Renormalization group analysis of $B \rightarrow \pi$ form factors with B -meson light-cone sum rules

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Within the framework of the B -meson light-cone sum rules, we review the calculation of radiative corrections to the three $B \rightarrow \pi$ transition form factors at leading power in Λ/m_b . To resum large logarithmic terms, we perform the complete renormalization group evolution of the correlation function. We employ the integral transformation which diagonalizes evolution equations of the jet function and the B -meson light-cone distribution amplitude to solve these evolution equations and obtain renormalization group improved sum rules for the $B \rightarrow \pi$ form factors. Results of the form factors are extrapolated to the whole physical q^2 region and are compared with that of other approaches. The effect of B -meson three-particle light-cone distribution amplitudes, which will contribute to the form factors at next-to-leading power in Λ/m_b at tree level, is not considered in this paper.

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I. INTRODUCTION

The knowledge of the heavy-to-light transition form factors is of great importance because it is crucial for the determination of parameters of the standard model and for the understanding of strong interaction dynamics. The $B \rightarrow \pi$ transition form factors, which are closely related to the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$, have been extensively studied in the literature. Because of the appearance of the end point singularity in the factorization of the heavy-to-light form factors, the form factors are regarded, in many approaches, as dominated by long-distance QCD dynamics and can be calculated only with nonperturbative methods, such as lattice QCD, (light-cone) QCD sum rules, etc. In Refs. [1–8], the light-cone sum rules (LCSRs) with pion light-cone distribution amplitudes (LCDAs) has been employed to study the $B \rightarrow \pi$ form factors, and next-to-leading-order (NLO) corrections to the twist-2 and the twist-3 terms as well as renormalization group (RG) evolution effects [9] have been considered. In this paper, we use the LCSR with B -meson LCDAs [10,11] to calculate the $B \rightarrow \pi$ form factors. The B -meson LCSR has also been established independently in the framework of the soft-collinear effective theory (SCET) [12,13], in which jet

functions encoding the “hard-collinear” dynamics have been calculated up to $\mathcal{O}(\alpha_s)$ [14,15]. An alternative approach to analyze NLO corrections to the sum rules is suggested in Ref. [16], in which the “method of regions” [17] was adopted to compute the vector form factor $f_{B\pi}^+(q^2)$ and the scalar form factor $f_{B\pi}^0(q^2)$ defined below:

$$\begin{aligned} \langle \pi(p) | \bar{u} \gamma^\mu b | \bar{B}(p_B) \rangle &= f_{B\pi}^+(q^2) \left[p_B + p - \frac{m_B^2 - m_\pi^2}{q^2} q \right]^\mu \\ &+ f_{B\pi}^0(q^2) \frac{m_B^2 - m_\pi^2}{q^2} q^\mu. \end{aligned} \quad (1)$$

Results of the B -meson LCSR were shown to be consistent with that of the SCET sum rules. There is another $B \rightarrow \pi$ transition form factor which is defined by the tensor current,

$$\begin{aligned} \langle \pi(p) | \bar{q} \sigma^{\mu\nu} q_\nu b | B(p_B) \rangle \\ = \frac{i f_{B\pi}^T(q^2)}{m_B + m_\pi} [(p_B + p)^\mu q^2 - (m_B^2 - m_\pi^2) q^\mu], \end{aligned} \quad (2)$$

and this form factor was not computed Ref. [16]. In the heavy-quark limit and at leading order in α_s , the three independent form factors are proportional to one universal form factor at large recoil. If loop corrections are included, differences between the three form factors appear. The hard spectator–scattering part can be factorized into a convolution of the perturbative function and the LCDAs of hadrons (B meson and pion) using the QCD factorization approach [18]. Both QCD corrections to the universal form factor and to symmetry-breaking hard-spectator interactions have been

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calculated in Refs. [14,15]. Using the method of regions, QCD corrections to the symmetry-conserving (universal) form factor and the symmetry-breaking part of the form factors are computed simultaneously.

The first step toward using the method of regions is identifying leading regions which are in principle determined by the analytic structure of the Feynman diagram [19]. Usually, leading regions are closely related to momentum modes of external lines. In this work, there are three momentum modes from external lines, namely, the hard (b -quark), the hard-collinear (interpolating current of pion), and the soft (light-spectator quark) modes. Thus, three momentum regions with scaling behaviors

$$\begin{aligned} P_\mu &\equiv (n \cdot P, \bar{n} \cdot P, P_\perp), & P_{h,\mu} &\sim \mathcal{O}(1, 1, 1), \\ P_{hc,\mu} &\sim \mathcal{O}(1, \lambda, \lambda^{1/2}), & P_{s,\mu} &\sim \mathcal{O}(\lambda, \lambda, \lambda), \end{aligned} \quad (3)$$

can contribute to the correlation function at leading power in λ , where $\lambda \sim \Lambda/m_b$ and n_μ and $\bar{n}_\mu$ are light-cone vectors, satisfying $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. In our calculation, only these three regions give leading-power contributions to the correlation function, which is in agreement with the general analysis. The momentum of the fast-moving pion is chosen to be along the $\bar{n}$ direction. This momentum is also chosen to be hard-collinear and in the Euclidean region ($p^2 < 0$) to ensure the light-cone operator-product expansion (OPE) of the correlation function [10,11]. The method of regions provides us a natural way to perform the factorization of the correlation function because contributions of different momentum regions are considered individually. It has been shown that the correlation function can be factorized into the convolution of the hard function, the jet function, and the B -meson LCDA, which describe dynamics of the hard, the hard-collinear, and the soft regions, respectively. This procedure is equivalent to the two-step matching in the SCET in which the hard (jet) function corresponds to the matching coefficient of SCET_I (SCET_{II}) [20,21].

The correlation function is factorization-scale independent; thus, the scale dependence of the hard function, the B -meson static decay constant, the jet function, and the B -meson LCDA must be canceled, which has been shown at one-loop level [15,16]. At present, there is still no complete analysis of RG evolutions of all of the relevant functions. In Ref. [16], RG evolutions of the hard function and the B -meson decay constant were performed. The factorization scale in that paper was chosen to be about 1.5 GeV, which is a typical hard-collinear scale. This choice is phenomenologically reasonable as the hard-collinear scale is not far from the nonperturbative scale. RG evolutions of the jet function and the B -meson LCDA are nontrivial since anomalous dimensions of these two functions are complicated. But on the conceptual side, a complete RG analysis is necessary.

