BPS sphalerons in the F_2 nonlinear sigma model

Yuki Amari^{1,2,*} and Nobuyuki Sawado^{1,†}

¹*Department of Physics, Tokyo University of Science, Noda, Chiba 278-8510, Japan*

²*Instituto de Física de São Carlos; IFSC/USP, Universidade de São Paulo—USP, Caixa Postal 369, CEP 13560-970 São Carlos, São Paulo, Brazil*

(Received 7 January 2018; published 19 March 2018)

We construct static and also time-dependent solutions in a nonlinear sigma model with target space being the flag manifold $F_2 = SU(3)/U(1)^2$ on the four-dimensional Minkowski spacetime by analytically solving the second-order Euler-Lagrange equation. We show that the static solutions saturate an energy lower bound and can be derived from coupled first-order equations though they are saddle-point solutions. We also discuss basic properties of the time-dependent solutions.

DOI: [10.1103/PhysRevD.97.065012](https://doi.org/10.1103/PhysRevD.97.065012)

I. INTRODUCTION

Nonlinear sigma ($NL\sigma$) models have been extensively studied in various branch of physics. Specifically, the $O(3)$ $NL\sigma$ model in two dimensions has been identified as a good toy model of the $SU(2)$ pure Yang-Mills theory in four dimensions, because they share many important features such as conformal invariance, dynamical mass gap and existence of the instanton solution. Moreover, the $O(3)$ $NL\sigma$ model (with higher differential terms) can be derived as a low energy effective theory for the $SU(2)$ Yang-Mills theory [1,2], Heisenberg model [3] etc.. The topological soliton solutions, like the instantons and the Hopfions, play a crucial role in the nonperturbative aspects in the theories [4,5].

It is of great importance to consider $SU(N)$ generalizations of the $O(3)$ $NL\sigma$ model because they share some basic properties with the $SU(N)$ Yang-Mills theory and may be derived from fundamental theories like the $SU(N)$ Yang-Mills theory or the $SU(N)$ Heisenberg model. There may be several variants and, in particular, two feasible cases have been widely examined—the CP^{N-1} and F_{N-1} $NL\sigma$ models. The models are $NL\sigma$ models whose target spaces are the complex projective space $CP^{N-1} = SU(N)/U(N-1)$ and the flag space $F_{N-1} = SU(N)/U(1)^{N-1}$, respectively.¹ Note that the $CP^1 (= F_1)$ $NL\sigma$ model is equivalent to the $O(3)$ $NL\sigma$ model at least in the classical level. Indeed, the

CP^{N-1} models arose as an effective theory from the $SU(N)$ Yang-Mills theory with the minimal case of the maximal Abelian gauge (MAG) [6], ferromagnetic $SU(N)$ Heisenberg model [7,8], and so on [9–13]. On the other hand, the F_{N-1} model was considered from the $SU(N)$ Yang-Mills theory by means of the maximal case of MAG [6,14]. Moreover, the F_2 model was derived from the $SU(3)$ antiferromagnetic Heisenberg model in $1+1$ dimensions [15,16] and in $2+1$ dimensions [17,18].

In this paper, we study classical solutions of the F_2 model. There are quite few studies for classical solutions of the F_{N-1} models, even in the $N=3$ case, though ones of the CP^{N-1} models have been studied for all N since the late 70s [19,20]. To our knowledge two seminal papers about classical solutions of the F_2 model are the ones. In [21], it was shown that the BPS equation is solved by the F_1 instantons embedded into the target space F_2 . In [22], the author studied classical solutions of the F_2 model with an additional term called the Kalb-Ramond field which is a nontopological term that contributes to the Euler-Lagrange equation. In some special choices of the coupling constant, the general static solutions were discussed in the context of the integrability [23]. Note that the Kalb-Ramond field can naturally appear when one derives the F_2 model from the $SU(3)$ spin chain [15,16]. In contrast, from the $SU(3)$ Yang-Mills theory or the $SU(3)$ Heisenberg model on lattice, the term may not be provided.

The main goal of this paper is to find genuine (i.e., nonembedding) solutions in the F_2 $NL\sigma$ model without the Kalb-Ramond term. The model in $(3+1)$ -dimensional Minkowski spacetime is defined by the Lagrangian density:

$$\mathcal{L} = -(\|Z_A^\dagger \partial_\mu Z_B\|^2 + \|Z_B^\dagger \partial_\mu Z_C\|^2 + \|Z_C^\dagger \partial_\mu Z_A\|^2). \quad (1)$$

Note that the double vertical line norm for (composite) vector fields denotes the Minkowski norm with the metric

*amari.yuki.ph@gmail.com

†sawado@ph.noda.tus.ac.jp

¹For $N \geq 4$, there are also intermediate options whose target space is the partial flag manifolds $SU(N)/S(U(n_1) \times \cdots \times U(n_k))$, where $\sum n_i = N$ including the Grassmannian manifolds.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$g_{\mu\nu} = (-, +, +, +)$; thus, the square of the norm is not positive definite. Here, the fields $Z_a (a = A, B, C)$ are three-component complex vectors which form a complete basis. Therefore, we can construct the $SU(3)$ matrix from the $U(3)$ matrix $U = (Z_A, Z_B, Z_C)$ and the fields satisfy the orthonormal condition and the completeness condition, i.e.,

$$Z_a^\dagger Z_b = \delta_{ab}, \quad (2)$$

$$Z_A \otimes Z_A^\dagger + Z_B \otimes Z_B^\dagger + Z_C \otimes Z_C^\dagger = \mathbf{1}_3, \quad (3)$$

where $\mathbf{1}_3$ is the 3×3 identity matrix. The Lagrangian is invariant under the left global $U(3)$ transformation such as $U \rightarrow gU, g \in U(3)$. In addition, it is also invariant under the local $U(1)^3$ transformation, i.e., three $U(1)$ phase rotations $Z_a \rightarrow Z'_a = e^{i\Theta_a} Z_a, a = A, B, C$.

The paper is organized as follows. In Sec. II, we briefly review the previous study of an embedding solution of the model. In Sec. III, the basic information for constructing the solutions, including the parametrization and the topological charges are given. Section IV presents both the static and the time-dependent solutions of the model. Some properties of the energy, for example an energy bound for the static solutions, are given in Sec. V. The summary and discussion are in Sec. VI.

II. EMBEDDING SOLUTIONS

As we mentioned, the authors of [21] found a BPS solution which is just an $F_1 (= CP^1)$ instanton embedded into the target space F_2 . Let us begin with a brief review of their work and an introduction of some notations.

