Mass gap in the weak coupling limit of $(2+1)$ -dimensional $SU(2)$ lattice gauge theory

Ramesh Anishetty^{*} and T. P. Sreeraj[†]

The Institute of Mathematical Sciences, C.I.T. campus, Taramani, Chennai 600113, India

(Received 19 February 2018; published 26 April 2018)

We develop the dual description of $(2+1)$ -dimensional $SU(2)$ lattice gauge theory as interacting “Abelian-like” electric loops by using Schwinger bosons. “Point splitting” of the lattice enables us to construct explicit Hilbert space for the gauge invariant theory which in turn makes dynamics more transparent. Using path integral representation in phase space, the interacting closed loop dynamics is analyzed in the weak coupling limit to get the mass gap.

DOI: [10.1103/PhysRevD.97.074511](https://doi.org/10.1103/PhysRevD.97.074511)

I. INTRODUCTION

There had been many attempts in the past to reformulate gauge theories in terms of gauge invariant Wilson loops [1–4] both in the continuum as well as on the lattice. Since all Wilson loops are not independent even on a finite lattice, such a description based on nonlocal Wilson loops is overcomplete. Constructing a complete algebra of observables in terms of loops and a complete loop basis for gauge theory Hilbert space is a nontrivial [2–4] task. Loop description of gauge theories on a lattice has been shown to be equivalent to a description based on integer local quantum numbers in dual space satisfying triangle inequalities [5] at each site. This is most conveniently realized in a Hamiltonian formulation using the prepotential representation of gauge theories on a square lattice. The matrix elements of the magnetic part of the Hamiltonian in such a basis involves higher Wigner coefficients [6] and are complicated to analyze.

In this paper, we construct a complete, gauge invariant, local description of $SU(2)$ lattice gauge theory (LGT) in $2+1$ dimensions by solving explicitly the gauge constraint on the “electric” configuration space which is then further simplified by envisaging what is called a “point split lattice” (see Sec. III). Furthermore, a manifest gauge invariant Hilbert space is constructed at every site on the split lattice. The local gauge invariant basis thus created is parametrized in terms of three quantum numbers at each site of the new lattice which are independent and have to satisfy triangle

inequalities. These quantum numbers give a measure of the gauge invariant electric flux lying on the links of the lattice. Therefore, gauge theory reduces to a theory of interacting electric flux loops. Matrix elements of the Hamiltonian are simpler to analyze in this basis especially in the weak coupling limit. Such a local, complete basis allows us to construct a gauge invariant path integral for $SU(2)$ LGT in the phase space. When analyzed in the weak coupling limit, the mean gauge invariant electric flux becomes large and the small spatial electric flux loops dominate in the vacuum state. Further, the triangle inequality constraints, which are very difficult to solve in general, becomes subdominant in the weak coupling limit.

The plan of the paper is as follows: In Sec. II, we give a brief review of the Kogut-Susskind Hamiltonian formulation [7] on a lattice. In Sec. III, we describe the procedure of modifying the lattice which we call “point splitting.” Further, we describe the prepotential representation of $SU(2)$ lattice gauge theory. In Sec. III B, the construction of a complete, gauge invariant local basis on the modified (point split) lattice is described. The Hilbert space of the point split lattice is described by three gauge invariant quantum numbers n_1, n_2, m per site. In Sec. III C, the Hamiltonian on the modified lattice is given in terms of the prepotential operators. In Sec. IV, we develop a gauge invariant path integral on the lattice by defining phases conjugate to n_1, n_2, m and using the usual time slicing method. A weak coupling expansion is developed in Sec. V. Normalization factors of the new basis are derived in Appendix A. The details of the construction of the path integral and the weak coupling expansion are given in Appendices B and C, respectively.

II. HAMILTONIAN FORMULATION OF $SU(2)$ LATTICE GAUGE THEORY

In this section, we review the essential features of Kogut-Susskind formulation [7] of $SU(2)$ lattice gauge theory. We

^{*}ramesha@imsc.res.in

[†]sreerajtp@imsc.res.in

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP³.

FIG. 1. (a) A general link operator $U_i(\vec{x})$ and the corresponding left and right electric fields $E_i(\vec{x})$ and $E_i(\vec{x} + \hat{i})$. (b) Gauss law at site $\vec{x}$.

consider a square lattice. The basic variables of Kogut Susskind Hamiltonian formulation [7] are the SU(2) link operators $U_i(\vec{x})$ and the conjugate left and right electric fields (see Fig. 1) $E_i(\vec{x})$ and $E_i(\vec{x} + \hat{i})$. $U_i(\vec{x})$ is an operator valued 2×2 SU(2) matrix lying on the link starting at site $\vec{x}$ along the i direction. The electric fields are SU(2) Lie algebra elements and can be expanded as $E_{i/\bar{i}}^a(\vec{x}) = E_{i/\bar{i}}^a(\vec{x}) \frac{\sigma^a}{2}$. They satisfy the following commutation relations:

$$[E_i^a(\vec{x}), U_i(\vec{x})] = -\left(\frac{\sigma^a}{2} U_i(\vec{x})\right),$$

$$[E_i^a(\vec{x} + \hat{i}), U_i(\vec{x})] = \left(U_i(\vec{x}) \frac{\sigma^a}{2}\right). \quad (1)$$

σ^a are the Pauli matrices. The different components of left and right electric fields satisfy the SU(2) algebra and the left electric field commutes with the right electric field. $E_i^a(\vec{x})$ and $E_i^a(\vec{x} + \hat{i})$ generate left and right SU(2) transformations on $U_i(\vec{x})$ and are related as follows:

$$E_i^a(\vec{x} + \hat{i}) = R_{ab}(U) E_i^a(\vec{x}). \quad (2)$$

Above, $R_{ab}(U) = \frac{1}{2} \text{Tr}(U \sigma^a U^\dagger \sigma^b)$. Therefore,

$$E_i^a(\vec{x}) E_i^a(\vec{x}) = E_i^a(\vec{x} + \hat{i}) E_i^a(\vec{x} + \hat{i}) = E^2(\vec{x}). \quad (3)$$

We call the above relation “link constraint.” The lattice electric fields are dimensionless and are related to their dimensionful continuum counterparts by powers of the lattice spacing a . Therefore, in the continuum the usual mutually commuting electric field components are recovered. Under a local gauge transformation $\Lambda(\vec{x})$, the link operator and the electric fields transform as follows:

FIG. 2. The three possible SU(2) addition schemes which can be used in the construction of a gauge invariant basis at each site. New links are introduced along the dotted lines to make the scheme manifest. Introduction of new links along with two new Gauss law constraints at the sites where dotted and solid lines meet and a link constraint for the new link electric fields keeps physics unchanged. We choose scheme (c) in this paper.

$$U_i(\vec{x}) \xrightarrow{\Lambda} \Lambda(\vec{x}) U_i(\vec{x}) \Lambda^\dagger(\vec{x} + \hat{i}),$$

$$E_i(\vec{x}) \xrightarrow{\Lambda} \Lambda(\vec{x}) E_i(\vec{x}) \Lambda^\dagger(\vec{x}),$$

$$E_i(\vec{x} + \hat{i}) \xrightarrow{\Lambda} \Lambda(\vec{x} + \hat{i}) E_i(\vec{x} + \hat{i}) \Lambda^\dagger(\vec{x} + \hat{i}). \quad (4)$$

The Hamiltonian is given by

$$H = \frac{\tilde{g}^2}{2} \sum_{\vec{x}, i} E_i^a(\vec{x}) E_i^a(\vec{x}) + \frac{1}{2g^2} \sum_p [2 - \text{Tr} U_p]. \quad (5)$$

In (5), p denotes plaquettes. We study the general case where the electric coupling is $\tilde{g}$ and magnetic coupling is g . Since the Hamiltonian has mass dimension of 1 in the natural units ($\hbar = c = 1$) and E^a and U are dimensionless it follows [3] from naive continuum limit that $\tilde{g}^2 \sim \frac{1}{a}$ and $g^2 \sim a$, where a is the lattice constant. The above Hamiltonian along with the following Gauss law constraints (6) (see Fig. 1) completely define the theory:

$$\mathcal{G}^a(\vec{x}) = \sum_{i=1}^2 [E_i^a(\vec{x}) + E_i^a(\vec{x})] = 0. \quad (6)$$

$\mathcal{G}^a(\vec{x})$ is the generator of gauge transformations at site $\vec{x}$.