B -meson LCDAs $\phi_B^-(\omega)$ and $\phi_B^+(\omega)$ are fundamental inputs of the B -meson LCSR. They are defined by the matrix element [22]

$$\begin{aligned} &\langle 0 | \bar{d}_\beta(\tau \bar{n}) [\tau \bar{n}, 0] b_\alpha(0) | \bar{B}(p+q) \rangle \\ &= -\frac{i \tilde{f}_B(\mu) m_B}{4} \left\{ \frac{1 + \not{v}}{2} [2 \tilde{\phi}_B^+(\tau) + (\tilde{\phi}_B^-(\tau) - \tilde{\phi}_B^+(\tau)) \not{v}] \gamma_5 \right\}_{\alpha\beta}, \end{aligned} \quad (4)$$

where $[\tau \bar{n}, 0]$ is the Wilson line along the $\bar{n}$ direction and v and $\tilde{f}_B(\mu)$ are the B -meson velocity vector and the B -meson static decay constant, respectively. The Fourier transformation of $\tilde{\phi}_B^\pm(\tau)$ leads to

$$\phi_B^\pm(\omega) = \int_{-\infty}^{+\infty} \frac{d\tau}{2\pi} e^{i\omega\tau} \tilde{\phi}_B^\pm(\tau - i0). \quad (5)$$

The scale dependence of $\phi_B^\pm(\omega)$ has been studied extensively [23,24]. $\phi_B^\pm(\omega)$ obeys the RG equation with the anomalous dimension being the Lange-Neubert (LN) kernel [23]. It is difficult to solve the Lange-Neubert equation in the momentum space. With the eigenfunction of the Lange-Neubert kernel found in Ref. [24], the RG equation of $\phi_B^\pm(\omega)$ can be diagonalized and readily solved. The RG equation of the jet function can be simplified in the same way since the scale dependence of the jet function should be partly canceled by that of the B -meson LCDA.

In this work, we only consider leading-power contributions of the $B \rightarrow \pi$ form factors. B -meson three-particle LCDAs can give subleading-power contributions to the form factors at leading order (LO) in α_s and give leading-power contributions at NLO [25]. Numerically, the subleading-power correction from three-particle LCDAs is only a few percent of the leading-power contribution from two-particle LCDAs [10]. Feynman diagrams related to three-particle LCDAs can be seen in Ref. [16]. The calculation of effects of three-particle LCDAs is expected to be rather complicated, and RG equations of the three-particle LCDAs are not completely available so far. Thus, we leave this part for the future work.

This paper is arranged as follows. In Sec. II, we briefly review the calculation of the $B \rightarrow \pi$ form factors at NLO and emphasize symmetry-breaking effects of these form factors. Then, the RG evolution of the correlation function is shown in details in the following section. In Sec. IV, we turn to the numerical analysis of the RG improved form factors. Concluding discussions are presented in Sec. V. The Appendix includes two parts, the jet function in the “dual” space and dispersion integrals used in the sum rules.

II. $B \rightarrow \pi$ FORM FACTORS WITH B -MESON LCSRS

To derive the sum rules for the $B \rightarrow \pi$ transition form factors, we start with the correlation function

$$\Pi^\mu(p, q) = i \int d^4x e^{ip \cdot x} \times \langle 0 | T \{ \bar{d}(x) \not{p} \gamma_5 u(x), \bar{u}(0) \Gamma^\mu b(0) \} | \bar{B}(P_B) \rangle, \quad (6)$$

where $\Gamma^\mu = \gamma^\mu (\sigma^{\mu\nu} q_\nu)$ denoting the vector (tensor) current. According to the Lorentz-structure analysis, the correlation function is parametrized as $\Pi^\mu(n \cdot p, \bar{n} \cdot p) = \Pi_n(n \cdot p, \bar{n} \cdot p) n^\mu + \Pi_{\bar{n}}(n \cdot p, \bar{n} \cdot p) \bar{n}^\mu$ for the vector current and $\Pi^\mu(p, q) = \Pi_T(n \cdot p, \bar{n} \cdot p) \epsilon^{\mu\nu} q_\nu$ for the tensor current, where the antisymmetric tensor $\epsilon^{\mu\nu} = (n^\mu \bar{n}^\nu - n^\nu \bar{n}^\mu)/2$. We work in the rest frame of the B meson, and the power-counting rule of the pion momentum reads

$$n \cdot p \sim \mathcal{O}(m_b), \quad \bar{n} \cdot p \sim \mathcal{O}(\Lambda). \quad (7)$$

In Euclidean region ($\bar{n} \cdot p < 0$), the light-cone OPE is employed to calculate the correlation function. Tree-level results are written as

$$\begin{aligned} \Pi_{\bar{n}}^{(0)}(n \cdot p, \bar{n} \cdot p) &= \Pi_T^{(0)}(n \cdot p, \bar{n} \cdot p) \\ &= \tilde{f}_B(\mu) m_B \int_0^\infty d\omega' \frac{\phi_B^-(\omega')}{\omega' - \bar{n} \cdot p - i0}, \\ \Pi_n^{(0)}(n \cdot p, \bar{n} \cdot p) &= 0. \end{aligned} \quad (8)$$

The correlation function can also be expressed in terms of the $B \rightarrow \pi$ form factors and the pion decay constant. For instance,

$$\begin{aligned} \Pi_T(p, q) &= \frac{i(n \cdot p)^2 f_\pi f_{B\pi}^T(q^2) m_B}{(m_\pi^2 - p^2)(m_B + m_\pi)} \\ &\quad + \int_{\omega_s}^\infty d\omega \frac{\rho_h(\omega)}{\omega - \bar{n} \cdot p - i\epsilon}, \end{aligned} \quad (9)$$

where $\rho_h(\omega)$ represents the contribution of excited and continuum states which have the same quantum numbers as the pion. The form factors are extracted by matching the partonic and the hadronic representations of the correlation function. After performing the Borel transformation which suppresses the contribution of excited and continuum states, we obtain the sum rules for the tensor form factor at LO:

$$f_{B\pi}^T(q^2) = \frac{f_B(m_B + m_\pi)}{n \cdot p f_\pi} e^{m_\pi^2/(n \cdot p \omega_M)} \int_0^{\omega_s} d\omega e^{-\omega/\omega_M} \phi_-^B(\omega). \quad (10)$$

FIG. 1. Diagrammatic representation of the correlation function $\Pi^\mu(n \cdot p, \bar{n} \cdot p)$ at next-to-leading order in α_s .

Large-recoil symmetry relations [18] indicate

$$f_{B\pi}^+(q^2) = \frac{m_B}{n \cdot p} f_{B\pi}^0(q^2) = \frac{m_B}{m_B + m_\pi} f_{B\pi}^T(q^2) = \xi(n \cdot p), \quad (11)$$

where $\xi(n \cdot p)$ is the Isgur-Wise function. These symmetry relations will be broken by QCD corrections.

The method of computing radiative corrections to the correlation function has been introduced in Ref. [16]. Adopting the diagrammatic factorization method [26], the hard-scattering kernel at NLO is determined by the matching condition

$$\Phi_{bd}^{(0)} \otimes T^{(1)} = \Pi_{bd}^{(1)} - \Phi_{bd}^{(1)} \otimes T^{(0)}, \quad (12)$$

where the first and the second terms on the right-hand side of the equation correspond to full-theory diagrams (Fig. 1) and effective diagrams (Fig. 2), respectively. The Lorentz index μ is suppressed in this equation. The definition of $\Phi_{bd}^{(0,1)}$ can be seen in Ref. [16]. It has been proven that soft dynamics are completely canceled between $\Pi_{bd}^{(1)}$ and $\Phi_{bd}^{(1)} \otimes T^{(0)}$. Thus, the hard-scattering kernel T contains only contributions from hard and hard-collinear regions at leading power in λ , with the hard-region and the hard-collinear-region contributions corresponding to the hard function and the jet function, respectively. The correlation function is factorized as

$$\begin{aligned} \Pi_a(p, q) &= \tilde{f}_B(\mu) m_B \sum_{k=\pm} C_a^{(k)}(n \cdot p, \mu) \int_0^\infty \frac{d\omega}{\omega - \bar{n} \cdot p} J_a^{(k)} \\ &\quad \times \left(\frac{\mu^2}{n \cdot p \omega}, \frac{\omega}{\bar{n} \cdot p} \right) \phi_B^k(\omega, \mu), \end{aligned} \quad (13)$$

