Effect of the dilaton on holographic complexity growth

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In this paper, we investigate the action growth in various backgrounds in Einstein-Maxwell-dilaton theory. We calculate the full time evolution of action growth in the anti-de Sitter dilaton black hole and find it approaches the late time bound from above. We investigate the black hole which is asymptotically Lifshitz and obtain its late time and full time behavior. We find the violation of Lloyd bound in the late time limit and show the full time behavior approaching the late time bound from above and exhibiting some new features for z sufficiently large.

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I. INTRODUCTION

The AdS/CFT correspondence shows that gravity theory in anti-de Sitter (AdS) space is dual to the conformal field theory on its boundary which implies a striking relation between the gauge theory and the gravity theory [1]. As it is a weak-strong duality, we are able to investigate various strong coupling effects in field theory from the gravity side. A systematic rule of correspondence between two sides is proposed which is the so-called holographic dictionary [2,3]. In 2006, Ryu and Takayanagi found that entanglement entropy in boundary field theory is dual to the minimal area of the boundary anchored surface in the bulk [4,5]. This work illuminates the relation between quantum information theory and gravity theory and implies that we can investigate the quantum gravity effect via the quantum information approach.

In recent years, there has been a growing interest in another field theory quantity called circuit complexity. In quantum computation theory, circuit complexity is defined to be the minimal number of simple gates required to build a typical state from a reference state within small tolerance ϵ . In order to interpret the growth of the size of Einstein-Rosen bridge (ERB) behind the horizon, the authors of Ref. [6] proposed that the ERB size is dual to the quantum complexity on the boundary, which is called

Complexity-Volume duality (CV) duality. Later, there was another proposal which says that complexity is dual to the on-shell action of the Wheeler-De Witt (WdW) patch [7,8], which is called Complexity-Action duality (CA) duality. By calculating the action growth [7–9], they find that the action obeys a bound called the Lloyd bound [10] which is obtained in quantum computation using the energy-time uncertainty principle. There are many works concerning this bound in various backgrounds [11–13]. Moreover recent study [14] shows the connection between the butterfly velocity and the complexity growth rate.

In the context of the string theory, as the low energy limit is reached, a scalar field called dilaton occurs in the action. Dilaton couples to other fields in various nontrivial ways. As an example, the authors of [15] investigate the case that dilaton couples to the Maxwell field acting as the coupling constant of Maxwell action. The appearance of the dilaton field changes the spacetime structure drastically and gives us more interesting black hole solutions. The authors of [15] get a dilaton black hole solution in asymptotically flat spacetime and [16] generalize this to asymptotically dS and AdS spacetime by introducing a Liouville-type potential. Much interest has been paid to investigate the thermodynamic behavior of these black holes [17,18], but the detailed analysis of the complexity behavior is still yet to be done.

Despite the success of AdS/CFT correspondence, there are also many investigations beyond AdS. In AdS, space and time coordinates scale isotropically, while in order to describe some condensed matter systems, an anisotropically scaling is needed. References [19–22] come up with many backgrounds which support the scaling,

$$t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x. \quad (1)$$

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Among them, the authors of Ref. [20] find that the Einstein-Maxwell-dilaton system has such a nontrivial black hole solution. While there are many investigations considering the thermodynamic and hydrodynamic properties of this solution [23,24], its complexity behavior is still under research.

In this work, we use the CA proposal and investigate the action growth of various black hole solutions in Einstein-Maxwell-dilaton theory. In Sec. II, we investigate a dilaton black hole in the AdS vacuum, while an electromagnetic field exists, and the Penrose diagram resembles the Schwarzschild case. While the authors of Ref. [12] investigated the late-time behavior of this black hole, we investigate the full-time evolution of it and show that it approaches the late-time bound from above as in Ref. [25]. In Sec. III, we go beyond the AdS case and investigate the Lifshitz-type black hole [20], and we calculate the on-shell action in the WdW patch of this black hole and find that it violates the Lloyd bound even in the late time limit. In Sec. IV, we discuss our result and give some interpretation.

II. FULL TIME EVOLUTION OF ACTION IN CHARGED DILATON BLACK HOLE IN ADS SPACE

A. Charged dilaton black hole

We consider the Einstein-Maxwell-dilaton theory with the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2(\partial\phi)^2 - V(\phi) - e^{-2\phi} F^2), \quad (2)$$

where the Liouville-type potential is

$$V(\phi) = -\frac{4}{l^2} - \frac{1}{l^2} [e^{2(\phi-\phi_0)} + e^{-2(\phi-\phi_0)}]. \quad (3)$$

When $\phi = \phi_0$, it reproduces the usual cosmological constant term in AdS space. Varying the action, we can get the equation of motion which is

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}V + 2e^{-2\phi}\left(F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2\right), \quad (4)$$

$$\partial_\mu(\sqrt{-g}e^{-2\phi}F^{\mu\nu}) = 0, \quad (5)$$

$$\partial^2\phi = \frac{1}{4}\frac{dV}{d\phi} - \frac{1}{2}e^{-2\phi}F^2, \quad (6)$$

and there exists a static spherically symmetric black hole solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r(r-2D)d\Omega^2, \quad (7)$$

where $f(r) = 1 - \frac{2M}{r} + \frac{r(r-2D)}{l^2}$ is the electromagnetic field, dilaton ϕ can be obtained via the equation of motion

$F_{tr} = \frac{Qe^{2\phi}}{r(r-2D)}$, $e^{2\phi} = e^{2\phi_0}(1 - \frac{2D}{r})$, ϕ_0 is the integration constant, and D is the conserved dilaton charge which is $D = \frac{Q^2 e^{2\phi_0}}{2M}$.

