HQET renormalization group improved Lagrangian at $\mathcal{O}(1/m^3)$ with leading logarithmic accuracy: Spin-dependent case

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(Received 25 February 2018; published 19 March 2018)

We obtain the renormalization group improved expressions of the Wilson coefficients associated to the $\mathcal{O}(1/m^3)$ spin-dependent heavy quark effective theory Lagrangian operators, with leading logarithmic approximation, in the case of zero light quarks. We have employed the Coulomb gauge.

DOI: [10.1103/PhysRevD.97.054018](https://doi.org/10.1103/PhysRevD.97.054018)

I. INTRODUCTION

The expansion in inverse powers of the heavy quark mass is a useful tool for the study of hadrons containing heavy quarks. This expansion can be formulated more systematically in terms of an effective theory described by an effective Lagrangian. For the one-heavy quark sector this theory is the heavy quark effective theory (HQET) [1]. The HQET Lagrangian is also a key object in the description of systems with more than one heavy quark, in particular in the description of the heavy quark-antiquark sector (i.e. heavy quarkonium), since the HQET Lagrangian corresponds to one of the building blocks of the nonrelativistic QCD (NRQCD) Lagrangian [2,3]. The Wilson coefficients of the HQET Lagrangian also enter into the Wilson coefficients of the operators (i.e. the potentials) of the pNRQCD Lagrangian [4,5], an effective field theory optimized for the description of heavy quarkonium systems near the threshold (for reviews see Refs. [6,7]). The Wilson coefficients we compute in this paper are necessary ingredients to obtain the pNRQCD Lagrangian with next-to-next-to-next-to-leading logarithmic (NNNLL) accuracy. It should be noted that the Wilson coefficients computed in this paper are not necessary to obtain the complete heavy quarkonium spectrum with NNNLL accuracy, nor the production and annihilation of heavy

quarkonium with NNLL precision, unlike their spin-independent counterparts (see Refs. [8,9]), but they may become important at the next order. At least, they must be studied. These results are also instrumental in the determination of higher order logarithms for NRQED bound states, such as hydrogen and muonic hydrogen-like atoms.

At present, the operator structure of the HQET Lagrangian, and the tree-level values of their Wilson coefficients, is known to $\mathcal{O}(1/m^3)$ in the case without massless quarks [10]. The inclusion of massless quarks has been considered in Ref. [11]. The Wilson coefficients with leading logarithmic (LL) accuracy were computed in Refs. [12–14] to $\mathcal{O}(1/m^2)$ and at next-to-leading order (NLO) in Ref. [10] to $\mathcal{O}(1/m^2)$ (without dimension 6 heavy-light operators). The LL running to $\mathcal{O}(1/m^3)$ without the inclusion of spectator quarks was considered in Refs. [15,16], which turned out to have internal discrepancies between their explicit single log results and their own anomalous dimension matrix. The computation was reconsidered in Ref. [17], where the spin-independent results were corrected. This work is a follow-up to Ref. [17] and, for this reason, is structured very similarly. Here we focus on spin-dependent operators and obtain the renormalization group improved Wilson coefficients of the HQET Lagrangian with LL accuracy to $\mathcal{O}(1/m^3)$. We do not include light quarks in our analysis.

The paper is divided as follows. In Sec. II we introduce the HQET Lagrangian. Section III is dedicated to the study of the spin-dependent part of Compton scattering, performed in order to find physical combinations of Wilson coefficients. In Sec. IV we find the renormalization group equations (RGE) for these Wilson coefficients. The QCD case is considered in Sec. IVA, and the QED case in Sec. IVB. The solution of these equations is

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studied in Sec. V. In Sec. VI we perform a detailed comparison between our results and the ones found in Refs. [15,16]. Our conclusions are summarized in Sec. VII. Finally, in Appendix A we present some new Feynman rules needed for the computation.

II. HQET LAGRANGIAN WITHOUT LIGHT FERMIONS

The HQET Lagrangian is defined uniquely up to field redefinitions. In this paper we use the HQET Lagrangian

density for a quark of mass $m \gg \Lambda_{\text{QCD}}$, in the special frame $v = (1, 0, 0, 0)$ given in Ref. [10],

$$\mathcal{L}_{\text{HQET}} = \mathcal{L}_g + \mathcal{L}_Q, \quad (1)$$

$$\mathcal{L}_g = -\frac{1}{4} G^{\mu\nu} G_{\mu\nu}^a + c_1^g \frac{g}{4m^2} f^{abc} G_{\mu\nu}^a G^{\mu b}{}_\alpha G^{\nu\alpha c} + \mathcal{O}\left(\frac{1}{m^4}\right), \quad (2)$$

$$\begin{aligned} \mathcal{L}_Q = Q^\dagger \bigg\{ & iD_0 + \frac{c_k}{2m} \mathbf{D}^2 + \frac{c_F}{2m} \boldsymbol{\sigma} \cdot g\mathbf{B} \\ & + \frac{c_D}{8m^2} (\mathbf{D} \cdot g\mathbf{E} - g\mathbf{E} \cdot \mathbf{D}) + i \frac{c_S}{8m^2} \boldsymbol{\sigma} \cdot (\mathbf{D} \times g\mathbf{E} - g\mathbf{E} \times \mathbf{D}) \\ & + \frac{c_4}{8m^3} \mathbf{D}^4 + ic_M g \frac{\mathbf{D} \cdot [\mathbf{D} \times \mathbf{B}] + [\mathbf{D} \times \mathbf{B}] \cdot \mathbf{D}}{8m^3} + c_{A_1} g^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8m^3} - c_{A_2} \frac{g^2 \mathbf{E}^2}{16m^3} \\ & + c_{W_1} g \frac{\{\mathbf{D}^2, \boldsymbol{\sigma} \cdot \mathbf{B}\}}{8m^3} - c_{W_2} g \frac{\mathbf{D}^i \boldsymbol{\sigma} \cdot \mathbf{B} \mathbf{D}^i}{4m^3} + c_{p'p} g \frac{\boldsymbol{\sigma} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \mathbf{D}}{8m^3} \\ & + c_{A_3} g^2 \frac{1}{N_c} \text{Tr} \left(\frac{\mathbf{B}^2 - \mathbf{E}^2}{8m^3} \right) - c_{A_4} g^2 \frac{1}{N_c} \text{Tr} \left(\frac{\mathbf{E}^2}{16m^3} \right) \\ & + ic_{B_1} g^2 \frac{\boldsymbol{\sigma} \cdot (\mathbf{B} \times \mathbf{B} - \mathbf{E} \times \mathbf{E})}{8m^3} - ic_{B_2} g^2 \frac{\boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{E})}{8m^3} \bigg\} Q + \mathcal{O}\left(\frac{1}{m^4}\right), \quad (3) \end{aligned}$$

where Q is the NR fermion field represented by a Pauli spinor. The components of the vector $\boldsymbol{\sigma}$ are the Pauli matrices. We define $iD^0 = i\partial^0 - gA^0$, $i\mathbf{D} = i\nabla + g\mathbf{A}$, $\mathbf{E}^i = G^{i0}$ and $\mathbf{B}^i = -\epsilon^{ijk} G^{jk}/2$, where ϵ^{ijk} is the three-dimensional totally antisymmetric tensor.¹ with $\epsilon^{123} = 1$ and $(\mathbf{a} \times \mathbf{b})^i \equiv \epsilon^{ijk} \mathbf{a}^j \mathbf{b}^k$. Note also that we have rescaled by a factor $1/N_c$ the coefficients $c_{A_{3,4}}$ following Ref. [17], as compared to the definitions given in Ref. [10].