A complete local electric basis can be constructed [5] at each site by quantum numbers of $(E_1)^2$, $(E_{\bar{1}})^2$, $(E_2)^2$, $(E_{\bar{2}})^2$, $(E_{\bar{1}} + E_1)^2$, $(E_{\bar{2}} + E_2)^2$, $(E_{\bar{1}} + E_1 + E_{\bar{2}} + E_2)^2$, $(E_{\bar{1}} + E_1 + E_{\bar{2}} + E_2)^{(3)}$. Above, (3) in the superscript denotes the third component. The Gauss law constraint (6) requires that the last two operators of the set of operators above vanish in the physical Hilbert space. This along with Eq. (3) implies that a complete gauge invariant basis at each site is given by $|j_i, j_{\bar{i}}, j_{\bar{\bar{i}}}\rangle$, where j_i , $j_{\bar{i}}$, $j_{\bar{\bar{i}}}$ are quantum numbers labeling the eigenstates of $(E_i)^2$, $(E_{\bar{i}})^2$ and $(E_1 + E_{\bar{1}})^2 = (E_2 + E_{\bar{2}})^2$, for the choice given in Fig. 2(c). This choice is not unique, there are two other such addition schemes possible leading to three different bases at each site

FIG. 3. (a). The point split lattice (ps-lattice) obtained after performing the splitting of point at each site. (b) Plaquette becomes an octagon after point splitting. Action of $\text{Tr}U_o$ increases n_i/m on each link on the octagon by ± 1 .

related by unitary transformations. These three schemes are graphically represented in Fig. 2.

III. POINT SPLITTING AND PREPOTENTIAL REPRESENTATION

Out of the three quantum numbers at each site, two give a measure of the electric flux on the links and the third gives a measure of the coupled electric flux according to the SU(2) addition scheme chosen. In this paper, we use the addition scheme (c) in Fig. 2. We now introduce new links and corresponding link operators and electric fields (and corresponding link constraints) along the dotted lines (see Fig. 2). This way, the angular momentum addition scheme used to solve the Gauss law becomes manifest. Now, each 4-vertex (i.e., sites where four links meet) is converted to two 3-vertices. Each of them are labeled by the same position index as the original 4-vertex. Therefore, the link operator on the dotted line connecting two 3-vertices labeled by $\vec{x}$ is denoted as $U_3(\vec{x})$ and the corresponding left and right electric fields are denoted as $E_3(\vec{x})$ and $E_{\bar{3}}(\vec{x})$. Each 3-vertex is associated with the Gauss law:

$$\begin{aligned} E_1(\vec{x}) + E_{\bar{1}}(\vec{x}) + E_3(\vec{x}) &= 0, \\ E_2(\vec{x}) + E_{\bar{2}}(\vec{x}) + E_{\bar{3}}(\vec{x}) &= 0. \end{aligned} \quad (7)$$

The lattice becomes a collection of 3-vertices connected by links (see Fig. 3). We call this process “point splitting” and the new lattice after point splitting a ps-lattice.

The ps-lattice describes the same physics as the original lattice. This equivalence can be seen as follows. Without any loss of generality, the Gauss law at one of the two new 3-vertexes can be used to gauge fix the new link operator (along the dotted line) to identity. This implies that [by Eq. (2)] after gauge fixing $E_i(\vec{x}) = -E_{\bar{i}}(\vec{x})$ on the new link.

Therefore, we get the old Gauss law involving four electric fields back, hence we are back to the old lattice. The degrees of freedom per site are easily counted on a ps-lattice to be three as expected. In this paper, we choose the splitting scheme (c) in Fig. 2 at each site of the lattice. This leads to a ps-lattice given in Fig. 3. The $\text{Tr}U_p$ in the Hamiltonian becomes the trace of the product of eight link operators around an octagon [see Fig. 3(b)]:

$$\text{Tr}U_p \rightarrow \text{Tr}U_o.$$

This can be easily shown to be true as $\text{Tr}U_o$ reduces to $\text{Tr}U_p$ by the gauge fixing discussed above. Only the electric fields at the solid (old x-links and y-links) links contribute to the Hamiltonian.

A. Prepotential representation of SU(2) LGT

The gauge invariant basis on the ps-lattice can be better analyzed in the prepotential [6,8] representation. Therefore, we briefly review the prepotential representation of SU(2) LGT. SU(2) lattice gauge theory can be reformulated in terms of harmonic oscillator doublets called Schwinger bosons. Left and right electric fields are written in terms of these Schwinger bosons as follows:

$$\begin{aligned} E_i^a(\vec{x}) &\equiv a_{i,\alpha}^\dagger(\vec{x}) \left(\frac{\sigma^a}{2} \right)_{\alpha\beta} a_{i,\beta}(\vec{x}), \\ E_{\bar{i}}^a(\vec{x} + \hat{i}) &\equiv a_{i,\alpha}^\dagger(\vec{x} + \hat{i}) \left(\frac{\sigma^a}{2} \right)_{\alpha\beta} a_{i,\beta}(\vec{x} + \hat{i}). \end{aligned} \quad (8)$$

Above, $a_{i,\alpha}^\dagger(\vec{x})$ and $a_{i,\alpha}^\dagger(\vec{x} + \hat{i})$ are the harmonic oscillator doublets at the left and right of the link operator $U_i(\vec{x})$. A similar prepotential representation can be made for the left and right electric fields of the dotted link also by putting

$i = 3$ and replacing $\vec{x} + \hat{l}$ with $\vec{x}$. The link constraint (3) when written in terms of Schwinger bosons reads

$$a_i^\dagger(\vec{x}) \cdot a_i(\vec{x}) = a_i^\dagger(\vec{x} + \hat{i}) \cdot a_i(\vec{x} + \hat{i}) \equiv \hat{N}. \quad (9)$$

The number of Schwinger bosons at the left end of a link equals that at the right end. The link operator can be written in terms of Schwinger bosons as

$$(U_i)_{\alpha\beta}(\vec{x}) = f(\hat{N})[\tilde{a}_{i,\alpha}^\dagger(\vec{x})a_{i,\beta}^\dagger(\vec{x} + \hat{i}) + a_{i,\alpha}(\vec{x})\tilde{a}_{i,\beta}(\vec{x} + \hat{i})]f(\hat{N}). \quad (10)$$

Above, $\tilde{a}^\dagger = \epsilon_{\alpha\gamma} a_\gamma^\dagger$ and $f(\hat{N}) = \frac{1}{\sqrt{\hat{N}+1}}$. $U_{\alpha\beta}$ changes the number of Schwinger bosons on the link by ± 1 . Since, $E^2 = \frac{\hat{N}}{2}(\frac{\hat{N}}{2} + 1)$, the electric part of the Hamiltonian counts the number of Schwinger bosons while the magnetic part ($\text{Tr} U_o$) fluctuates the number. Schwinger bosons at any 3-vertex transform as

$$a_\alpha \xrightarrow{\Lambda} \Lambda_{\alpha\beta} a_\beta,$$

where Λ is a gauge transformation at the 3-vertex. Note that $\tilde{a}_\alpha^\dagger$ transforms the same way as a_α .