Unlike the usual Reissner-Nordstrom (RN) case, although the system has an electromagnetic field, because of the introduction of the dilaton coupling, there is a new curvature singularity located at the $r = 2D$ (between the inner horizon and the outer horizon). It seems natural because of the instability of the inner horizon in the RN black hole. Therefore, the Penrose diagram looks like the AdS-Schwarzschild black hole rather than the AdS-RN black hole and is depicted in Fig. 1. The authors of [12] investigated the late time behavior of the complexity growth, and the result is

FIG. 1. WdW patch of the time before (left side) and after (right side) the critical time. We assume the boundary time satisfies the relation $t_L = t_R$, and at the right (left) boundary, bulk time flows in the same (opposite) direction as the boundary. In calculating the bulk contribution of the total action, we partition the spacetime into three regions.

$$\frac{dC}{dt} = 2M - \mu Q - D. \quad (8)$$

The authors of [12] proposed that the existence of dilaton will slow down the rate of complexity growth. Recent studies [25] say that in the AdS-Schwartzchild case, the bound proposed in [12] is approached from above so is violated in the early time. In the next section, we will show that the early time violation behavior also occurs in this situation.

B. General time dependence of the action

Reference [9] provides a method to calculate the action in the presence of the null boundary; the expression is as follows:

$$\begin{aligned} I = & \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^4x \sqrt{-g} (\mathcal{R} - 2(\partial\phi)^2 - V(\phi) - e^{-2\phi} F^2) \\ & + \frac{1}{8\pi G_N} \int_B d^3x \sqrt{|h|} K + \frac{1}{8\pi G_N} \int_{\Sigma} d^2x \sqrt{\sigma} \eta \\ & - \frac{1}{8\pi G_N} \int_{B'} d\lambda d^2\theta \sqrt{\gamma} \kappa + \frac{1}{8\pi G_N} \int_{\Sigma'} d^2x \sqrt{\sigma} a. \end{aligned} \quad (9)$$

Here, we follow the convention in [25,26]. Terms in the expression above are, respectively, the bulk term, Gibbons-Hawking-York (GHY) boundary term, Hayward joint term [27], null boundary term, and null joint term. We choose affine parametrization and set $\kappa = 0$ in the following, so the contribution of the null boundary vanishes.

We consider the full time evolution of action in this black hole (we set $G = 1$ in this section for simplicity), we assume the boundary time has the relation $t_L = t_R = \frac{t}{2}$, and the time dependence has two stages. First, the past null-boundary intersects the past-singularity, and we have the past GHY boundary term; after an amount of time, which we call the critical time, the two past-null boundaries intersect each other, and a null joint term replaces the GHY boundary term. The complexity behavior is different in the times below and above the critical time. So, we first determine the critical time t_c ,

$$\frac{t_c}{2} - r^*(\infty) = t - r^*(2D), \quad (10)$$

$$-\frac{t_c}{2} + r^*(\infty) = t + r^*(2D), \quad (11)$$

and the critical time is

$$t_c = 2(r^*(\infty) - r^*(2D)). \quad (12)$$

When $t < t_c$, the contribution contains three parts,

$$S = S_{\text{bulk}} + S_{\text{GHY}} + S_{\text{joint}}, \quad (13)$$

the bulk contribution is the Einstein-Hilbert action plus matter field action which is (2), and the surface comes from past and future singularity surfaces and the cutoff surface at r_{max} . The joint term is a null-spacelike or a null-timelike joint which occurs at a singularity or a cutoff surface, respectively. The bulk term is

$$S = \frac{1}{16\pi} \int d^4x \left[4r(r-2D) + (r-2D)^2 + r^2 - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right]. \quad (14)$$

It is convenient to divide the integral region into three portions, and the result is as follows:

$$\begin{aligned} S_1 = & -\frac{1}{4l^2} \int_{2D}^{rh} dr \left[4r(r-2D) + (r-2D)^2 + r^2 \right. \\ & \left. - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right] \left(\frac{t}{2} + r_{\infty}^* - r^*(r) \right), \end{aligned} \quad (15)$$

$$\begin{aligned} S_2 = & -\frac{1}{2l^2} \int_{rh}^{r_{\text{max}}} dr \left[4r(r-2D) + (r-2D)^2 + r^2 \right. \\ & \left. - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right] (r_{\infty}^* - r^*(r)), \end{aligned} \quad (16)$$

$$\begin{aligned} S_3 = & -\frac{1}{4l^2} \int_{2D}^{rh} dr \left[4r(r-2D) + (r-2D)^2 + r^2 \right. \\ & \left. - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right] \left(-\frac{t}{2} + r_{\infty}^* - r^*(r) \right). \end{aligned} \quad (17)$$