The discussion is based on an analogy with the BPS sector in the CP^2 model. For the sake of the reader's understanding, we introduce the covariant derivative,

$$D_\mu^{(a)} = \partial_\mu - (Z_a^\dagger \partial_\mu Z_a) \quad a = A, B, C, \quad (4)$$

and write the static energy per unit length as

$$E = \frac{1}{2} \int d^2x \sum_{a=A,B,C} \|D_i^{(a)} Z_a\|^2. \quad (5)$$

Each term in (5) can be viewed as the static energy of the CP^2 model linked by the orthonormal condition (2). Thus, by simply applying the BPS bound of the CP^2 model to the three terms, respectively, an energy lower bound may be derived as

$$E \geq \pi(|N_A| + |N_B| + |N_C|), \quad (6)$$

where N_a is the topological degrees of the map $Z_a: \mathbb{R}^2 \sim S^2 \rightarrow CP^2$ defined as

$$N_a = \frac{i}{2\pi} \int d^2x \epsilon^{ij} (D_i^{(a)} Z_a)^\dagger D_j^{(a)} Z_a. \quad (7)$$

Note that there is a constraint among the topological degrees of the form

$$N_A + N_B + N_C = 0. \quad (8)$$

The equality in (6) is satisfied if the following (implicitly) coupled first derivative equations are simultaneously satisfied:

$$D_i^{(a)} Z_a = \pm i \epsilon^{ij} D_j^{(a)} Z_a \quad \forall a = A, B, C. \quad (9)$$

Unfortunately, the solutions of (9) are given by only a very limited class of configurations, i.e., an F_1 instanton embedded into the target space F_2 , for which one of the vectors Z_a is fixed as a trivial one. By performing a proper global transformation, the solutions can generally be written as

$$\begin{aligned} Z_A &= \frac{1}{\sqrt{\Delta}} (1, p(z)/q(z), 0)^T, \\ Z_B &= \frac{1}{\sqrt{\Delta}} (p(\bar{z})/q(\bar{z}), -1, 0)^T, \\ Z_C &= (0, 0, 1)^T, \end{aligned} \quad (10)$$

with $\Delta \equiv 1 + |p(z)|^2/|q(z)|^2$. Here, we have introduced the standard complex variables,

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \quad (11)$$

and the functions $p(z)$ and $q(z)$ which are polynomials of z with no common roots. The solution (10) has apparently $N_C = 0$ and, therefore, the energy is given by

$$E = \pi(|N_A| + |N_B|) = 2\pi N_A = -2\pi N_B, \quad (12)$$

where $N_A (= -N_B)$ is given the highest power of z in either $p(z)$ or $q(z)$.

The embedding solutions are unique possibility of solutions in the Eq. (9). This can be understood as follows. Eq. (9) requires that all the vectors are holomorphic or antiholomorphic. Let us consider two arbitrary (anti)holomorphic vectors. Then, the orthogonality of the two vectors forces one of them to be equivalent to constant vector up to the local degrees of freedom. In addition, thanks to the symmetries we can assume the boundary condition $U \rightarrow \text{diag}(1, 1, 1)$ at infinity on the plane without loss of generality. The boundary condition determines the constant vector as two of the components being zero and another 1. Finally, according to the orthonormal condition (2), one can determine the other vectors, and find the set of the vectors is equivalent to the embedding solutions. If one

wants to get a more general solution, obviously a different strategy has to be implemented. Our approach for the problem is to solve the Euler-Lagrange equation, without touching first-order equations. For the solutions, of course, the energy bound (6) is no longer saturated.

The notable thing is that the nonembedding solutions satisfy a new coupled first-order equation different from (9). As a result, we obtain a new energy lower bound which, of course, is different from (6). Note that the energy is higher than that of embedding solutions and the new energy bound contains a nontopological term. The detailed analysis will be thoroughly discussed in Sec. V. Existence of such solutions will give us deeper understanding of the BPS structure of $NL\sigma$ models whose target space is nonsymmetric space.

III. PRELIMINARIES

This section provides preliminaries for analyzing the Euler-Lagrange equations and some properties of the energy functional.

A. Parametrization

In order to simplify the analysis, we parametrize the vectors Z_a in terms of complex scalar fields [24]. This can be realized by means of the isomorphism,

$$SU(3)/U(1)^2 \cong SL(3, \mathbb{C})/B_+, \quad (13)$$

where B_+ is the Borel subgroup of the upper triangular matrices with the determinant being equal to one. In the geometrical point of view, the complex scalar fields mean the local coordinates of the manifold F_2 . The coordinates can be introduced though a mapping $SU(3)/U(1)^2 \rightarrow SL(3, \mathbb{C})/B_+$, which is an extension of the stereographic projection, i.e., $S^2 \rightarrow \mathbb{R}^2$. By a formal result of the map, one can write an element of F_2 in terms of local coordinates u_i , $i = 1, 2, 3$ as the lower triangular matrix:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & u_3 & 1 \end{pmatrix}. \quad (14)$$

Since $\det X = 1$, X is an element of $SL(3, \mathbb{C})$. Note that, however, it is not necessary for X to be a unitary matrix.

Next, in order to obtain the $SU(3)$ matrix U or the vectors Z_a parametrized in terms of the scalar fields, we consider the inverse of the mapping, i.e., $SL(3, \mathbb{C})/B_+ \rightarrow SU(3)/U(1)^2$. Then, one can construct U from X by means of the so-called Iwasawa decomposition: any element of $SL(3, \mathbb{C})$ may be factorized into an element of $SU(3)$ and B_+ in a unique fashion, up to the torus elements of $U(1)^2$. The procedure is as follows. We regard X as an element of $SL(3, \mathbb{C})$, which can be expressed in terms of the column vectors, such as

$$X = (c_1, c_2, c_3) \in SL(3, \mathbb{C}) \quad (15)$$

with

$$c_1 = (1, u_1, u_2)^T, \quad c_2 = (0, 1, u_3)^T, \quad c_3 = (0, 0, 1)^T. \quad (16)$$