III. COMPTON SCATTERING

Reference [17] showed that it is possible for the Wilson coefficients associated to $1/m^3$ operators to be gauge dependent. For example, this is the case for c_{A_2} , which always appears in physical observables along with c_M (well-known to be gauge dependent because it is related with c_D through a reparametrization invariance, Ref. [10]) in such a way that the combination is gauge independent/physical. In order to explore the existence of other physical combinations involving the Wilson coefficients that we aim to calculate, i.e. c_{W_1} ,

c_{W_2} , $c_{p'p}$, c_{B_1} and c_{B_2} , we compute the amplitude for Compton scattering of a heavy quark with a gluon $Qg \rightarrow Qg$. In this section, we restrict to the spin-dependent part of this process in HQET. We compute it at tree level up to $\mathcal{O}(1/m^3)$ in the mass expansion and in the Coulomb gauge (though obviously the amplitude for Compton scattering is a gauge independent quantity). We take incoming and outgoing quarks to have four-momentum $p = (E_1, \mathbf{p})$ and $p' = (E'_1, \mathbf{p}')$. We take gluon four-momenta as outgoing and label them by k_1, i, a and k_2, j, b with respect to color and vector indices. This also implies the on shell condition $k_1^0 = -|\mathbf{k}_1|$ and $k_2^0 = |\mathbf{k}_2|$. We work in the incoming quark rest frame, i.e. $E_1 = 0$ and $\mathbf{p} = \mathbf{0}$, so $\mathbf{p}' = -(\mathbf{k}_1 + \mathbf{k}_2)$ and $E'_1 = -(k_1^0 + k_2^0)$. In addition, we define the unit vectors $\mathbf{n}_1 = \mathbf{k}_1/|\mathbf{k}_1|$ and $\mathbf{n}_2 = \mathbf{k}_2/|\mathbf{k}_2|$. The relation

$$|\mathbf{k}_2| = \frac{|\mathbf{k}_1|}{1 + \frac{|\mathbf{k}_1|}{m} (1 + \mathbf{n}_1 \cdot \mathbf{n}_2)} \quad (4)$$

holds from four-momenta conservation.

By inserting the appropriate Wilson coefficients up to $\mathcal{O}(1/m^3)$, the topologies of the diagrams we have to consider for this computation are listed in Fig. 1. The amplitude reads

¹In dimensional regularization several prescriptions are possible for the ϵ^{ijk} tensors and $\boldsymbol{\sigma}$, and the same prescription as for the calculation of the Wilson coefficients must be used.

FIG. 1. Topologies contributing to Compton scattering at tree level up to $\mathcal{O}(1/m^3)$. Diagrams are generated from these topologies by considering all possible vertices contributing up to $\mathcal{O}(1/m^3)$.

$$\begin{aligned}
\mathcal{A}^{ijab} = & (c_{B_1} - 2c_{W_1} - c_F^2 c_k - c_S c_k) \frac{g^2}{16m^3} |\mathbf{k}_1|^2 ((\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_2^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} \\
& + \boldsymbol{\sigma}^i (\mathbf{n}_1 \times \mathbf{n}_2)^j + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i) [T^a, T^b]_{\alpha\beta} \\
& - (2c_{W_1} - 2c_{W_2} + 2c_F c_k^2 + c_S c_k + c_S c_F) \frac{g^2}{16m^3} |\mathbf{k}_1|^2 ((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j - (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) [T^a, T^b]_{\alpha\beta} \\
& - (2c_{W_1} - 2c_{W_2} - c_S c_F + c_S c_k - 2c_F c_k^2) \frac{g^2}{16m^3} |\mathbf{k}_1|^2 ((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j + (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) \{T^a, T^b\}_{\alpha\beta} \\
& + (c_{B_2} + c_{B_1} - 2c_{W_1} - c_S c_F - c_S c_k) \frac{g^2}{8m^3} |\mathbf{k}_1|^2 \boldsymbol{\sigma}^k \epsilon^{ijk} [T^a, T^b]_{\alpha\beta} \\
& - c_{p'p} \frac{g^2}{16m^3} |\mathbf{k}_1|^2 [((\mathbf{n}_1 \times \mathbf{n}_2)^j \boldsymbol{\sigma}^i - (\mathbf{n}_1 \times \mathbf{n}_2)^i \boldsymbol{\sigma}^j + \epsilon^{ijk} (\mathbf{n}_1 - \mathbf{n}_2)^k (\boldsymbol{\sigma} \cdot (\mathbf{n}_1 + \mathbf{n}_2))) \{T^a, T^b\}_{\alpha\beta} \\
& - ((\mathbf{n}_1 \times \mathbf{n}_2)^j \boldsymbol{\sigma}^i + (\mathbf{n}_1 \times \mathbf{n}_2)^i \boldsymbol{\sigma}^j - \epsilon^{ijk} (\mathbf{n}_1 + \mathbf{n}_2)^k (\boldsymbol{\sigma} \cdot (\mathbf{n}_1 + \mathbf{n}_2))) [T^a, T^b]_{\alpha\beta}] \\
& + c_S c_k \frac{g^2}{8m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) \boldsymbol{\sigma}^k \epsilon^{ijk} [T^a, T^b]_{\alpha\beta} \\
& + c_F^2 \frac{g^2}{8m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) ((\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i) [T^a, T^b]_{\alpha\beta} \\
& - c_F c_k \frac{g^2}{4m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i [T^a, T^b]_{\alpha\beta} \\
& - \frac{g^2}{8m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) [c_S ((\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk}) + c_F^2 ((\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i) \\
& - 2c_F c_k (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i] \{T^a, T^b\}_{\alpha\beta} \\
& + \frac{g^2}{8m^2} |\mathbf{k}_1| [(2c_F c_k - c_S c_k) ((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j - (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) + c_S c_k ((\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_1^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk}) \\
& + c_F^2 ((\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_2^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^i (\mathbf{n}_1 \times \mathbf{n}_2)^j + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i)] \{T^a, T^b\}_{\alpha\beta} \\
& + c_F c_k \frac{g^2}{4m^2} |\mathbf{k}_1| ((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j + (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) [T^a, T^b]_{\alpha\beta} \\
& + c_F \frac{g^2}{2m} \boldsymbol{\sigma}^k \epsilon^{ijk} [T^a, T^b]_{\alpha\beta} \\
& + c_F \frac{g^2}{4m} \frac{1}{1 + \mathbf{n}_1 \cdot \mathbf{n}_2} ((\boldsymbol{\sigma} \times \mathbf{n}_2)^i \mathbf{n}_1^j - (\boldsymbol{\sigma} \times \mathbf{n}_1)^j \mathbf{n}_2^i + \boldsymbol{\sigma}^i (\mathbf{n}_1 \times \mathbf{n}_2)^j + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i \\
& + 2(\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j - 2(\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) [T^a, T^b]_{\alpha\beta}.
\end{aligned} \tag{5}$$

Note that c_D does not appear explicitly. One can also observe that two combinations always appear in the observable: $\bar{c}_W \equiv c_{W_1} - c_{W_2}$ and $\bar{c}_{B_1} \equiv c_{B_1} - 2c_{W_1}$. These, together with c_{B_2} and $c_{p'p}$, are physical combinations, i.e. they are gauge independent. This implies that the renormalization group equations (RGE) of these physical combinations can only depend on physical combinations of Wilson coefficients. Later on we will see that this is indeed the case. We suspect that

individually c_{W_1} , c_{W_2} and c_{B_1} are gauge dependent quantities, since we are in agreement with Ref. [15], where the calculation was done in Feynman gauge, at the level of single logs for physical combinations but we disagree for each of these three individually.

For QED we obtain

$$\begin{aligned}
\mathcal{A}^{ij} = & \frac{g^2}{4m^2} |\mathbf{k}_1| [(2c_F c_k - c_S)((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j - (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) + c_S((\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_1^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk}) \\
& + c_F^2((\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_2^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^i (\mathbf{n}_1 \times \mathbf{n}_2)^j + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i)] \\
& - (2c_{W_1} - 2c_{W_2} - c_S c_F + c_S c_k - 2c_F c_k^2) \frac{g^2}{8m^3} |\mathbf{k}_1|^2 ((\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j + (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i) \\
& - c_{p'p} \frac{g^2}{8m^3} |\mathbf{k}_1|^2 ((\mathbf{n}_1 \times \mathbf{n}_2)^j \boldsymbol{\sigma}^i - (\mathbf{n}_1 \times \mathbf{n}_2)^i \boldsymbol{\sigma}^j + \epsilon^{ijk} (\mathbf{n}_1 - \mathbf{n}_2)^k (\boldsymbol{\sigma} \cdot (\mathbf{n}_1 + \mathbf{n}_2))) \\
& - \frac{g^2}{4m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) [c_S((\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk}) + c_F^2((\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i) \\
& - 2c_F c_k (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i].
\end{aligned} \tag{6}$$

Note that there is no $\mathcal{O}(1/m^0, 1/m)$ contribution. Setting the Wilson coefficients to their tree level values we obtain

$$\begin{aligned}
\mathcal{A}^{ij} = & \frac{g^2}{4m^2} |\mathbf{k}_1| [(\boldsymbol{\sigma} \times \mathbf{n}_1)^i \mathbf{n}_1^j - (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i + (\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_1^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk} \\
& + (\boldsymbol{\sigma} \cdot \mathbf{n}_1) \mathbf{n}_2^k \epsilon^{ijk} + (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + \boldsymbol{\sigma}^i (\mathbf{n}_1 \times \mathbf{n}_2)^j + \boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i] \\
& + \frac{g^2}{4m^3} |\mathbf{k}_1|^2 (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) (-\boldsymbol{\sigma}^j (\mathbf{n}_1 \times \mathbf{n}_2)^i - (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_1^k \epsilon^{ijk} + (\boldsymbol{\sigma} \times \mathbf{n}_2)^j \mathbf{n}_2^i - (\boldsymbol{\sigma} \cdot \mathbf{n}_2) \mathbf{n}_2^k \epsilon^{ijk}).
\end{aligned} \tag{7}$$

This expression agrees with Eq. (19) in Ref. [18].