We see the time dependences cancel each other and the bulk term is independent of time. Then, we calculate the surface term at singularity. We choose

$$n_{\alpha} = -|f(r)|^{-1/2} \partial_{\alpha} r, \quad (18)$$

and the extrinsic curvature is

$$K = \nabla_{\alpha} n^{\alpha} = \frac{1}{U^2} \frac{d}{dr} (U^2 n^r), \quad (19)$$

where $U^2 = r(r-2D)$. So the GHY action is

$$I_{\text{future}} = \frac{1}{2} |f|^{1/2} \frac{d}{dr} (r(r-2D) |f|^{1/2}) \left(\frac{t}{2} + r_{\infty}^* - r^*(r) \right) \Big|_{r=2D}, \quad (20)$$

$$I_{\text{past}} = \frac{1}{2} |f|^{1/2} \frac{d}{dr} (r(r-2D) |f|^{1/2}) \left(-\frac{t}{2} + r_{\infty}^* - r^*(r) \right) \Big|_{r=2D}, \quad (21)$$

$$I_{\text{cutoff}} = |f|^{1/2} \frac{d}{dr} (r(r-2D)|f|^{1/2})(r_\infty^* - r^*(r)) \Big|_{r=r_{\text{max}}}. \quad (22)$$

We see the cancellation between surfaces at the past and the future singularities, and from [28] we know that the joint terms are independent of time. Combining the above results, we see the action is constant until $t = t_c$.

When $t > t_c$, the null joint forms at $r = r_m$ and there is no surface term from past singularity. r_m is obtained by equation

$$\frac{t - t_c}{2} + r^*(r_m) - r^*(2D) = 0. \quad (23)$$

The null joint term depends on time implicitly through r_m .

The bulk action is

$$I_{\text{bulk}} = I_{\text{bulk}}^0 - \frac{1}{4L^2} \int_{2D}^{r_m} dr \left[4r(r-2D) + (r-2D)^2 + r^2 - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right] \left(\frac{t}{2} - r_\infty^* + r^*(r) \right). \quad (24)$$

So the change of the bulk action compared with the $t < t_c$ case is

$$\delta I_{\text{bulk}} = -\frac{1}{4L^2} \int_{2D}^{r_m} dr \left[4r(r-2D) + (r-2D)^2 + r^2 - \frac{2Q^2 l^2 e^{2\phi_0}}{r^2} \right] \left(\frac{\delta t}{2} + r^*(r) - r^*(2D) \right), \quad (25)$$

where $\delta t = t - t_c$ because of the lack of a surface term of past singularity, and the surface contribution also depends on t ,

$$I_{\text{surf}} = I_0 - I_{\text{past}}, \quad (26)$$

so

$$\delta I_{\text{surf}} = \frac{1}{2} \frac{d}{dr} (r(r-2D)|f|^{1/2}) |f|^{1/2} \times \left(\frac{\delta t}{2} + r^*(r) - r^*(2D) \right) \Big|_{r=2D}. \quad (27)$$

For the null joint term

$$k_R = -\alpha dt + \alpha \frac{dr}{f(r)}, \quad (28)$$

$$k_L = \alpha dt + \alpha \frac{dr}{f(r)}, \quad (29)$$

FIG. 2. There are two parameters: $y = \frac{\mu Q}{2M}$, $z = r_+/L$. For this picture, we fix $z = 1$, the green line corresponds to $y = 0.1$, the blue line $y = 0.2$, and the red line $y = 0.3$. We find the complexity growth rate approaches the late time bound from above.

$$a = \log \left(-\frac{1}{2} k_R \cdot k_L \right) = -\log \left(\frac{|f|}{\alpha^2} \right), \quad (30)$$

so the joint term reads

$$I_{\text{jnt}} = \frac{1}{8\pi} \int_{\Sigma'} d^2 x \sqrt{\sigma} a = -\frac{r_m(r_m - 2D)}{2} \log \frac{|f(r_m)|}{\alpha^2}. \quad (31)$$

Combining the above result (25), (27), and (31), taking the derivative with respect to t , and recalling that $\frac{dr_m}{dt} = -\frac{1}{2} f(r_m)$, we finally get the action growth rate at time $t > t_c$,

$$\frac{dI_{\text{tot}}}{dt} = 2M - \mu_m Q - D + \frac{1}{2} (r_m - D) f(r_m) \log \frac{|f(r_m)|}{\alpha^2}, \quad (32)$$

where $\mu_m = \frac{Q e^{2\phi_0}}{r_m}$ at late time limit $r_m \rightarrow r_+$, and we see that μ_m becomes the chemical potential and the last term vanishes. So we recover the late time result in [12]. The rate of growth of action for $t > t_c$ is plotted as Fig. 2.

III. ACTION GROWTH IN LIFSHITZ-LIKE BLACK BRANE

A. Lifshitz black brane in EMD theory

In many condensed matter systems, near the critical point, anisotropic scaling symmetry is expected. The gravitational systems which have the same scaling behavior are constructed in various situations. The authors of Ref. [19] add high order form field, while the authors of Ref. [20] realize Lifshitz spacetime using massive vector field and Einstein-Maxwell-dilaton (EMD) theory. Here we

choose the solution constructed in Ref. [20] by coupling the dilaton field to an Abelian Maxwell field. The thermodynamic behavior of Lifshitz spacetime is investigated in Ref. [23].