FIG. 2. One-loop-level diagrams of the B -meson LCDA $\Phi_{bu}^{ab}(\omega')$.

where $a = n, \bar{n}, T$. Results of $C_n, C_{\bar{n}}, J_n$, and $J_{\bar{n}}$ have been given in Ref. [16]. C_T and J_T can be calculated following the same method. Nevertheless, using symmetry relations between the form factors, we are able to obtain C_T and J_T without repeating the calculation. Symmetry relations in Eq. (11) are broken by loop corrections. The difference between C_T (J_T) and $C_{\bar{n}}$ ($J_{\bar{n}}$) comes from the symmetry-breaking effect. At one-loop level, large-recoil relations can be written as [18]

$$\begin{aligned} f_{B\pi}^0 &= f_{B\pi}^+ \left[1 + \frac{\alpha_s}{4\pi} \left(2 + \frac{2r}{\bar{r}} \ln r \right) \right] r + \frac{\alpha_s}{4\pi} \Delta f_0, \\ f_{B\pi}^T &= f_{B\pi}^+ \left[1 + \frac{\alpha_s}{4\pi} \left(2 \ln \frac{m_b^2}{\mu^2} - \frac{2r}{\bar{r}} \ln r \right) \right] r + \frac{\alpha_s}{4\pi} \Delta f_T, \end{aligned} \quad (14)$$

where $r = n \cdot p / m_b$. The first line of above equations has been confirmed in Refs. [14, 16].

The first kind of symmetry-breaking effects, which is shown in the square brackets of Eq. (14), arises from the hard function of the weak-vertex correction, Fig. 1(a). The weak tensor current $\bar{u}(0) i\sigma^{\mu\nu} q_\nu b(0)$ is not a conserved current; thus, there exists an operator-renormalization contribution to the correction function of the tensor current. This contribution produces a term additional to the hard function $C_T^{(-)}(n \cdot p, \mu)$

$$C_T^{(-)}(n \cdot p, \mu, \nu) = C_T^{(-)}(n \cdot p, \mu) + \delta C_T^{(-)}(n \cdot p, \mu, \nu), \quad (15)$$

where ν is the renormalization scale. $\delta C_T^{(-)}(n \cdot p, \mu, \nu)$ corresponds to the $\ln \frac{m_b^2}{\mu^2}$ term (μ should be changed to ν) in Eq. (14). Inserting the hard function of $f_{B\pi}^+$ [16] into Eq. (15), we obtain

$$\begin{aligned} C_T^{(-)}(n \cdot p, \mu, \nu) &= 1 - \frac{\alpha_s C_F}{4\pi} \left[2 \ln \frac{\nu}{m_b} + 2 \ln^2 \frac{\mu}{m_b} - (4 \ln r - 5) \ln \frac{\mu}{m_b} \right. \\ &\quad \left. + 2 \ln^2 r + 2 \text{Li}_2(\bar{r}) - \frac{4r-2}{r-1} \ln r + \frac{\pi^2}{12} + 6 \right]. \end{aligned} \quad (16)$$

The second kind of symmetry-breaking effects Δf_T , which corresponds to Figs. 1(b) and 1(c), comes from the “hard-spectator” contribution in the QCD factorization. Since only the hard-collinear region contributes to Figs. 1(b) and 1(c) at leading power in λ , Δf_T is related to jet functions. The jet function $J_T^{(-)}$, which equals $J_{\bar{n}}^{(-)}$, is the symmetry-conserving term. Only $J_T^{(+)}$ corresponds to the symmetry-breaking effect. Comparing Δf_T with Δf_0 , and employing the result of $J_{n,\bar{n}}^{(+)}$, we find

$$J_T^{(+)} = -\frac{\alpha_s C_F}{4\pi} (1+r) \left(1 + \frac{1}{\eta} \right) \ln(1+\eta), \quad (17)$$

where $\eta = -\omega/\bar{n} \cdot p$.

III. RG EVOLUTION

The hard and the jet functions contain logarithmic terms such as $\ln^2 \frac{\mu}{n \cdot p}$ and $\ln \frac{\mu}{n \cdot p}$, which become large when the factorization scale μ is much smaller than $n \cdot p$. The RG equation approach can be used to resum large logarithms to all orders in α_s . In the following, we will present details of the RG evolution of $\Pi_T(n \cdot p)$, and the approach can be easily generalized to $\Pi_n(n \cdot p)$ and $\Pi_{\bar{n}}(n \cdot p)$.

Equation (16) contains the operator renormalization of the tensor current, so we need to perform evolutions to both the renormalization and the factorization scales. RG equations governing the renormalization-scale and the factorization-scale dependence are given by

$$\begin{aligned} \frac{d}{d \ln \nu} C_T^{(-)}(n \cdot p, \mu, \nu) &= \gamma_T(\alpha_s) C_T^{(-)}(n \cdot p, \mu, \nu), \\ \frac{d}{d \ln \mu} C_T^{(-)}(n \cdot p, \mu, \nu) &= \Gamma_C(\alpha_s) C_T^{(-)}(n \cdot p, \mu, \nu), \end{aligned} \quad (18)$$

where

$$\Gamma_C(\alpha_s) = -\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{n \cdot p} + \gamma_h(\alpha_s). \quad (19)$$

Solutions to Eq. (18) are written by

$$\begin{aligned} C_T^{(-)}(n \cdot p, \mu, \nu) &= \exp \left[\int_{\alpha_s(m_b)}^{\alpha_s(\nu)} d\alpha_s \frac{\gamma_T(\alpha_s)}{\beta(\alpha_s)} \right] C_T^{(-)}(n \cdot p, \mu, m_b) \\ C_T^{(-)}(n \cdot p, \mu, m_b) &= \exp \left\{ \int_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha_s \left[\frac{\gamma_h(\alpha_s)}{\beta(\alpha_s)} + \frac{\Gamma_{\text{cusp}}(\alpha_s)}{\beta(\alpha_s)} \left(\ln \frac{n \cdot p}{\mu_h} - \int_{\alpha_s(\mu_h)}^{\alpha_s} \frac{d\alpha'_s}{\beta(\alpha'_s)} \right) \right] \right\} C_T^{(-)}(n \cdot p, \mu_h, m_b). \end{aligned} \quad (20)$$