The above analysis gives us the set of Wilson coefficients and their combinations that appear in physical observables: $\{\bar{c}_W, \bar{c}_{B_1}, c_{B_2}, c_{p'p}\}$. We compute the anomalous dimensions for these, but also for the unphysical set: $\{c_{W_1}, c_{W_2}, c_{B_1}, c_{B_2}, c_{p'p}\}$, in Coulomb gauge, as it can be important for future research investigating the possible gauge dependence of these Wilson coefficients.

IV. ANOMALOUS DIMENSIONS FOR $1/m^3$ SPIN-DEPENDENT OPERATORS

In this section we determine the anomalous dimensions of the Wilson coefficients associated to $1/m^3$ spin-dependent operators with $\mathcal{O}(\alpha)$ accuracy. In principle, one would like to only compute irreducible diagrams. However, as indicated in Ref. [17], this would involve considering a more extensive basis of operators, including those that vanish on shell. Instead, since we want to work in a minimal basis of operators, we will also need to consider reducible diagrams in a computation that resembles that of an S-matrix element. In particular, we will compute the divergent part of the amplitude for elastic scattering of the heavy quark with a transverse gluon at one-loop. These divergences cancel with the divergences of the Wilson coefficients determining the

anomalous dimension. The computation is organized in powers of $1/m$, up to $\mathcal{O}(1/m^3)$, by considering all possible insertions of the HQET Lagrangian operators. As a cross-check, we will also compute the elastic scattering of a heavy quark with a longitudinal gluon, which allows us to determine the anomalous dimension of the combination $c_{B_1} + c_{B_2}$. Furthermore, we compute the one transverse gluon-matrix element of the heavy quark, which allows us to cross-check the anomalous dimensions of c_{W_1} , c_{W_2} and $c_{p'p}$. In the latter only irreducible diagrams enter the calculation. Note that c_{B_1} has not been cross-checked in an independent calculation because one would need to consider irreducible and reducible diagrams with at least three external transverse gluons. Such a calculation is very hard and arguably not worth it because obtaining the correct structure of the c_{B_1} vertex is nontrivial enough to be considered as a strong cross-check. In general, external gluons and heavy quarks will be considered to be on shell i.e. free asymptotic states, so the free equations of motion (EOM) will be used throughout.

Note that we keep explicit the Wilson coefficients of the kinetic term for tracking purposes even though they are protected by reparametrization invariance ($c_k = c_4 = 1$ to any order in perturbation theory) [19]. Also, $c_{p'p} = c_F - 1$ and the physical combination $\bar{c}_W = c_{W_1} - c_{W_2} = 1$ are

fixed by a reparametrization invariance [10]. We will check by explicit calculation that these relations are satisfied at LL.

The Coulomb gauge will be used throughout this paper. On the one hand, this significantly reduces the number of diagrams but, on the other hand, the complexity of each one of them increases. It also makes it difficult to use standard

routines for computations of diagrams designed for Feynman gauges and relativistic setups. However, since we are only looking for the pole, the calculation is feasible. The normalization of the heavy quark field, gluon fields, the strong coupling g and the Wilson coefficients c_F and c_S are needed. In the Coulomb gauge, they read (we define $D = 4 + 2\epsilon$)

$$\begin{aligned} Z_{A^0}^{-1/2} &= Z_g = 1 + \frac{11}{6} C_A \frac{\alpha}{4\pi\epsilon} - \frac{2}{3} T_F n_f \frac{\alpha}{4\pi\epsilon}, & Z_A^{\frac{1}{2}} &= 1 - \frac{1}{2} C_A \frac{\alpha}{4\pi\epsilon} - \frac{2}{3} T_F n_f \frac{\alpha}{4\pi\epsilon}, \\ Z_g^2 Z_A &= 1 + \frac{8}{3} C_A \frac{\alpha}{4\pi\epsilon}, & Z_l &= 1 + C_F \frac{\alpha}{4\pi\epsilon}, & Z_h &= 1 + \frac{\mathbf{p}^2}{m^2} \frac{4}{3} C_F \frac{\alpha}{4\pi\epsilon}, \\ c_{F,B} &= c_{F,R} - c_{F,R} C_A \frac{\alpha}{4\pi\epsilon}, & c_{S,B} &= c_{S,R} - 2c_{F,R} C_A \frac{\alpha}{4\pi\epsilon}, \end{aligned} \quad (8)$$

where

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \quad C_A = N_c = 3. \quad (9)$$

The subscript B stands for bare and R for renormalized quantities. Often the subscript R will be removed in the following when it is understood.

A. QCD

Let us consider the general case of QCD. First of all, for the pure gluonic sector, we have that c_1^g is NLL, so it can be neglected.

The running of the set: $\{c_{W_1}, c_{W_2}, c_{B_1}, c_{B_2}, c_{p'p}\}$ is determined from the topologies drawn in Fig. 2. From these, we generate all possible diagrams up to order $1/m^3$ by considering all possible vertices to the appropriate order in $1/m$ and/or kinetic insertions, which correspond to the expansion of the nonstatic heavy quark propagator. Note that diagrams of a lower order than $1/m^3$ must also be considered, at least those that depend on the energy, as the use of the heavy quark EOM, $E = c_k \frac{\mathbf{p}^2}{2m}$, adds extra powers of $1/m$. This generates around 200 diagrams (without taking into account permutations and crossing) in both cases: the elastic scattering with a transverse gluon and, similarly, with a longitudinal gluon.

In the case of scattering with a transverse gluon, for diagrams proportional to $1/m^3$ operators, only the irreducible ones need to be considered. Note that this is not true for the case of scattering with a longitudinal gluon because the Coulomb vertex does not add extra powers of $1/m$. When one considers diagrams proportional to iterations of $1/m^2$ and/or $1/m$ operators one also has to consider reducible diagrams in both cases. One has to keep in mind that Taylor expanding reducible diagrams in the energy can produce nonlocal terms which cancel at the end of the calculation and all divergences can be absorbed by local counterterms

that correspond to operators of the Lagrangian.² It is also worth mentioning that we find that the sum of all reducible diagrams whose subirreducible diagram is $1/m$ or below ($1/m^2$ or below in the case of the scattering with a longitudinal gluon) cancel with the renormalization of the tree level reducible diagrams. Therefore, nonlocal terms coming from expanding these diagrams in the energy vanish at all orders in the expansion.

Let us consider the calculation of the one transverse gluon exchange, which has a peculiarity which deserves a comment. This calculation allows us to determine the anomalous dimensions of c_{W_1} , c_{W_2} , $c_{p'p}$ and c_S . The necessary topologies to produce the diagrams are shown in Fig. 3. They produce around 50–60 diagrams without counting inverted ones. Note that we can only draw irreducible diagrams in this case. What is interesting in this calculation is that one obtains a structure which does not look like any structure of the $1/m^3$ operators, i.e. the $1/m^3$ vertices with a single transverse gluon. So at first sight, it would look like a problem, since the divergence proportional to this structure could not be absorbed by any operator in the theory (leading one to suspect that there might be operators missing). However, this is not the case. The explanation is the following: in principle, one would consider c_S as an $\mathcal{O}(1/m^2)$ operator. Nevertheless, the vertex with an external transverse gluon is proportional to k^0 , so it becomes $\mathcal{O}(1/m^3)$ after using the EOM. Therefore, in order to determine the running of c_S through the calculation of the one gluon exchange one must consider this operator as an $\mathcal{O}(1/m^3)$ operator. Only in this way is the correct running of c_S (expected from reparametrization invariance) obtained. So everything must be made physical, meaning put on shell, in order to arrive to proper results. Note that the running of c_S will appear also in the

²If we only compute irreducible diagrams we would need a larger number of operators, in particular those that vanish on shell.