We consider the following action in $(d+2)$ -dimensional spacetime:

$$I = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \right], \quad (33)$$

where Λ is the cosmological constant and the matter fields are a massless scalar and an Abelian gauge field. We can get the equations of motion

$$\partial_\mu (\sqrt{-g} e^{\lambda\phi} F^{\mu\nu}) = 0, \quad (34)$$

$$\partial_\mu (\sqrt{-g} \partial^\mu \phi) - \frac{\lambda}{4} \sqrt{-g} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} = 0, \quad (35)$$

$$R_{\mu\nu} = \frac{2}{d} \Lambda g_{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{\lambda\phi} F_{\mu\rho} F_{\nu}{}^\rho - \frac{1}{4d} g_{\mu\nu} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu}, \quad (36)$$

which has the following asymptotic Lifshitz black hole solution:

$$ds^2 = L^2 \left(-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^d dx_i^2 \right), \quad (37)$$

$$f(r) = 1 - \frac{r_+^{z+d}}{r^{z+d}}, \quad (38)$$

$$\begin{aligned} F_{rt} &= q e^{-\lambda\phi} r^{z-d-1}, & e^{\lambda\phi} &= r^{\lambda\sqrt{2(z-1)d}}, \\ \lambda^2 &= \frac{2d}{z-1}, & q^2 &= 2L^2(z-1)(z+d), \\ \Lambda &= -\frac{(z+d-1)(z+d)}{2L^2}. \end{aligned} \quad (39)$$

We can obtain the temperature via Euclidean path integral

$$T_H = \frac{(z+d)r_+^z}{4\pi}, \quad (40)$$

and the black hole entropy

$$S_{\text{BH}} = \frac{\Omega_d L^d}{4G_{d+2}} \left(\frac{4\pi}{z+d} \right)^{\frac{d}{z}} T_z^{\frac{d}{z}}, \quad (41)$$

where Ω_d denotes the volume of the d -dimensional spatial coordinates.

B. Late time behavior in Lifshitz black brane

The volume contribution is

$$S_V = \int_V \left(R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} d^{n+2}x. \quad (42)$$

We will use the null coordinates u and v , defined by

$$du := dt + \frac{1}{r^{z+1}f} dr, \quad dv := dt - \frac{1}{r^{z+1}f} dr. \quad (43)$$

Integrating these relations yields the “infalling” null coordinate $u = t + r^*(r)$ and the “outgoing” null coordinate $v = t - r^*(r)$, where $r^*(r) := \int \frac{1}{r^{z+1}f} dr$. The metric becomes

$$ds^2 = L^2 \left[-r^{2z} f du^2 + 2r^{z-1} du dr + r^2 \sum_{i=1}^d dx_i^2 \right] \quad (44)$$

or

$$ds^2 = L^2 \left[-r^{2z} f dv^2 - 2r^{z-1} dv dr + r^2 \sum_{i=1}^d dx_i^2 \right] \quad (45)$$

when expressed in terms of the null coordinates. For the three choices (t, r) , (u, r) , and (v, r) we have

$$\int \sqrt{-g} d^{d+2}x = \Omega_d L^{d+2} \int r^{d+z-1} dr dw, \quad (46)$$

where $w = \{t, u, v\}$.

As shown in Fig. 3, the region V_1 is bounded by the null surfaces $u = u_0$, $u = u_0 + \delta t$, $v = v_0 + \delta t$, and $r = \epsilon$. The volume integral is best performed in the (u, r) coordinate system; in this system the surface $v = v_0 + \delta t$ is described by $r = \rho_0(u)$, with $r^*(\rho_0) = \frac{1}{2}(v_0 + \delta t - u)$. Making use of equations of motion, we get

$$\begin{aligned} R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} &= \frac{4\Lambda}{d} - \frac{1}{2d} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \\ &= \frac{-2(z+d)}{L^2}, \end{aligned} \quad (47)$$

and with this we have

$$\begin{aligned} S_{V_1} &= -2(d+z)L^d \Omega_d \int_{u_0}^{u_0+\delta t} du \int_{\epsilon}^{\rho_0(u)} r^{d+z-1} dr \\ &= -2L^d \Omega_d \int_{u_0}^{u_0+\delta t} du [\rho_0^{d+z}(u)], \end{aligned} \quad (48)$$

where we neglect the ϵ^{n+1} term in the integral as $\epsilon \rightarrow 0$. The region V_2 is bounded by the null surfaces $u = u_0$, $u = u_1$,

FIG. 3. WdW patch of Lifshitz black hole.

$v = v_0$, and $v = v_0 + \delta t$. In this case, the volume integral is most easily performed in the (v, r) coordinates, in which the surfaces $u = u_{0,1}$ are described by $r = \rho_{0,1}(v)$, with $r^*(\rho_{0,1}) = \frac{1}{2}(v - u_{0,1})$. Then we have

$$\begin{aligned} S_{V_2} &= -2(d+z)L^d\Omega_d \int_{v_0}^{v_0+\delta t} dv \int_{\rho_1(v)}^{\rho_0(v)} r^{d+z-1} dr \\ &= -2L^d\Omega_d \int_{v_0}^{v_0+\delta t} dv [\rho_0^{d+z}(v) - \rho_1^{d+z}(v)]. \end{aligned} \quad (49)$$