As the hard function has been calculated at one-loop level, evolution functions of the hard function are also required to be expanded to $\mathcal{O}(\alpha_s)$. The beta function appearing in Eq. (20) reads

$$\beta(\alpha_s) = -2\alpha_s \sum_{n=0} \beta_n \left(\frac{\alpha_s}{4\pi} \right)^{n+1}; \quad (21)$$

thus, anomalous dimensions γ_h and γ_T need to be expanded to two-loop level,

$$\begin{aligned} \gamma_h(\alpha_s) &= \frac{\alpha_s C_F}{4\pi} \left[\gamma_h^{(0)} + \left(\frac{\alpha_s}{4\pi} \right) \gamma_h^{(1)} + \dots \right], \\ \gamma_T(\alpha_s) &= \frac{\alpha_s C_F}{4\pi} \left[-2 + \frac{\alpha_s}{4\pi} \left(19C_F - \frac{257}{9}C_A + \frac{52}{9}n_f' T_F \right) + \dots \right], \end{aligned} \quad (22)$$

where n_f' is the number of the active quark flavors. While the cusp anomalous dimension $\Gamma_{\text{cusp}}(\alpha_s)$ should be expanded to three-loop level,

$$\begin{aligned} \Gamma_{\text{cusp}}(\alpha_s) &= \frac{\alpha_s C_F}{4\pi} \left[\Gamma_{\text{cusp}}^{(0)} + \left(\frac{\alpha_s}{4\pi} \right) \Gamma_{\text{cusp}}^{(1)} + \left(\frac{\alpha_s}{4\pi} \right)^2 \Gamma_{\text{cusp}}^{(2)} + \dots \right], \end{aligned} \quad (23)$$

since the factor $\int_{\alpha_s(\mu_h)}^{\alpha_s} d\alpha' / \beta(\alpha')$ starts at $\mathcal{O}(\alpha_s^{-1})$.

Note that the ν dependence of the form factor must be canceled by that of the Wilson coefficient of the tensor current. For phenomenological applications, this renormalization scale can be fixed at $\nu = m_b$. Then, the corresponding evolution kernel reduces to 1, and the Wilson coefficient should be evolved to m_b . We rewrite the second equation of Eq. (20) as

$$C_T^{(-)}(n \cdot p, \mu) = U_1(n \cdot p, \mu_h, \mu) C_T^{(-)}(n \cdot p, \mu_h), \quad (24)$$

where the specific expression of $U_1(n \cdot p, \mu_h, \mu)$ at $\mathcal{O}(\alpha_s)$ can be found in the Appendix of Ref. [27].

RG equations of the jet function and the B -meson LCDA have the following forms:

$$\begin{aligned} \frac{d}{d \ln \mu} J_T^{(-)} \left(\frac{\mu^2}{n \cdot p \omega}, \frac{\omega}{\bar{n} \cdot p} \right) &= \left[\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{n \cdot p \omega} \right] J_T^{(-)} \left(\frac{\mu^2}{n \cdot p \omega}, \frac{\omega}{\bar{n} \cdot p} \right) \\ &+ \int_0^\infty d\omega' \omega \Gamma(\omega, \omega', \mu) J_T^{(-)} \left(\frac{\mu^2}{n \cdot p \omega'}, \frac{\omega'}{\bar{n} \cdot p} \right), \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{d}{d \ln \mu} \phi_B^-(\omega, \mu) &= - \left[\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{\omega} + \gamma_+(\alpha_s) \right] \phi_B^-(\omega, \mu) \\ &- \int_0^\infty d\omega' \omega \Gamma(\omega, \omega', \mu) \phi_B^-(\omega', \mu). \end{aligned} \quad (26)$$

These evolution kernels are nondiagonal. It is thus difficult to solve the above RG equations in the momentum space. An alternative method was suggested in Ref. [24], in which the LN kernel can be diagonalized by translating to the dual space. It was found in Ref. [28] that the basis of the dual space is the eigenfunction of the generator of special conformal transformations. The specific form of the transformation can be written by

$$\rho_B^-(\omega', \mu) = \int_0^\infty \frac{d\omega}{\omega'} J_0 \left(2\sqrt{\frac{\omega}{\omega'}} \right) \phi_B^-(\omega, \mu). \quad (27)$$

The dual-space LCDA $\rho_B^-(\omega', \mu)$ satisfies a simpler RG equation,

$$\frac{d}{d \ln \mu} \rho_B^-(\omega', \mu) = \Gamma_\rho(\mu) \rho_B^-(\omega', \mu), \quad (28)$$

where $\Gamma_\rho(\mu) = -\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{\hat{\omega}'} - \gamma_+(\alpha_s)$, and $\hat{\omega}' = e^{-2\gamma_E} \omega'$. Solving the RG equation of $\rho_B^-(\mu)$, one has

$$\rho_B^-(\omega', \mu) = e^{V(\mu, \mu_0)} \left(\frac{\mu_0}{\hat{\omega}'} \right)^{-g(\mu, \mu_0)} \rho_B^-(\omega', \mu_0), \quad (29)$$

where

$$\begin{aligned} V(\mu, \mu_0) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \left[\Gamma_{\text{cusp}}(\alpha) \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} + \gamma_+(\alpha) \right], \\ g(\mu, \mu_0) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}. \end{aligned} \quad (30)$$

Using the orthogonality of the Bessel function, we can express $\phi_B^-(\omega, \mu)$ in terms of $\rho_B^-(\omega', \mu)$:

$$\phi_B^-(\omega, \mu) = \int_0^\infty \frac{d\omega'}{\omega'} J_0 \left(2\sqrt{\frac{\omega}{\omega'}} \right) \rho_B^-(\omega', \mu). \quad (31)$$

Substituting Eq. (31) into Eq. (13), we obtain the factorization formula of the correlation function in the dual space,

$$\begin{aligned} \Pi_T(\mu, n \cdot p) &= \tilde{f}_B(\mu) m_B C_T^{(-)}(n \cdot p, \mu) \int_0^\infty \frac{d\omega'}{\omega'} j_T^{(-)}(\hat{\omega}', \mu) \rho_B^-(\omega', \mu) \\ &+ \tilde{f}_B(\mu) m_B C_T^{(+)}(n \cdot p, \mu) \int_0^\infty \frac{d\omega}{\omega - \bar{n} \cdot p} J_T^{(+)} \\ &\times (\omega, \mu) \phi_B^+(\omega, \mu), \end{aligned} \quad (32)$$

where $j_T^{(-)}(\hat{\omega}', \mu)$ is the jet function in the dual space. In the second term of the rhs, $J_T^{(+)}(\omega)$ and $\phi_B^+(\omega)$ are not transformed to the dual space because it is not necessary to perform the RG evolution to this term, which will be explained in the first comment in the next subsection. We rewrite $j^-(\hat{\omega}', \mu)$ as $j(\hat{\omega}', \mu)$, which satisfies the following RG equation:

$$\frac{d}{d \ln \mu} j(\hat{\omega}', \mu) = \Gamma_j(\mu) j(\hat{\omega}', \mu). \quad (33)$$

