, x_2, x_3) &= [e^{(z-z')V_1^2(x_1)} V_{-1}^2(x_1) e^{(z'-z)V_1^2(x_1)}] \\ &\quad \times e^{(z-1)V_1^2(x_1)} e^{-V_1^2(x_2)} v_3(j_i|x_i), \\ u_2(z; z'; x_1, x_2, x_3) &= [e^{-zV_1^2(x_1)} V_{-1}^2(x_2) e^{zV_1^2(x_1)}] \\ &\quad \times e^{(z-1)V_1^2(x_1)} e^{-V_1^2(x_2)} v_3(j_i|x_i) \end{aligned} \quad (4.35)$$

with $j_1 = j_2 = j$, $j_3 = p$, and

$$\begin{aligned} u_3(z'; x_1, x_2, x_3) &= \lim_{w \rightarrow \infty} w^{-2q} [e^{(w-z')V_1^2(x_1)} V_{-1}^2(x_1) e^{(z'-w)V_1^2(x_1)}] \\ &\quad \times e^{(w-1)V_1^2(x_1)} v_3(j_i|x_i) \end{aligned} \quad (4.36)$$

with $j_1 = j_2 = q$, $j_3 = -1 - p$. Then, the integrals corresponding to diagrams (d) and (f) are

$$\begin{aligned} C \left(\frac{6}{c}\right)^2 \int_1^z dz_2 \int_1^\infty dz_1 \int_0^\infty dx u_1(z; z_2; 0, 0, x) u_3(z_1; 0, 0, x) \\ \times \langle J^{(2)}(z_1) J^{(2)}(z_2) \rangle, \\ C \left(\frac{6}{c}\right)^2 \int_1^0 dz_2 \int_1^\infty dz_1 \int_0^\infty dx u_2(z; z_2; 0, 0, x) u_3(z_1; 0, 0, x) \\ \times \langle J^{(2)}(z_1) J^{(2)}(z_2) \rangle, \end{aligned} \quad (4.37)$$

respectively. The sum of all contributions is computed as

$$\begin{aligned} \langle G_4(j_i|z_i) \rangle|_{\mathcal{O}(c^{-1})} &= \frac{6z^{2j-p-1}}{c} \left\{ {}_2F_1(-p, -p; -2p; z) [z \log(z) (2j(j+1) - p(p+1)) \right. \\ &\quad + \log(1-z) (2jq(z-2) + p^2(z-1) + p(z-1)) + z(2j-p)(p-2q)] \\ &\quad \left. - {}_2F_1(1-p, -p; -2p; z) \log(1-z) p(z-1) (2j+2q+1) \right\} \end{aligned} \quad (4.38)$$

up to the terms proportional to (4.32). Here, we neglected the integral corresponding to diagram (c) since it can be shown to give only terms proportional to (4.32) or $h_p \log(w)$ with $w \rightarrow \infty$.

Let us compare the $1/c$ order expression with the known result in (2.20). First, we examine the $\log(z)$ term in (4.38). This type of contribution comes from the shifts of conformal weights as

$$\begin{aligned} z^{-2h_j+h_p} {}_2F_1(h_p, h_p; 2h_p; z) \\ = z^{2j-p} g_p(z) + \frac{1}{c} (-2h_j^{(1)} + h_p^{(1)}) \log(z) z^{2j-p} g_p(z) \\ + \frac{1}{c} z^{2j-p} r_p(z) + \mathcal{O}(c^{-2}) \end{aligned} \quad (4.39)$$

with

$$\begin{aligned} r_p(z) &= -12p(p+1) \sum_{n=0}^\infty \frac{[(-p)_n]^2 z^n}{(-2p)_n n!} \\ &\quad \times \sum_{k=1}^n \left(\frac{1}{k-p-1} - \frac{1}{k-2p-1} \right). \end{aligned} \quad (4.40)$$

We can see that the term proportional to $\log(z)$ successfully reproduces the corresponding term in (4.38). There is another type of contribution summarized as

$$\frac{jq}{c} f_a(-p, z) - \frac{j+q}{c} f_b(-p, z) + \frac{1}{c} f_c(-p, z). \quad (4.41)$$

We read off the functions f_a, f_b, f_c from (4.38) as

$$\begin{aligned} f_a(-p, z) &= 12z^{2j-p-1} {}_2F_1(-p, -p; -2p; z) \\ &\quad \times (\log(1-z)(z-2) - 2z), \\ f_b(-p, z) &= -12pz^{2j-p-1} ({}_2F_1(-p, -p; -2p; z) \\ &\quad - (z-1) \log(1-z) {}_2F_1(1-p, -p; -2p; z)), \\ f_c(-p, z) &= 6z^{2j-p-1} [{}_2F_1(-p, -p; -2p; z) \\ &\quad \times ((p^2(z-1) + p(z-1)) \log(1-z) - p^2z) \\ &\quad - p(z-1) \log(1-z) {}_2F_1(1-p, -p; -2p; z)] \\ &\quad - z^{2j-p} r_p(z). \end{aligned} \quad (4.42)$$

The functions f_a, f_b reproduce those in Ref. [8]. For the other function f_c , the terms proportional to z^{2j-p+l} ($l = 2, 3, 4, 5$) were obtained in Ref. [8], and the z expansion of the above result correctly reproduces them. In other words, our method enables us to obtain the all-order expression in z at the next nontrivial order in $1/c$, which was not available before.

One may wonder how this becomes possible. In Ref. [8], the authors evaluated the operator product expansion (OPE) blocks in the sense of Ref. [41] in terms of open Wilson lines and then computed the four-point Virasoro blocks

using the expression of the OPE blocks. The procedure requires the integration over z_5, z_6 for the two-point function of the intermediate state as $\langle \mathcal{O}_p(z_5) \mathcal{O}_p(z_6) \rangle$, while we need to integrate over only one parameter x except for the positions z_i of the currents inserted. We think that this is a main reason for the simplicity.

3. General W_3 blocks

So far, we have analyzed general Virasoro blocks, but we also would like to make some comments on general W_3 blocks. We are interested in the expectation value of $G_n(\Lambda_j|z_j)$ in (4.19), where we can obtain the vertex of open Wilson lines by solving the singlet condition in (4.18). Solutions to the condition were obtained in Ref. [12] for $n = 2, 3, 4$, which may be denoted as

$$\left[\prod_{j=1}^n \langle X_j | \right] |S\rangle = \mathcal{E}_n(\Lambda_j|X_j) \quad (4.43)$$

for the n -point function. For $n = 3$, the representation $\Lambda = (\lambda_1, \lambda_2)$ of an external operator is restricted to be degenerated as $\lambda_1 = 0$ or $\lambda_2 = 0$. A solution for $n = 4$ is written as an integral of the product of two $n = 3$ solutions, so there are also similar restrictions on the representations of external operators.

We first indicate that our formulation reduces to the one in Ref. [12] at the leading order in $1/c$. At this order, the expression in (4.19) reduces to

$$\langle G_3(\Lambda_j|z_j) \rangle|_{\mathcal{O}(c^0)} = \left[\prod_{j=1}^3 e^{z_j V_1^2(X_j)} \right] \mathcal{E}_n(\Lambda_j|X_j) \Big|_{X_j=0}, \quad (4.44)$$

where we have set $z_0 = 0$ by using the independence of z_0 . Utilizing the formula (B8), we can see that the leading order expression is obtained from $\mathcal{E}_n(\Lambda_j|X_j)$ by replacing $X_j = (x_j, y_j, w_j)$ with $(z_j, z_j, z_j^2/2)$. Just like the $N = 2$ case, the recipe to obtain the large c limit of the W_3 block is the same as the one in Ref. [12]. As a result of this fact, we can conclude that our formulation with Wilson lines reproduces the large c limit of W_3 blocks as obtained in Ref. [12]; see also Ref. [42]. Similar analysis with Wilson lines was done in Ref. [7] with a different technique for algebraic calculations. Our method with the X -basis makes computations easier for generic representations as seen above, but their method seems to be convenient for a simple representation, such as, $\Lambda = f$. In Appendix C, we compute the $1/c$ correction of a simple nonidentity W_3 block from the Wilson line network (3.7) by utilizing their analysis.

We can also show that the analysis of the two-point function reduces to the previous one in Sec. III C. For $n = 2$, the solution to the singlet condition (4.18) was obtained in Ref. [12] as

$$\begin{aligned} \mathcal{E}_2(\Lambda, \bar{\Lambda}|X_1, X_2) &= 2^{\lambda_1+\lambda_2} \rho_{12}^{\lambda_1} \rho_{21}^{\lambda_2}, \\ \rho_{ij} &= y_i(x_i - x_j) - (w_i - w_j). \end{aligned} \quad (4.45)$$

For $z_0 = z_2$, the expression in (4.19) becomes

$$G_2(\Lambda, \bar{\Lambda}|z_1, z_2) = W_\Lambda(z_1; z_2; X_1) \langle X_1 | \text{hw} \rangle|_{X_1=0} \quad (4.46)$$

as for (3.16) with (3.17). Here, we have used

$$\rho_{12}^{\lambda_1} \rho_{21}^{\lambda_2}|_{X_2=0} = \langle X_1 | \text{hw} \rangle, \quad (4.47)$$

where $\langle X | \text{hw} \rangle$ was defined in (B3) with $\Lambda = (\lambda_1, \lambda_2)$. Similarly, for $z_0 = z_1$, we obtain

$$G_2(\Lambda, \bar{\Lambda}|z_1, z_2) = W_{\bar{\Lambda}}(z_2; z_1; X_2) \langle X_2 | \text{hw} \rangle|_{X_2=0}, \quad (4.48)$$

but now $\langle X | \text{hw} \rangle$ is with $\bar{\Lambda} = (\lambda_2, \lambda_1)$.