FIG. 2. Topologies contributing to the anomalous dimensions of the Wilson coefficients associated to $1/m^3$ spin-dependent operators in QCD. The double-line represents the heavy fermion, whereas the curly line represents either a transverse or a longitudinal gluon. Both external gluons are transverse or longitudinal depending on the kind of scattering we are considering. All diagrams are generated from these topologies by considering all possible vertices up to $\mathcal{O}(1/m^3)$. Tree level diagrams should be understood to be multiplied by the Wilson coefficient, field and strong coupling counterterms.

determination of the running of Wilson coefficients at higher orders in $1/m$ if it is done through the calculation of the one transverse gluon matrix element of a heavy quark, because the EOM have corrections in $1/m$. In particular, it

will appear at $\mathcal{O}(1/m^5)$. This is important to keep in mind in future calculations.

The renormalization group equations for the unphysical set $\{c_{W_1}, c_{W_2}, c_{B_1}, c_{B_2}, c_{p'p}\}$ in Coulomb gauge read

$$\begin{aligned}
 \nu \frac{d}{d\nu} c_{W_1} = & \frac{\alpha}{\pi} \left(\frac{13}{12} c_{W_1} C_A + \frac{7}{12} c_{W_2} C_A - \frac{1}{4} c_{B_1} C_A - \frac{1}{8} c_{B_2} C_A + \frac{1}{24} c_{p'p} C_A \right. \\
 & \left. + \frac{7}{24} c_S c_k C_A - \frac{1}{6} c_S c_F C_A - \frac{1}{12} c_F c_k^2 (16 C_F + 15 C_A) + \frac{7}{8} c_F^2 c_k C_A \right), \quad (10)
 \end{aligned}$$

FIG. 3. Topologies contributing to the one transverse gluon exchange. All diagrams are generated from these topologies by considering all possible vertices up to $\mathcal{O}(1/m^3)$. While the external gluon is transverse, internal gluons must be understood as either longitudinal or transverse.

$$\begin{aligned} \nu \frac{d}{d\nu} c_{W_2} = & \frac{\alpha}{\pi} \left(\frac{7}{12} c_{W_1} C_A + \frac{13}{12} c_{W_2} C_A - \frac{1}{4} c_{B_1} C_A - \frac{1}{8} c_{B_2} C_A + \frac{1}{24} c_{p'p} C_A \right. \\ & \left. - \frac{5}{24} c_S c_k C_A - \frac{1}{6} c_S c_F C_A - \frac{1}{12} c_F c_k^2 (16 C_F + 3 C_A) + \frac{7}{8} c_F^2 c_k C_A \right), \end{aligned} \quad (11)$$

$$\begin{aligned} \nu \frac{d}{d\nu} c_{B_1} = & \frac{\alpha}{\pi} \left(\frac{1}{6} c_{W_1} C_A + \frac{1}{6} c_{W_2} C_A + c_{B_1} C_A - \frac{1}{3} c_{B_2} C_A + \frac{7}{12} c_{p'p} C_A \right. \\ & \left. + \frac{1}{12} c_S c_k C_A - \frac{1}{4} c_S c_F C_A + \frac{7}{6} c_F c_k^2 C_A + \frac{7}{6} c_F^2 c_k C_A + \frac{3}{2} c_F^3 C_A \right), \end{aligned} \quad (12)$$

$$\begin{aligned} \nu \frac{d}{d\nu} c_{B_2} = & \frac{\alpha}{\pi} \left(c_{W_2} C_A - \frac{1}{2} c_{B_1} C_A + \frac{7}{6} c_{B_2} C_A \right. \\ & \left. - \frac{4}{3} c_S c_k (4 C_F + C_A) - \frac{1}{6} c_S c_F C_A + \frac{4}{3} c_F c_k^2 (2 C_F - C_A) + \frac{2}{3} c_F^2 c_k C_A - \frac{3}{2} c_F^3 C_A \right), \end{aligned} \quad (13)$$

$$\nu \frac{d}{d\nu} c_{p'p} = \frac{\alpha}{\pi} \left(\frac{1}{2} c_{p'p} C_A - \frac{1}{2} c_S c_k C_A + c_F c_k^2 C_A \right). \quad (14)$$

The renormalization group equations for the physical set $\{\bar{c}_W, \bar{c}_{B_1}, c_{B_2}, c_{p'p}\}$ read

$$\nu \frac{d}{d\nu} \bar{c}_W = \frac{\alpha}{\pi} \left(\frac{1}{2} \bar{c}_W C_A + \frac{1}{2} c_S c_k C_A - c_F c_k^2 C_A \right) = 0, \quad (15)$$

$$\begin{aligned} \nu \frac{d}{d\nu} \bar{c}_{B_1} = & \frac{\alpha}{\pi} \left(\frac{3}{2} \bar{c}_{B_1} C_A + \bar{c}_W C_A - \frac{1}{12} c_{B_2} C_A + \frac{1}{2} c_{p'p} C_A \right. \\ & \left. - \frac{1}{2} c_S c_k C_A + \frac{1}{12} c_S c_F C_A + \frac{1}{3} c_F c_k^2 (8 C_F + 11 C_A) - \frac{7}{12} c_F^2 c_k C_A + \frac{3}{2} c_F^3 C_A \right), \end{aligned} \quad (16)$$

$$\begin{aligned} \nu \frac{d}{d\nu} c_{B_2} = & \frac{\alpha}{\pi} \left(-\frac{1}{2} \bar{c}_{B_1} C_A - \bar{c}_W C_A + \frac{7}{6} c_{B_2} C_A \right. \\ & \left. - \frac{4}{3} c_S c_k (4 C_F + C_A) - \frac{1}{6} c_S c_F C_A + \frac{4}{3} c_F c_k^2 (2 C_F - C_A) + \frac{2}{3} c_F^2 c_k C_A - \frac{3}{2} c_F^3 C_A \right), \end{aligned} \quad (17)$$

$$\nu \frac{d}{d\nu} c_{p'p} = \frac{\alpha}{\pi} \left(\frac{1}{2} c_{p'p} C_A - \frac{1}{2} c_S c_k C_A + c_F c_k^2 C_A \right), \quad (18)$$

where $\bar{c}_W$ and $\bar{c}_{B_1}$ come from the definitions given in Sec. III. The last equality in Eq. (15) can be easily deduced by using the relations between Wilson coefficients imposed by reparametrization invariance. When writing the counter-term of each Wilson coefficient it is enough to know that the scaling with the renormalization scale is ν^{2e} . It is quite remarkable that the RG equations depend only on gauge-independent combinations of Wilson coefficients: $\bar{c}_W$, $\bar{c}_{B_1}$ and c_{B_2} . This is quite a strong check, as at intermediate steps we get contributions from c_{W_1} , c_{W_2} and c_{B_1} , which only at the end of the computation arrange themselves in gauge-independent combinations.

B. QED

In this section we analyze the purely Abelian case of QED. To do this, we just need to take the appropriate limit of the results found in Sec. IVA i.e. to take $C_F = 1$, $C_A = 0$ and $n_f = 0$. Note that the operators proportional to c_{B_1} and c_{B_2} do not appear now. The diagrams that contribute are the same drawn in Figs. 2 and 3 disregarding non-Abelian diagrams.

The renormalization group equations read

$$\begin{aligned} \nu \frac{d}{d\nu} c_{W_1} &= -\frac{4}{3} c_F c_k^2 C_F \frac{\alpha}{\pi}, & \nu \frac{d}{d\nu} c_{W_2} &= -\frac{4}{3} c_F c_k^2 C_F \frac{\alpha}{\pi}, \\ \nu \frac{d}{d\nu} c_{p'p} &= 0. \end{aligned} \quad (19)$$

This result is in agreement with the explicit single logs given in Refs. [15,16], where the calculation was done in the Feynman gauge. This is not that strange; for instance, the running of c_D in QED happens to be equal in the Coulomb and Feynman gauges (see the discussion in Ref. [20]). The analysis done in Sec. III suggests that the physical object is still $\bar{c}_W$, though. The results obtained are in agreement with reparametrization invariance relations given in Ref. [10] (recall that c_F has no running in QED).