In this case, the Iwasawa decomposition can be proved by the Gramm-Schmit orthogonalization process for (16). Then one obtains mutually orthogonal vectors,

$$\begin{aligned} e_A &= c_1, \\ e_B &= c_2 - \frac{(c_2, e_A)}{(e_A, e_A)} e_A, \\ e_C &= c_3 - \frac{(c_3, e_B)}{(e_B, e_B)} e_B - \frac{(c_3, e_A)}{(e_A, e_A)} e_A, \end{aligned} \quad (17)$$

where the inner product is defined as

$$(c_i, c_j) = c_j^\dagger c_i. \quad (18)$$

By normalizing the vectors (17), one obtains an orthonormal basis. Therefore, we can define $Z_a = \frac{1}{\sqrt{(e_a, e_a)}} e_a$, $a = A, B, C$, and they can explicitly be written as

$$\begin{aligned} Z_A &= \frac{1}{\sqrt{\Delta_1}} \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix}, \\ Z_B &= \frac{1}{\sqrt{\Delta_1 \Delta_2}} \begin{pmatrix} -u_1^* - u_2^* u_3 \\ 1 - u_1 u_2^* u_3 + |u_2|^2 \\ -u_1^* u_2 + u_3 + u_3 |u_1|^2 \end{pmatrix}, \\ Z_C &= \frac{1}{\sqrt{\Delta_2}} \begin{pmatrix} u_1^* u_3^* - u_2^* \\ -u_3^* \\ 1 \end{pmatrix} \end{aligned} \quad (19)$$

where

$$\begin{aligned} \Delta_1 &= 1 + |u_1|^2 + |u_2|^2, \\ \Delta_2 &= 1 + |u_3|^2 + |u_1 u_3 - u_2|^2. \end{aligned} \quad (20)$$

For later convenience, we express the off-diagonal components of the one-form $Z_a^\dagger dZ_b$ as

$$\begin{aligned} Z_B^\dagger dZ_A &= \frac{1}{\Delta_1 \sqrt{\Delta_2}} (P_{11} du_1 + P_{12} du_2 + P_{13} du_3), \\ Z_C^\dagger dZ_A &= \frac{1}{\sqrt{\Delta_1 \Delta_2}} (P_{21} du_1 + P_{22} du_2 + P_{23} du_3), \\ Z_C^\dagger dZ_B &= \frac{1}{\sqrt{\Delta_1 \Delta_2}} (P_{31} du_1 + P_{32} du_2 + P_{33} du_3) \end{aligned} \quad (21)$$

where

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} = \begin{pmatrix} 1 - u_1^* u_2 u_3^* + |u_2|^2 & -u_1 u_2^* + u_3^* + u_3^* |u_1|^2 & 0 \\ -u_3 & 1 & 0 \\ -u_3(u_1^* + u_2^* u_3) & u_1^* + u_2^* u_3 & -\Delta_1 \end{pmatrix}. \quad (22)$$

B. Topological properties

We consider static configurations in two space dimensions (x^1, x^2) and also with wave components traveling along the x^3 axis with the speed of light. The vectors (19) provide a mapping from $\mathbb{R}^2$ (or $\mathbb{R}^2 \times \mathbb{R}^2$) into F_2 . However, for finiteness of the energy, the vectors should go to a constant at infinity on the $x^1 x^2$ plane. Thus, the plane can be identified as S^2 . Since both the static and time-dependent configurations localize only on the plane, they are topologically classified into homotopy classes of a mapping $S^2 \rightarrow F_2$, i.e., $\pi_2(F_2)$. In addition, there exists a useful theorem (see, e.g., [19]); $\pi_2(G/H) = \pi_1(H)_G$, where $\pi_1(H)_G$ is the subset of $\pi_1(H)$ formed by closed paths in H which can be contracted to a point in G . Therefore, these configurations are characterized by the second homotopy group,

$$\pi_2(F_2) = \pi_2(SU(3)/U(1)^2) = \pi_1(U(1)^2) = \mathbb{Z} + \mathbb{Z}, \quad (23)$$

which guarantees that the solutions are classified by two independent integers. Note that there are three topological degrees defined in (7), but only two independent degrees due to the identity in (8).

We can also define a topological invariant by means of the Kähler structure of the flag space F_2 , though F_2 is not a Kähler manifold. Note that the nonsymmetric manifolds, unlike symmetric manifolds, possess a family of invariant metrics including both Kähler and non-Kähler metrics. In order to simplify the discussion, we consider the Kähler case. Then the corresponding Kähler potential can be defined as

$$K = K_1 + K_2, \quad (24)$$

where $K_j = m_j \log \Delta_j$, $j = 1, 2$, with m_j being a positive constant. Since the manifold is now the Kähler one, it possesses the closed two-form Ω , i.e., $d\Omega = 0$. Here, $d = \partial + \bar{\partial} = du_\alpha \frac{\partial}{\partial u_\alpha} + du_\beta^* \frac{\partial}{\partial u_\beta^*}$ denotes the exterior derivative, while the operators ∂ and $\bar{\partial}$ are called the Dolbeault operators. The condition $d\Omega = 0$ is equivalent to

$$\Omega = i\partial\bar{\partial}K. \quad (25)$$

If one divides the two-form into two parts $\Omega_j = i\partial\bar{\partial}K_j$, they are given by

$$\Omega_1 = \frac{im_1}{\Delta_1^2} \{ (1 + |u_2|^2) du_1 \wedge du_1^* - u_2 u_1^* du_1 \wedge du_2^* - u_1 u_2^* du_2 \wedge du_1^* + (1 + |u_1|^2) du_2 \wedge du_2^* \}, \quad (26)$$

$$\begin{aligned} \Omega_2 = \frac{im_2}{\Delta_2^2} \{ & (1 + |u_3|^2) d(u_1 u_3 - u_2) \wedge d(u_1^* u_3^* - u_2^*) \\ & - u_3(u_1^* u_3^* - u_2^*) d(u_1 u_3 - u_2) \wedge du_3^* \\ & - u_3^*(u_1 u_3 - u_2) du_3 \wedge d(u_1^* u_3^* - u_2^*) \\ & + (1 + |u_1 u_3 - u_2|^2) du_3 \wedge du_3^* \}. \end{aligned} \quad (27)$$

The topological invariant Q is defined in terms of the integral of the Kähler two-form:

$$Q \equiv Q_1 + Q_2, \quad Q_j = \int \Omega_j. \quad (28)$$

The relation of the topological invariant and the winding numbers is directly calculable by the integration

$$Q_1 = -im_1 \int d^2 x \varepsilon^{ij} (D_i^{(A)} Z_A)^\dagger (D_j^{(A)} Z_A) = 2\pi m_1 N_A, \quad (29)$$

$$Q_2 = im_2 \int d^2 x \varepsilon^{ij} (D_i^{(C)} Z_C)^\dagger (D_j^{(C)} Z_C) = -2\pi m_2 N_C. \quad (30)$$

Consequently, after an appropriate normalization, i.e., $m_1 = m_2 = \frac{1}{2\pi}$, the topological invariant is characterized by just two integers, N_A and N_C , as

$$Q = N_A - N_C, \quad (31)$$

which, of course, is consistent with the formal argument (23).

There are exact expressions of the topological invariants corresponding to (23). However, the field configurations in this model are characterized by two integers, and it is not so straightforward to find the proper form of the topological charge interpreted as the net number of solitons. The definition (31) seems suitable for the net number of solitons for the parametrization (19).