B. Gauge invariant local basis on the ps-lattice

The ps-lattice is a collection of 3-vertices connected together by links. At any 3-vertex, there are three prepotentials (see Fig. 4) denoted as $a_{i,\alpha}^\dagger, a_{\bar{i},\alpha}^\dagger, a_{3,\alpha}^\dagger$, where $i = 1, 2$ according to whether the vertex contains x-links or y-links. On every 3-vertex, $(a_i^\dagger \cdot \tilde{a}_i^\dagger)$, $(a_i^\dagger \cdot \tilde{a}_3^\dagger)$, $(a_3^\dagger \cdot \tilde{a}_i^\dagger)$ are locally gauge invariant. So, a complete, orthonormal, gauge invariant local basis at a site is given by

$$|l_{\bar{i}}, l_{\bar{3}}, l_{3i}\rangle = \frac{(a_i^\dagger \cdot \tilde{a}_i^\dagger)^{l_{\bar{i}}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l_{\bar{3}}} (a_3^\dagger \cdot \tilde{a}_i^\dagger)^{l_{3i}}}{\sqrt{(l_{\bar{i}} + l_{\bar{3}} + l_{3i} + 1)! (l_{\bar{i}})! (l_{\bar{3}})! (l_{3i})!}} |0\rangle. \quad (11)$$

In (11), $|0\rangle$ is the strong coupling vacuum: $a_\alpha |0\rangle = 0$ and $l_{\bar{i}}, l_{\bar{3}}, l_{3i}$ are positive integers. The normalization factor in (11) is calculated in Appendix A. The above states can also be labeled by the eigenvalues of the number operators $\hat{N}_i = a_i^\dagger \cdot a_i$, $\hat{N}_2 = a_i^\dagger \cdot a_{\bar{i}}$, $\hat{N}_3 = a_3^\dagger \cdot a_3$. That is,

$$|l_{\bar{i}}, l_{\bar{3}}, l_{3i}\rangle \equiv |n_i, n_{\bar{i}}, m \equiv n_3\rangle,$$

where $n_i = l_{\bar{i}} + l_{3i}$, $n_{\bar{i}} = l_{\bar{i}} + l_{\bar{3}}$, $m \equiv n_3 = l_{\bar{3}} + l_{3i}$. The inverse relations are $l_{\bar{i}} = n_i + n_{\bar{i}} - n_3$, $l_{\bar{3}} = n_{\bar{i}} + n_3 - n_i$, $l_{3i} = n_3 + n_i - n_{\bar{i}}$. Since, $E^2 = \frac{\hat{N}}{2}(\frac{\hat{N}}{2} + 1)$, the number of prepotentials on a link is a measure of the electric flux through the link. The state $|n_i, n_{\bar{i}}, m\rangle$ is precisely the gauge invariant coupled states obtained by the diagonalization procedure described in the Introduction. The n_i, m quantum numbers of the links meeting at a 3-vertex satisfy triangle inequalities. For example, at the 3-vertex shown in Fig. 4,

FIG. 4. A basic 3-vertex on the PS-lattice. Three prepotentials associated with the 3-vertex are also shown. The three links are labeled by $i, \bar{i}, 3$.

$$|n_i - n_{\bar{i}}| \leq m \leq n_i + n_{\bar{i}}. \quad (12)$$

In other words, $m(\vec{x})$ is limited by the value of $n_i(\vec{x})$ which in turn can take values freely between 0 and ∞ . Alternatively, $l_{\bar{i}}, l_{\bar{3}}, l_{3i}$ at a site can be varied independently. The link constraint (9) can be translated in terms of the l_{ij} quantum numbers. The physical state space can be described in terms of l_{ij} at each site and the link constraints (9) on every link. This implies a collection of overlapping, closed electric flux loops on the ps-lattice. Equivalently, they can also be labeled by link variables n_i and m on the ps-lattice satisfying local triangle inequalities at each vertex. This equivalence can be realized only on the ps-lattice where the coordination number of any vertex is always three.

C. Hamiltonian on a ps-lattice

The Hamiltonian on the ps-lattice in the prepotential representation is given by

$$H = \frac{g^2}{2} \sum_{\vec{x}} \frac{\hat{N}_i(\vec{x})}{2} \left(\frac{\hat{N}_i(\vec{x})}{2} + 1 \right) + \frac{1}{2g^2} \sum_{\text{oct}} [2 - \text{Tr} U_o]. \quad (13)$$

Above, $\vec{x}$ denotes 3-vertices, $i = 1, 2$ according as the 3-vertex under consideration and $\text{Tr} U_o = \text{Tr}(\hat{P}(31)_a \hat{P}(\bar{1}3)_b \hat{P}(32)_c \hat{P}(\bar{2}3)_d \hat{P}(3\bar{1})_e \hat{P}(13)_f \hat{P}(3\bar{2})_g \hat{P}(23)_h)$ where a, b, c, d, e, f, g, h are the eight 3-vertices [see Fig. 3(b)] in the octagon and

$$\hat{P}(ij) = \begin{pmatrix} \frac{1}{\sqrt{\hat{N}_i(\hat{N}_j+1)}} (a_i^\dagger \cdot \tilde{a}_j^\dagger) & \frac{1}{\sqrt{\hat{N}_i(\hat{N}_j+1)}} (a_i^\dagger \cdot a_j) \\ \frac{1}{\sqrt{(\hat{N}_i+2)(\hat{N}_j+1)}} (a_i \cdot a_j^\dagger) & \frac{1}{\sqrt{(\hat{N}_i+2)(\hat{N}_j+1)}} (\tilde{a}_i \cdot a_j) \end{pmatrix}. \quad (14)$$

At each 3-vertex along the octagon, there are two links which lie on the octagon. These links are labeled by i and j in (14). $\hat{P}(ij)_{11}/\hat{P}(ij)_{22}$ increases/decreases both n_i, n_j by 1, $\hat{P}(ij)_{12}$ increases n_i but decreases n_j and $\hat{P}(ij)_{21}$ increases n_j but decreases n_i . $\text{Tr} U_o$ has 2^8 terms. Each term changes n_i/m at the eight links of the octagon by ± 1 thereby exhausting the 2^8 possible ways to do it. The action of $\text{Tr} U_o$ on the state

$|n_i, n_{\bar{i}}, m\rangle$ is illustrated in Fig. 3(b). This kind of construction can be done for any of the splitting schemes. We note that in [8], where Fig. 2(a) type of splitting was envisaged and the corresponding Hamiltonian was studied in the strong coupling expansion, it was found that it agrees with the naive strong coupling expansion. The matrix element of the Hamiltonian in the $|n_1, n_2, m\rangle$ basis is easily computed using (11) and (14) [see Eq. (B1)] and can be used to perform numerical analysis.

IV. GAUGE INVARIANT PATH INTEGRAL

The gauge invariant reformulation is described locally and completely in terms of n_i, m fields without any redundant fields. However, the dynamics is still unconventionally complicated. To further simplify, we introduce the phase operators [9] $e^{i\hat{\phi}_i}, e^{i\hat{\chi}}$, conjugate to n_i, m . For this, we first extend the Hilbert space by increasing the domain of n_i, m to $(-\infty, \infty)$. Phase operators are defined such that they satisfy the following commutation relations:

$$\begin{aligned} [n_i, e^{i\hat{\phi}_i}] &= e^{i\hat{\phi}_i}, \\ [m, e^{i\hat{\chi}}] &= e^{i\hat{\chi}}. \end{aligned} \quad (15)$$

$e^{i\hat{\phi}_i}/e^{-i\hat{\phi}_i}, e^{i\hat{\chi}}/e^{-i\hat{\chi}}$ acts as step operators on the number states increasing/decreasing n_i, m by 1. We then define an eigenbasis of the phase operator $e^{i\hat{\phi}_i}, e^{i\hat{\chi}}$:

$$\begin{aligned} e^{i\hat{\phi}_i}|\phi_1, \phi_2, \chi\rangle &= e^{i\phi_i}|\phi_1, \phi_2, \chi\rangle, \\ e^{i\hat{\chi}}|\phi_1, \phi_2, \chi\rangle &= e^{i\chi}|\phi_1, \phi_2, \chi\rangle. \end{aligned} \quad (16)$$