The factorization-scale independence of the correlation function indicates

$$\Gamma_j(\mu) = -\Gamma_C(\mu) - \Gamma_\rho(\mu) - \tilde{\gamma}(\alpha_s(\mu)), \quad (34)$$

where $\tilde{\gamma}(\alpha_s)$ is the anomalous dimension of $\tilde{f}_B(\mu)$. The RG equation of $\tilde{f}_B(\mu)$ is

$$\frac{d}{d \ln \mu} \tilde{f}_B(\mu) = \tilde{\gamma}(\alpha_s) \tilde{f}_B(\mu), \quad (35)$$

where

$$\begin{aligned} \tilde{\gamma}(\alpha_s) &= \frac{\alpha_s C_F}{4\pi} \left[\tilde{\gamma}^{(0)} + \left(\frac{\alpha_s}{4\pi} \right) \tilde{\gamma}^{(1)} + \dots \right], \\ \tilde{\gamma}^{(0)} &= 3, \quad \tilde{\gamma}^{(1)} = \frac{127}{6} + \frac{14\pi^2}{9} - \frac{5}{3} n_f, \end{aligned} \quad (36)$$

and n_f is the number of light-quark flavors. The evolution factor of $\tilde{f}_B(\mu)$ is

$$\begin{aligned} U_2(\mu_{h2}, \mu) &= \text{Exp} \left[\int_{\alpha_s(\mu_{h2})}^{\alpha_s(\mu)} d\alpha_s \frac{\tilde{\gamma}(\alpha_s)}{\beta(\alpha_s)} \right] \\ &= z^{-\frac{\tilde{\gamma}^{(0)}}{2\beta_0} C_F} \left[1 + \frac{\alpha_s(\mu_{h2}) C_F}{4\pi} \left(\frac{\tilde{\gamma}^{(1)}}{2\beta_0} - \frac{\tilde{\gamma}^{(0)} \beta_1}{2\beta_0^2} \right) (1-z) + \mathcal{O}(\alpha_s^2) \right], \end{aligned} \quad (37)$$

where $z = \alpha_s(\mu)/\alpha_s(\mu_{h2})$.

The anomalous dimension of the jet function can be expressed as

$$\Gamma_j = \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{n \cdot p \hat{\omega}'} + \gamma_{hc}(\alpha_s), \quad (38)$$

where $\gamma_{hc}(\alpha_s) = \frac{\alpha_s C_F}{4\pi} [\gamma_{hc}^{(0)} + (\frac{\alpha_s}{4\pi}) \gamma_{hc}^{(1)} + \dots]$. At one-loop level, $\gamma_{hc}^{(0)} = 0$. There has been no calculation about the two-loop anomalous dimension $\gamma_{hc}^{(1)}$ until now. This parameter could not be determined by the factorization-scale independence of the correlation function ($\gamma_{hc}^{(1)} = -\gamma_h^{(1)} + \gamma_+^{(1)} - \tilde{\gamma}^{(1)}$), since $\gamma_+^{(1)}$ is yet unknown. However, it has been checked numerically that the form factors are insensitive to $\gamma_{hc}^{(1)}$. In addition, NLO corrections should not be very large for the convergence of the α_s expansion. We choose $\gamma_{hc}^{(1)} = 0$ in our calculation. The solution of Eq. (33) can be obtained straightforwardly,

$$j(\hat{\omega}', \mu) = e^{-2V_{hc}(\mu, \mu_{hc})} \left(\frac{\mu_{hc}^2}{\hat{\omega}' \bar{n} \cdot p} \right)^{g(\mu, \mu_{hc})} j(\hat{\omega}', \mu_{hc}), \quad (39)$$

with $j(\hat{\omega}', \mu_{hc})$ given by

$$\begin{aligned} j(\hat{\omega}', \mu_{hc}) &= 2K_0 \left(2\sqrt{\frac{1}{\eta'}} \right) \left\{ 1 + \frac{\alpha_s C_F}{4\pi} \left[\ln^2 \frac{\mu_{hc}^2}{-p^2} - \frac{\pi^2}{3} - 1 - \frac{1}{2} \ln \hat{\eta}' \left(2 \ln \frac{\mu_{hc}^2}{-p^2} + 3 \right) + \frac{1}{4} \ln^2 \hat{\eta}' \right] \right\} \\ &\quad + \frac{\alpha_s C_F}{2\pi} K_0^{(2,0)} \left(2\sqrt{\frac{1}{\eta'}} \right) + \frac{\alpha_s C_F}{\pi} \int_{2\sqrt{\frac{1}{\eta'}}}^{\infty} \frac{d\beta}{\beta} K_0(\beta), \end{aligned} \quad (40)$$

where $\hat{\eta}' = e^{-2\gamma_E} \eta' = -\hat{\omega}'/\bar{n} \cdot p$. The detailed derivation of $j(\hat{\omega}', \mu_{hc})$ is given in Appendix A. Collecting evolution factors of the hard function, the jet function, the B -meson LCDA, and the B -meson decay constant together, we obtain the RG improved correlation function

$$\begin{aligned} \Pi_T(\bar{n} \cdot p) &= m_B [U_1(n \cdot p, \mu_{h1}, \mu) U_2(\mu_{h2}, \mu)] [\tilde{f}_B(\mu_{h2}) C_T^{(-)}(n \cdot p, \mu_{h1})] \\ &\quad \times \int_0^\infty \frac{d\omega'}{\omega'} [U_j(n \cdot p, \mu_{hc}, \mu) U_\rho(n \cdot p, \mu_0, \mu)] j(\hat{\omega}', \mu_{hc}) \rho_B^-(\omega', \mu_0) \\ &\quad + m_B [U_1(n \cdot p, \mu_{h1}, \mu) U_2(\mu_{h2}, \mu)] \tilde{f}_B(\mu_{h2}) \int_0^\infty d\omega \frac{\phi_B^+(\omega, \mu)}{\omega - \bar{n} \cdot p} J_T^{(+)} \left(\frac{\omega}{\bar{n} \cdot p} \right), \end{aligned} \quad (41)$$

where

$$U_j(n \cdot p, \mu_{hc}, \mu) = e^{-2V_{hc}(\mu, \mu_{hc})} \left(\frac{\mu_{hc}^2}{\hat{\omega}' \bar{n} \cdot p} \right)^{g(\mu, \mu_{hc})},$$

$$U_\rho(n \cdot p, \mu_0, \mu) = e^{V(\mu, \mu_0)} \left(\frac{\mu_0}{\hat{\omega}'} \right)^{-g(\mu, \mu_0)}. \quad (42)$$

We are now in the position to construct the sum rules for the three $B \rightarrow \pi$ form factors at NLO. Useful dispersion integrals are collected in Appendix B. Performing the Borel transformation, we obtain the sum rules for the $B \rightarrow \pi$ form factors. The tensor form factor reads

$$f_\pi e^{-m_\pi^2 n \cdot p / \omega_M^2} f_{B\pi}^T(q^2) = [U_1(n \cdot p, \mu_{h1}, \mu) U_2(\mu_{h2}, \mu)] \tilde{f}_B(\mu_{h2})$$

$$\times \int_0^{\omega_s} d\omega e^{-\omega / \omega_M} [C_T^{(-)}(n \cdot p, \mu_{h1}) \rho_{\text{eff}}^-(\omega) - \phi_{B,\text{eff}}^+(\omega)], \quad (43)$$

where

$$\phi_{B,\text{eff}}^+(\Omega, \mu) = \frac{\alpha_s C_F}{4\pi} \int_\Omega^\infty \frac{d\omega}{\omega} \phi_B^+(\omega, \mu),$$

$$\rho_{\text{eff}}^-(\Omega, \mu) = \int_0^\infty \frac{d\omega'}{\omega'} \left\{ \left[1 + \frac{\alpha_s C_F}{4\pi} \left(\ln^2 \frac{\mu^2}{n \cdot p \Omega} - 2 \ln \frac{\mu^2}{n \cdot p \Omega} \ln \frac{\hat{\omega}'}{\Omega} \right. \right. \right.$$

$$\left. \left. + \frac{1}{2} \ln^2 \frac{\hat{\omega}'}{\Omega} - \frac{3}{2} \ln \frac{\hat{\omega}'}{\Omega} + \frac{\pi^2}{2} - 1 \right) \right] J_0 \left(2\sqrt{\frac{\Omega}{\omega'}} \right)$$