As we shall see in Sec. V, the static energy of the solutions is proportional to (31) like standard BPS-type solutions and then we call (31) the topological charge.

IV. SOLUTIONS OF THE EULER-LAGRANGE EQUATION

For finding genuine solutions, we solve the Euler-Lagrange equation of the Lagrangian (1), instead of first-order coupled equations. The Euler-Lagrange equation is given by

$$\partial^\mu J_\mu = 0, \quad J_\mu := U \mathcal{B}_\mu U^\dagger, \quad (32) \quad \mathcal{A}_\mu = U^\dagger \partial_\mu U. \quad (35)$$

with

$$\mathcal{B}_\mu = \begin{pmatrix} 0 & Z_A^\dagger \partial_\mu Z_B & Z_A^\dagger \partial_\mu Z_C \\ Z_B^\dagger \partial_\mu Z_A & 0 & Z_B^\dagger \partial_\mu Z_C \\ Z_C^\dagger \partial_\mu Z_A & Z_C^\dagger \partial_\mu Z_B & 0 \end{pmatrix}. \quad (33)$$

Here J_μ is the Noether current associated to the global $SU(3)$ symmetry. Equation (32) can be rewritten as

$$\partial^\mu \mathcal{B}_\mu + [\mathcal{A}^\mu, \mathcal{B}_\mu] = 0, \quad (34)$$

where $\mathcal{A}_\mu$ is a flat connection defined by

From (34) with (33), we obtain the coupled equations,

$$\begin{aligned} \partial^\mu (Z_B^\dagger \partial_\mu Z_A) + (Z_B^\dagger \partial^\mu Z_B - Z_A^\dagger \partial^\mu Z_A) Z_B^\dagger \partial_\mu Z_A &= 0, \\ \partial^\mu (Z_C^\dagger \partial_\mu Z_A) + (Z_C^\dagger \partial^\mu Z_C - Z_A^\dagger \partial^\mu Z_A) Z_C^\dagger \partial_\mu Z_A &= 0, \\ \partial^\mu (Z_C^\dagger \partial_\mu Z_B) + (Z_C^\dagger \partial^\mu Z_C - Z_B^\dagger \partial^\mu Z_B) Z_C^\dagger \partial_\mu Z_B &= 0, \end{aligned} \quad (36)$$

and also their complex conjugations. Plugging the explicit form of the vectors (19), (20) and the one-form (21) into the equations, we finally obtain the equations for the scalar fields,

$$\begin{aligned} \Delta_1 \Delta_2 (P_{11} \partial^\mu \partial_\mu u_1 + P_{12} \partial^\mu \partial_\mu u_2) - 2 \Delta_2 (P_{11} \partial_\mu u_1 + P_{12} \partial_\mu u_2) (u_1^* \partial^\mu u_1 + u_2^* \partial^\mu u_2) \\ - \Delta_1 [(u_1 + u_2 u_3^*) (u_3^* \partial^\mu u_1^* - \partial^\mu u_2^*) + \Delta_1 \partial^\mu u_3^*] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0, \end{aligned} \quad (37)$$

$$\begin{aligned} \Delta_1 \Delta_2 (P_{21} \partial^\mu \partial_\mu u_1 + P_{22} \partial^\mu \partial_\mu u_2 - \partial^\mu u_3 \partial_\mu u_1) - (P_{21} \partial_\mu u_1 + P_{22} \partial_\mu u_2) [\Delta_2 \{u_1^* \partial^\mu u_1 + u_2^* \partial^\mu u_2\} \\ + \Delta_1 \{(u_1^* u_3^* - u_2^*) \partial^\mu (u_1 u_3 - u_2) + u_3^* \partial^\mu u_3\}] = 0, \end{aligned} \quad (38)$$

$$\begin{aligned} \Delta_1 \Delta_2 \{P_{31} \partial^\mu \partial_\mu u_1 + P_{32} \partial^\mu \partial_\mu u_2 + P_{33} \partial^\mu \partial_\mu u_3 - 2(u_1^* + u_2^* u_3) \partial_\mu u_1 \partial^\mu u_3\} \\ - 2 \Delta_1 (P_{31} \partial_\mu u_1 + P_{32} \partial_\mu u_2 + P_{33} \partial_\mu u_3) \{(u_1^* u_3^* - u_2^*) \partial^\mu (u_1 u_3 - u_2) + u_3^* \partial^\mu u_3\} \\ - \Delta_2 [(1 - u_1 u_2^* u_3 + |u_2|^2) \partial_\mu u_1^* + (-u_1^* u_2 + u_3 + u_3 |u_1|^2) \partial_\mu u_2] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0, \end{aligned} \quad (39)$$

where the explicit forms of Δ_1 , Δ_2 and P_{ij} , $j = 1, 2, 3$ are given in (20) and (22), respectively.

Since the equations are quite complicated and highly nonlinear, for the moment we introduce the simplified ansatz; all the complex scalars u_i depend only on z and y_+ ($y_\pm$ is defined as $y_\pm = x^0 \pm x^3$), i.e.,

$$u_i = u_i(z, y_+) \quad \text{and} \quad u_i^* = u_i^*(\bar{z}, y_+). \quad (40)$$

It is easy to see that the ansatz satisfies

$$\partial^\mu \partial_\mu u_i = 0, \quad \partial^\mu u_i \partial_\mu u_j = 0. \quad (41)$$

One can directly check that the second equation (38) is automatically satisfied by the ansatz (40). Equations (37) and (39) are certainly simplified:

$$[P_{32}^* (u_3^* \partial^\mu u_1^* - \partial^\mu u_2^*) + \Delta_1 \partial^\mu u_3^*] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0, \quad (42)$$

$$[P_{11}^* \partial^\mu u_1^* + P_{12}^* \partial^\mu u_2^*] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0. \quad (43)$$

Therefore, if we find the field configurations satisfying

$$u_3 \partial_\mu u_1 - \partial_\mu u_2 = 0, \quad (44)$$

they exactly solve (37)–(39).

First we consider the static solutions within the ansatz (40) which satisfy (44). Note that if one scalar field is set to a constant, there are only trivial and embedding solutions. In the intriguing case where all the scalars are not constants, the solutions can generally be written as

$$u_1 = \frac{p_1(z)}{q_1(z)}, \quad u_2 = \frac{p_2(z)}{q_2(z)}, \quad u_3 = \frac{\partial_z u_2}{\partial_z u_1}, \quad (45)$$

where $p_i(z)$ and $q_i(z)$ are irreducible polynomials of z and u_1 is not proportional to u_2 . The winding number of the scalar field u_i is equal to the number of the poles of the scalar field including those at infinity and, thereby,

$$\begin{aligned} n_i &= \max\{\deg(p_i), \deg(q_i)\}, \quad i = 1, 2 \\ n_3 &= n_2 - n_1. \end{aligned} \quad (46)$$

According to the derivation of the topological charge in the CP^N model [19], the topological degrees N_A and N_C are given by the pair of winding numbers n_i as

$$N_A = \max(0, n_1, n_2) - \min(0, n_1, n_2), \quad (47)$$

$$N_C = \max(-n_2, -n_3, 0) - \min(-n_2, -n_3, 0). \quad (48)$$

TABLE I. The values of (N_A, N_C) for the winding numbers n_1, n_2, n_3 . As was shown in Sec. III B, the topological charge is given by $Q = N_A - N_C$.