The gauge invariant path integral is then constructed as the probability amplitude to go from the state $|\phi_1, \phi_2, \chi\rangle$ to $|\bar{\phi}_1, \bar{\phi}_2, \bar{\chi}\rangle$ in time t . The details of the construction are described in Appendix B. The domain of n_i, m is then restricted to positive values of n_i, m within the path integral to get back to the original Hilbert space. The Euclidean path integral thereby constructed is given by

$$\begin{aligned} Z &= \int D\phi_1 D\phi_2 D\chi \\ &\times \sum_{n_1, n_2, m} e^{-\int dt (\frac{g^2}{2} \sum_{\vec{x}} (\frac{n_i}{2} [\frac{m_i}{2} + 1]) + \frac{1}{2g^2} \sum_{\text{oct}} [2 - \text{Tr}\{P_{\text{oct}}\}] + \sum_{\vec{x}} \{in_1\dot{\phi}_1 + in_2\dot{\phi}_2 + im\dot{\chi} + \lambda(\Theta(-n_i) + \Theta(-m) + \Theta((n_i - n_i)^2 - m^2) + \Theta(m^2 - (n_i + n_i)^2))\})}. \end{aligned} \quad (17)$$

Above, $P_{\text{oct}} = \bar{P}(31)_a P(\bar{1}3)_b \bar{P}(32)_c P(\bar{2}3)_d \bar{P}(3\bar{1})_e P(13)_f \bar{P}(3\bar{2})_g P(23)_h$, where $\bar{P}(ij) = \langle n_j, n_{\bar{j}}, m | \hat{P}(ij) | \phi_j, \phi_{\bar{j}}, \chi \rangle$ and $P(ij) = \langle \phi_j, \phi_{\bar{j}}, \chi | \hat{P}(ij) | n_j, n_{\bar{j}}, m \rangle$ are given by

$$\begin{aligned} P(ij) &= \begin{pmatrix} \sqrt{\frac{(n_i + n_j + n_{\bar{j}} + 4)(n_i + n_j - n_{\bar{j}} + 2)}{4(n_i + 1)(n_j + 2)}} e^{\frac{i}{2}(\phi_i + \phi_j)} & \sqrt{\frac{(n_i - n_j + n_{\bar{j}} + 2)}{2(n_i + 1)(n_j)}} e^{\frac{i}{2}(\phi_i - \phi_j)} \\ \sqrt{\frac{(n_j - n_i + n_{\bar{j}} + 2)}{2(n_i + 1)(n_j + 2)}} e^{-\frac{i}{2}(\phi_i - \phi_j)} & \sqrt{\frac{(n_i + n_j + n_{\bar{j}} + 2)(n_i + n_j - n_{\bar{j}})}{4(n_i + 1)(n_j)}} e^{-\frac{i}{2}(\phi_i + \phi_j)} \end{pmatrix}; \quad n_j \neq 0, n_i \neq 0 \\ \bar{P}(ij) &= \begin{pmatrix} \sqrt{\frac{(n_i + n_j + n_{\bar{j}} + 2)(n_i + n_j - n_{\bar{j}})}{4(n_i)(n_j + 1)}} e^{\frac{i}{2}(\phi_i + \phi_j)} & \sqrt{\frac{(n_i - n_j + n_{\bar{j}})}{2(n_i)(n_j + 1)}} e^{\frac{i}{2}(\phi_i - \phi_j)} \\ \sqrt{\frac{(n_j - n_i + n_{\bar{j}})}{2(n_i + 2)(n_j + 1)}} e^{-\frac{i}{2}(\phi_i - \phi_j)} & \sqrt{\frac{(n_i + n_j + n_{\bar{j}} + 4)(n_i + n_j - n_{\bar{j}} + 2)}{4(n_i + 2)(n_j + 1)}} e^{-\frac{i}{2}(\phi_i + \phi_j)} \end{pmatrix}; \quad n_i \neq 0, n_j \neq 0. \end{aligned} \quad (18)$$

Equation (14) implies, when $n_j = 0$, the second column of $P(ij)$ is 0 and when $n_i = 0$, the second row of $P(ij)$ is 0. Similarly, when $n_i = 0$ the first row of $\bar{P}(ij)$ is 0 and when $n_j = 0$ the first column is 0. There is a matrix $P(ij)$ and $\bar{P}(ij)$ associated with each 3-vertex along an octagon under consideration on the split lattice. Our convention is, at a 3-vertex where the links $i, j, \bar{j}$ meet, i, j are the links in the direction of the octagon and $\bar{j}$ is the spectator. In (17), the step function Θ is used to implement the positivity condition of n_i, m as well as the triangle inequality of n_i, m at each 3-vertex by taking $\lambda \rightarrow \infty$. $i = 1, 2$ in n_i according to the 3-vertex under consideration. n_i, m and ϕ_i, χ fields are functions of time t and lattice points.

V. WEAK COUPLING, CONTINUUM LIMIT

The deduction in the previous sections has been exact. Now we turn to making a useful ansatz which allows us to make a weak coupling expansion. We remind ourselves that the magnetic term is strictly positive and its lowest value occurs when $U(\vec{x})$ reach **1**, this in the gauge invariant configuration space leads to $U_o \rightarrow 1$ and $P(ij), \bar{P}(ij) \rightarrow 1$. Furthermore, all the integer fields n_i, m are always positive and hence $\langle n_i \rangle$ and $\langle m \rangle$ are nonzero. If we assume $\langle n_i \rangle = N$ and insist that $P(ij), \bar{P}(ij) \sim 1$, From Eq. (18), we note that $\langle m \rangle = 2N$ and N is large with $\phi_i, \chi \rightarrow 0$, as the only solution. This mean field ansatz is indeed possible only in the point splitting Fig. 2(c). In the other possible schemes,

such a mean field with corresponding $P(ij), \bar{P}(ij) \sim 1$ is not possible. We rescale $\phi_i, \chi \rightarrow g\phi_i, g\chi$ and make the substitution, $n_i \equiv N + \tilde{n}_i, m \equiv 2N + \tilde{m}$. $\text{Tr}P_{\text{oct}}$ consists of 2^8 terms corresponding to all possible fluctuations generated by the plaquette term along the octagon [see Fig. 3(b)]. In each of the terms, the off-diagonal terms come in pairs. This is due to the link constraint (3). The off-diagonal terms of $P(ij)$ and $\bar{P}(ij)$ are at least of the order of $\frac{1}{\sqrt{N}}$. Therefore, the terms can be classified according to the number of off-diagonal terms occurring. Such a classification corresponds to an expansion in $\frac{1}{N}$. The leading term $\frac{1}{N^0}$ is the term which

does not contain any off-diagonal terms and corresponds to the term which either increases (+) or decreases (−) the field n_i, m at every link along the octagon. Every other term consists of several flips $+\rightarrow -$ or $-\rightarrow +$ along the octagon and each flip brings in a $\frac{1}{\sqrt{N}}$. A consistent diagram necessarily has an even number of flips only. The second leading term has two flips and hence is of the order of $\frac{1}{N}$. The next term $\frac{1}{N^2}$ consists of four such flips. After a straightforward tedious calculation, keeping the terms up to the order $\frac{1}{N}$ and quadratic in ϕ_i, χ , we get

$$\text{Tr}U_o \approx 2 - \left(\frac{1}{4N^2} \tilde{m}^2 + V(\phi_1, \phi_2, \chi) \right), \quad (19)$$

$$V(\phi_1, \phi_2, \chi) = \frac{g^2}{2} \left\{ \left[\left(\Delta_1 \left(\phi_2 - \frac{1}{2} \Delta_2 \chi \right) - \Delta_2 \left(\phi_1 + \frac{1}{2} \Delta_1 \chi \right) \right) \right]^2 + \frac{4}{N} \left[16 \left[\left(\phi_1 + \frac{1}{2} \Delta_1 \chi \right)^2 + \left(\phi_2 - \frac{1}{2} \Delta_2 \chi \right)^2 + \chi^2 \right] - \left[\Delta_1 \left(\phi_2 - \frac{1}{2} \Delta_2 \chi \right) - \Delta_2 \left(\phi_1 + \frac{1}{2} \Delta_1 \chi \right) + \Delta_1 \Delta_2 \chi \right]^2 - (\Delta_1 \Delta_2 \chi)^2 \right] \right\}. \quad (20)$$