V. HEAVY-LIGHT CORRELATOR

The W_N minimal model includes light operators of order c^0 and heavy operators of order c as explained in Sec. II. Heavy operators correspond to nontrivial geometries, and correlators with heavy operators are important for examining their quantum effects. In this section, we examine a heavy-light correlator for the simplest case with $N = 2$, and in Sec. VI, we make some comments on the general cases with $N \geq 3$.

For the analysis in this section, it is convenient to adopt the usual convention for the label of primary operators by two positive integers (r, s) ; see Ref. [30] for a review. In Sec. II, we have used the numbers of boxes in Young diagrams as (n_1^+, n_1^-) , and the relation is $r = n_1^+ + 1$, $s = n_1^- + 1$. The conformal weight of primary operator $\mathcal{O}_{r,s}$ is given by (2.3) as

$$\begin{aligned} h_{r,s} &= \frac{(r(k+3) - s(k+2))^2 - 1}{4(k+2)(k+3)} \\ &= -\frac{(s^2 - 1)c}{24} + \frac{1}{24}(-12rs + 13s^2 - 1) \\ &\quad + \mathcal{O}(c^{-1}), \end{aligned} \quad (5.1)$$

which is of order c for $n_1^- = s - 1 \neq 0$. We study the four-point correlator in (2.24) with $m = 1$, $n = s - 1$ for $N = 2$, the expression with finite k of which can be found in (2.25).

It is also useful to work on the cylindrical coordinate w , which is related to the planar one as $z = \exp(iw)$. In particular, the conformal weight is shifted from $h_{r,s}$ in (5.1) to $h_{r,s} - \frac{c}{24}$. The expression of the four-point function in (2.25) becomes¹¹

¹¹We have changed the phase factor for convenience, but it does not matter when we consider the product with the anti-holomorphic sector.

$$\begin{aligned}
& \langle \mathcal{O}_{1,s}(-i\infty) \mathcal{O}_{2,1}(0) \mathcal{O}_{2,1}(w) \mathcal{O}_{1,s}(i\infty) \rangle \\
&= \frac{(2 \sin \frac{sw}{2})(-2i \sin \frac{w}{2})^{9/c}}{s} \\
&\quad - \frac{6i}{c} e^{-\frac{i\pi w}{2}} \sum_{l=1}^{s-1} \frac{(-1)^l (1 - e^{iw})^{l+1} \Gamma(s)}{\Gamma(s-l) \Gamma(l+2)} \left(\sum_{j=1}^l \frac{1}{j} - \frac{2l}{l+1} \right) \\
&\quad + \mathcal{O}(c^{-2}) \tag{5.2}
\end{aligned}$$

in the $1/c$ expansion. In the rest of this section, we shall reproduce it from the bulk viewpoint.

A. Conical spaces in AdS₃ gravity

As explained in Sec. II, $\mathcal{O}_{1,s}$ and $\mathcal{O}_{2,1}$ are supposed to be dual to a conical geometry and a bulk scalar field, respectively. For the heavy-light correlator in (5.2), we evaluate the expectation value of the open Wilson line from 0 to w in the background of a conical defect sitting at the center of AdS₃. In this subsection, we review the analysis on the quantum effects of conical spaces in Ref. [13], which will be used in the next subsection for the bulk computation of the heavy-light correlator.

In the Chern-Simons formulation, a geometry is realized as a configuration of the gauge field. We express the gauge field in the $\mathfrak{sl}(2)$ Chern-Simons theory with the cylindrical coordinate as

$$a(w) = V_1^2 - \frac{6}{c} J^{(2)}(w) V_{-1}^2 \tag{5.3}$$

instead of (3.4).¹² The conical defect at the center of AdS₃ corresponds to the classical configuration as [13] ($s = 1, 2, 3, \dots$)

$$L_0 = -\frac{s^2 c}{24}, \quad L_{n \neq 0} = 0 \tag{5.4}$$

with the mode expansion of $J^{(2)}(w)$ as

$$J^{(2)}(w) = \sum_n L_n e^{-inw}. \tag{5.5}$$

In particular, the AdS vacuum is given by the case with $s = 1$.

Quantization of the conical spaces is analyzed in Ref. [13] utilizing the coadjoint orbits of the Virasoro algebra [14,15]. The conical defect at the center of AdS₃ is represented by operators inserted at $w = i\infty$ and $w = -i\infty$, and the corresponding states are denoted as $|0\rangle_s$ and ${}_s\langle 0|$. The states $|0\rangle_s$ and ${}_s\langle 0|$ are defined as the Fock vacua of quantum operators A_m as

$$A_m |0\rangle_s = 0, \quad {}_s\langle 0| A_{-m} = 0, \quad {}_s\langle 0|0\rangle_s = 1 \tag{5.6}$$

with $m > 0$. Here, A_m satisfies

$$[A_m, A_n] = \frac{\delta_{m+n}}{m(m^2 - s^2)}. \tag{5.7}$$

The asymptotic symmetry is given by the Virasoro algebra generated by L_n , which satisfies

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m^3 \delta_{m+n}. \tag{5.8}$$

These generators are written in terms of A_m as

$$\begin{aligned}
L_0 &= -\frac{s^2 c}{24} + n_0 + \sum_{m=1}^{\infty} m^2 (m^2 - s^2) A_{-m} A_m \\
&\quad + \mathcal{O}(c^{-1/2}); \tag{5.9}
\end{aligned}$$

see Ref. [13] for the others. Requiring that the generators satisfy (5.8), the number n_0 is determined as

$$n_0 = \frac{1}{24} (s-1)(1+13s). \tag{5.10}$$

In particular, we have

$$\begin{aligned}
L_m |0\rangle_s &= 0, \quad {}_s\langle 0| L_{-m} = 0, \\
{}_s\langle 0| L_0 |0\rangle_s &= -\frac{s^2 c}{24} + n_0 + \mathcal{O}(c^{-1/2}) \tag{5.11}
\end{aligned}$$

with $m > 0$. The L_0 eigenvalue is the same as $h_{1,s} - \frac{c}{24}$ in (5.1) up to the next nontrivial order in $1/c$, which confirms the identification as $|0\rangle_s = \mathcal{O}_{1,s}(i\infty)|0\rangle$ and ${}_s\langle 0| = \langle 0| \mathcal{O}_{1,s}(-i\infty)$.

B. Open Wilson line in conical space

We consider the expectation value of the open Wilson line in the conical geometry with label s . As explained in Sec. III A, we compute

$$W_s(w) = P \exp \left[\int_0^w dw' \left(V_1^2 - \frac{6}{c} J^{(2)}(w') V_{-1}^2 \right) \right] x \Big|_{x=0}, \tag{5.12}$$

where the background configuration for $J^{(2)}(w')$ is set as in (5.4). The generators V_n^2 are given by (3.15) with $h_0 = -1/2$. Here, we set the coupling in (3.11) as $c_2 = 1$, which is enough for the analysis in this subsection.

We evaluate its expectation value by expanding in $1/c$. Since $J^{(2)}(w')$ is now of order c , we define a new operator,

$$\hat{J}^{(2)}(w') \equiv J^{(2)}(w') + \frac{s^2 c}{24}, \tag{5.13}$$

¹²The relative sign in front of $J^{(2)}$ (and the shift of conformal weight) can be explained from the transformation of the energy momentum tensor under the change of coordinates as (D8) with (D9).

which is of order $c^{1/2}$ as in (5.27) below. As before, we find

$$\begin{aligned} \frac{d}{dw} [e^{-(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} W_s(w)] \\ = -\frac{6}{c} \hat{J}^{(2)}(w) (V_{-1}^2)_s(-w) [e^{-(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} W_s(w)] \end{aligned} \quad (5.14)$$

with

$$(V_{-1}^2)_s(w) \equiv e^{(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} V_{-1}^2 e^{-(V_1^2 + \frac{s^2}{4} V_{-1}^2)w}. \quad (5.15)$$

Using the equation, the $1/c$ expansion is given by

$$\begin{aligned} W_{h_0}(w) = \sum_{n=0}^{\infty} \left(-\frac{6}{c}\right)^n \int_0^w dw_n \cdots \\ \times \int_0^{w_2} dw_1 \left[\prod_{j=1}^n \hat{J}^{(2)}(w_j) \right] f_s(w_n, \dots, w_1) \end{aligned} \quad (5.16)$$

with

$$\begin{aligned} f_s(w_n, \dots, w_1) \\ = (V_{-1}^2)_s(w_{*n}) \cdots (V_{-1}^2)_s(w_{*1}) e^{(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} x \Big|_{x=0}, \\ w_{*i} \equiv w - w_i. \end{aligned} \quad (5.17)$$

From (see (7.6) and (7.7) of Ref. [7])

$$e^{(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} = e^{c_1(w)V_1^2} [c_0(w)]^{2V_0^2} e^{c_{-1}(w)V_{-1}^2} \quad (5.18)$$

with

$$\begin{aligned} c_1(w) &= \frac{2}{s} \tan \frac{sw}{2}, & c_0(w) &= \cos \frac{sw}{2}, \\ c_{-1}(w) &= \frac{s}{2} \tan \frac{sw}{2}, \end{aligned} \quad (5.19)$$

we find

$$e^{(V_1^2 + \frac{s^2}{4} V_{-1}^2)w} x = c_0(w)(x + c_1(w)) \quad (5.20)$$

and

$$(V_{-1}^2)_s(w) = c_0(w)^2 [V_{-1}^2 + 2c_1(w)V_0^2 + c_1(w)^2 V_1^2]. \quad (5.21)$$

In particular, the leading order expression in $1/c$ is obtained as

$$W_s(w)|_{\mathcal{O}(c^0)} = \frac{2}{s} \sin \frac{sw}{2}, \quad (5.22)$$

which reproduces (5.2).