$$+ \frac{\alpha_s C_F}{4\pi} \left(\ln \frac{\hat{\omega}'}{\Omega} + \frac{3}{2} \right) \pi N_0 \left(2\sqrt{\frac{\Omega}{\omega'}} \right)$$

$$\left. + \frac{\alpha_s C_F}{2\pi} \left[J_0^{(2,0)} \left(2\sqrt{\frac{\Omega}{\omega'}} \right) + \frac{\Omega}{\omega'} {}_2F_3 \left(1, 1; 2, 2, 2; -\frac{\Omega}{\omega'} \right) - \ln \frac{\Omega}{\hat{\omega}'} \right] \right\}$$

$$\times U_j(\mu_{hc}, \mu) U_\rho(\mu_0, \mu) \rho_B^{(-)}(\omega', \mu_0), \quad (44)$$

with ${}_pF_q(a_1 \dots a_p; b_1 \dots b_q; z)$ being the generalized hypergeometric function. Vector and scalar form factors $f_{B\pi}^{+,0}$ have similar forms,

$$f_\pi e^{-m_\pi^2 / (n \cdot p \omega_M)} \{r' f_{B\pi}^+(q^2), f_{B\pi}^0(q^2)\} = [U_1(n \cdot p, \mu_{h1}, \mu) U_2(\mu_{h2}, \mu)] \tilde{f}_B(\mu_{h2}) \int_0^{\omega_s} d\omega' e^{-\omega' / \omega_M}$$

$$\times \{[C_{\bar{n}}^{(-)} - \bar{r}' C_n^{(-)}](n \cdot p, \mu_{h1}) \rho_{B,\text{eff}}^-(\omega', \mu) + (r \mp \bar{r}') \phi_{B,\text{eff}}^+(\omega', \mu)\}, \quad (45)$$

where $r' = n \cdot p / m_B$. For the above results, several comments are as follows:

- (i) We perform the complete RG evolution of ϕ_B^- terms. For ϕ_B^+ terms, we only apply the RG evolution to the hard function and the B -meson decay constant, not to $J_i^{(+)}$ and ϕ_B^+ since, first, the anomalous dimension of $J_i^{(+)}$ is yet unknown and, second, evolution effects of the jet function and the B -meson LCDA will partially cancel each other. In principle, we should resum logarithmic terms in $J_i^{(+)}$ and ϕ_B^+ to leading-logarithm level because the jet function starts at $\mathcal{O}(\alpha_s)$. While due to the factorization-scale independence of the correlation function, we set μ to be a

hard-collinear scale in the numerical analysis, which means that there are no large logarithms in $J_i^{(+)}$ and ϕ_B^+ . Also, uncertainties, arising from the variation of μ , of these ϕ_B^+ terms should be small since these terms are suppressed by α_s . As we can see from the numerical analysis, the factorization-scale dependence of these terms is negligible.

- (ii) Dispersion integrals of the correlation function in the momentum space is nontrivial for the appearance of both pole and branch-cut singularities, which can be seen in the Appendix of Ref. [16]. However, in the dual space, the imaginary part of the jet function in Eq. (44) can be obtained much more easily; i.e., we can simply make the replacement $\bar{n} \cdot p \rightarrow \Omega e^{i\pi}$.

IV. NUMERICAL ANALYSIS

In this section, we perform the numerical analysis of the $B \rightarrow \pi$ form factors. B -meson LCDAs serve as fundamental ingredients of the B -meson LCSR approach. Nevertheless, there is very limited knowledge about these LCDAs so far. Several phenomenological models of the B -meson LCDA ϕ_B^+ are suggested. We employ three typical models [22,29,30]:

$$\begin{aligned}\phi_{B1}^+(\omega, \mu_0) &= \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0}, \\ \phi_{B2}^+(\omega, \mu_0) &= \frac{\omega}{2\omega_0} \theta(2\omega_0 - \omega), \\ \phi_{B3}^+(\omega, \mu_0) &= \frac{3}{4\omega_1^3} \theta(2\omega_1 - \omega) \omega(2\omega_1 - \omega), \quad \omega_1 = \frac{3}{2}\omega_0.\end{aligned}\quad (46)$$

Neglecting the contribution from the three-particle Fock state, $\phi_B^-(\omega, \mu_0)$ is determined by the Wandzura-Wilczek approximation [18]

$$\phi_B^-(\omega, \mu_0) = \int_0^1 \frac{d\xi}{\xi} \phi_B^+\left(\frac{\omega}{\xi}, \mu_0\right). \quad (47)$$

The parameter ω_0 equals the inverse moment of B -meson LCDAs, i.e.,

$$\lambda_B^{-1}(\mu) = \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega, \mu) = \frac{1}{\omega_0}. \quad (48)$$

λ_B is closely related to exclusive B -meson decays. The experimental data can only give a very rough constraint on this parameter. In the previous work [16], the inverse moment is determined by fixing $f_{B\pi}^+(0) = 0.28 \pm 0.03$, which is the prediction of the pion LCSR [8]. In this paper, we adopt the same value as in Ref. [16] for comparison, i.e., $\omega_0 = 0.354_{-0.030}^{+0.038}$ (GeV).

Because the RG equation of ϕ_B^- is evaluated in the dual space, the corresponding dual-space expression of this LCDA is required. ρ_B^- has a similar meaning with Gegenbauer moments of the light-meson LCDAs. Explicit forms of ρ_B^- are

$$\begin{aligned}\rho_{B1}^-(\omega', \mu) &= \rho_{B1}^+(\omega', \mu) = \frac{1}{\omega'} e^{-\omega_0/\omega'}, \\ \rho_{B2}^-(\omega', \mu) &= \rho_{B2}^+(\omega', \mu) = \frac{1}{2\omega_0} J_2\left(2\sqrt{\frac{2\omega_0}{\omega'}}\right), \\ \rho_{B3}^-(\omega', \mu) &= \rho_{B3}^+(\omega', \mu) = \frac{3}{4\omega_1} \sqrt{\frac{\omega'}{2\omega_1}} J_3\left(2\sqrt{\frac{2\omega_1}{\omega'}}\right).\end{aligned}\quad (49)$$

FIG. 3. Shapes of B -meson LCDAs in the dual space. Black, red, and blue lines correspond to $\rho_{B1}^-(\omega', \mu)$, $\rho_{B2}^-(\omega', \mu)$, and $\rho_{B3}^-(\omega', \mu)$, respectively.

To give a more intuitive picture of LCDAs in the dual space, we plot the ω' dependence of them in Fig. 3. The dual-space LCDAs are factorization-scale dependent. The RG evolution effect can modify the behavior of the original model [31], and this effect has been considered in our calculation of the form factors.