V. SOLUTION AND NUMERICAL ANALYSIS

The RG equations obtained in the previous section depend on a list of five Wilson coefficients $\mathbf{A} = \{c_{W_1}, c_{W_2},$

$c_{B_1}, c_{B_2}, c_{p'p}\}$. The running of $c_{p'p}$ has been found to be the same as that of c_F , as predicted by reparametrization invariance, so we will not give an explicit expression. The remaining equations can be written more compactly in a matrix form

$$\nu \frac{d}{d\nu} \mathbf{A} = \frac{\alpha}{\pi} (\mathbf{M} \mathbf{A} + \mathbf{F}(\alpha)). \quad (20)$$

The matrix $\mathbf{M}$ follows from the results of the previous section. We only need the running of α with LL accuracy,

$$\nu \frac{d}{d\nu} \alpha \equiv \beta(\alpha_s) = -2\alpha \left\{ \beta_0 \frac{\alpha}{4\pi} + \dots \right\}, \quad (21)$$

where

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \quad (22)$$

and n_f is the number of dynamical (active) quarks i.e. the number of light quarks. Recall that $n_f = 0$ for the case we are studying.

In this approximation, the above equation can be simplified to

$$\frac{d}{d\alpha} \mathbf{A} = -\frac{2}{\beta_0 \alpha} (\mathbf{M} \mathbf{A} + \mathbf{F}(\alpha)). \quad (23)$$

It is useful to define $z \equiv \left(\frac{\alpha(\nu)}{\alpha(m)} \right)^{\frac{1}{\beta_0}} \simeq 1 - \frac{1}{2\pi} \alpha(\nu) \ln\left(\frac{\nu}{m}\right)$ and write the equation above as

$$\frac{d\mathbf{A}}{dz} = -\frac{2}{z} (\mathbf{M} \mathbf{A} + \mathbf{F}(z)). \quad (24)$$

We also need the initial matching conditions at the hard scale, at tree-level. They have been determined in Ref. [10] and read $c_k = c_F = c_S = c_{W_1} = c_{B_1} = 1$ and $c_{W_2} = c_{p'p} = c_{B_2} = 0$.

Note that the matching coefficient of the kinetic term is protected by reparametrization invariance ($c_k = 1$ to any order in perturbation theory) [19]. Nevertheless, although we set them to 1 when solving the RG equations, we keep them explicit in the RG equations for tracking purposes.

After solving the RG equations we obtain the LL running of the Wilson coefficients associated to the $1/m^3$ spin-dependent operators of the HQET Lagrangian. We obtain the following analytic results for the unphysical set in Coulomb gauge:

$$\begin{aligned} c_{W_1} &= \frac{47}{82} - \frac{160C_F}{287C_A} + \left(\frac{1}{3} + \frac{10C_F}{3C_A} \right) z^{-C_A} - z^{-2C_A} + \frac{9}{2} z^{-3C_A} + \left(5 + \frac{22C_F}{7C_A} \right) z^{-\frac{7}{3}C_A} \\ &+ \left(-\frac{517}{123} + \frac{14}{41} \sqrt{\frac{2}{5}} - \sqrt{\frac{5}{2}} + \frac{50}{123} \sqrt{10} - \frac{364C_F}{123C_A} - \frac{452C_F}{123C_A} \sqrt{\frac{2}{5}} \right) z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\ &+ \left(-\frac{517}{123} - \frac{14}{41} \sqrt{\frac{2}{5}} + \sqrt{\frac{5}{2}} - \frac{50}{123} \sqrt{10} - \frac{364C_F}{123C_A} + \frac{452C_F}{123C_A} \sqrt{\frac{2}{5}} \right) z^{-\frac{1}{6}(16+\sqrt{10})C_A}, \end{aligned} \quad (25)$$

$$\begin{aligned}
c_{W_2} = & -\frac{35}{82} - \frac{160C_F}{287C_A} + \left(\frac{1}{3} + \frac{10C_F}{3C_A}\right)z^{-C_A} - z^{-2C_A} + \frac{9}{2}z^{-3C_A} + \left(5 + \frac{22C_F}{7C_A}\right)z^{-\frac{7}{3}C_A} \\
& + \left(-\frac{517}{123} + \frac{14}{41}\sqrt{\frac{2}{5}} - \sqrt{\frac{5}{2}} + \frac{50}{123}\sqrt{10} - \frac{364C_F}{123C_A} - \frac{452C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
& + \left(-\frac{517}{123} - \frac{14}{41}\sqrt{\frac{2}{5}} + \sqrt{\frac{5}{2}} - \frac{50}{123}\sqrt{10} - \frac{364C_F}{123C_A} + \frac{452C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{-\frac{1}{6}(16+\sqrt{10})C_A}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
c_{B_1} = & \frac{55}{123} - \frac{1184C_F}{861C_A} + \left(-\frac{19}{9} + \frac{44C_F}{9C_A}\right)z^{-C_A} - z^{-2C_A} - 6z^{-3C_A} + \left(10 + \frac{44C_F}{7C_A}\right)z^{-\frac{7}{3}C_A} \\
& + \left(-\frac{62}{369} - \frac{1034}{369}\sqrt{10} - \frac{1808C_F}{369C_A} - \frac{728C_F}{369C_A}\sqrt{10}\right)z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
& + \left(-\frac{62}{369} + \frac{1034}{369}\sqrt{10} - \frac{1808C_F}{369C_A} + \frac{728C_F}{369C_A}\sqrt{10}\right)z^{-\frac{1}{6}(16+\sqrt{10})C_A}, \tag{27}
\end{aligned}$$

$$\begin{aligned}
c_{B_2} = & -\frac{24}{41} - \frac{192C_F}{41C_A} + \left(\frac{11}{3} + \frac{32C_F}{3C_A}\right)z^{-C_A} + z^{-2C_A} + 18z^{-3C_A} \\
& + \left(-\frac{1358}{123} + \frac{64}{41}\sqrt{\frac{2}{5}} + \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} + \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
& + \left(-\frac{1358}{123} - \frac{64}{41}\sqrt{\frac{2}{5}} - \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} - \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{-\frac{1}{6}(16+\sqrt{10})C_A}. \tag{28}
\end{aligned}$$

The solution for the physical set of Wilson coefficients read

$$\bar{c}_W = 1, \tag{29}$$

$$\begin{aligned}
\bar{c}_{B_1} = & -\frac{86}{123} - \frac{32C_F}{123C_A} - \left(\frac{25}{9} + \frac{16C_F}{9C_A}\right)z^{-C_A} + z^{-2C_A} - 15z^{-3C_A} \\
& + \left(\frac{3040}{369} - \frac{5077}{369}\sqrt{\frac{2}{5}} + \frac{376C_F}{369C_A} - \frac{928C_F}{369C_A}\sqrt{\frac{2}{5}}\right)z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
& + \left(\frac{3040}{369} + \frac{5077}{369}\sqrt{\frac{2}{5}} + \frac{376C_F}{369C_A} + \frac{928C_F}{369C_A}\sqrt{\frac{2}{5}}\right)z^{-\frac{1}{6}(16+\sqrt{10})C_A}, \tag{30}
\end{aligned}$$

$$\begin{aligned}
c_{B_2} = & -\frac{24}{41} - \frac{192C_F}{41C_A} + \left(\frac{11}{3} + \frac{32C_F}{3C_A}\right)z^{-C_A} + z^{-2C_A} + 18z^{-3C_A} \\
& + \left(-\frac{1358}{123} + \frac{64}{41}\sqrt{\frac{2}{5}} + \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} + \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
& + \left(-\frac{1358}{123} - \frac{64}{41}\sqrt{\frac{2}{5}} - \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} - \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right)z^{-\frac{1}{6}(16+\sqrt{10})C_A}. \tag{31}
\end{aligned}$$

As can be seen, Eq. (29) satisfies reparametrization invariance. We have proven by explicit calculation that, at LL, $c_{p'p}$ and $\bar{c}_W$ satisfy the relations imposed by reparametrization invariance given in Ref. [10].

If we expand the above solutions in powers of α we can explicitly write the single log (it can also be obtained by trivial inspection of the RG equations in Sec. IV). We obtain for the unphysical set that

$$c_{W_1} = 1 + \left(-\frac{4}{3}C_F + \frac{7}{12}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \tag{32}$$

$$c_{W_2} = \left(-\frac{4}{3}C_F + \frac{7}{12}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \tag{33}$$

$$c_{B_1} = 1 + \frac{29}{6}C_A\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \tag{34}$$

FIG. 4. Running of the physical $1/m^3$ spin-dependent Wilson coefficients. The continuous line is the LL result with $n_f = 0$ and the dashed line is the single LL result which does not depend on n_f .

$$c_{B_2} = -\left(\frac{8}{3}C_F + \frac{25}{6}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \quad (35)$$

and for the physical set, that

$$\bar{c}_W = 1 + \mathcal{O}(\alpha^2), \quad (36)$$

$$\bar{c}_{B_1} = -1 + \left(\frac{8}{3}C_F + \frac{11}{3}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \quad (37)$$

$$c_{B_2} = -\left(\frac{8}{3}C_F + \frac{25}{6}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2). \quad (38)$$

In Fig. 4 one can see the above results when applied to the bottom heavy quark case, illustrating the importance of incorporating large logarithms in heavy quark physics. Only physical combinations and specific combinations that appear in physical observables, like Compton scattering, are represented. We run the Wilson coefficients from the heavy quark mass to 1 GeV. For illustrative purposes, we take $m_b = 4.73$ GeV and $\alpha(m_b) = 0.215943$.