At the next leading order in $1/c$, there are two types of contributions with one and two insertions of $\hat{J}^{(2)}$. With one insertion, we compute

$$\begin{aligned} H_s^{(1)}(w) &= -\frac{6}{c} \int_0^w dw_1 f_s(w_1)_s \langle 0 | \hat{J}^{(2)}(w_1) | 0 \rangle_s \\ &= \frac{12n_0}{c} \left[\frac{2 \sin(\frac{sw}{2}) - sw \cos(\frac{sw}{2})}{s^3} \right], \end{aligned} \quad (5.23)$$

where we have used

$$_s \langle 0 | \hat{J}^{(2)}(w_1) | 0 \rangle_s = n_0 + \mathcal{O}(c^{-1}); \quad (5.24)$$

see (5.11). Here, we do not need to worry about the regularization issue. In the following, we examine the case of two insertions with special care on the issue.

With the two insertions of $\hat{J}^{(2)}$, the integral is

$$\left(-\frac{6}{c}\right)^2 \int_0^w dw_2 \int_0^{w_2} dw_1 f_s(w_2, w_1)_s \langle 0 | \hat{J}^{(2)}(w_2) \hat{J}^{(2)}(w_1) | 0 \rangle_s \quad (5.25)$$

with

$$\begin{aligned} f_s(w_2, w_1) &= \frac{8}{s^3} \sin\left(\frac{sw_1}{2}\right) \sin\left(\frac{s(w-w_2)}{2}\right) \\ &\times \sin\left(\frac{s(w_2-w_1)}{2}\right). \end{aligned} \quad (5.26)$$

Moreover, we find

$$\begin{aligned} _s \langle 0 | \hat{J}^{(2)}(w_1) \hat{J}^{(2)}(w_2) | 0 \rangle_s |_{\mathcal{O}(c)} \\ = \sum_{m,n>0} e^{-i(mw_1 - nw_2)} _s \langle 0 | [L_m, L_{-n}] | 0 \rangle_s |_{\mathcal{O}(c)} \\ = \frac{c}{12} \sum_{m>0} e^{-im(w_1 - w_2)} (m^3 - s^2 m) \\ = \frac{c}{16} \left(\frac{1}{2 \sin^4(\frac{w_1 - w_2}{2})} + \frac{s^2 - 1}{3 \sin^2(\frac{w_1 - w_2}{2})} \right). \end{aligned} \quad (5.27)$$

However, the integration over w_1, w_2 diverges in (5.25), and we have to properly regularize it.

In order to make the expression simpler, we introduce new parameters as

$$\begin{aligned} z &= e^{iw}, & z_1 &= e^{iw_1}, & z_2 &= e^{iw_2}, \\ y &= e^{isw}, & y_1 &= e^{isw_1}, & y_2 &= e^{isw_2}. \end{aligned} \quad (5.28)$$

Then, the expression in (5.27) can be rewritten as

$$\begin{aligned} _s \langle 0 | \hat{J}^{(2)}(w_1) \hat{J}^{(2)}(w_2) | 0 \rangle_s |_{\mathcal{O}(c)} \\ = \frac{c}{2} \frac{z_1^2 z_2^2}{(z_1 - z_2)^4} - \frac{c}{12} \frac{(s^2 - 1) z_1 z_2}{(z_1 - z_2)^2}. \end{aligned} \quad (5.29)$$

The integral (5.25) diverges due to $(y_1 - y_2)^{-4}$ singularity in (5.29) as discussed in Ref. [17], so we separate the singular part in (5.29) as

$${}_s\langle 0|\hat{J}^{(2)}(w_1)\hat{J}^{(2)}(w_2)|0\rangle_s|_{\mathcal{O}(c)} = g_1[w_1, w_2] + g_2[w_1, w_2],$$

$$\begin{aligned} \frac{1}{s^4(z_1 z_2)^{2s}} g_1[w_1, w_2] &= \frac{c(z_1 z_2)^{2-2s}}{2s^4(z_2 - z_1)^4} \\ &\quad - \frac{c(s^2 - 1)(z_1 z_2)^{1-2s}}{12s^4(z_2 - z_1)^2} \\ &\quad - \frac{c}{2(y_1 - y_2)^4}, \\ \frac{1}{s^4(z_1 z_2)^{2s}} g_2[w_1, w_2] &= \frac{c}{2(y_1 - y_2)^4}. \end{aligned} \quad (5.30)$$

The integral with $g_1[w_1, w_2]$ was performed in Ref. [17], which yields a finite result. The integral with $g_2[w_1, w_2]$ diverges, so we need to regularize it in a proper way.

The integral with $g_1[w_1, w_2]$ can be written as

$$\begin{aligned} H_s^{(2)}(w) &= \frac{6^2 i z^{-s/2}}{c s^3} \int_1^z dz_2 \int_1^{z_2} dz_1 z_1^{s-1} \\ &\quad \times z_2^{s-1} (1 - z_1^s) (z_2^s - z_1^s) (z_1^s - z_2^s) \\ &\quad \times \left(\frac{(z_1 z_2)^{2-2s}}{2(z_2 - z_1)^4} - \frac{(s^2 - 1)(z_1 z_2)^{1-2s}}{12(z_2 - z_1)^2} - \frac{s^4}{2(z_1^s - z_2^s)^4} \right) \end{aligned} \quad (5.31)$$

by using the explicit form of $f_s(w_2, w_1)$ in (5.26). The expression with general complex s after the integration over z_1, z_2 was obtained in Ref. [17]. However, several terms diverge for positive integer s , and we should take care of cancellations among them. Instead of doing so, we evaluate the integral directly with the explicit values of s . As for illustration, we write down the expressions for $s = 2, 3, 4$ as

$$\begin{aligned} \frac{c s^3}{6^2 i z^{-s/2}} H_2^{(2)}(w) &= \frac{1}{8} ((1 - 7z^2) \log(z) - (z^2 - 1)(-8 \log(z + 1) - 3 + 8 \log(2))), \\ \frac{c s^3}{6^2 i z^{-s/2}} H_3^{(2)}(w) &= \frac{1}{12} \left((7 - 47z^3) \log(z) + z \left(z^2 \left(\frac{49}{3} - 27 \log(3) \right) - 9z + 9 \right) \right. \\ &\quad \left. + 27(z^3 - 1) \log(z^2 + z + 1) - \frac{49}{3} + 27 \log(3) \right), \\ \frac{c s^3}{6^2 i z^{-s/2}} H_4^{(2)}(w) &= \frac{1}{12} \left(-96(z^4 - 1) \log(2) + (19 - 125z^4) \log(z) + 48(z^4 - 1) \log(z + 1) \right. \\ &\quad \left. + 48(z^4 - 1) \log(z^2 + 1) + \frac{1}{6} (z - 1)(z + 1)(z(223z - 128) + 223) \right). \end{aligned} \quad (5.32)$$

In order to compare with (5.2), we should set $z = \exp(iw)$ at the end of computation.