Before presenting numerical results of the form factors, we first show behaviors of evolution factors of the hard function [$U_1(\mu, \mu_{h1})$], the jet function [$U_j(\mu, \mu_{hc})$], the B -meson LCDA [$U\rho(\mu, \mu_0)$], and the B -meson decay constant [$U_2(\mu, \mu_{h2})$], in Fig. 4. ω' is fixed at 1.0 GeV when plotting this figure. This choice leads to large

FIG. 4. Evolution factors of the hard function, the B -meson decay constant, the jet function, and the B -meson LCDA, which are denoted using dotted, solid, dashed, and dot-dashed lines, respectively.

logarithmic terms in evolution kernels of the LCDA and the jet function; hence, evolution effects of these two functions are significant. But there is a strong cancellation between these two effects due to different signs of slopes of their curves. This cancellation is important to guarantee the scale invariance of the form factors. To illustrate the effect of the RG evolution, we plot in Fig. 5 the scale dependence of $f_{B\pi}^{(+)}(0)$, where the first type of B -meson LCDAs $\phi_{B1}^{\pm}(\omega)$ and $\rho_{B1}^-(\omega')$ are employed. It is obvious that after the complete RG evolution the theoretical prediction of the form factor is almost independent of the factorization scale as expected. Results of $f_{B\pi}^{(+)}(0)$ with leading-logarithm (LL) resummation and next-to-leading-logarithm (NLL) resummation are both displayed for a comparison. It can be seen that the scale dependence is mild in both cases, but the NLL resummation reduces the value of the form factor about 3% compared to the LL-resummation value. For terms containing ϕ_B^+ , the RG evolution has not been performed (suppressed by the coupling constant). This figure also indicates that ϕ_B^+ terms of the form factors are almost factorization-scale independent. In the numerical analysis, we set the factorization scale to be a hard-collinear scale ($\mu = 1.5 \pm 0.5$ GeV), as there are no large logarithmic terms in ϕ_B^+ terms at this scale [16].

Since B -meson LCDAs are most important inputs of the B -meson LCSR, we need to test the LCDA-model dependence of the form factors. In Fig. 6, the vector form factor computed with three different B -meson LCDA models is displayed. Central values of the inverse moment are fitted as 0.392 in ρ_{B2}^- and 0.382 in ρ_{B3}^- . From this figure, we can see that the model of the B -meson LCDA has a tiny

FIG. 5. The factorization-scale dependence of $f_{B\pi}^{+}(0)$. Solid, dotted, dot-dashed, and dashed lines stand for values of the form factor with full RG evolution at the NLL level, at the LL level, with RG evolution only with respect to the hard coefficient and the B -meson decay constant, and without RG evolution, respectively.

FIG. 6. The LCDA-model dependence of the vector form factor. Black, blue, and red lines stand for the form factor computed with $\rho_{B1}^-(\omega', \mu)$, $\rho_{B2}^-(\omega', \mu)$, and $\rho_{B3}^-(\omega', \mu)$, respectively.

influence on the shape of the form factor. Hereafter, we will take ρ_{B1}^- as the default model.

In the LCSR approach, the form factors should be insensitive to the Borel parameter and the effective threshold. These parameters are constrained following conditions in Ref. [16], where the contribution from excited and continuum states should be less than 40% and the rate of change $\frac{\partial \ln f_{B\pi}^+(q^2)}{\partial \ln \omega_M} \leq 35\%$. We fix $q^2 = 0$ to study ω_M and s_0 dependence of the form factors. The above constraints lead to a region $0.24 \leq \omega_M \leq 0.36$ (corresponding to $1.0 \leq M^2/\text{GeV}^2 \leq 1.5$) for all the three form factors. We plot the Borel mass dependence of the form factor $f_{B\pi}^{+}(0)$ in Fig. 7; a manifest platform at $M^2 \in [1.0 \text{ GeV}^2, 1.5 \text{ GeV}^2]$ guarantees that our calculation is insensitive to this unphysical parameter. The form factors are also almost independent on the effective threshold s_0 when it is adopted as $s_0 = (0.7 \pm 0.05) \text{ GeV}^2$.

It has been argued that the $B \rightarrow \pi$ form factors calculated using the B -meson LCSR can be trusted at $q^2 \leq q_{\text{max}}^2 = 8 \text{ GeV}^2$ (see Ref. [11] for more detailed discussions). To extrapolate the form factors calculated with the B -meson LCSR at large recoil toward large momentum transfer, we follow the same vein as Ref. [16], in which the z -series parametrization was employed. In this parametrization, the cut q^2 plane (the branch cut is the $q^2 > t_+$ region of the real axis) is mapped onto the unit disk $|z(q^2, t_0)| < 1$ via the conformal transformation

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}, \quad (50)$$

where $t_+ = (m_B + m_\pi)^2$ denotes the threshold of continuum states in the B^* -meson channel. The free parameter

FIG. 7. The Borel parameter and the effective threshold dependence of $f_{B\pi}^+(0)$. Solid, dot-dashed, and dotted lines in the left (right) figure correspond to $s_0 = 0.7, 0.75$, and 0.65 GeV^2 ($M^2 = 1.25, 1.0$, and 1.5 GeV^2), respectively.

$t_0 \in (-\infty, t_+)$ determines the value of q^2 mapped onto the origin in the z plane. One can adjust the value of t_0 to minimize the corresponding z interval in the region $q_{\min}^2 \leq q^2 \leq q_{\max}^2$, in order that the z -series expansion converges rapidly. Here, we choose the same value as that in Ref. [8],

$$t_0 = t_+^2 - \sqrt{t_+ - t_-} \sqrt{t_+ - q_{\min}^2}, \quad (51)$$

where $q_{\min}^2 = -6.0 \text{ GeV}^2$ and $t_- \equiv (m_B - m_\pi)^2$. Using the z -series expansion and taking into account the threshold t_+ behavior, one can obtain the parametrization of each form factor.

Parametrizations of the vector and the scalar form factors have been given in Refs. [8,32]. The parametrization of the tensor form factor is similar to that of $f_{B\pi}^+(q^2)$ [33],

$$f_{B\pi}^T(q^2) = \frac{f_{B\pi}^T(0)}{1 - q^2/m_{B^*}^2} \left\{ 1 + \sum_{k=1}^{N-1} b_k^T (z(q^2, t_0)^k - z(0, t_0)^k) - (-1)^{N-k} \frac{k}{N} [z(q^2, t_0)^N - z(0, t_0)^N] \right\}, \quad (52)$$

where the expansion coefficient(s) b_k^T is (are) determined by matching the large-recoil $f_{B\pi}^T(q^2)$ onto Eq. (52). As the interval in the z plane is constrained in a small region, it is reasonable to truncate the z series at $N = 2$ in the practical calculation. Slope parameters b_1 , $\tilde{b}_1$, and b_1^T are collected in Table I. Uncertainties from different sources, including the inverse moment, the model of B -meson LCDA, the Borel parameter, the effective threshold, quark masses, etc., are taken into account in our numerical analysis. In Table I, we collect parameters which cause large uncertainties.

In Fig. 8, the q^2 dependence of form factors $f_{B\pi}^{0,+}(q^2)$ is shown, where lattice results are from the HPQCD Collaboration [34], Fermilab/MILC Collaboration [35], and RBC/UKQCD Collaboration [36]. Our results of $f_{B\pi}^0$ within errors are in agreement with the lattice data. However, our results of $f_{B\pi}^+$ is larger than the lattice data. Results of $f_{B\pi}^{0,+}(q^2)$ from the pion LCSR are also shown. It is manifest that slopes of these two form factors with the B -meson LCSR are greater than that of the pion LCSR. The difference between the B -meson and other approaches can be understood through the following points:

- (1) As we can see from Table I, the parameter λ_B brings huge uncertainty to results of the form factors.