Concerning the numerical analysis, we observe that the effect due to the logarithms is large in general (not for QED however, where the only physical combination that appears, $\bar{c}_W$, does not run). This is because the coefficients multiplying the logs are large, in particular those that multiply the non-Abelian coefficient C_A . We also observe that the LL resummation is saturated by the single log in all cases except in the combination $\bar{c}_{B_1} + c_{B_2}$. Let us now

discuss in more detail each individual Wilson coefficient. We observe the following: $\bar{c}_{B_1}$ changes from -1 to -2.3 after running. The case of c_{B_2} is rather similar, it goes from 0 to 1.45 after running. In general, the effect of the resummation of logarithms is not quite large, but certainly sizable. It introduces a change of approximately 0.3 with respect to the single log result. For the combination $\bar{c}_{B_1} + c_{B_2}$ the effect is very small, it goes from -1 to -0.99 even though it has a maximum in which the value is -0.95 . In this case the resummation of logs is important because the behavior is not saturated by the single log.

VI. COMPARISON WITH EARLIER WORK

The LL running of the Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian without considering light fermion effects was first addressed in Refs. [15,16], where expressions for the anomalous dimension matrix and explicit expressions for the Wilson coefficients with single log accuracy are given. As in Ref. [17] for the spin-independent case, we find that these results are mutually inconsistent, as the anomalous dimension matrix produces different expressions for the single log result compared to the explicit single log expressions written in these references.

The basis of operators these results were obtained from is different from the basis used in this paper, so in order to compare our results, we have to change the operator basis. This is done via field redefinitions, which at the order we are working in, is equivalent to using the full equations of

motion to order $1/m$. To this purpose, we use the HQET Lagrangian in a general frame, Eq. (8) in Ref. [10]³ We obtain the following relations between the spin-dependent $1/m^3$ Wilson coefficients in the two bases:

$$c_5^{(3)} = -c_{B_2} - c_k c_F^2 - c_S c_F, \quad (39)$$

$$c_6^{(3)} = -c_{W_1} - c_{p'p} + c_k^2 c_F + \frac{1}{2} c_D c_F + \frac{1}{2} c_S c_k, \quad (40)$$

$$c_7^{(3)} = 2c_{W_2} - 2c_{p'p} - c_{B_1} + c_k c_F^2 + c_S c_F, \quad (41)$$

$$c_8^{(3)} = -c_{W_1} - c_{p'p} + c_k^2 c_F + \frac{1}{2} c_D c_F + \frac{1}{2} c_S c_k, \quad (42)$$

$$c_9^{(3)} = -c_{B_1} + c_k c_F^2 + c_S c_F, \quad (43)$$

$$c_{10}^{(3)} = c_{p'p} + c_{B_1} - c_k c_F^2 - c_S c_F, \quad (44)$$

$$c_{11}^{(3)} = c_{p'p} + c_{B_1} - c_k c_F^2 - c_S c_F. \quad (45)$$

Note that $c_6^{(3)} = c_8^{(3)}$ and $c_{10}^{(3)} = c_{11}^{(3)}$. This is to be expected since it is well-known from Ref. [10] that there are five spin-dependent operators and five different Wilson coefficients, whereas in Refs. [15,16] there are seven operators and seven different Wilson coefficients. Thereby, one must find that two Wilson coefficients have to be equal to another two. In Ref. [17] these relations between the Wilson coefficients at $\mathcal{O}(1/m)$ and $\mathcal{O}(1/m^2)$ were also found,

$$c_1^{(1)} = c_k, \quad c_2^{(1)} = c_F, \quad c_1^{(2)} = -c_D, \quad c_2^{(2)} = c_S. \quad (46)$$

In Ref. [17] expressions for the correct anomalous dimension matrix for the spin-independent Wilson coefficients in

the basis used in Refs. [15,16] were given. However the situation for the spin-dependent case turns out to be more complicated, since Wilson coefficients are gauge dependent. Because the calculation in these references was done in the Feynman gauge and in this paper in the Coulomb gauge, we can not give a prediction for the anomalous dimension matrix in Refs. [15,16] in the Feynman gauge (note that in the spin-independent case there were also gauge dependent Wilson coefficients, but all gauge dependence came from c_D and c_M , for which expressions are well-known in the Feynman gauge). Instead, to compare results, we compute the RG equations in our basis for physical quantities from the anomalous dimension matrices given in Ref. [15]. To do this, one needs the inverse relations between the Wilson coefficients in the two bases,

$$c_{p'p} = c_9^{(3)} + c_{10}^{(3)}, \quad (47)$$

$$c_{W_1} = -c_6^{(3)} - c_9^{(3)} - c_{10}^{(3)} + c_1^{(1)2} c_2^{(1)} - \frac{1}{2} c_1^{(2)} c_2^{(1)} + \frac{1}{2} c_2^{(2)} c_1^{(1)}, \quad (48)$$

$$c_{W_2} = \frac{1}{2} c_7^{(3)} + \frac{1}{2} c_9^{(3)} + c_{10}^{(3)}, \quad (49)$$

$$c_{B_1} = -c_9^{(3)} + c_1^{(1)} c_2^{(1)2} + c_2^{(2)} c_2^{(1)}, \quad (50)$$

$$c_{B_2} = -c_5^{(3)} - c_1^{(1)} c_2^{(1)2} - c_2^{(2)} c_2^{(1)}. \quad (51)$$

Firstly, note that the anomalous dimension matrix in Ref. [15] gives $c_6^{(3)} \neq c_8^{(3)}$. This already disagrees with our results and with the explicit single log results given in Table II of that reference. We continue with the comparison nonetheless. We take the expression for $c_6^{(3)}$, which is the one which minimizes the discrepancies. The RG equations read

$$\nu \frac{d}{d\nu} c_{p'p} = \frac{\alpha}{\pi} \left(C_A c_F c_k^2 + \frac{1}{2} C_A c_{p'p} - \frac{1}{2} C_A c_S c_k \right), \quad (52)$$

$$\nu \frac{d}{d\nu} \bar{c}_W = \frac{\alpha}{\pi} \left(\frac{1}{6} C_A c_D c_F - \frac{5}{12} C_A c_F^3 - \frac{3}{4} C_A c_F^2 c_k - \frac{1}{12} C_A c_F c_k^2 - \frac{4}{3} C_F c_F c_k^2 - \frac{1}{4} C_A c_S c_k + \frac{1}{2} C_A \bar{c}_W \right), \quad (53)$$

$$\begin{aligned} \nu \frac{d}{d\nu} \bar{c}_{B_1} = & \frac{\alpha}{\pi} \left(\frac{3}{2} C_A \bar{c}_{B_1} - \frac{1}{12} C_A c_{B_2} - \frac{1}{3} C_A c_D c_F + \frac{5}{6} C_A c_F^3 + \frac{35}{12} C_A c_F^2 c_k + \frac{11}{6} C_A c_F c_k^2 \right. \\ & \left. + \frac{16}{3} C_F c_F c_k^2 + \frac{1}{2} C_A c_{p'p} - \frac{17}{12} C_A c_F c_S + 1 C_A c_S c_k + C_A \bar{c}_W \right), \end{aligned} \quad (54)$$

³We take this opportunity to correct a misprint in the term proportional to $c_{p'p}$, where the minus sign appearing there should be a plus sign in order to reproduce the Lagrangian Eqs. (6) and (7) in Ref. [10].

$$\nu \frac{d}{d\nu} c_{B_2} = \frac{\alpha}{\pi} \left(-\frac{1}{2} C_A \bar{c}_{B_1} + \frac{7}{6} C_A c_{B_2} - \frac{4}{3} C_A c_F^2 c_k - \frac{4}{3} C_A c_F c_k^2 + \frac{8}{3} C_F c_F c_k^2 + \frac{4}{3} C_A c_F c_S \right. \\ \left. - \frac{4}{3} C_A c_S c_k - \frac{16}{3} C_F c_S c_k - C_A \bar{c}_W + 0 c_F^3 \right), \quad (55)$$

where numbers in bold indicate a discrepancy with respect to our results. In general we find disagreement for all RG equations (even in QED) except for Eq. (52), which satisfies reparametrization invariance. Conceptually, the disagreement with Eqs. (53)–(54) is important, as these equations do not depend only on physical combinations of Wilson coefficients due to the explicit appearance of c_D , which is gauge dependent. In addition, Eq. (53) does not satisfy reparametrization invariance.