Using a variable change, $u = u_0 + v_0 + \delta t - v$, we can see the terms involving $\rho_0(u)$ and $\rho_0(v)$ cancel out. We are left with

$$S_{V_1} - S_{V_2} = -2L^d \Omega_d \int_{v_0}^{v_0 + \delta t} dv [\rho_1^{d+z}(v)] \quad (50)$$

with the function $\rho_1(v)$ varying from r_B to $r_B + O(\delta t)$, and hence the volume contribution to δS is simply

$$S_{V_1} - S_{V_2} = -2L^d \Omega_d r_B^{d+z} \delta t. \quad (51)$$

The surface contributions to δS are given by $-2 \int_S K d\Sigma$, where S is the boundary segment given by the spacelike hypersurface $r = \epsilon$. The (future-directed) unit normal is given by $n_\alpha = \frac{r}{\epsilon} |f|^{-1/2} \partial_\alpha r$. The extrinsic curvature is then

$$K = \nabla_\alpha n^\alpha = -\frac{1}{L^{d+2} r^{z+d-1}} \frac{d}{dr} (L^{d+1} r^{z+d} |f|^{1/2}), \quad (52)$$

and the volume element

$$d\Sigma = \Omega_d L^{d+1} |f|^{1/2} r^{z+d} dt. \quad (53)$$

Letting $r = \epsilon \ll r_+$ and then approximating $f \simeq -(r_+/r)^{z+d}$; $K \simeq -\frac{z+d}{2L}(r_+/r)^{\frac{z+d}{2}}$; $d\Sigma \simeq \Omega_d L^{d+1}(r_+r)^{\frac{z+d}{2}} dt$, we find that

$$-2 \int_S K d\Sigma = (z + d) \Omega_d L^d r_+^{z+d} \delta t. \quad (54)$$

It is finite and independent of ϵ .

We then calculate joint terms at B, B' . The null joint rule states that [9]

$$a = \ln\left(-\frac{1}{2}k \cdot \bar{k}\right). \quad (55)$$

We choose the vectors k^α and $\bar{k}^\alpha$ to be affinely parametrized and read

$$\begin{aligned} k_\alpha &= -c\partial_\alpha v = -c\partial_\alpha(t - r^*), \\ \bar{k}_\alpha &= \bar{c}\partial_\alpha u = \bar{c}\partial_\alpha(t + r^*), \end{aligned} \quad (56)$$

where c and $\bar{c}$ are arbitrary (positive) constants. With these choices, we have that $k \cdot \bar{k} = 2c\bar{c}/(fL^2r^{2z})$, and then

$$a = \ln[-c\bar{c}/(fL^2r^{2z})]. \quad (57)$$

With the above expression, we find that

$$2 \oint_{B'} adS - 2 \oint_B adS = 2\Omega_d[h(r_{B'}) - h(r_B)], \quad (58)$$

where $h(r) := r^d L^d \ln[-c\bar{c}/(fL^2 r^{2z})]$.

Then we perform a Taylor expansion of $h(r)$ about $r = r_B$. Because the displacement is in a direction of increasing v , we have that $du = 0$, $dv = \delta t$, and $dr = -\frac{1}{2}fr^{z+1}\delta t$. This gives us

$$\begin{aligned}
& h(r_{B'}) - h(r_B) \\
&= -\frac{1}{2} f r^{z+1} \frac{dh}{dr} \Big|_{r=r_B} \delta t \\
&= -\frac{1}{2} L^d r^{z+d} \left[-r \frac{df}{dr} - 2zf + f d \ln \left(\frac{c\bar{c}}{-f L^2 r^{2z}} \right) \right] \Big|_{r=r_B} \delta t,
\end{aligned} \tag{59}$$

and then

$$\begin{aligned}
& 2 \oint_{B'} adS - 2 \oint_B adS \\
& = -\Omega_d L^d r^{\tilde{z}+d} \left[-r \frac{df}{dr} - 2\tilde{z}f + f d \ln \left(\frac{c\tilde{c}}{-fL^2 r^{2\tilde{z}}} \right) \right] \Big|_{r=r_B} \delta t.
\end{aligned} \tag{60}$$

Making use of the explicit expression of f , and taking the late time limit $r_B \rightarrow r_+$, the Log term vanishes, and the result is

$$2 \oint_{B'} adS - 2 \oint_B adS = (z+d)L^d \Omega_d r_+^{d+z} \delta t. \quad (61)$$

Combining Eqs. (51), (54), and (61), we have

$$\delta S = (2z + 2d - 2)L^d \Omega_d r_+^{d+z} \delta t \quad (62)$$

in the late time limit. From Ref. [24], we know the mass of the Lifshitz spacetime, which is

$$M = \frac{r_+^{z+d} d L^d \Omega_d}{16\pi G_{d+2}}. \quad (63)$$

This mass is the Komar integral subtracting the zero temperature background to remove the infinite volume divergence [20]. So we have

$$\frac{dS}{dt} = 32\pi G_{d+2} \frac{z+d-1}{d} M, \quad (64)$$

and in a more usual convention,

$$\frac{dI}{dt} = 2 \frac{z+d-1}{d} M. \quad (65)$$

We can see that, when $z = 1$,

$$\frac{dI}{dt} = 2M, \quad (66)$$

it is just the result in Ref. [9]. When $z > 1$, we have $\frac{dI}{dt} > 2M$, and the bound in [7,8] is violated even in the late time limit.