		(N_A, N_C)	
		$n_1 < 0$	$n_1 > 0$
$n_3 > 0$	$n_2 > 0$	i. $(n_3, -n_3)$	ii. $(n_2, -n_2)$
	$n_2 < 0$	iii. $(n_1, -n_1)$	iv. $(n_2, -n_2)$
$n_3 < 0$	$n_2 > 0$	v. $(n_1, -n_1)$	vi. $(n_2, -n_2)$
	$n_2 < 0$	vii. $(n_1, -n_1)$	viii. $(n_2, -n_2)$

As we summarize in Table I, the topological degrees (N_A, N_C) are, thus, defined by the pair of winding numbers with opposite signs.

The most simple choice of the static solution (45) seems

$$u_1 = c_1 z^{n_1}, \quad u_2 = c_2 z^{n_2}, \quad u_3 = \frac{c_2 n_2}{c_1 n_1} z^{n_2 - n_1}, \quad (49)$$

where c_i are nonzero complex constants. In Fig. 1, we plot the topological charge density of (49). In the case of $(n_1, n_2) = (1, 2)$, it is lump shaped with the peak located at the origin, while for higher winding numbers, the solutions always exhibit a craterlike structure. An anisotropic configuration can also be examined. The solution

$$u_1 = c_1 z, \quad u_2 = z^2 - 1, \quad u_3 = \frac{2}{c_1} z \quad (50)$$

exhibits the noncircular shape of the density (see Fig. 2). By increasing the parameter c_1 , the lumps collide and scatter with the right angle.

Following the prescription for construction of the CP^N vortices [25–27], we examine a class of time-dependent solutions that is made by a product of the localizing

component on the $x^1 x^2$ plane and the traveling wave being parallel to x^3 with the speed of light, i.e., an ansatz of the form $u_i = f_i(z) w_i(y_+)$, $i = 1, 2, 3$. Again, the nonembedding time-dependent solutions are not allowed when one or more of the functions f_i and w_i remain constant. When all f_i and w_i are the functions, by substituting the ansatz into (44), one easily can obtain the conditions

$$f_2 = \alpha f_1^C, \quad f_3 = \alpha A f_1^{C-1}, \quad (51)$$

$$w_2 = \beta w_1^C, \quad w_3 = \beta B w_1^{C-1}, \quad (52)$$

where α, β, A , and B are nonzero constants, and $C = A/B$. One of the solutions that keeps the single-valuedness and finiteness of the energy per unit length is given as

$$\begin{aligned} u_1 &= c_1 z^{n_1} e^{i n_1 k y_+}, \\ u_2 &= c_2 z^{n_2} e^{i n_2 k y_+}, \\ u_3 &= \frac{c_2 n_2}{c_1 n_1} z^{n_2 - n_1} e^{i (n_2 - n_1) k y_+}, \end{aligned} \quad (53)$$

where k is a constant being proportional to the inverse of a wavelength.

Note that under the constraint (44), there is a sure relation among the vectors Z_a ; $D_\mu^{(A)} Z_A$ is proportional to Z_B and, similarly, $D_\mu^{(B)} Z_B$ is to Z_C . Consequently, the vectors correspond to the tower of the general solution in the CP^2 model generated through the Bäcklund transformation [20,28]. It implies that for the solutions (45) and (53), Z_A can be recognized as an instanton, Z_C as an anti-instanton, and Z_B as an instanton–anti-instanton bound state. Similar types of solutions are already discussed in [22], but the model the author employed is different from ours because it contains a nonzero Kalb-Ramond field which contributes to the Euler-Lagrange equation. We have shown that the extra term has no effect, at least on the existence of the analytical solutions. Further, as far as we know, any time-dependent solutions of the F_2 model have not been discussed previously.

FIG. 1. The topological charge density for the solution $(u_1, u_2, u_3) = (z^{n_1}, z^{n_2}, \frac{n_2}{n_1} z^{n_2 - n_1})$ with the winding numbers $(n_1, n_2) = (1, 2), (1, 3), (2, 3)$, from left to right.

FIG. 2. The topological charge density for $(u_1, u_2, u_3) = (c_1 z, z^2 - 1, 2z/c_1)$ with the parameter $c_1 = 0.8, 1.0, 1.3, 1.5, 1.8, 2.5$, from upper left to lower right.

V. PROPERTY OF THE ENERGY

In this section, we discuss the energy of the static solutions (45) and also of the wave solution (53). We show that the static solutions saturate to an energy lower bound different from (6). It is proved that the solutions (45) are obtained through coupled first-derivative equations which correspond to the saturation condition. For the traveling wave solution (53), the energy can be written by the topological charge plus Noether charges which are relevant to some $U(1)$ symmetries.

Using the completeness condition (3), the static energy per unit length (5) can be written as

$$E = \int d^2x (\|D_i^{(A)} Z_A\|^2 + \|D_i^{(C)} Z_C\|^2 - \|Z_C^\dagger \partial_i Z_A\|^2). \quad (54)$$

We again emphasize that the first and second term in (54) are the static energy of the CP^2 model. Applying the BPS bound of the CP^2 model to the two terms, we obtain the energy bound

$$E = \frac{1}{2} \int d^2x \|D_i^{(A)} Z_A \mp i \varepsilon^{ij} D_j^{(A)} Z_A\|^2 + \frac{1}{2} \int d^2x \|D_i^{(C)} Z_C \mp i \varepsilon^{ij} D_j^{(C)} Z_C\|^2 \pm i \int d^2x \varepsilon^{ij} \{ (D_i^{(A)} Z_A)^\dagger D_j^{(A)} Z_A + (D_i^{(C)} Z_C)^\dagger D_j^{(C)} Z_C \} - \int d^2x \|Z_C^\dagger \partial_i Z_A\|^2 \geq 2\pi(|N_A| + |N_C|) - \int d^2x \|Z_C^\dagger \partial_i Z_A\|^2. \quad (55)$$