On the other hand, it is remarkable that using the single log results given in Table II of Ref. [16] one finds agreement with our single log results for the physical quantities $\bar{c}_W$, $\bar{c}_{B_1}$, c_{B_2} , given in Eqs. (36)–(38), and $c_{p'p}$, not presented explicitly. However, we find disagreement for the unphysical quantities c_{W_1} , c_{W_2} and c_{B_1} , given in Eqs. (32)–(34). If we trust the explicit single logs presented in this reference, this is a clear indication these Wilson coefficients are gauge dependent.

VII. CONCLUSIONS

We have computed the LL running of the Wilson coefficients associated to the spin-dependent $1/m^3$ operators of the HQET Lagrangian without light fermion effects. We observe that reparametrization invariance relations are satisfied and that the running of physical quantities depend only on gauge-independent quantities, as expected. Numerically, we observe that the running produces a large effect, except for the combination $\bar{c}_{B_1} + c_{B_2}$, which also appears in Compton scattering. However, in this case, the resummation of large logarithms also happens to be important because the behavior of this combination is not saturated by the single log.

We have compared our results with the previous work done in Refs. [15,16]. For the gauge invariant combinations we have computed in our paper, the anomalous dimension matrix given in Ref. [15] yields different RG equations than those we found in Sec. IV A and also different single logs as those given explicitly in that reference. Nevertheless, it is

remarkable that we find agreement with the explicit single logs given in these references.

These results are necessary building blocks for the determination of the pNRQCD Lagrangian with NNNLL accuracy. Even though they appear not to be necessary to obtain the heavy quarkonium spectrum with NNNLL accuracy, nor the production and annihilation of heavy quarkonium with NNLL precision, they are necessary at higher orders. Moreover, it may be important for future research in the determination of higher order logarithms for NRQED bound states, like hydrogen and muonic hydrogenlike atoms.

ACKNOWLEDGMENTS

We thank Antonio Pineda for useful discussions and for reading over the manuscript. This work was supported in part by the Spanish Grants No. FPA2014-55613-P and No. SEV-2016-0588. RPB was supported for much of the duration of this work by the Barcelona Institute of Science and Technology (BIST). RPB is currently supported by a Clarendon Scholarship from the University of Oxford.

APPENDIX: HQET FEYNMAN RULES

Here we collect some Feynman rules in Coulomb gauge that are needed for our computation, and complement those that can be found in Refs. [7] and [17].

The heavy quark four-momentum $p = (E_1, \mathbf{p})$ is incoming with associated color index β , whereas the heavy quark four-momentum $p' = (E'_1, \mathbf{p}')$ is outgoing with associated color index α . All four-momenta of gluons, k_i , are outgoing. If more than one gluon appears, let us say n , then they are labeled with four-momentum k_i ($i = 1, \dots, n$) and, by four-momentum conservation, $k = \sum_{i=1}^n k_i = p - p'$. We start labeling transverse gluons first and longitudinal gluons after with the labels $a, b, c, \dots$ referring to color indices in the adjoint representation, $i, j, k, \dots$ referring to space vector indices and $k_1, k_2, k_3, \dots$ referring to four-momentum.

1. Proportional to c_{W_1}

$$\mathcal{V}_{c_{W_1}}^{ia} = -c_{W_1} \frac{g}{8m^3} (\mathbf{p}^2 + \mathbf{p}'^2) (\boldsymbol{\sigma} \times \mathbf{k})^i (T^a)_{\alpha\beta} \quad (A1)$$

$$\mathcal{V}_{c_{W_1}}^{ijab} = -c_{W_1} \frac{g^2}{8m^3} [(\boldsymbol{\sigma}^k \epsilon^{kij} (\mathbf{p}^2 + \mathbf{p}'^2) - (\boldsymbol{\sigma} \times \mathbf{k}_1)^i \mathbf{k}_1^j + \mathbf{k}_2^i (\boldsymbol{\sigma} \times \mathbf{k}_2)^j) [T^a, T^b]_{\alpha\beta} \\ - ((\boldsymbol{\sigma} \times \mathbf{k}_1)^i (\mathbf{p} + \mathbf{p}')^j + (\mathbf{p} + \mathbf{p}')^i (\boldsymbol{\sigma} \times \mathbf{k}_2)^j) \{T^a, T^b\}_{\alpha\beta}] \quad (A2)$$

$$\begin{aligned}
\mathcal{V}_{c_{w_1}}^{ijkabc} = & c_{w_1} \frac{g^3}{8m^3} \sigma^m (\epsilon^{mjk} (\{T^a, [T^b, T^c]\}_{\alpha\beta} (\mathbf{p} + \mathbf{p}')^i - [T^a, [T^b, T^c]]_{\alpha\beta} (\mathbf{k}_2 + \mathbf{k}_3)^i) \\
& + \epsilon^{mki} (\{T^b, [T^c, T^a]\}_{\alpha\beta} (\mathbf{p} + \mathbf{p}')^j - [T^b, [T^c, T^a]]_{\alpha\beta} (\mathbf{k}_1 + \mathbf{k}_3)^j) \\
& + \epsilon^{mij} (\{T^c, [T^a, T^b]\}_{\alpha\beta} (\mathbf{p} + \mathbf{p}')^k - [T^c, [T^a, T^b]]_{\alpha\beta} (\mathbf{k}_1 + \mathbf{k}_2)^k) \\
& - \epsilon^{mri} \delta^{jk} \{T^a, \{T^b, T^c\}\}_{\alpha\beta} \mathbf{k}_1^r - \epsilon^{mrj} \delta^{ik} \{T^b, \{T^c, T^a\}\}_{\alpha\beta} \mathbf{k}_2^r \\
& - \epsilon^{mrk} \delta^{ij} \{T^c, \{T^a, T^b\}\}_{\alpha\beta} \mathbf{k}_3^r
\end{aligned} \tag{A3}$$

2. Proportional to c_{w_2}

$$\mathcal{V}_{c_{w_2}}^{ia} = c_{w_2} \frac{g}{4m^3} (\mathbf{p} \cdot \mathbf{p}') (\boldsymbol{\sigma} \times \mathbf{k})^i (T^a)_{\alpha\beta} \tag{A4}$$

$$\begin{aligned}
\mathcal{V}_{c_{w_2}}^{ijab} = & -c_{w_2} \frac{g^2}{8m^3} [((\boldsymbol{\sigma} \times \mathbf{k}_2)^j \mathbf{k}_2^i - (\boldsymbol{\sigma} \times \mathbf{k}_1)^i \mathbf{k}_1^j - 2\sigma^k \epsilon^{kij} (\mathbf{p} \cdot \mathbf{p}')) [T^a, T^b]_{\alpha\beta} \\
& + ((\boldsymbol{\sigma} \times \mathbf{k}_1)^i (\mathbf{p} + \mathbf{p}')^j + (\mathbf{p} + \mathbf{p}')^i (\boldsymbol{\sigma} \times \mathbf{k}_2)^j) \{T^a, T^b\}_{\alpha\beta}]
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\mathcal{V}_{c_{w_2}}^{ijkabc} = & -c_{w_2} \frac{g^3}{4m^3} \sigma^l (-\epsilon^{lij} ([T^a, T^b] T^c)_{\alpha\beta} \mathbf{k}^k - \epsilon^{lki} ([T^c, T^a] T^b)_{\alpha\beta} \mathbf{k}^j \\
& - \epsilon^{ljk} ([T^b, T^c] T^a)_{\alpha\beta} \mathbf{k}^i + \epsilon^{lij} \{T^c, [T^a, T^b]\}_{\alpha\beta} \mathbf{p}^k + \epsilon^{lki} \{T^b, [T^c, T^a]\}_{\alpha\beta} \mathbf{p}^j \\
& + \epsilon^{ljk} \{T^a, [T^b, T^c]\}_{\alpha\beta} \mathbf{p}^i - \epsilon^{lrj} \delta^{ki} (T^a T^b T^c + T^c T^b T^a)_{\alpha\beta} \mathbf{k}_2^r \\
& - \epsilon^{lrk} \delta^{ij} (T^a T^c T^b + T^b T^c T^a)_{\alpha\beta} \mathbf{k}_3^r - \epsilon^{lri} \delta^{jk} (T^b T^a T^c + T^c T^a T^b)_{\alpha\beta} \mathbf{k}_1^r)
\end{aligned} \tag{A6}$$