In the above expression, Δ_i is the forward difference operator defined by $\Delta_i f(\vec{x}) = f(\vec{x} + \hat{i}) - f(\vec{x})$. The details of the above weak coupling expansion are described in Appendix C. The path integral becomes

$$Z = \int_{-\frac{\pi}{g}}^{\frac{\pi}{g}} D\phi_1 D\phi_2 D\chi \sum_{\substack{\tilde{n}_1 = -N \\ \tilde{n}_2 = -N}}^{\infty} \sum_{\tilde{m} = -M_1}^{M_2} e^{-\int dt \sum_{\text{sites}} \left[\frac{\tilde{g}^2}{8} (\tilde{n}_1^2 + \tilde{n}_2^2 + 2N^2) + \frac{1}{8g^2 N^2} \tilde{m}^2 + i\tilde{n}_1 g \dot{\phi}_1 + i\tilde{m} g \dot{\chi} + \frac{1}{2g^2} V(\phi_1, \phi_2, \chi) \right]}. \quad (21)$$

Above, $M_1 = \min(\tilde{n}_1 + \tilde{n}_1, \tilde{n}_2 + \tilde{n}_2)$, $M_2 = \max(2N - |\tilde{n}_1 - \tilde{n}_1|, 2N - |\tilde{n}_2 - \tilde{n}_2|)$. This is due to the fact that $\tilde{m}$ is limited by the triangle inequality (13). In the weak coupling limit, $M_1 \approx 0$, $M_2 \approx 2N$. Therefore, triangle inequality becomes irrelevant up to the leading term in the weak coupling limit and theta function terms used to implement the triangle inequality and the positivity condition can be removed. Using Euler-Maclaurin formula to convert the summation over $\tilde{n}$ and $\tilde{m}$ to integration and performing the integral, we get up to an irrelevant constant C:

$$Z = C \int D\phi_1 D\phi_2 D\chi e^{-W[N, \phi_i, \dot{\phi}_i, \chi, \dot{\chi}]} \quad (22)$$

$$W[N, \phi_i, \dot{\phi}_i, \chi, \dot{\chi}] = \int dt \sum_{\text{sites}} \left[2g^2 \frac{\dot{\phi}_1^2 + \dot{\phi}_2^2}{\tilde{g}^2} + g^2 \frac{\dot{\chi}^2}{1/(2g^2 N^2)} + \frac{1}{2g^2} V(\phi_1, \phi_2, \chi) \right]. \quad (23)$$

We now make the following transformation:

$$\phi_i = \frac{1}{\sqrt{-\Delta^2}} (\Delta_i \eta + \epsilon_{ij} \delta_j \psi) + \frac{1}{2} \epsilon_{ji} \Delta_j \chi. \quad (24)$$

In (24), Δ^2 is the lattice Laplacian and δ_j is the backward difference operator defined as $\Delta^2 = \sum_i (\Delta_i - \delta_i)$; $\delta_j f(\vec{x}) = f(\vec{x}) - f(\vec{x} - \hat{j})$. Ignoring constant terms, the path integral becomes

$$Z = \int D\psi D\eta D\chi e^{-\int dt \sum_{\text{sites}} \left[\frac{2g^2}{\tilde{g}^2} (\dot{\eta}^2 + \dot{\psi}^2) + 2g^4 N^2 \dot{\chi}^2 + \frac{2g^2}{\tilde{g}^2} \left(\frac{1}{4} (\Delta_1 \dot{\chi})^2 + (\Delta_2 \dot{\chi})^2 \right) + \dot{\chi} \frac{1}{\sqrt{-\Delta^2}} [(\delta_1 \Delta_1 - \delta_2 \Delta_2) \dot{\eta} + 2\delta_1 \delta_2 \dot{\psi}] \right] + \frac{1}{2g^2} V(\psi, \eta, \chi) \right]}. \quad (25)$$

$$V(\psi, \eta, \chi) = 2g^2 \left\{ \frac{1}{4} (\Delta \psi)^2 + \frac{1}{N} \left[16(\eta^2 + \psi^2 + \chi^2) - (\Delta \psi)^2 - 2(\Delta_1 \Delta_2 \chi)^2 + \Delta^2 \chi \frac{\Delta_1 \Delta_2}{\sqrt{-\Delta^2}} \chi \right] \right\}. \quad (26)$$

Above, $(\Delta\psi)^2 \equiv (\Delta_1\psi)^2 + (\Delta_2\psi)^2$. In order to cast the ψ terms in the canonical form, we rescale $\psi \rightarrow \sqrt{2}\psi$. Now, the velocity of light being 1 requires

$$\frac{g^2}{\tilde{g}^2} = \frac{1}{8}a^2. \quad (27)$$

The continuum limit is now taken by making

$$\frac{64}{N} = M^2 a^2, \quad (28)$$

where M is the mass in the continuum and a is the lattice constant. Since $g^2 = a$, $\tilde{g}^2 = \frac{8}{a}$, $N = \frac{64}{M^2 g^4} = \frac{\tilde{g}^4}{M^2}$. Consequently, the Euclidean inverse propagators in the energy-momentum space to the leading order are

$$\begin{aligned} \psi: & p_0^2 + M^2 + \vec{p}^2 + O(a^2), \\ \eta: & p_0^2 + M^2 + O(a^4), \\ \chi: & \#a^2 \vec{p}^2 p_0^2 + M^2 + O(a^4). \end{aligned} \quad (29)$$

Above, $\#$ denotes a real positive constant. Therefore, ψ is a relativistic particle with mass M and χ do not propagate. η may propagate due to higher order corrections.

VI. SUMMARY AND DISCUSSION

A complete gauge invariant Hilbert space could be constructed in the electric space [5] by solving the local Gauss law at each site on the lattice. This leads to complicated dynamics as the matrix element of the Hamiltonian in this basis is given by higher Wigner coefficients [5,6]. In order to construct a gauge invariant basis where the dynamics is more transparent, it is convenient to break down the link operators into simpler objects which transform under the fundamental representation of the gauge group namely, the Schwinger bosons. This leads to the prepotential representation. However, the local gauge invariant objects constructed at each site by contracting various Schwinger bosons are not all independent. This is the analogue in electric space of the Mandelstam constraints [2] in the Wilson loop space. In order to construct an independent basis, we point split each site of the lattice.

Advantages of point splitting are manifold. Point splitting allows us to construct a local set of independent operators. This in turn allows us to construct a complete, local, orthonormal, gauge invariant basis (11) of SU(2) LGT. As a consequence, the Mandelstam constraints mentioned above are completely bypassed. Second, on a split lattice, the matrix elements of $\text{Tr}U_p$ [see (B1)] become much simpler to analyze, especially in the weak coupling limit. At each split site (3-vertex), the matrix element becomes a simple algebraic expression [see (B1)] involving

n_i , m fields, as opposed to the 6j coefficients on the unsplit lattice. Therefore, we are in effect writing down 6j symbols as the product of two simpler gauge invariant expressions which can be thought of as the gauge invariant projection of standard 3j symbols. Further, the dynamics is local and the exact matrix elements in the ps-basis (11) is easily written down [see (B1)]. These matrix elements could be much more convenient in a numerical analysis as opposed to the higher Wigner coefficients [5,6] which occur on the unsplit lattice. More importantly, point splitting enables us to analyze a theory of interacting closed loops in terms of local quantum numbers satisfying triangle inequalities.