According to the discussion in the CP^2 model, the equality in (55) is satisfied when the scalar fields are functions of only z (or $\bar{z}$), i.e., $u_i = u_i(z)$. Thus, the static solutions (45) obviously saturate the new bound (55). Unlike the standard BPS bound, (55) has a nontopological term [the last term in (55)]. Since the maximum of the nontopological term (without minus sign) gives the equivalence between the two bounds (6) and (55), the value of the term runs

$$0 \leq \int d^2x \|Z_C^\dagger \partial_i Z_A\|^2 \leq \pi(|N_A| - |N_B| + |N_C|). \quad (56)$$

One can find from (21) that $Z_C^\dagger \partial_i Z_A$ is proportional to $u_3 \partial_i u_1 - \partial_i u_2$. Therefore, for the solutions of (44), this nontopological term identically vanishes [then takes the lower bound of the inequality of (56)]. Consequently, the static energy of the nonembedding solutions (45) is given by

$$E = 2\pi(|N_A| + |N_C|) = 2\pi Q. \quad (57)$$

On the other hand, for the embedding solutions where u_1 and u_3 are constants, the term satisfies the lower bound of

the inequality (56) and then corresponds to the energy bound (6) with $N_B = 0$. Thus, we find that the static energy is 2 times greater than that of the embedding solution that belongs to the same topological sector. This is consistent with the interpretation that the nonembedding solutions include an instanton–anti-instanton bound state which is described by the harmonic map solutions in the CP^2 model.

Note that, from the saturation conditions of the energy bound (55) and the lower inequality in (56), we obtain the following coupled first-order equations corresponding to the solutions (45),

$$\begin{aligned} D_i^{(A)} Z_A &= i\epsilon^{ij} D_j^{(A)} Z_A, \\ D_i^{(C)} Z_C &= -i\epsilon^{ij} D_j^{(C)} Z_C, \end{aligned} \quad (58)$$

and

$$Z_C^\dagger \partial_i Z_A = 0. \quad (59)$$

Equations (58) are equivalent to the Cauchy-Riemann equations for the complex scalar fields u_i , and (59) exactly is (44). This clearly means that the solutions (45) of the Euler-Lagrange equations (36) can be obtained by the first-order equations (58) and (59). Note that even when the saturation of the energy bound (55) is attained, the energy does not take a stationary point due to the presence of the nontopological term. According to the above discussion, however, if both (58) and (59) are satisfied, the energy is stationary, i.e., maximal energy in the bound (55). Therefore, we conclude that the solutions (45) that satisfy both saturation conditions are saddle point.

The solutions (45) possess basic properties as BPS solutions, such that they saturate an energy lower bound and are the solutions of first-order equations. They seem to be unstable, because they are not of the global minima in a given topological sector, which is fulfilled by the embedding solutions. It is worth to check behavior of the energy for changing the parameters of the configuration and

confirm the stability nature of our solutions. In Fig. 3, we plot the energy per unit length for the configuration

$$u_1 = \frac{1-a_1}{a_1} z^3, \quad u_2 = \frac{1-a_2}{a_2} z^4, \quad u_3 = \frac{1-a_3}{a_3} z, \quad (60)$$

with the parameters $a_i \in [0, 1]$. The nonembedding solutions locate on the ridge line of the surface, while the embedding solutions are on the left front corner. There is a zero mode along the ridge, but it is essentially unstable. A natural question occurs where the nonembedding solutions might eventually decay into the embedding ones. In order to see this in detail, we plot the cuts of the energy surface at several values of a_1 for fixed $a_2 = 0.5$ in Fig. 4. For the configuration (60), unless $u_1 u_3 = u_2$ or $u_2 = 0$, at infinities of the $x^1 x^2$ plane, the fields Z_a approach the value

$$Z_A \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Z_B \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z_C \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (61)$$

up to phase factors. The configuration (60) with $u_1 u_3 = u_2$ or $u_2 = 0$ does not support the boundary condition (61), so it belongs to a different topological sector from the nonembedding and embedding solutions. It clearly indicates that the solutions cannot continuously deform into the configuration which satisfies $u_1 u_3 = u_2$ or $u_2 = 0$ while keeping the finiteness of the energy. One can find that the topological barrier associated with $u_1 u_3 = u_2$ lies between the nonembedding and the embedding solutions. It is a common feature among all topological classifications in Table I that such a barrier lies between the nonembedding and embedding solutions. For configurations corresponding to the domain *i.* or *ii.* in the Table, as in the above example, the barrier exists on the line corresponding to $u_1 u_3 = u_2$, which does not obey the boundary condition of the solutions at infinity. For the domain *iv.* or *v.*, the condition $u_1 u_3 = u_2$ does not satisfy the boundary condition of solutions at the origin. In this case, one can also

FIG. 3. Energy surface for $u_1 = \frac{1-a_1}{a_1} z^3$, $u_2 = \frac{1-a_2}{a_2} z^4$, $u_3 = \frac{1-a_3}{a_3} z$. From left to right $a_3 = 0.5$, $a_2 = 0.5$, $a_1 = 0.5$

FIG. 4. The cuts of the total energy (per 2π) corresponding to Fig. 3 with $a_2 = 0.5$. The red lines denote a barrier satisfying $u_1 u_3 = u_2$ which is prohibited by the topological reason. There are always dips on the barrier and maxima near the line, but neither of them are stationary points.

easily prove that the condition $u_1 u_3 = u_2$ lies between the nonembedding and embedding solutions. The situation for the domain *iii.* or *vi.* is a little more complicated than for the others. Nonembedding solutions in the domain exist in region $c_1 c_2 c_3 < 0$ and, on the other hand, embedding solutions are derived as $c_1, c_2 \rightarrow \infty$ while keeping the condition $c_1 = c_2 c_3$, which gives $c_1 c_2 c_3 > 0$. Therefore, during deformation from a nonembedding solution to an embedding solution, at least one of the coefficients have to change sign. At the point $c_i = 0, i = 1, 2, 3$, the boundary condition of the solutions is broken and, therefore, the barrier appears between the solutions. This observation indicates that the nonembedding solution cannot decay into the embeddings in the classical level, at least, through a deformation described by variations of the parameter in (60). However, quantum mechanically, the tunneling between these solutions may occur, making the decay process possible. Some readers may expect the other possibility. One can find the lower energy region behind the ridges in Fig. 4, but it is a fake effect from fixing one of the parameters that the ridges divide the region from an embedding solution. If we vary all three parameters, all configurations may finally decay into the embedding solutions through the tunneling effect.