A main problem here is to regularize the divergence in the integral with $g_2[w_1, w_2]$, which can be written as

$$\frac{18iy^{-1/2}}{cs} \int_1^y dy_2 \int_1^{y_2} dy_1 \frac{(1 - y_1)(y_2 - y)(y_1 - y_2)}{(y_2 - y_1)^4} \quad (5.33)$$

in terms of y, y_1, y_2 in (5.28). Notice that this is proportional to the $1/c$ correction in $\langle W_{-1/2}(y, 1) \rangle|_{\mathcal{O}(c^{-1})}$ with the AdS_3 background. Applying the regularization for the case, we find

$$\begin{aligned} \frac{18iy^{-1/2}}{cs} \int_1^y dy_2 \int_1^{y_2} dy_1 \frac{(1 - y_1)(y_2 - y)(y_1 - y_2)}{(y_2 - y_1)^{4-2\epsilon}} \\ = \frac{18iy^{-1/2}}{cs} \left[\frac{1 - y}{4\epsilon} - \frac{1}{4} (y - 1)(2 \log(y - 1) + 1) \right] \\ + \mathcal{O}(\epsilon). \end{aligned} \quad (5.34)$$

However, this computation is too naive since we are actually working on the cylindrical coordinate w but not

on $y (\equiv \exp(isw))$. Using the relation (see Appendix D for some discussions)

$$\langle W_{-1/2}(w, 0) \rangle|_{\mathcal{O}(c^{-1})} = \left[\left(\frac{\partial w}{\partial y} \right)^{-h_{1,2}} \langle W_{-1/2}(y, 1) \rangle \right] \Big|_{\mathcal{O}(c^{-1})}, \quad (5.35)$$

we find

$$\begin{aligned} \frac{18iy^{-1/2}}{cs} \int_1^y dy_2 \int_1^{y_2} dy_1 \frac{(1 - y_1)(y_2 - y)(y_1 - y_2)}{(y_2 - y_1)^4} \\ \sim -\frac{9iy^{-1/2}}{cs} (y - 1) \left(\log(y - 1) - \frac{1}{2} \log(y) + \delta \right) \\ \equiv H_s^{(3)}(w). \end{aligned} \quad (5.36)$$

Here, we have removed the divergent term by changing the overall normalization of the open Wilson line. Moreover, we introduce δ by using the ambiguity at the ϵ^0 order.

For the comparison with (5.2), we should set $y = \exp(isw)$ as above.

The $1/c$ order correction to the tree level result in (5.22) is given by the sum of three contributions as $\sum_{a=1}^3 H_s^{(a)}(w)$, which are given in (5.23), (5.31), and (5.36). We have checked for explicit values of s (i.e., for $s = 2, 3, \dots, 50$) that the sum reproduces the $1/c$ order expression of (5.2) by setting the parameter as $\delta = -\log(s)$.

VI. CONCLUSION AND OPEN PROBLEMS

We evaluated the networks of open Wilson lines in the $\mathfrak{sl}(N)$ Chern-Simons gauge theory with $N = 2, 3$ including quantum corrections. We compared the results with the $1/c$ expansions of correlators and conformal blocks in the W_N minimal model (1.1). In order to treat divergences associated with loop diagrams, we adopted the renormalization prescription proposed in Ref. [10]. In the previous paper, we examined the two- and three- point functions in (1.3) with general light operators for $N = 2$ and with specific operators for $N = 3$ by applying the prescription. We first extended the analysis by evaluating the correlators in (1.3) with general light operators for $N = 3$. Applying the method, the identity W_3 block was computed from the product of two open Wilson lines, and the result reproduced the CFT one obtained in Appendix A. We further studied general conformal blocks with $N = 2$. In particular, we obtained the all-order expressions in z expansion at the $1/c$ order, which are available in Ref. [8] except for a part. We also made some comments on the $N = 3$ extensions. For a heavy-light correlator, we examined an open Wilson line in a conical geometry, where we utilized the analysis in Ref. [13] for the quantum effects of the geometry. The analysis may be understood as a bulk interpretation of the CFT computations in Ref. [17]. We concentrated on the Virasoro case for the $1/c$ corrections of general light-light blocks and heavy-light correlators, but it is desired to analyze the W_3 case in a systematic way.

For the general W_3 blocks at the leading order in $1/c$, we explained that our formulation nicely combines previous analyses in Ref. [12] from CFT and in Ref. [7] with open Wilson lines. We thought that it is straightforward to extend the analysis beyond the leading order just like the $N = 2$ examples, but it does not seem to be the case. A complication may arise from the use of the X -basis as, for instance, the bra and ket states are expressed in a quite different way as in (B2) and (B3). In Appendix C, we analyze the $1/c$ correction of a simple example using the technique of Ref. [7], but we would like to develop our method with the X -basis for more systematic analysis.

The analysis of a heavy-light correlator is also much simpler with $N = 2$. For general N , we may use the results in Ref. [43] on the quantum effects of conical spaces. However, it is nontrivial to regularize divergences from loops associated with the open Wilson line in a nontrivial

background. For $N = 2$, a four-point function in the heavy-light limit can be mapped to a two-point function by applying a coordinate transformation as explained in Refs. [17,44]. Thus, we can apply our regularization prescription in the new coordinate system and come back to the original setup; see Appendix D. However, we do not know whether we have such a nice transformation for general N .

The current analysis can be applied to other setups. As a next project, we would like to consider supersymmetric extensions. These extensions are important to connect higher spin theory to superstring theory as in Refs. [45–47] with $\mathcal{N} = 3$ supersymmetry and Refs. [48,49] for $\mathcal{N} = 4$ supersymmetry. As an example, the $\mathcal{N} = 2$ version of Ref. [1] was proposed in Ref. [50], and the large c regime of the $\mathcal{N} = 2$ W_{N+1} minimal model is claimed to be described by the Chern-Simons theory based on $\mathfrak{sl}(N+1|N)$ [51]. Therefore, superconformal blocks are expected to be computed by the networks of open Wilson lines in a Chern-Simons theory based on a superalgebra.

An important point of our formulation is to use open Wilson lines instead of bulk scalar propagators. It is better to use bulk scalars if we want to do higher-dimensional extensions; see, e.g., Ref. [52] for a recent argument. However, Wilson lines are gauge invariant objects and should be useful in examining the properties of geometry in terms of Chern-Simons theory. In particular, a Wilson line was used to compute the entanglement entropy of boundary theory in a holographic way [53,54], which is a higher spin extension of Refs. [55,56]; see also Ref. [57]. We expect that our formulation is useful to extract nontrivial information on the quantum effects of nontrivial geometry.

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APPENDIX A: CFT CALCULATIONS

In this Appendix, we first provide the computation of the four-point function (1.4) with $\Lambda = n\omega_m$ and $\Xi = f$, i.e.,

$$G(z) \equiv \langle \bar{\mathcal{O}}_{(n\omega_m;0)}(\infty) \mathcal{O}_{(n\omega_m;0)}(1) \bar{\mathcal{O}}_{(f;0)}(z) \mathcal{O}_{(f;0)}(0) \rangle. \quad (A1)$$

We adopt the Coulomb gas method in Refs. [19,28,30]; see also Refs. [26,27,29]. We then compute the three-point functions in (1.3) as

$$\langle \bar{\mathcal{O}}_{(n\omega_m;0)}(z_1) \mathcal{O}_{(n\omega_m;0)}(z_2) J^{(s)}(z_3) \rangle \quad (A2)$$

from the conformal block decomposition of the four-point function (A1) by following the strategy in Refs. [10,21].

1. Coulomb gas method

We start from $N - 1$ free scalar fields $\phi \equiv \phi^i (i = 1, \dots, N - 1)$ with the normalization of two-point functions as

$$\langle \phi^i(z, \bar{z}) \phi^j(0, 0) \rangle = -2\delta^{ij} \log |z|^2. \quad (\text{A3})$$

We use the vertex operators

$$V_{\mathbf{b}} =: e^{i\mathbf{b} \cdot \phi} :, \quad \mathbf{b} = -\alpha_+ \Lambda_+ - \alpha_- \Lambda_- \quad (\text{A4})$$

with the conformal weight $h_{\mathbf{b}} = \mathbf{b}^2$ and $\alpha_{\pm}$ in (2.2). The $\text{SU}(N)$ -weight vectors $\Lambda_{\pm}$ are decomposed as

$$\Lambda_{\pm} = \sum_{i=1}^{N-1} \lambda_i^{\pm} \omega_i \quad (\text{A5})$$

with the Dynkin labels $\lambda_i^{\pm}$ and the fundamental weights ω_i . The Dynkin labels are related to the numbers $n_i^{\pm}$ introduced in Sec. II as $\lambda_i^{\pm} = n_i^{\pm} - n_{i+1}^{\pm}$. In particular, we set

$$\mathbf{b}_{(\mathbf{f};0)} = -\alpha_+ \omega_1, \quad \mathbf{b}_{(n\omega_m;0)} = -\alpha_+ n\omega_m \quad (\text{A6})$$

for our computations.

We introduce a background charge, which modifies the energy momentum tensor as

$$T(z) = -\frac{1}{4} \partial \phi \cdot \partial \phi + i\alpha_0 \cdot \partial^2 \phi. \quad (\text{A7})$$

Here, $\alpha_0 \equiv \alpha_0 \rho$ with $\alpha_0 = \alpha_+ + \alpha_-$, and $\rho (= \sum_i \omega_i)$ is the Weyl vector of $\text{SU}(N)$. Then, the conformal weight $h_{\mathbf{b}}$ is shifted as

$$h_{\mathbf{b}} = \mathbf{b} \cdot (\mathbf{b} - 2\alpha_0). \quad (\text{A8})$$

This is invariant under the exchange $\mathbf{b} \leftrightarrow 2\alpha_0 - \bar{\mathbf{b}}$,¹³ which implies that we should identify the operator $V_{\mathbf{b}}$ with $V_{2\alpha_0 - \bar{\mathbf{b}}}$. In terms of weight vectors, the conformal weight becomes (2.3), and in particular,

$$\begin{aligned} h(\mathbf{f};0) &= \frac{N-1}{2N} \left(1 + \frac{N+1}{k+N} \right), \\ h(n\omega_m;0) &= \frac{nm(N-m)}{2N} \left(n + \frac{N+n}{k+N} \right) \end{aligned} \quad (\text{A9})$$

with finite k, N .