There does not seem to be any serious hindrance in the construction of a point splitting scheme in higher dimensions and SU(N) [6] group. Construction of a complete local gauge invariant basis of the Hilbert space similar to what is described in this paper can be envisaged for these cases as well.

On inclusion of fermions, the local Gauss law is modified at each site. Therefore, a modified point splitting scheme involving fermions has to be constructed which leads to a new complete, gauge invariant, local ps-basis. Such a construction leads to new singlet operators involving fermions along with the usual singlets involving only Schwinger bosons.

In the weak coupling $g \rightarrow 0$ limit, $U \rightarrow \mathbf{1}$. In Sec. V, we found that writing the fields n_i , m around a large mean value N , $2N$ and small ϕ, χ , makes $U \rightarrow \mathbf{1}$. This ansatz is also consistent with the triangle inequality at each 3-vertex on the ps-lattice. In the weak coupling limit, such a vacuum is dominated by small electric flux loops in space carrying large fluxes. In other schemes, there is no mean field ansatz consistent with triangle inequalities at each 3-vertex which takes $U \rightarrow \mathbf{1}$. Therefore, even though other splitting schemes lead to a different basis of the same Hilbert space and are therefore completely equivalent in an exact analysis, $g \rightarrow 0$ cannot be achieved by a simple mean field ansatz. However, in other splitting schemes, mean field ansatz can be constructed where $U \rightarrow c\mathbf{1}$, where $c < 1$. Such an ansatz is suitable to study g small but not 0. It will be interesting to see whether such schemes lead to a Lorentz covariant fixed point at a finite but small value of g .

The dispersion relations (29) were deduced analytically on the lattice up to lowest order in the weak coupling expansion. It is not evident that everything is Lorentz covariant. ψ mode turns out to be a relativistic scalar particle satisfying relativistic dispersion relation and χ is a massive mode (on the lattice) which does not survive the continuum limit. η could become a propagating mode due to higher order terms. The lowest excited state ψ being nondegenerate is consistent with the results from the existing literature [3,10]. In order for the mass gap to be compared, it has to be written in terms of string tension, i.e., we need to introduce heavy SU(2) charged fermions. As described earlier inclusion of fermions changes the Gauss

law at sites and hence the point splitting scheme and the construction of the gauge invariant Hilbert space. Interaction terms in the higher orders in the weak coupling expansion have to be investigated.

ACKNOWLEDGMENTS

We thank Manu Mathur for useful discussions.

APPENDIX A: NORMALIZATION OF LOCAL BASIS STATES AT A 3-VERTEX

A general gauge invariant state at a 3-vertex shown in Fig. 4 is

$$|l_{\bar{i}\bar{i}}, l_{i3}, l_{3i}\rangle = (a_i^\dagger \cdot \tilde{a}_{\bar{i}}^\dagger)^{l_{\bar{i}\bar{i}}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l_{i3}} (a_3^\dagger \cdot \tilde{a}_{\bar{i}}^\dagger)^{l_{3i}} |0\rangle. \quad (\text{A1})$$

The inner product between two such states is given by

$$\begin{aligned} \langle l_{\bar{i}\bar{i}}, l_{i3}, l_{3i} | l'_{\bar{i}\bar{i}}, l'_{i3}, l'_{3i} \rangle &= \langle 0 | (a_{\bar{i}} \cdot \tilde{a}_3)^{l_{\bar{i}\bar{i}}} (a_i \cdot \tilde{a}_3)^{l_{i3}}, \\ (a_i \cdot \tilde{a}_{\bar{i}})^{l_{i\bar{i}}} (a_i^\dagger \cdot \tilde{a}_{\bar{i}}^\dagger)^{l'_{\bar{i}\bar{i}}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l'_{i3}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l'_{3i}} |0\rangle. \end{aligned} \quad (\text{A2})$$

Using the relation

$$[a_n \cdot \tilde{a}_m, a_n^\dagger \cdot \tilde{a}_m^\dagger] = \hat{N}_{\bar{n}} + \hat{N}_m + 2 \quad (\text{A3})$$

repeatedly to shift $(a_i \cdot \tilde{a}_{\bar{i}})$, $(a_i \cdot \tilde{a}_3)$ and $(a_{\bar{i}} \cdot \tilde{a}_3)$ to the right, we get

$$\begin{aligned} \langle l_{\bar{i}\bar{i}}, l_{i3}, l_{3i} | l'_{\bar{i}\bar{i}}, l'_{i3}, l'_{3i} \rangle \\ = \delta_{l_{\bar{i}\bar{i}} l'_{\bar{i}\bar{i}}} \delta_{l_{i3} l'_{i3}} \delta_{l_{3i} l'_{3i}} (l_{\bar{i}\bar{i}} + l_{i3} + l_{3i} + 1)! (l_{\bar{i}\bar{i}})! (l_{i3})! (l_{3i})!. \end{aligned} \quad (\text{A4})$$

Therefore, the normalized states are

$$|l_{\bar{i}\bar{i}}, l_{i3}, l_{3i}\rangle = \frac{(a_i^\dagger \cdot \tilde{a}_{\bar{i}}^\dagger)^{l_{\bar{i}\bar{i}}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l_{i3}} (a_i^\dagger \cdot \tilde{a}_3^\dagger)^{l_{3i}}}{\sqrt{(l_{\bar{i}\bar{i}} + l_{i3} + l_{3i} + 1)! (l_{\bar{i}\bar{i}})! (l_{i3})! (l_{3i})!}} |0\rangle. \quad (\text{A5})$$

A couple of comments are in order here. In general, we can associate a sign [8] for the states. However, in our calculation as long as the same sign convention is applied to both the bras and kets, all our expectation values are independent of the sign. States defined by (A5), are a special case of what occurs in the context of addition of angular momentum. In the latter, normalization depends on the azimuthal quantum numbers as well. Because of local gauge invariance, in our context (A5), they depend only on the Casimirs.

APPENDIX B: PATH INTEGRAL FORMULATION

The commutation relations (15) implies that $e^{i\phi_i}$, $e^{i\chi}$ increases the n_i , m quantum numbers respectively by one. Hence, when $\hat{P}(ij)$ acts on the number state, $(a_i^\dagger \cdot \tilde{a}_j^\dagger)$, $(a_i^\dagger \cdot a_j)$, $(a_i \cdot \tilde{a}_j^\dagger)$, $(\tilde{a}_i \cdot a_j^\dagger)$ can be replaced by n dependent factors times $e^{\frac{i}{2}(\phi_i + \phi_j)}$, $e^{\frac{i}{2}(\phi_i - \phi_j)}$, $e^{\frac{i}{2}(\phi_i - \phi_j)}$, $e^{\frac{i}{2}(\phi_i + \phi_j)}$. The factor of $\frac{1}{2}$ comes from the fact that each n is shared by two 3-vertices. Therefore, for $n_j \neq 0$, $n_i \neq 0$,

$$\hat{P}(ij) |n_j, n_{\bar{j}}, m\rangle = \begin{pmatrix} e^{\frac{i}{2}(\hat{\chi} + \hat{\phi}_j)} \sqrt{\frac{(\hat{N}_j + \hat{N}_{\bar{j}} + \hat{N}_i + 4)(\hat{N}_j - \hat{N}_{\bar{j}} + \hat{N}_i + 2)}{4(\hat{N}_i + 1)(\hat{N}_j + 2)}} & e^{\frac{i}{2}(\hat{\chi} - \hat{\phi}_j)} \sqrt{\frac{(\hat{N}_j - \hat{N}_{\bar{j}} + \hat{N}_i + 2)}{2(\hat{N}_i + 1)(\hat{N}_j)}} \\ e^{-\frac{i}{2}(\hat{\chi} - \hat{\phi}_j)} \sqrt{\frac{(\hat{N}_j + \hat{N}_{\bar{j}} - \hat{N}_i + 2)}{2(\hat{N}_i + 1)(\hat{N}_j + 2)}} & e^{-\frac{i}{2}(\hat{\chi} + \hat{\phi}_j)} \sqrt{\frac{(\hat{N}_j + \hat{N}_{\bar{j}} + \hat{N}_i + 2)(\hat{N}_j - \hat{N}_{\bar{j}} + \hat{N}_i)}{4(\hat{N}_i + 1)(\hat{N}_j)}} \end{pmatrix} |n_j, n_{\bar{j}}, m\rangle. \quad (\text{B1})$$

From (14), for $n_j = 0$, the second column of $\hat{P}(ij)$ becomes 0 and for $n_i = 0$, the second row becomes zero.