Next, we consider the energy of the traveling wave solution (53). From a simple calculation, one can get the energy as

$$E_{\text{wave}} = 2\pi Q + 8\pi k^2 \left\{ \mathcal{I}(n_1, n_2, c_1, c_2) + \mathcal{I}\left(n_2 - n_1, n_2, \frac{c_2 n_2}{c_1 n_1}, \frac{c_2(n_2 - n_1)}{c_1 n_1}\right) \right\}, \quad (62)$$

with

$$\mathcal{I}(n, m, a, b) = \int_0^\infty r dr \frac{n^2 |ar^n|^2 + m^2 |br^m|^2 + (n-m)^2 |abr^{n+m}|^2}{(1 + |ar^n|^2 + |br^m|^2)^2}, \quad (63)$$

where $n \neq m$ and $a, b \neq 0$. By counting the powers of r , one finds the integral (63) converges if

$$\begin{aligned} n > 1 + m > 1, & \quad n > 1, m < 0, & \quad -1 > n > m, \\ m > 1 + n > 1, & \quad m > 1, n < 0, & \quad -1 > m > n. \end{aligned}$$

We represent combinations of winding numbers for (in) finite energy solutions (62) in Fig. 5.

The energy (62) is associated with Noether charges. The model (1) possesses two $U(1)$ symmetries corresponding to the transformations,

$$(u_1, u_2, u_3) \rightarrow (e^{i\alpha_1} u_1, u_2, e^{-i\alpha_1} u_3), \quad (64)$$

$$(u_1, u_2, u_3) \rightarrow (u_1, e^{i\alpha_2} u_2, e^{i\alpha_2} u_3), \quad (65)$$

where $\alpha_i, i = 1, 2$ are constant parameters. The corresponding Noether currents are given by

FIG. 5. Combination of winding numbers corresponding to the solutions in the form (53), where the infinite energy (per unit length) solutions are represented by red squares, and black dots correspond to finite energy solutions. Points with no genuine solutions are shown by cross marks.

$$\begin{aligned} \mathcal{J}_{(1)}^\mu = & \frac{4i}{\Delta_1^2} [(1 + |u_2|^2)u_1 \partial^\mu u_1^* - |u_1|^2 u_2 \partial^\mu u_2^*] \\ & - \frac{4i}{\Delta_2^2} [(1 + |u_1 u_3 - u_2|^2)u_3 \partial^\mu u_3^* \\ & - |u_3|^2 (u_1 u_3 - u_2) \partial^\mu (u_1^* u_3^* - u_2^*)] \\ & - \frac{i}{\Delta_1 \Delta_2} u_1 u_3 (u_3^* \partial^\mu u_1^* - \partial^\mu u_2^*) - \{u_i \leftrightarrow u_i^*\}, \end{aligned} \quad (66)$$

$$\begin{aligned} \mathcal{J}_{(2)}^\mu = & \frac{4i}{\Delta_1^2} [(1 + |u_1|^2)u_2 \partial^\mu u_2^* - |u_2|^2 u_1 \partial^\mu u_1^*] \\ & + \frac{4i}{\Delta_2^2} [(1 + |u_3|^2)(u_1 u_3 - u_2) \partial^\mu (u_1^* u_3^* - u_2^*) \\ & - |u_1 u_3 - u_2|^2 u_3 \partial^\mu u_3^*] \\ & + \frac{i}{\Delta_1 \Delta_2} u_2 (u_3^* \partial^\mu u_1^* - \partial^\mu u_2^*) - \{u_i \leftrightarrow u_i^*\}. \end{aligned} \quad (67)$$

The Noether charges per unit length are obtained as

$$Q_{(i)} = \int dx^2 \mathcal{J}_{(i)}^0. \quad (68)$$

Then, for the time-dependent solutions (53), the energy (62) coincides with

$$E = 2\pi Q + k(n_1 Q_{(1)} + n_2 Q_{(2)}). \quad (69)$$

It is worth commenting on the scaling property, like Derrick's theorem, for the traveling wave solutions (53). Because the traveling wave solutions localize only on the $x_1 x_2$ plane, we consider the rescaling $x_i \rightarrow \lambda x_i$, $i = 1, 2$. Then, the energy contribution from the x_3 derivative term is

scaled like a potential term. On the other hand, as discussed in the Q -lump case [29], the contribution from the time-derivative term behaves like a quadratic term in x_i derivatives. The scaling property of the terms is opposite, but when $\lambda = 1$, the contributions from the x_3 derivative term and the time-derivative term are equal. Therefore, the traveling wave solutions satisfy the Derrick-type condition.

VI. SUMMARY AND DISCUSSION

In this paper, we have constructed both static and time-dependent solutions of codimension two in the F_2 NL σ model by analytically solving the second-order Euler-Lagrange equation. The static solutions possess novel features; the solutions saturate an energy lower bound and satisfy coupled first-order differential equations corresponding to the saturation condition, so, in this sense, they are BPS solutions. On the other hand, because a nontopological term is contained in the energy bound, they are saddle-point solutions, i.e., sphalerons. As far as we know, our solutions are a rare example of “BPS sphalerons.” The time-dependent solutions are given by a product of the localizing component on the x^1 - x^2 plane and the traveling wave being parallel to x^3 with the speed of light. The energy of the wave solutions is given by the topological charge and the Noether charges associated to two $U(1)$ transformations. Both the static and wave solutions consist of a triple in the tower of the CP^2 solutions generated through the Bäcklund transformations, i.e., a holomorphic, antiholomorphic, and harmonic map, and therefore can be recognized as a complex of the CP^2 instantons and anti-instantons. We are currently investigating the generalization to the F_N case. Moreover, the static solution may be able to generalize the F_2 model on the cylinder $\mathbb{R} \times S^1$ with twisted boundary conditions by means of a conformal mapping technique [30].

Since we have found several analytical solutions of the model, it is certainly worth examining in the context of integrable theories in higher dimensions based on discussion of the integrable submodels possessing an infinite number of local conservation laws [31,32] as a generalization of the previous work [23,33], which studied the integrability of two-dimensional NL σ models on homogeneous spaces.

The existence and stability of the solutions that we obtained implies that instanton–anti-instanton creation or annihilation may occur, through the quantum tunneling effect, in condensed matters described by the $SU(3)$ anti-ferromagnetic Heisenberg model, such as cold atoms in an optical lattice [34] and rotating dense quark matter [35]. In addition, we hope this work gives a hint for finding finite-energy nonembedding solutions of the $SU(3)$ pure Yang-Mills theory on the four-dimensional Euclidean space.

Apart from the BPS and the analytical study, it is also important to discuss solutions of the model that break the

scale invariance by a higher derivative order term and a potential term. Such “baby-Skyrmion” solutions may be more easy to find in several physical applications. We will report results of these issues in a forthcoming article.