C. General time behavior Lifshitz black brane

In this subsection, we investigate the full time evolution of action growth and see whether the late time result is approached from below or above. We divide the Wheeler-DeWitt patch into three parts as in Fig. 1. When $t < t_c$, the action is composed of the following three parts:

$$I_{\text{tot}} = I_{\text{bulk}} + I_{\text{GHY}} + I_{\text{joint}}, \quad (67)$$

where I_{bulk} is the bulk term, I_{GHY} is the GHY surface term at $r = \epsilon \rightarrow 0$ and $r = r_{\text{max}} \rightarrow \infty$, and the joint term is the joint formed by null and spacelike (timelike) surface at $r = \epsilon$ ($r = r_{\text{max}}$). For $t < t_c$, we have

$$I_{\text{bulk}}^1 = - \int_{\epsilon_0}^{r_+} \frac{(z+d)L^d \Omega_d}{8\pi G} \left(\frac{t}{2} + r^*(\infty) - r^*(r) \right) r^{z+d-1} dr, \quad (68)$$

$$I_{\text{bulk}}^2 = - \int_{r_+}^{r_{\text{max}}} \frac{(z+d)L^d \Omega_d}{8\pi G} 2(r^*(\infty) - r^*(r)) r^{z+d-1} dr, \quad (69)$$

$$I_{\text{bulk}}^3 = - \int_{\epsilon_0}^{r_+} \frac{(z+d)L^d \Omega_d}{8\pi G} \left(-\frac{t}{2} + r^*(\infty) - r^*(r) \right) r^{z+d-1} dr, \quad (70)$$

we see total cancellation, and the bulk term is independent of time.

The GHY term is calculated as

$$I_{\text{surf}}^{\text{future}} = \frac{r_+^{z+d} L^d \Omega_d}{8\pi G} \left(\frac{t}{2} + r^*(\infty) - r^*(r) \right) \times \left[(z+d)|f| + \frac{r}{2} \partial_r |f| \right] \Big|_{r=\epsilon_0}, \quad (71)$$

$$I_{\text{surf}}^{\text{past}} = \frac{r_+^{z+d} L^d \Omega_d}{8\pi G} \left(-\frac{t}{2} + r^*(\infty) - r^*(r) \right) \times \left[(z+d)|f| + \frac{r}{2} \partial_r |f| \right] \Big|_{r=\epsilon_0}, \quad (72)$$

$$I_{\text{surf}}^{\text{UV cutoff}} = \frac{r_+^{z+d} L^d \Omega_d}{8\pi G} (2(r^*(\infty) - r^*(r))) \times \left[(z+d)|f| + \frac{r}{2} \partial_r |f| \right] \Big|_{r=r_{\text{max}}}, \quad (73)$$

and it can be seen that the total boundary terms are also independent of time.

Joint terms from null boundaries that intersect with the surfaces of past and future singularities vanish, and joint terms from null boundaries that intersect with UV cutoff is independent of time [28]. So it has no contribution to complexity growth.

So we conclude that $\frac{dI}{dt} = 0$ when $t < t_c$.

When $t > t_c$, the only difference is the intersections between the null boundaries with past singularity surface change to intersections between two null boundaries. We calculate in the same way

$$I_{\text{bulk}}^1 = - \int_{\epsilon_0}^{r_+} \frac{(z+d)L^d \Omega_d}{8\pi G} \left(\frac{t}{2} + r^*(\infty) - r^*(r) \right) r^{z+d-1} dr, \quad (74)$$

$$I_{\text{bulk}}^2 = - \int_{r_+}^{r_{\text{max}}} \frac{(z+d)L^d \Omega_d}{8\pi G} 2(r^*(\infty) - r^*(r)) r^{z+d-1} dr, \quad (75)$$

$$I_{\text{bulk}}^3 = - \int_{r_m}^{r_+} \frac{(z+d)L^d \Omega_d}{8\pi G} \left(-\frac{t}{2} + r^*(\infty) - r^*(r) \right) r^{z+d-1} dr, \quad (76)$$

and combining them all we get

$$I_{\text{bulk}} - I_{\text{bulk}}^0 = - \int_0^{r_m} \frac{(z+d)L^d \Omega_d}{4\pi G} \times \left(\frac{t}{2} - r^*(\infty) + r^*(r) \right) r^{z+d-1} dr, \quad (77)$$

where we have included the factor of 2 coming from two sides of the WdW patch. For GHY boundary terms, in the absence of past surface at singularity, it also depends on time,

$$I_{\text{surf}}^{\text{past}} = \frac{r^{z+d} L^d \Omega_d}{8\pi G} \left(-\frac{t}{2} + r^*(\infty) - r^*(r) \right) \times \left[(z+d)|f| + \frac{r}{2} \partial_r |f| \right] \Big|_{r=\epsilon_0}, \quad (78)$$

$$I_{\text{surf}} - I_{\text{surf}}^0 = -2I_{\text{past}} = - \frac{r^{z+d} L^d \Omega_d}{4\pi G} \left(-\frac{t}{2} + r^*(\infty) - r^*(r) \right) \times \left[(z+d)|f| + \frac{r}{2} \partial_r |f| \right] \Big|_{r=\epsilon_0}, \quad (79)$$

where the factor of 2 comes from the two sides of the WdW patch.