¹³The conjugation operation $\bar{\mathbf{b}}$ replaces $\lambda_i^{\pm}$ by $\lambda_{N-i}^{\pm}$.

In the free theory, the correlator of the form

$$\langle V_{\mathbf{b}_1}(z_1) \dots V_{\mathbf{b}_n}(z_n) \rangle \quad (\text{A10})$$

is nonzero when the charges are conserved. In the presence of background charge, the conservation rule is shifted as

$$\sum_{i=1}^n \mathbf{b}_i = 2\alpha_0. \quad (\text{A11})$$

In order to satisfy this condition and compute a correlation function, we introduce the screening operator

$$Q \equiv \oint dz V_{\tilde{\mathbf{b}}}(z), \quad (\text{A12})$$

where $\tilde{\mathbf{b}}$ is chosen to satisfy $h_{\tilde{\mathbf{b}}} = 1$. The condition comes from the fact that the dimension of Q should be zero in order not to break conformal properties of the correlation function. From this condition, we find

$$\tilde{\mathbf{b}} = \alpha_{\pm} \mathbf{e}_i, \quad (\text{A13})$$

where $\mathbf{e}_i$ is one of the simple roots of $\text{SU}(N)$. For our calculations, it is useful to use the following relations about the inner products as

$$\begin{aligned} \omega_i \cdot \omega_j &= \frac{(N-i)j}{N} (i \geq j), & \omega_i \cdot \mathbf{e}_j &= \delta_{ij}, \\ \mathbf{e}_i \cdot \rho &= 1, & \mathbf{e}_i \cdot \mathbf{e}_j &= A_{ij}, \end{aligned} \quad (\text{A14})$$

where A_{ij} is the Cartan matrix of $\text{SU}(N)$.

2. Four-point function

We consider the four-point function in (A1). From the mapping (A6), the correlator is written as

$$\begin{aligned} G(z) &= \left\langle V_{2\alpha_0 + \alpha_+ n\omega_m}(z_{\infty}) V_{-\alpha_+ n\omega_m}(1) V_{-\alpha_+ \omega_{N-1}}(z) V_{-\alpha_+ \omega_1}(0) \right. \\ &\quad \times \left. \left(\prod_{i=1}^{N-1} \oint dt_i V_{\alpha_+ \mathbf{e}_i}(t_i) \right) \right\rangle. \end{aligned} \quad (\text{A15})$$

Here, we have introduced z_{∞} , which will be set as $z_{\infty} = \infty$ at the end of the computation. We have changed the first operator $V_{\mathbf{b}} \rightarrow V_{2\alpha_0 - \bar{\mathbf{b}}}$ and inserted the screening operators to satisfy (A11). Using the simple form of the free field correlator as

$$\langle V_{\mathbf{b}_1}(z_1) \dots V_{\mathbf{b}_n}(z_n) \rangle = \prod_{i < j} |z_i - z_j|^{4\mathbf{b}_i \cdot \mathbf{b}_j}, \quad (\text{A16})$$

the holomorphic part of $G(z)$ is written as

$$\begin{aligned}
G(z) = & (z_\infty - 1)^{\gamma_1} (z_\infty - z)^{\gamma_z} z_\infty^{\gamma_0} (1 - z)^{2\alpha_+^{\frac{nm}{N}} z^{2\alpha_+^{\frac{1}{N}}}} \oint dt_1 \cdots \oint dt_{m-1} t_1^{-2\alpha_+^2} \left[\prod_{i=1}^{m-1} (t_i - z_\infty)^{4\alpha_0\alpha_+} (t_i - t_{i+1})^{-2\alpha_+^2} \right] \\
& \times \oint dt_m (t_m - z_\infty)^{4\alpha_0\alpha_+ + 2n\alpha_+^2} (t_m - 1)^{-2n\alpha_+^2} \oint dt_{m+1} \cdots \oint dt_{N-1} (t_{N-1} - z)^{-2\alpha_+^2} \\
& \times \left[\prod_{i=m+1}^{N-1} (t_i - z_\infty)^{4\alpha_0\alpha_+} \right] \left[\prod_{i=m}^{N-2} (t_i - t_{i+1})^{-2\alpha_+^2} \right]. \tag{A17}
\end{aligned}$$

Here, we can compute $\gamma_1, \gamma_z, \gamma_0$ explicitly, but they are not necessary for our purpose. Using the relation

$$\begin{aligned}
\oint dt_i \prod_{j=1}^3 (t_i - s_j)^{2\beta_j} & \propto (s_1 - s_2)^{\beta_1 + \beta_2 - \beta_3} (s_2 - s_3)^{\beta_2 + \beta_3 - \beta_1} \\
& \times (s_3 - s_1)^{\beta_3 + \beta_1 - \beta_2} \tag{A18}
\end{aligned}$$

for $\sum_{j=1}^3 \beta_j = -2$, we can integrate over all variables without t_m . After the integration, we arrive at

$$G(z) \propto \left| z^{2\alpha_+^{\frac{1}{N}}} (1 - z)^{2\alpha_+^{\frac{nm}{N}}} \oint dt_m t_m^a (t_m - 1)^b (t_m - z)^c \right|^2 \tag{A19}$$

with

$$\begin{aligned}
a = m - 1 - 2m\alpha_+^2, \quad b = -2n\alpha_+^2, \\
c = N - m - 1 - 2(N - m)\alpha_+^2. \tag{A20}
\end{aligned}$$

As in (9.88) of Ref. [30], the final result should be of the form

$$\begin{aligned}
& \left| \oint dy y^a (y - 1)^b (y - z)^c \right|^2 \\
& = \frac{\sin(\pi b) \sin(\pi(a + b + c))}{\sin(\pi(a + c))} |I_1(z)|^2 \\
& + \frac{\sin(\pi a) \sin(\pi c)}{\sin(\pi(a + c))} |I_2(z)|^2, \tag{A21}
\end{aligned}$$

where $I_i(z)$ are

$$\begin{aligned}
I_1(z) & = \frac{\Gamma(-a - b - c - 1) \Gamma(b + 1)}{\Gamma(-a - c)} \\
& \times {}_2F_1(-c, -a - b - c - 1; -a - c; z), \\
I_2(z) & = z^{1+a+c} \frac{\Gamma(a + 1) \Gamma(c + 1)}{\Gamma(a + c + 2)} \\
& \times {}_2F_1(-b, a + 1; a + c + 2; z). \tag{A22}
\end{aligned}$$

Applying this result and using the relation

$$\begin{aligned}
{}_2F_1(a, b; c; z) \\
= (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z), \tag{A23}
\end{aligned}$$

we obtain (2.22) with (2.23).

3. Identity block expansion and higher spin charges

We consider the holomorphic part of the identity block as

$$z^{-2h(f;0)} (1 - z)^{nm \frac{k+N+1}{N(k+N)}} {}_2F_1\left(n + \frac{n}{k+N}, -\frac{m}{k+N}; -\frac{N}{k+N}; z\right). \tag{A24}$$

We obtain the three-point functions in (A2) including $1/c$ corrections by applying the method in Refs. [10,21]. We first decompose the four-point function as

$$|z|^{4h(f;0)} G(z) = \mathcal{V}_0(z) + \sum_{s=3}^{\infty} C_f^{(s)} C_{n\omega_m}^{(s)} \mathcal{V}_s(z) + \cdots \tag{A25}$$

Here, $\mathcal{V}_0(z)$ and $\mathcal{V}_s(z)$ are the identity Virasoro block and the Virasoro block of the spin s current, and their expansions in $1/c$ are given in (2.19) and (2.20) with $h_p = s$. The coefficients $C_\Lambda^{(s)}$ are defined as

$$C_\Lambda^{(s)} \equiv \frac{\langle \tilde{\mathcal{O}}_{(\Lambda;0)} \mathcal{O}_{(\Lambda;0)} J^{(s)} \rangle}{\langle J^{(s)} J^{(s)} \rangle^{1/2} \langle \tilde{\mathcal{O}}_{(\Lambda;0)} \mathcal{O}_{(\Lambda;0)} \rangle}, \tag{A26}$$

which is expanded in $1/c$ as

$$C_\Lambda^{(s)} = c^{-1/2} [C_{\Lambda,0}^{(s)} + c^{-1} C_{\Lambda,1}^{(s)} + \mathcal{O}(c^{-2})]. \tag{A27}$$

On the other hand, from (2.5), the expansion of (A24) in $1/c$ is obtained as

$$\begin{aligned}
& |z|^{4h(f;0)} G(z) \\
& \sim 1 + \frac{(1 - N^2)}{c} \sum_{l=1}^{\infty} \left(-\frac{nm}{l} + \frac{n}{\Gamma(m)} \frac{\Gamma(1+N)\Gamma(l+m)}{\Gamma(l+N)} \right) z^l \\
& + \frac{1}{c^2} \sum_{l=2}^{\infty} f_{c;n\omega_m}^{(l)} z^l, \tag{A28}
\end{aligned}$$

where we have defined

$$\begin{aligned} \frac{f_{c;n\omega_m}^{(l)}}{(1-N^2)^2} &= \frac{nN^2\Gamma(N)\Gamma(l+m)}{l\Gamma(m)\Gamma(l+N)} \left[\sum_{j=1}^l \frac{N}{N+j-1} - \left(m \sum_{j=1}^l \frac{1}{j+m-1} - nH_{l-1} \right) - \frac{2N^2+2N+1}{N(N+1)} \right] \\ &\quad - n^2m \frac{N\Gamma(N)}{\Gamma(m)} \sum_{j=1}^{l-1} \frac{1}{l-j} \frac{\Gamma(j+m)}{j\Gamma(j+N)} + \frac{nm}{l} \left(2N + \frac{1}{N+1} + nmH_{l-1} \right). \end{aligned} \quad (\text{A29})$$

Comparing the $1/c$ and z expansions in the both sides of (A25), we obtain constraints for the coefficients $C_\Lambda^{(s)}$.