3. Proportional to $c_{p'p}$

$$\mathcal{V}_{c_{p'p}}^{ia} = c_{p'p} \frac{g}{8m^3} \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{p}') (\mathbf{p} \times \mathbf{p}')^i (T^a)_{\alpha\beta} \tag{A7}$$

$$\begin{aligned}
\mathcal{V}_{c_{p'p}}^{ijab} = & c_{p'p} \frac{g^2}{16m^3} \{ [((\mathbf{p} + \mathbf{p}') \times \mathbf{k}_1)^i \sigma^j + \sigma^i ((\mathbf{p} + \mathbf{p}') \times \mathbf{k}_2)^j + \epsilon^{ijk} (\mathbf{k}_1 - \mathbf{k}_2)^k (\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{p}'))] \{T^a, T^b\}_{\alpha\beta} \\
& + [(\mathbf{k}_1 \times \mathbf{k}_2)^i \sigma^j + \sigma^i (\mathbf{k}_1 \times \mathbf{k}_2)^j - \epsilon^{ijk} (\mathbf{k}^k (\boldsymbol{\sigma} \cdot \mathbf{k}) + 2((\boldsymbol{\sigma} \cdot \mathbf{p}) \mathbf{p}^k + (\boldsymbol{\sigma} \cdot \mathbf{p}') \mathbf{p}^k))] [T^a, T^b]_{\alpha\beta} \}
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\mathcal{V}_{c_{p'p}}^{ijkabc} = & c_{p'p} \frac{g^3}{8m^3} [(\epsilon^{ljk} \sigma^i \mathbf{p}^l + \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p})) (T^a [T^b, T^c])_{\alpha\beta} + (\epsilon^{ljk} \sigma^i \mathbf{p}^l + \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p}')) ([T^b, T^c] T^a)_{\alpha\beta} \\
& + (\epsilon^{lik} \sigma^j \mathbf{p}^l - \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p})) (T^b [T^a, T^c])_{\alpha\beta} + (\epsilon^{lik} \sigma^j \mathbf{p}^l - \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p}')) ([T^a, T^c] T^b)_{\alpha\beta} \\
& + (\epsilon^{lij} \sigma^k \mathbf{p}^l + \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p})) (T^c [T^a, T^b])_{\alpha\beta} + (\epsilon^{lij} \sigma^k \mathbf{p}^l + \epsilon^{ijk} (\boldsymbol{\sigma} \cdot \mathbf{p}')) ([T^a, T^b] T^c)_{\alpha\beta} \\
& - (\epsilon^{ljk} \sigma^i - \epsilon^{lij} \sigma^k) \mathbf{k}_2^l (T^a T^b T^c + T^c T^b T^a)_{\alpha\beta} \\
& + (\epsilon^{lik} \sigma^j + \epsilon^{ljk} \sigma^i) \mathbf{k}_3^l (T^b T^c T^a + T^a T^c T^b)_{\alpha\beta} \\
& - (\epsilon^{lij} \sigma^k + \epsilon^{lik} \sigma^j) \mathbf{k}_1^l (T^c T^a T^b + T^b T^a T^c)_{\alpha\beta}]
\end{aligned} \tag{A9}$$

4. Proportional to c_{B_1}

$$\mathcal{V}_{c_{B_1}}^{ab} = -c_{B_1} \frac{g^2}{8m^3} \boldsymbol{\sigma} \cdot (\mathbf{k}_1 \times \mathbf{k}_2) [T^a, T^b]_{\alpha\beta} \tag{A10}$$

$$\mathcal{V}_{c_{B_1}}^{iab} = -c_{B_1} \frac{g^2}{8m^3} k_1^0 (\boldsymbol{\sigma} \times \mathbf{k}_2)^i [T^a, T^b]_{\alpha\beta} \tag{A11}$$

$$\mathcal{V}_{c_{B_1}}^{ijab} = c_{B_1} \frac{g^2}{16m^3} (\epsilon^{ijk}(\boldsymbol{\sigma} \cdot \mathbf{k}_1)\mathbf{k}_2^k + \epsilon^{ijk}(\boldsymbol{\sigma} \cdot \mathbf{k}_2)\mathbf{k}_1^k + \boldsymbol{\sigma}^i(\mathbf{k}_1 \times \mathbf{k}_2)^j + \boldsymbol{\sigma}^j(\mathbf{k}_1 \times \mathbf{k}_2)^i - 2\boldsymbol{\sigma}^k \epsilon^{kij} k_1^0 k_2^0) [T^a, T^b]_{\alpha\beta} \quad (\text{A12})$$

$$\mathcal{V}_{c_{B_1}}^{iabc} = -c_{B_1} \frac{g^3}{8m^3} ((\boldsymbol{\sigma} \times \mathbf{k}_2)^i [T^b, [T^a, T^c]]_{\alpha\beta} + (\boldsymbol{\sigma} \times \mathbf{k}_3)^i [T^c, [T^a, T^b]]_{\alpha\beta}) \quad (\text{A13})$$

$$\mathcal{V}_{c_{B_1}}^{ijabc} = c_{B_1} \frac{g^3}{8m^3} \boldsymbol{\sigma}^k \epsilon^{kij} (k_1^0 [T^a, [T^b, T^c]]_{\alpha\beta} - k_2^0 [T^b, [T^a, T^c]]_{\alpha\beta}) \quad (\text{A14})$$

$$\begin{aligned} \mathcal{V}_{c_{B_1}}^{ijkabc} = & -c_{B_1} \frac{g^3}{8m^3} \{ (\epsilon^{ijk} \boldsymbol{\sigma} \cdot \mathbf{k}_1 - \epsilon^{ljk} \boldsymbol{\sigma}^l \mathbf{k}_1^l) [T^a, [T^b, T^c]]_{\alpha\beta} \\ & - (\epsilon^{ijk} \boldsymbol{\sigma} \cdot \mathbf{k}_2 - \epsilon^{ilk} \boldsymbol{\sigma}^l \mathbf{k}_2^l) [T^b, [T^a, T^c]]_{\alpha\beta} \\ & + (\epsilon^{ijk} \boldsymbol{\sigma} \cdot \mathbf{k}_3 - \epsilon^{lij} \boldsymbol{\sigma}^l \mathbf{k}_3^l) [T^c, [T^a, T^b]]_{\alpha\beta} \} \end{aligned} \quad (\text{A15})$$

5. Proportional to c_{B_2}

$$\mathcal{V}_{c_{B_2}}^{ab} = -c_{B_2} \frac{g^2}{8m^3} \boldsymbol{\sigma} \cdot (\mathbf{k}_1 \times \mathbf{k}_2) [T^a, T^b]_{\alpha\beta} \quad (\text{A16})$$

$$\mathcal{V}_{c_{B_2}}^{iab} = -c_{B_2} \frac{g^2}{8m^3} k_1^0 (\boldsymbol{\sigma} \times \mathbf{k}_2)^i [T^a, T^b]_{\alpha\beta} \quad (\text{A17})$$

$$\mathcal{V}_{c_{B_2}}^{ijab} = -c_{B_2} \frac{g^2}{8m^3} \boldsymbol{\sigma}^k \epsilon^{kij} k_1^0 k_2^0 [T^a, T^b]_{\alpha\beta} \quad (\text{A18})$$

$$\mathcal{V}_{c_{B_2}}^{iabc} = -c_{B_2} \frac{g^3}{8m^3} ((\boldsymbol{\sigma} \times \mathbf{k}_2)^i [T^b, [T^a, T^c]]_{\alpha\beta} + (\boldsymbol{\sigma} \times \mathbf{k}_3)^i [T^c, [T^a, T^b]]_{\alpha\beta}) \quad (\text{A19})$$

$$\mathcal{V}_{c_{B_2}}^{ijabc} = c_{B_2} \frac{g^3}{8m^3} \boldsymbol{\sigma}^k \epsilon^{kij} (k_1^0 [T^a, [T^b, T^c]]_{\alpha\beta} - k_2^0 [T^b, [T^a, T^c]]_{\alpha\beta}) \quad (\text{A20})$$

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