ACKNOWLEDGMENTS

We thank the anonymous referee for his or her careful reading and useful comments. The authors would

like to thank Professor L. A. Ferreira for many helpful discussions and the kind hospitality at his institute where part of this work was done. We are very grateful to D. Bykov, S. Krusch, and R. Sasaki for careful reading of the manuscript and helpful comments. We are also very grateful to M. Nitta, T. Misumi, K. Toda, H. T. Ueda, and W. Zakrzewski for valuable comments and discussions.

-
- [1] L. D. Faddeev and A. J. Niemi, Partially Dual Variables in $SU(2)$ Yang-Mills Theory, *Phys. Rev. Lett.* **82**, 1624 (1999).
 - [2] H. Gies, Wilsonian effective action for $SU(2)$ Yang-Mills theory with Cho-Faddeev-Niemi-Shabanov decomposition, *Phys. Rev. D* **63**, 125023 (2001).
 - [3] F. D. M. Haldane, Nonlinear Field Theory of Large Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy Axis Neel state, *Phys. Rev. Lett.* **50**, 1153 (1983).
 - [4] A. M. Polyakov and A. A. Belavin, Metastable states of two-dimensional isotropic ferromagnets, *Pis'ma Zh. Eksp. Teor. Fiz.* **22**, 503 (1975) [*JETP Lett.* **22**, 245 (1975)].
 - [5] L. D. Faddeev and A. J. Niemi, Knots and particles, *Nature (London)* **387**, 58 (1997).
 - [6] K.-I. Kondo, T. Shinohara, and T. Murakami, Reformulating $SU(N)$ Yang-Mills theory based on change of variables, *Prog. Theor. Phys.* **120**, 1 (2008).
 - [7] I. Affleck, The quantum Hall effect, σ models at $\theta = \pi$ and quantum spin chains, *Nucl. Phys.* **B257**, 397 (1985).
 - [8] N. Read and S. Sachdev, Valence-Bond and Spin-Peierls Ground States of Low-Dimensional Quantum Antiferromagnets, *Phys. Rev. Lett.* **62**, 1694 (1989).
 - [9] M. Eto, E. Nakano, and M. Nitta, Effective world-sheet theory of color magnetic flux tubes in dense QCD, *Phys. Rev. D* **80**, 125011 (2009).
 - [10] M. Eto, Y. Hirono, M. Nitta, and S. Yasui, Vortices and other topological solitons in dense quark matter, *Prog. Theor. Exp. Phys.* (2014) 012D01.
 - [11] M. Hindmarsh, Existence and Stability of Semilocal Strings, *Phys. Rev. Lett.* **68**, 1263 (1992).
 - [12] J. Garaud, J. Carlström, E. Babaev, and M. Speight, Chiral CP^2 skyrmions in three-band superconductors, *Phys. Rev. B* **87**, 014507 (2013).
 - [13] D. Bykov, The worldsheet low-energy limit of the $AdS_4 \times CP^3$ superstring, *Nucl. Phys.* **B838**, 47 (2010).
 - [14] L. D. Faddeev and A. J. Niemi, Partial duality in $SU(N)$ Yang-Mills theory, *Phys. Lett. B* **449**, 214 (1999).
 - [15] D. Bykov, Haldane limits via Lagrangian embeddings, *Nucl. Phys.* **B855**, 100 (2012).
 - [16] M. Lajkó, K. Wamer, F. Mila, and I. Affleck, Generalization of the Haldane conjecture to $SU(3)$ chains, *Nucl. Phys.* **B924**, 508 (2017).
 - [17] J. N. Bernatska and P. I. Holod, Topological excitations in a two-dimensional spin system with high spin $s \geq 1$, *Theor. Math. Phys.* **160**, 878 (2009).
 - [18] A. Smerald and N. Shannon, Theory of spin excitations in a quantum spin-nematic state, *Phys. Rev. B* **88**, 184430 (2013).
 - [19] A. D’Adda, M. Luscher, and P. Di Vecchia, A $1/n$ expandable series of nonlinear sigma models with instantons, *Nucl. Phys.* **B146**, 63 (1978).
 - [20] A. M. Din and W. J. Zakrzewski, General classical solutions in the $CP^{(n-1)}$ model, *Nucl. Phys.* **B174**, 397 (1980).
 - [21] H. T. Ueda, Y. Akagi, and N. Shannon, Quantum solitons with emergent interactions in a model of cold atoms on the triangular lattice, *Phys. Rev. A* **93**, 021606 (2016).
 - [22] D. Bykov, Classical solutions of a flag manifold σ -model, *Nucl. Phys.* **B902**, 292 (2016).
 - [23] D. Bykov, Integrable properties of sigma-models with non-symmetric target spaces, *Nucl. Phys.* **B894**, 254 (2015).
 - [24] R. F. Picken, The Duistermaat-Heckman integration formula on flag manifolds, *J. Math. Phys.* **31**, 616 (1990).
 - [25] L. A. Ferreira, Exact vortex solutions in an extended Skyrme-Faddeev model, *J. High Energy Phys.* **05** (2009) 001.
 - [26] L. A. Ferreira and P. Klimas, Exact vortex solutions in a CP^N Skyrme-Faddeev type model, *J. High Energy Phys.* **10** (2010) 008.
 - [27] L. A. Ferreira, P. Klimas, and W. J. Zakrzewski, Some $(3+1)$ dimensional vortex solutions of the CP^N model, *Phys. Rev. D* **83**, 105018 (2011).
 - [28] L. A. Ferreira, P. Klimas, and W. J. Zakrzewski, Properties of some $(3+1)$ dimensional vortex solutions of the CP^N model, *Phys. Rev. D* **84**, 085022 (2011).
 - [29] R. S. Ward, Topological Q solitons, *J. Math. Phys.* **44**, 3555 (2003).
 - [30] S. Bolognesi and W. Zakrzewski, Clustering and decomposition for non-BPS solutions of the CP^{N-1} models, *Phys. Rev. D* **89**, 065013 (2014).
 - [31] O. Alvarez, L. A. Ferreira, and J. Sanchez Guillen, A new approach to integrable theories in any dimension, *Nucl. Phys.* **B529**, 689 (1998).

- [32] L. A. Ferreira and E. E. Leite, Integrable theories in any dimension and homogeneous spaces, *Nucl. Phys.* **B547**, 471 (1999).
- [33] D. Bykov, Complex structures and zero-curvature equations for σ -models, *Phys. Lett. B* **760**, 341 (2016).
- [34] C. Honerkamp and W. Hofstetter, Ultracold Fermions and the $SU(n)$ Hubbard Model, *Phys. Rev. Lett.* **92**, 170403 (2004).
- [35] M. Kobayashi, E. Nakano, and M. Nitta, Color magnetism in non-Abelian vortex matter, *J. High Energy Phys.* **06** (2014) 130.