The path integral is given by

$$\begin{aligned} Z &= \langle \bar{\phi}_1, \bar{\phi}_2, \bar{\chi} | e^{-Ht} | \phi_1, \phi_2, \chi \rangle \\ &= \prod_{i=1}^{n-1} \int d\phi_{1i} d\phi_{2i} d\chi_i \langle \phi_{1i+1}, \phi_{2i+1}, \chi_{i+1} | e^{-H\delta t} | \phi_{1i}, \phi_{2i}, \chi_i \rangle e^{-\lambda(\Theta(-n_i) + \Theta(-m) + \Theta(-(m^2 - (n_i - n_{\bar{i}})^2) + \Theta((n_i + n_{\bar{i}})^2 - m^2))}. \end{aligned} \quad (\text{B2})$$

Above, $t = (n + 1)\delta t$ and λ is a real parameter. The step function Θ is used to restrict n_i , m to positive values satisfying triangle inequality by taking the limit $\lambda \rightarrow \infty$. Now,

$$\begin{aligned} \langle \phi_{1i+1}, \phi_{2i+1}, \phi_{3i+1} | e^{-H\delta t} | \phi_{1i}, \phi_{2i}, \phi_{3i} \rangle &= \sum_{n_1, n_2, n_3} \langle \phi_{1i+1}, \phi_{2i+1}, \phi_{3i+1} | e^{-H\delta t} | n_1, n_2, n_3 \rangle \langle n_1, n_2, n_3 | \phi_{1i}, \phi_{2i}, \phi_{3i} \rangle \\ &= \sum_{n_1, n_2, n_3} e^{-\delta t [\sum_s [i(n_1 \dot{\phi}_{1i} + n_2 \dot{\phi}_{2i} + m \dot{\chi}_i) + \frac{g^2}{2}(n_1^2(s) + n_2^2(s))] + \frac{1}{2g^2} \sum_{\text{oct}} [2 - \text{Tr} P_{\text{oct}}]]}. \end{aligned} \quad (\text{B3})$$

Above, $\dot{\phi}_{1i} = \frac{(\phi_{1i+1}-\phi_{1i})}{\delta t}$, $\dot{\phi}_{2i} = \frac{(\phi_{2i+1}-\phi_{2i})}{\delta t}$, $\dot{\phi}_{3i} = \frac{(\phi_{3i+1}-\phi_{3i})}{\delta t}$, $P_{\text{oct}} = \bar{P}(31)_a P(\bar{1}3)_b \bar{P}(32)_c P(\bar{2}3)_d \bar{P}(3\bar{1})_e P(13)_f \bar{P}(3\bar{2})_g P(23)_h$ where, $\bar{P}(ij) = \langle n_j, n_{\bar{j}}, m | \hat{P}(ij) | \phi_j, \phi_{\bar{j}}, \chi \rangle$ and $P(ij) = \langle \phi_j, \phi_{\bar{j}}, \chi | \hat{P}(ij) | n_j, n_{\bar{j}}, m \rangle$ given by (18). Plugging (B3) back into (B2) and taking the limit $\delta t \rightarrow 0$ gives the path integral (17). We have used the λ term in (B2) to avoid some technical difficulties alluded to in Ref. [9]. In our context, in the path integral all the n_i, m fields are restricted to be positive and satisfy local triangle inequality as expected.

APPENDIX C: WEAK COUPLING EXPANSION OF $\text{Tr}U_o$

As described in Sec. V, in the continuum limit ϕ_i, χ becomes small and N becomes large. This motivates us to rescale $\phi_i, \chi \rightarrow g\phi_i, g\chi$ and make the substitution, $n_i = N + \tilde{n}_i, m = 2N + \tilde{m}$. This allows us to expand $\text{Tr}U_o$ as powers of the small parameter $\frac{1}{N}$. We write the n dependent terms $f(n)$ in $P(ij)$ and $\bar{P}ij$ as $e^{\ln(f(n))}$ and expand the $\ln$ in powers of $\frac{1}{N}$. This gives, for example,

$$\begin{aligned} P_{11}(31) &= e^{\frac{\tilde{n}_1+\tilde{n}_1+\tilde{n}_3+4}{8N}-\frac{1}{4}\left(\frac{\tilde{n}_1+\tilde{n}_1+\tilde{n}_3+4}{4N}\right)^2+\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3+2}{(4N)}-\frac{1}{4}\left(\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3+2}{2N}\right)^2-\frac{\tilde{n}_3+1}{4N}+\frac{1}{4}\left(\frac{\tilde{n}_3+1}{2N}\right)^2-\frac{\tilde{n}_1+2}{2N}+\frac{1}{4}\left(\frac{\tilde{n}_1+2}{N}\right)^2+\frac{i}{2}(\phi_3+\phi_1)}, \\ P_{12}(31) &= \frac{1}{\sqrt{N}} e^{\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3+2}{4N}-\frac{1}{4}\left(\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3+2}{2N}\right)^2-\frac{\tilde{n}_3+1}{4N}+\frac{1}{4}\left(\frac{\tilde{n}_3+1}{2N}\right)^2-\frac{\tilde{n}_1}{2N}+\frac{1}{4}\left(\frac{\tilde{n}_1}{N}\right)^2+\frac{i}{2}(\phi_3-\phi_1)}, \\ P_{21}(31) &= \frac{1}{\sqrt{N}} e^{\frac{\tilde{n}_1+\tilde{n}_1-\tilde{n}_3+2}{8N}-\frac{1}{2}\frac{1}{4}\left(\frac{\tilde{n}_1+\tilde{n}_1-\tilde{n}_3+2}{4N}\right)^2-\frac{\tilde{n}_3+1}{4N}+\frac{1}{4}\left(\frac{\tilde{n}_3+1}{2N}\right)^2-\frac{\tilde{n}_1+2}{2N}+\frac{1}{4}\left(\frac{\tilde{n}_1+2}{N}\right)^2-\frac{i}{2}(\phi_3-\phi_1)}, \\ P_{22}(31) &= e^{\frac{\tilde{n}_1+\tilde{n}_1+\tilde{n}_3+2}{8N}-\frac{1}{4}\left(\frac{\tilde{n}_1+\tilde{n}_1+\tilde{n}_3+2}{4N}\right)^2+\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3}{(4N)}-\frac{1}{4}\left(\frac{\tilde{n}_1-\tilde{n}_1+\tilde{n}_3}{2N}\right)^2-\frac{\tilde{n}_3+1}{4N}+\frac{1}{4}\left(\frac{\tilde{n}_3+1}{2N}\right)^2-\frac{\tilde{n}_1}{2N}+\frac{1}{4}\left(\frac{\tilde{n}_1}{N}\right)^2-\frac{i}{2}(\phi_3+\phi_1)}, \end{aligned} \quad (C1)$$

and similar expressions for $P(ij)$ and $\bar{P}(ij)$. We now make the following field redefinition:

$$\begin{aligned} n_1(\vec{x})/\phi_1(\vec{x}) &\rightarrow (-1)^{x_1+x_2} n_1(\vec{x})/\phi_1(\vec{x}), \\ n_2(\vec{x})/\phi_2(\vec{x}) &\rightarrow (-1)^{x_1+x_2+1} n_2(\vec{x})/\phi_2(\vec{x}), \\ m(\vec{x})/\chi(\vec{x}) &\rightarrow (-1)^{x_1+x_2+1} m(\vec{x})/\chi(\vec{x}). \end{aligned} \quad (C2)$$

The above redefinition implies $P(ij) \approx \bar{P}(ij)$. Expanding the exponential in $\text{Tr}U_o$ and keeping terms up to $O(1/N^2)$ and $O(g^2)$, we arrive at Eq. (19).