Joint terms from null boundaries that intersect with surfaces of past and future singularities vanish, and joint terms from null boundaries that intersect with UV cutoff is independent of time. So we only need to consider the intersection of two past null boundaries at $r = r_m$, where r_m satisfy the equation

$$\frac{t}{2} + r^*(r_m) - r^*(\infty) = 0, \quad (80)$$

and for $\delta t = t - t_c$, we can rewrite the above equation as

$$\frac{\delta t}{2} = r^*(0) - r^*(r_m), \quad (81)$$

and varying it with t , we get

$$\frac{dr_m}{dt} = -\frac{1}{2} r_m^{z+1} f(r_m). \quad (82)$$

We choose the normal vectors of null boundaries to be

$$k_L = \alpha \left(dt + \frac{dr}{r^{z+1} f} \right), \quad (83)$$

$$k_R = -\alpha \left(dt - \frac{dr}{r^{z+1} f} \right). \quad (84)$$

The joint term is

$$I_{\text{joint}} = \frac{1}{8\pi G} \oint ads = \frac{1}{8\pi G} \oint \log \left(-\frac{1}{2} k_L \cdot k_R \right) ds = \frac{L^d \Omega_d r^d}{8\pi G} \log \left(\frac{-\alpha^2}{L^2 r_m^{2z} f} \right) \Big|_{r=r_m}. \quad (85)$$

Combining Eqs. (77), (79), (82), and (85), we can get

$$\begin{aligned} \frac{dI}{dt} &= \frac{dI_{\text{bulk}}}{dt} + \frac{dI_{\text{surf}}}{dt} + \frac{dI_{\text{joint}}}{dt} \\ &= -\frac{L^d \Omega_d}{8\pi G} r_m^{z+d} + \frac{(z+d)r_m^{z+d} L^d \Omega_d}{16\pi G} \\ &\quad + \frac{L^d \Omega_d}{8\pi G} \left[-\frac{d}{2} f(r_m) r_m^{z+d} \log \left(\frac{-\alpha^2}{L^2 r_m^{2z} f(r_m)} \right) \right. \\ &\quad \left. + z r_m^{d+z} + \frac{d-z}{2} r_+^{z+d} \right] \\ &= \frac{L^d \Omega_d}{8\pi G} r_m^{z+d} (z-1) + \frac{dr_+^{z+d} L^d \Omega_d}{8\pi G} \\ &\quad - \frac{dL^d \Omega_d}{16\pi G} \left[f(r_m) r_m^{z+d} \log \left(\frac{-\alpha^2}{L^2 r_m^{2z} f(r_m)} \right) \right]. \end{aligned} \quad (86)$$

At the late time situation, $r_m \rightarrow r_+$ and $f(r_m) \rightarrow 0$, so the last term of (86) is zero. Then the complexity growth can be simplified to

$$\frac{dI}{dt} = \frac{(d+z-1)r_+^{z+d} L^d \Omega_d}{8\pi G} \quad (87)$$

FIG. 4. The relation between complexity growth rate and boundary time. We choose the normalization factor $\alpha = Lr_+^z$, and green, blue, and red lines correspond, respectively, to $z = 3$, $z = 2$, and $z = 1$. We find it approaches the late time bound from above, and for z sufficiently large, the complexity growth rate experiences a decreasing period during early time.

which recover the late time result. To see whether the limit is approached from above or below, we plot the relation between complexity growth rate and time in four dimensions for various z in Fig. 4.

IV. CONCLUSION AND DISCUSSION

In this paper, we investigate the effect of dilaton that is coupled to Maxwell field on the holographic complexity growth. In Sec. II, we investigate the black hole solution which is asymptotically AdS. In this case we find the presence of dilaton charge can slow down the computation rate, and when we take into account the full time evolution, we find the complexity growth rate approaches the late time bound from above like Ref. [25]. In Sec. III, we investigate the case which is asymptotically Lifshitz; in this case, dilaton is essential to support anisotropic vacuum structure. We find that in this case, the Lloyd bound is violated even in the late time limit. We investigate the full time evolution of the holographic complexity growth. It approaches the late time bound from above, moreover, for z sufficiently large, and it exhibits an interesting behavior in time which is above critical time but much earlier than late time.

Although the above two cases are both solutions in EMD theory, the roles dilaton play are very different. For the AdS dilaton example, dilaton charge and electrocharge are free parameters and appear in the black hole solutions. While in the Lifshitz case, dilaton and Maxwell field are added to maintain the anisotropic scaling background. It does not contribute to the black hole solution, which means that the black hole only has one free parameter M . Holographically, eternal AdS black hole is dual to thermofield double state $|\text{TFD}\rangle$, in the Schwarzschild case,

$$|\text{TFD}\rangle = \sum_n e^{-\beta E_n} |n_L\rangle |n_R\rangle. \quad (88)$$

In the RN case

$$|\text{TFD}\rangle = \sum_n e^{-\beta(E_n - \mu Q)} |n_L\rangle |n_R\rangle \quad (89)$$

what appears on the exponential depends on the thermodynamic relation. In the Lifshitz case, because the charge is fixed constant, the first law of thermodynamics is just $dM = T dS$. So from the boundary respect, there is no chemical potential term in the thermofield double state. In other words, Ref. [8] gives the complexity growth inequality

$$\frac{dC}{dt} \leq \int_{gs}^s T dS, \quad (90)$$

so from thermodynamical relation $dM = T dS$, we conclude that the complexity growth rate will only depend on

the mass despite the presence of matter field in our Lifshitz case.

Reference [29] proves that the action growth obeys the bound $2M$ under the following two conditions: (1) the matter field locates outside the outmost killing horizon; and (2) the strong energy condition is obeyed. In our two cases, the matter field extends into the killing horizon, so it does not satisfy the condition required in [29]. In the AdS dilaton case, the bound $2M$ is satisfied, while in the Lifshitz case, it is violated. So the requirements in [29] are too restrictive, and our calculation shows that a weaker condition is needed to prove the late time bound from the bulk side and distinguish these two cases.