We find the explicit form of the coefficients $C_\Lambda^{(s)}$ by solving the constraint equations. From the coefficients of z^l , constraints at the leading order in $1/c$ are given as

$$\left(-\frac{nm}{l} + \frac{n}{\Gamma(m)} \frac{\Gamma(1+N)\Gamma(l+m)}{l\Gamma(l+N)} \right) = \frac{(\Gamma(l))^2}{1-N^2} \sum_{s=2}^l \frac{\Gamma(2s)C_{f,0}^{(s)}C_{n\omega_m,0}^{(s)}}{(\Gamma(s))^2\Gamma(s+l)(l-s)!}, \quad (\text{A30})$$

where

$$C_{f,0}^{(2)}C_{n\omega_m,0}^{(2)} = \frac{nm}{2}(1-N)(m-N) \quad (\text{A31})$$

is read off from (2.8). The first few solutions to the constraint equations are

$$\begin{aligned} C_{f,0}^{(3)}C_{n\omega_m,0}^{(3)} &= \frac{nm(1-N)(m-N)(2m-N)}{6(N+2)}, \\ C_{f,0}^{(4)}C_{n\omega_m,0}^{(4)} &= \frac{nm(1-N)(m-N)(5m^2-5mN+N^2+1)}{20(N+2)(N+3)}, \\ C_{f,0}^{(5)}C_{n\omega_m,0}^{(5)} &= \frac{nm(1-N)(m-N)(2m-N)(7m^2-7mN+N^2+5)}{70(N+2)(N+3)(N+4)}. \end{aligned} \quad (\text{A32})$$

Using the definition of the coefficients $C_{\Lambda,0}^{(s)}$ in (A27) with (A26) and the normalization of currents in (2.11) with (2.12), we rewrite the three-point functions as

$$w^{(s)}(\Lambda, 0)|_{\mathcal{O}(c^0)} \equiv \langle \bar{\mathcal{O}}_{(\Lambda;0)} \mathcal{O}_{(\Lambda;0)} J^{(s)} \rangle|_{\mathcal{O}(c^0)} = \frac{\Gamma(N+s)}{(1-N)\Gamma(N+2)} C_{f,0}^{(s)} C_{\Lambda,0}^{(s)}, \quad (\text{A33})$$

which leads to

$$\begin{aligned} w^{(3)}(n\omega_m, 0)|_{\mathcal{O}(c^0)} &= \frac{1}{6} mn(m-N)(2m-N), \\ w^{(4)}(n\omega_m, 0)|_{\mathcal{O}(c^0)} &= \frac{1}{20} mn(m-N)(5m^2-5mN+N^2+1), \\ w^{(5)}(n\omega_m, 0)|_{\mathcal{O}(c^0)} &= \frac{1}{70} mn(m-N)(2m-N)(7m^2-7mN+N^2+5). \end{aligned} \quad (\text{A34})$$

We can see that the results with $s = 3, 4$ and general n, m are the same as (2.17) and (C.3) in Ref. [25].

At the next order in $1/c$, constraint equations for $s = 3, 4, 5$ are given as

$$\begin{aligned} f_{c;n\omega_m}^{(3)} &= f_{c;n\omega_m}^{(2)} + (C_{f,0}^{(3)}C_{n\omega_m,1}^{(3)} + C_{f,1}^{(3)}C_{n\omega_m,0}^{(3)}), \\ f_{c;n\omega_m}^{(4)} &= f_{c;n\omega_m}^{(2)} \frac{9}{10} + \frac{(1-N)^2}{8(1+N)^2} + \frac{1-N}{10(1+N)^2} + \frac{1}{50(1+N)^2} + (C_{f,0}^{(3)}C_{n\omega_m,1}^{(3)} + C_{f,1}^{(3)}C_{n\omega_m,0}^{(3)}) \frac{3}{2} + (C_{f,0}^{(4)}C_{n\omega_m,1}^{(4)} + C_{f,1}^{(4)}C_{n\omega_m,0}^{(4)}), \\ f_{c;n\omega_m}^{(5)} &= f_{c;n\omega_m}^{(2)} \frac{4}{5} + \frac{(1-N)^2}{4(1+N)^2} + \frac{1-N}{5(1+N)^2} + \frac{1}{25(1+N)^2} + (C_{f,0}^{(3)}C_{n\omega_m,1}^{(3)} + C_{f,1}^{(3)}C_{n\omega_m,0}^{(3)}) \frac{12}{7} + (C_{f,0}^{(4)}C_{n\omega_m,1}^{(4)} + C_{f,1}^{(4)}C_{n\omega_m,0}^{(4)}) \cdot 2 \\ &\quad + (C_{f,0}^{(5)}C_{n\omega_m,1}^{(5)} + C_{f,1}^{(5)}C_{n\omega_m,0}^{(5)}) + C_{f,0}^{(3)}C_{n\omega_m,0}^{(3)} \left[\frac{1}{2} \frac{1-N}{1+N} + \frac{6}{7(1+N)} + \frac{18}{49(1-N^2)} \right]. \end{aligned} \quad (\text{A35})$$

From the z^3 order constraint, we obtain

$$\frac{C_{n\omega_m,1}^{(3)}}{C_{n\omega_m,0}^{(3)}} = N^3 + 3nN^2 - 3N - \frac{6}{N+2} - 3n + 4, \quad (\text{A36})$$

which reproduces the $1/c$ order term in (2.17). From the z^4 order constraint, we find

$$\begin{aligned} \frac{C_{n\omega_m,1}^{(4)}}{C_{n\omega_m,0}^{(4)}} = & \frac{1}{40nm(N-1)(N+1)^2(N+2)(N+3)(m-N)(5m^2-5mN+N^2+1)} \\ & \times [4(-25nm^4(N-1)^2(N+1)^3(n(N-8)(N+2)(N+3)-2(N(2N(N+8)+29)+6))) \\ & + 50nm^3(N-1)^2N(N+1)^3(n(N-8)(N+2)(N+3)-2(N(2N(N+8)+29)+6)) \\ & - 5nm^2(N-1)^2(N+1)^3(n(N+2)(N+3)(N^2(5N-52)-2)-2(N(6N(N(2N(N+8)+29)+7)+17)+6)) \\ & - 10nm(N-1)^2N(N+1)^3(n(N+2)(N+3)(6N^2+1)+N(N(N(2N(N+8)+29)+12)+17)+6) \\ & + (N+2)^2(N+3)^2(7-5N)^2] \\ & + \frac{1}{40} \left[-40N^3 - 290N^2 - 60N - \frac{3780}{N-3} + \frac{320}{N-2} - \frac{47}{N-1} + \frac{12}{(N-1)^2} + \frac{27}{N+1} + \frac{12}{(N+1)^2} + \frac{240}{N+2} + \frac{1440}{N+3} - 1610 \right]. \end{aligned} \quad (\text{A37})$$

In the same way, we can obtain the expression of $C_{n\omega_m,1}^{(5)}/C_{n\omega_m,0}^{(5)}$, but the result is too complicated to write down here.