-
- [1] K. G. Wilson, Confinement of quarks, *Phys. Rev. D* **10**, 2445 (1974); T. T. Wu and C. N. Yang, Concept of non-integrable phase factors and global formulation of gauge fields, *Phys. Rev. D* **12**, 3845 (1975); A. M. Polyakov, Gauge fields as rings of glue, *Nucl. Phys. B* **164**, 171 (1979).
[2] S. Mandelstam, Charge-monopole duality and the phases of non-Abelian gauge theories, *Phys. Rev. D* **19**, 2391 (1979); R. Gambini and J. Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity* (Cambridge University Press, Cambridge, England, 2000); Y. M. Makeenko and A. A. Migdal, Quantum chromodynamics as dynamics of loops, *Nucl. Phys. B* **188**, 269 (1981); A. Jevicki and B. Sakita, Loop-space representation and the large- N behavior of the one-plaquette Kogut-Susskind Hamiltonian, *Phys. Rev. D* **22**, 467 (1980); B. Brüggmann, Method of loops applied to lattice gauge theory, *Phys. Rev. D* **43**, 566 (1991); R. Giles, Reconstruction of gauge potentials from Wilson loops, *Phys. Rev. D* **24**, 2160 (1981); W.

- Furmanski and A. Kolawa, Yang-Mills vacuum: An attempt at lattice loop calculus, *Nucl. Phys. B* **291**, 594 (1987); R. Gambini, L. Leal, and A. Trias, Loop calculus for lattice gauge theories, *Phys. Rev. D* **39**, 3127 (1989); C. Bartolo, R. Gambini, and L. Leal, Hamiltonian lattice gauge theories in a loop-dependent magnetic representation, *Phys. Rev. D* **39**, 1756 (1989); R. Loll, Independent SU(2) loop variables and the reduced configuration space of SU(2) lattice gauge theory, *Nucl. Phys. B* **368**, 121 (1992); Yang-Mills theory without Mandelstam constraints, *Nucl. Phys. B* **400**, 126 (1993); N. J. Watson, Solution of the SU(2) Mandelstam constraints, *Phys. Lett. B* **323**, 385 (1994); Gauge-invariant variables and Mandelstam constraints in SU(2) gauge theory, *Nucl. Phys. B, Proc. Suppl.* **39**, 224 (1995); A. A. Migdal, Loop equations and $1/N$ expansion, *Phys. Rep.* **102**, 199 (1983).
[3] V. F. Muller and W. Ruhl, Energy gap of SU(2) lattice gauge theory in 2 + 1 dimensions, *Nucl. Phys. B* **230**, 49 (1984);

- D. Robson and D. M. Webber, Gauge covariance in lattice field theories, *Z. Phys. C* **15**, 199 (1982); N. E. Ligterink, N. R. Walet, and R. F. Bishop, Toward a many-body treatment of Hamiltonian lattice $SU(N)$ gauge theory, *Ann. Phys. (N.Y.)* **284**, 215 (2000).
- [4] M. Mathur and T. P. Sreeraj, Canonical transformations and loop formulation of $SU(N)$ lattice gauge theories, *Phys. Rev. D* **92**, 125018 (2015); From lattice gauge theories to hydrogen atoms, *Phys. Lett. B* **749**, 137 (2015); Lattice gauge theories and spin models, *Phys. Rev. D* **94**, 085029 (2016).
- [5] R. Anishetty and H. S. Sharatchandra, Duality Transformation for Non-Abelian Lattice Gauge Theories, *Phys. Rev. Lett.* **65**, 813 (1990); H. S. Sharatchandra, Local observables in non-Abelian gauge theories, *Nucl. Phys.* **B196**, 62 (1982); R. Anishetty, Local dynamics on gauge invariant basis of non-Abelian gauge theories, *Phys. Rev. D* **44**, 1895 (1991).
- [6] M. Mathur, Harmonic oscillator prepotentials in $SU(2)$ lattice gauge theory, *J. Phys. A* **38**, 10015 (2005); Loop approach to lattice gauge theories, *Nucl. Phys.* **B779**, 32 (2007); The loop states in lattice gauge theories, *Phys. Lett. B* **640**, 292 (2006); M. Mathur, I. Raychowdhury, and R. Anishetty, $SU(N)$ irreducible Schwinger bosons, *J. Math. Phys. (N.Y.)* **51**, 093504 (2010); R. Anishetty, M. Mathur, and I. Raychowdhury, Prepotential formulation of $SU(3)$ lattice gauge theory, *J. Phys. A* **43**, 035403 (2010); M. Mathur and I. Raychowdhury, $SU(N)$ coherent states and irreducible Schwinger bosons, *J. Phys. A* **44**, 035203 (2011); R. Anishetty, M. Mathur, and I. Raychowdhury, Irreducible $SU(3)$ Schwinger bosons, *J. Math. Phys. (N.Y.)* **50**, 053503 (2009); R. Anishetty, H. Gopalkrishna Gadiyar, M. Mathur, and H. S. Sharatchandra, Color invariant additive fluxes for $SU(3)$ gauge theory, *Phys. Lett. B* **271**, 391 (1991); G. Burgio, R. De Pietri, H. A. Morales-Tecotl, L. F. Urrutia, and J. D. Vergara, The basis of the physical Hilbert space of lattice gauge theories, *Nucl. Phys.* **B566**, 547 (2000); M. Mathur, I. Raychowdhury, and T. P. Sreeraj, Invariants, projection operators and $SU(N) \times SU(N)$ irreducible Schwinger bosons, *J. Math. Phys. (N.Y.)* **52**, 113505 (2011).
- [7] J. Kogut and L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, *Phys. Rev. D* **11**, 395 (1975).
- [8] R. Anishetty and I. Raychowdhury, $SU(2)$ lattice gauge theory: local dynamics on non-intersecting electric flux loops, *Phys. Rev. D* **90**, 114503 (2014).
- [9] P. Carruthers and M. M. Nieto, Phase and Angle variables in Quantum mechanics, *Rev. Mod. Phys.* **40**, 411 (1968).
- [10] D. Karabali and V. P. Nair, A gauge-invariant Hamiltonian analysis for non-Abelian gauge theories in $(2+1)$ dimensions, *Nucl. Phys.* **B464**, 135 (1996); D. Karabali, C. Kim, and V. P. Nair, Planar Yang-Mills theory: Hamiltonian, regulators and mass gap, *Nucl. Phys.* **B524**, 661 (1998); Manifest covariance and the Hamiltonian approach to mass gap in $2+1$ dimensional Yang-Mills theory, *Phys. Rev. D* **64**, 025011 (2001); R. G. Leigh, D. Minic, and A. Yelnikov, Glueball spectrum of pure Yang-Mills theory in $2+1$ dimensions, *Phys. Rev. D* **76**, 065018 (2007); V. P. Nair and A. Yelnikov, On the invariant measure for the Yang-Mills configuration space in $(3+1)$ dimensions, *Nucl. Phys.* **B691**, 182 (2004); L. Freidel, R. G. Leigh, and D. Minic, Towards a solution of pure Yang-Mills theory in $3+1$ dimensions, *Phys. Lett. B* **641**, 105 (2006); M. J. Teper, $SU(N)$ lattice gauge theories in $2+1$ dimensions, *Phys. Rev. D* **59**, 014512 (1998).