Recent study [13] investigates the action growth of the hyperscaling violation background in EMD theory. They found that it depends on two parameters, the “dynamical exponent” z and hyperscaling violation exponent θ . They showed when $\theta = 0$, their results match our Lifshitz result.

There are several interesting directions to pursue: the reason for the violation of the Lloyd bound is still unclear. The EMD theory is not expected to be a UV complete theory of quantum gravity, so it would be interesting to take into account the stringy effect near the singularity and recalculate the action growth. Moreover, for full time evolution of complexity growth in Lifshitz black brane, for z sufficiently large, such as $z = 3$, the complexity growth rate exhibits new behaviors that are not found in [25]; it is interesting to investigate what it means physically when z becomes large [30]. Although the CA conjecture proposed in [7,8] passes many nontrivial tests including switchback effect [30], it is still unproved. Recent studies [25] implies that although CA conjecture captures some essence in complexity, it needs to be revised. Our work’s result on Lifshitz black hole also gives the evidence that the revision is needed in CA duality.

Apart from CA duality, there are also other proposals that relate complexity to other bulk quantities, such as [6,31–34]. It is interesting to calculate complexity growth rate in these proposals and investigate if our results still hold. We will discuss the thermodynamic volume proposal [31] in the following.

Reference [31] related the complexity on the boundary to the spacetime volume of the WdW patch and gave the proposal complexity-volume duality 2.0. It is

$$C \sim \frac{1}{\hbar} P(\text{spacetime volume}). \quad (91)$$

In our Lifshitz case, because of the following relation:

$$R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda \phi} F_{\mu\nu} F^{\mu\nu} = \frac{-2(z+d)}{L^2}, \quad (92)$$

the bulk action contribution is proportional to the spacetime volume

$$I_{\text{bulk}} = -\frac{z+d}{8\pi GL^2} \int d^{d+2}x \sqrt{-g} \quad (93)$$

so there is no essential difference between the volume and bulk action of the WdW patch. We calculate the complexity growth rate using the CV 2.0 proposal explicitly. In extended phase space, the pressure is identified as the cosmological constant

$$P = -\frac{\Lambda}{8\pi G} = \frac{(z+d)(z+d-1)}{16\pi GL^2}. \quad (94)$$

The spacetime volume inside the WdW patch is calculated from (74). And when $t > t_c$, the time dependence of the volume is

$$\frac{dV}{dt} = \frac{L^{d+2}\Omega_d r_m^{z+d}}{z+d}. \quad (95)$$

After taking the late time limit ($r_m \rightarrow r_+$), the complexity growth rate is

$$\frac{dC}{dt} = \frac{(z+d-1)M}{d}. \quad (96)$$

We see in the complexity-volume 2.0 proposal, the result is half of the CA proposal, the Lloyd bound $2M$ is satisfied when z is not large enough, but as in Ref. [31] the Lloyd bound can be altered up to a prefactor

$$\frac{dC}{dt} \leq \frac{\alpha E}{\pi\hbar}. \quad (97)$$

Reference [31] shows that in the AdS-RN case, this bound is satisfied for $\alpha = 1$. In the CV 2.0 proposal if we set $\alpha = 1$ to be the Lloyd bound, this bound is violated in the Lifshitz case at late time because of the anisotropic scaling z as in the CA proposal.

Moreover, as conjectured in Ref. [31], in various cases the late time value of the time derivative of the spacetime volume of the WdW patch reduces to the thermodynamical volume (or the difference between two thermodynamic volumes in the case of two horizons). In the Lifshitz case, this late time limit of the spacetime volume derivative reads

$$V = \frac{L^{d+2}\Omega_d r_+^{z+d}}{z+d}. \quad (98)$$

When $z = 1$ in four dimensions, it reduces to the familiar thermodynamic volume of Schwartzchild case $\frac{4\pi r_+^3}{3}$. While when $z \neq 1$, because the profile of matter field depends on the cosmological constant nontrivially, it is hard to use Iyer-Wald formalism [35] to derive the explicit expression of the thermodynamic volume. So in further work, it will be interesting to derive the thermodynamic volume in the Lifshitz case and see if it is the same as our late time limit result. It will provide convincing evidence for the conjecture in Ref. [31] that thermodynamic volume is essential for complexity growth.

Although there have been many discussions on the complexity in the holographic side, a concrete definition of complexity in quantum field theory still lacks. Recently, there are many works concerning this question [36,37] that propose some definitions about complexity in the field theory side using the Finsler geometry and Fubini-Study metric. Reference [38] discretizes the Euclidean path integral and defines the “path integral complexity” in terms of Liouville action. For concrete calculation in the field theory side, see also [39–41]. Once a concrete and calculable definition of complexity is given, the problem of the CA duality conjecture will be clarified to a large extent. Moreover, apart from the problems from CA conjecture, the assumptions made in the process to derive the Lloyd bound should also be checked [42].

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Note added.—Recently this problem is investigated in Ref. [43] by adding a counterterm to remove the normalization ambiguity in the joint term. There is no strong motivation to add this term to remove the ambiguity unless field theory suggests us to do that. However, as Ref. [36] suggests, this normalization ambiguity is somehow related to the choice of reference state in the field theory construction of complexity.

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