APPENDIX B: SL(3) GENERATORS

One of the aims in this paper is to generalize the analysis in Ref. [10] by examining correlators with general light operators for $N = 3$. We parametrize the representation of the operator by Dynkin labels as $(\lambda_1, \lambda_2; 0)$. For the generalization, we need the expressions of $\text{sl}(3)$ generators in terms of $X = (x, y, w)$ and the highest- and lowest-weight states denoted as $\langle \text{lw}|X\rangle$ and $\langle X|\text{hw}\rangle$, respectively. For the $\text{sl}(3)$ generators, we use the expressions in Ref. [30] as

$$\begin{aligned} f_0^1 &= -\frac{\partial}{\partial x}, \quad f_0^2 = -\frac{\partial}{\partial y} - x\frac{\partial}{\partial w}, \quad f_0^3 = -\frac{\partial}{\partial w}, \\ h_0^1 &= -\lambda_2 + 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}, \\ h_0^2 &= -\lambda_1 - x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}, \\ e_0^1 &= -\lambda_2 x + w\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w} - xy\frac{\partial}{\partial y}, \\ e_0^2 &= -\lambda_1 x - w\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \\ e_0^3 &= -(\lambda_1 + \lambda_2)w + \lambda_1 xy + xw\frac{\partial}{\partial x} + yw\frac{\partial}{\partial y} + w^2\frac{\partial}{\partial w} - xy^2\frac{\partial}{\partial y}. \end{aligned} \quad (\text{B1})$$

The general states are expressed by monomials of the form $x^f y^s w^t$. The lowest-weight state is represented as $\langle X|\text{lw}\rangle = 1$. The state is annihilated by the lowering operators f_0^a ($a = 1, 2, 3$), and the eigenvalues of the Casimir generators h_0^i ($i = 1, 2$) are $(-\lambda_2, -\lambda_1)$. This implies that we should use (see also Refs. [8,10])

$$\langle \text{lw}|X\rangle = \delta(x)\delta(y)\delta(w). \quad (\text{B2})$$

The highest-weight states are annihilated by e_0^a ($a = 1, 2, 3$), and the eigenvalues of h_0^i are (λ_1, λ_2) . These conditions can be solved by

$$\langle X|\text{hw}\rangle = 2^{\lambda_1+\lambda_2}(xy-w)^{\lambda_1}w^{\lambda_2}, \quad (\text{B3})$$

where the overall factor is chosen such that the expression in (3.30) becomes simpler.

For the application to higher spin gravity, it is important to decompose the $\text{sl}(3)$ generators in terms of embedded $\text{sl}(2)$. Here, we choose

$$V_1^2 = -f_0^1 - f_0^2, \quad V_0^2 = h_0^1 + h_0^2, \quad V_{-1}^2 = 2(e_0^1 + e_0^2), \quad (\text{B4})$$

and

$$\begin{aligned} V_2^3 &= f_0^3, \quad V_1^3 = -\frac{1}{2}(f_0^1 - f_0^2), \quad V_0^3 = \frac{1}{3}(h_0^1 - h_0^2), \\ V_{-1}^3 &= -e_0^1 + e_0^2, \quad V_{-2}^3 = 4e_0^3. \end{aligned} \quad (\text{B5})$$

We can show that the commutation relations

$$[V_m^2, V_n^2] = (m-n)V_{m+n}^2, \quad [V_m^2, V_n^3] = (2m-n)V_{m+n}^3, \\ [V_m^3, V_n^3] = -\frac{1}{12}(m-n)(2m^2+2n^2-mn-8)V_{m+n}^2 \quad (\text{B6})$$

are satisfied.

For the manipulation of Wilson line operators, we need some algebraic computations. In (3.17), we may use

$$e^{zV_1^2} \langle X | l w \rangle = 2^{\lambda_1 + \lambda_2} \left((x+z)(y+z) - \left(\frac{1}{2} z^2 + zx + w \right) \right)^{\lambda_1} \\ \times \left(\frac{1}{2} z^2 + zx + w \right)^{\lambda_2}, \quad (\text{B7})$$

where we have utilized the following formula:

$$e^{zV_1^2} f(X) = e^{z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x \frac{\partial}{\partial w})} f(X) \\ = e^{(\frac{z^2}{2} + zx) \frac{\partial}{\partial w}} e^{z \frac{\partial}{\partial x}} e^{z \frac{\partial}{\partial y}} f(X). \quad (\text{B8})$$

Similarly, we can compute

$$e^{zV_1^2} V_{-1}^2 e^{-zV_1^2} = -2 \left[\lambda_2 x_z + \lambda_1 y_z + (x_z y_z - w_z - y_z^2) \frac{\partial}{\partial y} \right. \\ \left. + (w_z - x_z^2) \left(\frac{\partial}{\partial x} - z \frac{\partial}{\partial w} \right) - x_z w_z \frac{\partial}{\partial w} \right], \\ e^{zV_1^2} V_{-2}^3 e^{-zV_1^2} = -4 \left[(\lambda_1 + \lambda_2) w_z - \lambda_1 x_z y_z - x_z w_z \left(\frac{\partial}{\partial x} - z \frac{\partial}{\partial w} \right) \right. \\ \left. + (x_z y_z^2 - y_z w_z) \frac{\partial}{\partial y} - w_z^2 \frac{\partial}{\partial w} \right] \quad (\text{B9})$$

with

$$x_z = x + z, \quad y_z = y + z, \\ w_z = w + zx + \frac{1}{2} z^2, \quad (\text{B10})$$

which are used for (3.17).

APPENDIX C: SIMPLE EXAMPLE OF GENERAL W_3 BLOCK

In the main context, we examined general blocks with the Wilson line method for $N = 2$ up to the next leading order in $1/c$ and for $N = 3$ at the leading order in $1/c$. In this Appendix, we examine the $1/c$ correction of the general W_3 block for (2.21) with $N = 3$ and $n = m = 1$. For the simple case, it is not difficult to apply the technique in Ref. [7] for algebraic calculations, even though our approach with the X -basis should be useful for more complicated examples with generic representations just as the leading order analysis in Sec. IV B 3.

Similarly to the $N = 2$ case, we start by examining the three-point function

$$\langle \mathcal{O}_{(\bar{f};0)}(z) \mathcal{O}_{(f;0)}(0) \mathcal{O}_{(\text{adj};0)}(1) \rangle \propto \frac{1}{(1-z)^{h_{\text{adj}} z^2 h_f - h_{\text{adj}}}}, \quad (\text{C1})$$

where adj represents the adjoint representation of $\text{su}(3)$. The conformal dimensions are

$$h_f \equiv h(f; 0) = -1 - \frac{32}{c} + \mathcal{O}(c^{-2}), \\ h_{\text{adj}} \equiv h(\text{adj}; 0) = -2 - \frac{72}{c} + \mathcal{O}(c^{-2}); \quad (\text{C2})$$

see (2.7). In the Wilson line network of (3.7) with $n = 3$, we set $(z_1, z_2, z_3) = (z, 0, 1)$ and $(\Lambda_1, \Lambda_2, \Lambda_3) = (\bar{f}, f, \text{adj})$. Putting $z_0 = z_3 = 1$, we compute the $1/c$ corrections from the diagrams in Fig. 5. For the leading order in $1/c$, we use

$$q_j^{(i)} \equiv \langle e_j | e^{z_{0i} V_1^2} | e_1 \rangle_i = \delta_j^1 - \sqrt{2} z_{0i} \delta_j^2 + z_{0i}^2 \delta_j^3, \\ \bar{q}_{(i)}^j \equiv \langle \bar{e}_j | e^{z_{0i} V_1^2} | \bar{e}_3 \rangle_i = \delta_j^3 + \sqrt{2} z_{0i} \delta_j^2 + z_{0i}^2 \delta_j^1 \quad (\text{C3})$$

by adopting the conventions in Refs. [2, 7]. For examples, $|e_j\rangle_i$ and $|\bar{e}_j\rangle_i$ denote the states in the fundamental and antifundamental representations, which are used to construct the general representation Λ_i . As a singlet $\langle S |$, we first pick up the adjoint representation among $\Lambda_1 \otimes \Lambda_2$ and then construct a singlet with $\Lambda_3 = \text{adj}$. This choice of singlet leads to

$$\langle G_3(\Lambda_i | z_i) \rangle |_{\mathcal{O}(c^0)} = \bar{q}_{(1)}^i q_j^{(2)} \bar{q}_{(3)}^j q_i^{(3)} = (1-z)^2, \quad (\text{C4})$$

which reproduces the leading order term in (C1). For the next leading order in $1/c$, we define

$$q_j^{(i;s)}(z') \equiv \frac{6}{c} \langle e_j | e^{(z_0 - z') V_1^2} V_{-s+1}^s e^{(z' - z_0) V_1^2} e^{z_{0i} V_1^2} | e_1 \rangle_i, \\ \bar{q}_{(i;s)}^j(z') \equiv \frac{6}{c} \langle \bar{e}_j | e^{(z_0 - z') V_1^2} V_{-s+1}^s e^{(z' - z_0) V_1^2} e^{z_{0i} V_1^2} | \bar{e}_3 \rangle_i, \\ q_j^{(i;s,s)}(z', z'') \equiv \left(\frac{6}{c} \right)^2 \langle e_j | e^{(z_0 - z') V_1^2} V_{-s+1}^s e^{(z' - z_0) V_1^2} e^{(z_0 - z'') V_1^2} \\ \times V_{-s+1}^s e^{(z'' - z_0) V_1^2} e^{z_{0i} V_1^2} | e_1 \rangle_i, \\ \bar{q}_{(i;s,s)}^j(z', z'') \equiv \left(\frac{6}{c} \right)^2 \langle \bar{e}_j | e^{(z_0 - z') V_1^2} V_{-s+1}^s e^{(z' - z_0) V_1^2} e^{(z_0 - z'') V_1^2} \\ \times V_{-s+1}^s e^{(z'' - z_0) V_1^2} e^{z_{0i} V_1^2} | \bar{e}_3 \rangle_i. \quad (\text{C5})$$

Just like the leading order, contributions at order $1/c$ come from the integrals as

$$\begin{aligned}
& \int_z^1 dz_2 \int_z^{z_2} dz_1 \bar{q}_{(1;s,s)}^i(z_2, z_1) q_j^{(2)} \bar{q}_{(3)}^j q_i^{(3)} \langle J^{(s)}(z_2) J^{(s)}(z_1) \rangle, \\
& \int_1^0 dz_2 \int_1^{z_2} dz_1 \bar{q}_{(1)}^i q_j^{(2;s,s)}(z_1, z_2) \bar{q}_{(3)}^j q_i^{(3)} \langle J^{(s)}(z_2) J^{(s)}(z_1) \rangle, \\
& \int_z^1 dz_2 \int_1^0 dz_1 \bar{q}_{(1;s)}^i(z_2) q_j^{(2;s)}(z_1) \bar{q}_{(3)}^j q_i^{(3)} \langle J^{(s)}(z_2) J^{(s)}(z_1) \rangle
\end{aligned} \tag{C6}$$

with $s = 2, 3$ and (3.9) with (3.32). The sum of them is computed as

$$\langle G_3(\Lambda_i|z_i) \rangle|_{\mathcal{O}(c^{-1})} = \frac{1}{c} (1-z)^2 (72 \log(1-z) - 8 \log(z)) \tag{C7}$$

for the ϵ -independent part. With the shifts of conformal weight in (C2), we can see that this is the result expected from the conformal symmetry as in (C1).

The four-point block for (2.21) with $N = 3$ and $n = m = 1$ can be examined in a similar manner. Since we have already examined the identity block in Sec. IV A, we focus on the W_3 block with the exchange of adjoint operator in (2.22). The expression in $1/c$ expansion is

$$\begin{aligned}
& (1-z)^2 - \frac{z^2}{3} + \frac{1}{c} \left[\left((1-z)^2 - \frac{z^2}{3} \right) (64 \log(1-z) - 8 \log(z)) \right. \\
& \quad \left. + 24z(1-z)^2 - 8(1-z)^2 \sum_{n=2}^{\infty} (n-1) \left(3H_{n-2} + \frac{1}{n} - 3 \right) z^n \right] \\
& \quad + \mathcal{O}(c^{-2}).
\end{aligned} \tag{C8}$$

In the Wilson line network (3.7) with $n = 4$, we set $(z_1, z_2, z_3, z_4) = (z, 0, \infty, 1)$ and $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (\bar{f}, f, \bar{f}, f)$. We may fix $z_0 = z_4 = 1$ as in Fig. 6. For the singlet $\langle S \rangle$, we first pick up the adjoint representations in $\Lambda_1 \otimes \Lambda_2$ and $\Lambda_3 \otimes \Lambda_4$ and then construct a singlet from the product of the two adjoint representations. At the leading order in $1/c$, we find

$$\begin{aligned}
\langle G_4(\Lambda_i|z_i) \rangle|_{\mathcal{O}(c^0)} &= \bar{q}_{(1)}^i q_j^{(2)} \bar{q}_{(3)}^j q_i^{(4)} - \frac{1}{3} \bar{q}_{(1)}^i q_i^{(2)} \bar{q}_{(3)}^j q_j^{(4)} \\
&= (1-z)^2 - \frac{z^2}{3},
\end{aligned} \tag{C9}$$

which reproduces the leading order term in (C8). Here, we should notice that the products $\bar{q}_{(1)}^i q_j^{(2)} \bar{q}_{(3)}^j q_i^{(4)}$ and $\bar{q}_{(1)}^i q_i^{(2)} \bar{q}_{(3)}^j q_j^{(4)}$ are the same as the identity blocks for the $z \rightarrow 1$ and $z \rightarrow 0$ channels, respectively. Since the structure of contractions among indices does not change even at the next leading orders in $1/c$, the computation for the Wilson line network reduces to

$$\begin{aligned}
\langle G_4(\Lambda_i|z_i) \rangle &= \langle W_f(z; 1) W_f(\infty; 0) \rangle \\
&\quad - \frac{1}{3} \langle W_f(z; 0) W_f(\infty; 1) \rangle
\end{aligned} \tag{C10}$$

at least up to order $1/c$. Using (4.6) and (4.9), the $1/c$ order terms of the quantity are obtained as

$$\begin{aligned}
\langle G_4(\Lambda_i|z_i) \rangle|_{\mathcal{O}(c^{-1})} &= \frac{1}{c} \left\{ \left[64(1-z)^2 \log(1-z) \right. \right. \\
&\quad \left. + 2(1-z)^4 {}_2F_1(2, 2; 4; 1-z) \right. \\
&\quad \left. - \frac{2}{15} (1-z)^5 {}_2F_1(3, 3; 6; 1-z) \right] \\
&\quad - \frac{1}{3} \left[64z^2 \log(z) + 2z^4 {}_2F_1(2, 2; 4; z) \right. \\
&\quad \left. \left. - \frac{2}{15} z^5 {}_2F_1(3, 3; 6; z) \right] \right\}.
\end{aligned} \tag{C11}$$

The difference between the $1/c$ order terms in (C8) and (C11) is proportional to the leading order terms in (C8), which can be removed by changing the overall factor of the conformal block. Thus, we conclude that the Wilson line computation reproduces the CFT one in (2.22) up to the $1/c$ order for the simple example.

APPENDIX D: CONFORMAL TRANSFORMATION OF CORRELATORS

In Appendix A of Ref. [7], it was analyzed how global conformal transformations act on correlators from Wilson line networks at the leading order in $1/c$. Here, we would like to extend the analysis to local conformal transformations.

We start from the gauge field

$$a(z) = V_1^2 + \frac{6}{c} T(z) V_{-1}^2 \tag{D1}$$

as in (3.4) with $N = 2$ and $T(z) = J^{(2)}(z)$. For the current analysis, we neglect the shift of coupling c_2 in (3.11). We consider the $\text{su}(2)$ gauge transformation as

$$a'(z) = L(z) a(z) L(z)^{-1} + L(z) \frac{d}{dz} L(z)^{-1} \tag{D2}$$

with

$$L(z) = e^{c_{-1}(z) V_{-1}^2} e^{2 \log c_0(z) V_0^2} e^{c_1(z) V_1^2}. \tag{D3}$$

We require that the $a'(z)$ is of the form (D1), i.e., the coefficient of V_0^2 to be zero. We consider the following solution to this condition:

$$c_1(z) = 0, \quad c_{-1}(z) = -c_0(z) c'_0(z). \tag{D4}$$

This solution reduces to (A.6) of Ref. [7] for global transformation induced by $c_0(z) = cz + d$.

Under the gauge transformation, the open Wilson line becomes

$$P \exp \left\{ \int_{z_i}^{z_f} dz \left[\frac{1}{c_0(z)^2} V_1^2 + c_0(z)^2 \left(\frac{6}{c} T(z) + \frac{c_0''(z)}{c_0(z)} \right) V_{-1}^2 \right] \right\}. \quad (\text{D5})$$

In order to set the coefficient of V_1^2 to be 1, we change the coordinate z to w with

$$c_0(z) = \left(\frac{dz}{dw} \right)^{\frac{1}{2}}. \quad (\text{D6})$$

The open Wilson line is now rewritten as

$$P \exp \left(\int_{w_i}^{w_f} dw a(w) \right), \quad a(w) = V_1^2 + \frac{6}{c} T(w) V_{-1}^2 \quad (\text{D7})$$

with

$$T(w) = \left(\frac{dw}{dz} \right)^{-2} \left(T(z) - \frac{c}{12} S(w, z) \right). \quad (\text{D8})$$

Here, we have defined the Schwarzian derivative as

$$S(w, z) = \frac{d^3 w / dz^3}{(dw/dz)} - \frac{3}{2} \left(\frac{d^2 w / dz^2}{dw/dz} \right)^2. \quad (\text{D9})$$

Let us consider how a two-point function changes under the transformation. We first adopt the regularization in Ref. [8], where we take the normal ordered prescription of $T(z)$ in the same open Wilson line and replace the quantum number $h_0 = -j$ by the exact conformal weight h_j . Then, we find

$$\begin{aligned} \langle 1w | P \exp \left(\int_{z_i}^{z_f} dz a(z) \right) | hw \rangle \\ = \langle 1w | L^{-1}(z_f) P \exp \left(\int_{z_i}^{z_f} dz a'(z) \right) L(z_i) | hw \rangle \\ = \left(\frac{dz_f}{dw_f} \right)^{-h_j} \left(\frac{dz_i}{dw_i} \right)^{-h_j} \langle 1w | P \exp \left(\int_{w_i}^{w_f} dw a(w) \right) | hw \rangle; \end{aligned} \quad (\text{D10})$$

see (A.10) of Ref. [7]. This is consistent with the local conformal transformation of the two-point function as

$$\langle \mathcal{O}_j(z_f) \mathcal{O}_j(z_i) \rangle = \left(\frac{dz_f}{dw_f} \right)^{-h_j} \left(\frac{dz_i}{dw_i} \right)^{-h_j} \langle \mathcal{O}_j(w_f) \mathcal{O}_j(w_i) \rangle. \quad (\text{D11})$$

With our prescription, we regularized the open Wilson line for the two-point function when the gauge field corresponds to the AdS_3 background and the coordinate z is the planar one. For other cases, we offer a way to regularize it such that (D11) is satisfied, and this is the prescription adopted in Sec. V.

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