TABLE I. z -parameter fitted values of $f_{B\pi}^+(0)$, $f_{B\pi}^T(0)$, b_1 , $\tilde{b}_1$, and b_1^T [$f_{B\pi}^0(0)$ is not listed here because $f_{B\pi}^0(0) = f_{B\pi}^+(0)$]. The notation “default” means that all of the parameters are taken as central values.

Parameter	Default	$\lambda_B(1 \text{ GeV})$	ω_M	$\overline{m}_b(\overline{m}_b)$	s_0^B	LCDA
$f_{B\pi}^+(0)$	0.254	+0.022 -0.024	+0.008 -0.012	+0.009 -0.011	+0.011 -0.006	+0.075 -0.000
$f_{B\pi}^T(0)$	0.254	+0.022 -0.024	+0.007 -0.012	+0.009 -0.012	+0.011 -0.006	+0.076 -0.000
b_1	-4.13	+0.08 -0.09	+0.02 -0.05	+0.00 -0.01	+0.00 -0.01	+0.64 -0.00
$\tilde{b}_1$	-5.56	+0.11 -0.12	+0.03 -0.06	+0.00 -0.01	+0.00 -0.00	+0.79 -0.00
b_1^T	-4.42	+0.08 -0.11	+0.02 -0.10	+0.00 -0.01	+0.00 -0.01	+0.71 -0.00

FIG. 8. q^2 dependence of the form factor $f_{B\pi}^0(q^2)$ and the rescaled form factor $f_{B\pi}^+(q^2)$. Black and red curves are results of the B -meson and the pion LCSRs, respectively. Lattice QCD results are taken from the HPQCD Collaboration [34] (blue squares), Fermilab/MILC Collaboration [35] (blue band), and RBC/UKQCD Collaboration [36] (green and blue triangles).

Values of the form factors at $q^2 = 0 \text{ GeV}^2$ are significantly influenced by the changing of λ_B . But the value of this parameter is not determined yet.

- (2) We only calculate leading-power contributions of the form factors in this work, while power-suppressed contributions, which are induced by higher-twist pion LCDAs, are taken into account in calculations of the pion LCSR. The subleading-power effect in the B -meson LCSR may influence both values of the form factors at $q^2 = 0 \text{ GeV}^2$ and slopes of the form factors.

$B \rightarrow \pi$ form factors are very important phenomenologically. Here, we briefly discuss two applications of our result. The CKM matrix element $|V_{ub}|$ can be determined from the (partial) branching fraction of $B \rightarrow \pi \ell \nu_\ell$. If we neglect mass of leptons, the integrated decay width is written by

$$\int_0^{q_0^2} dq^2 \frac{d\Gamma}{dq^2}(B \rightarrow \pi \ell \nu) \equiv |V_{ub}|^2 \Delta\zeta(0, q_0^2), \quad (53)$$

where $|\vec{p}_\pi|$ is the magnitude of the pion 3-momentum in the B -meson rest frame, $\ell = e, \mu$, and

$$\Delta\zeta(0, q_0^2) = \frac{G_F^2}{24\pi^3} \int_0^{q_0^2} dq^2 |\vec{p}_\pi|^3 |f_{B\pi}^+(q^2)|^2. \quad (54)$$

A straightforward extraction of $|V_{ub}|$ can be performed using the relation

$$|V_{ub}|^2 = \frac{\Delta\mathcal{BR}(0, q_0^2) \tau_{B^0}}{\Delta\zeta(0, q_0^2)}, \quad (55)$$

where $\Delta\mathcal{BR}(0, q_0^2)$ is the integrated branching ratio and the mean lifetime of the B^0 meson $\tau_{B^0} = (1.519 \pm 0.005) \text{ ps}$

[37]. Experimental measurements of $\Delta\mathcal{BR}(0, q_0^2)$ of the semileptonic $\bar{B}^0 \rightarrow \pi^+ \mu \nu_\mu$ decay [38,39] are given by

$$\Delta\mathcal{BR}(0, 12 \text{ GeV}^2) = (0.83 \pm 0.03 \pm 0.04) \times 10^{-4} \quad (\text{BABAR 2012}),$$

$$\Delta\mathcal{BR}(0, 12 \text{ GeV}^2) = (0.808 \pm 0.062) \times 10^{-4} \quad (\text{Belle 2013}). \quad (56)$$

Utilizing the result of the form factor $f_{B\pi}^+(q^2)$, which is computed with the B -meson LCSR and extrapolated with the z -series parametrization, we can obtain

$$\begin{aligned} \Delta\zeta(0, 12 \text{ GeV}^2) &= 4.93_{-0.05}^{+0.30} \bigg|_{\omega_0}^{+0.36}_{-0.41} \bigg|_{\omega_M}^{+0.44}_{-0.23} \bigg|_{s_0}^{+2.79}_{-0} \bigg|_{\phi_B} \text{ ps}^{-1} \\ &= 4.93_{-0.97}^{+2.99} \text{ ps}^{-1}. \end{aligned} \quad (57)$$

Then, the extracted CKM matrix element

$$|V_{ub}| = (3.33_{-0.74}^{+0.37})_{\text{th}} \pm 0.09_{\text{exp}} \times 10^{-3}, \quad (58)$$

where the theoretical uncertainty comes from uncertainties of $\Delta\zeta(0, 12 \text{ GeV}^2)$ as displayed in Eq. (57). This $|V_{ub}|$ is larger compared to that in Ref. [16], since the RG evolution reduces the value of $f_{B\pi}^+(q^2)$. In Fig. 9, we display the normalized differential q^2 distribution of $B \rightarrow \pi \mu \nu_\mu$. Black curves represent the prediction of this work, where the solid line is the central value and dashed curves correspond to uncertainties. Because of the cancellation of the uncertainty of $f_{B\pi}^+(q^2)$, the uncertainty of the normalized differential distribution of $B \rightarrow \pi \mu \nu_\mu$ is small. Our prediction is in agreement with the experimental data from BABAR [38,40,41] and Belle [39,42].

FIG. 9. The normalized differential q^2 distribution of $B \rightarrow \pi\mu\nu_\mu$. The black solid curve represents the central value of our prediction, and black dashed curves correspond to uncertainties. Experimental data bins are from Refs. [38] (blue stars), [39] (red stars), [40] (blue circles), [41] (blue triangles), and [42] (red circles).

V. CONCLUSION AND DISCUSSION

We reviewed the method of calculating the $B \rightarrow \pi$ tensor form factor with the B -meson LCSR. In this framework, the method of regions was employed, and contributions from different momentum regions were separated naturally. Precise soft cancellation guarantees the factorization theorem. The correlation function was factorized into the convolution of the hard function, the jet function, and the B -meson LCDA, which correspond to contributions from hard, hard-collinear, and soft regions, respectively. We obtained one loop-level hard and jet functions through the analysis of symmetry-breaking effects.

To resum large logarithmic terms in the form factors, we carried out the complete RG evolution of the factorized

correlation function, including evolutions of the jet function and the B -meson LCDA. The B -meson LCDAs, defined via the heavy quark effective theory, obey the Lange-Neubert equation, which contains the nondiagonal anomalous dimension. Following the approach in Ref. [24], we diagonalized the RG equation of the B -meson LCDA in the dual space and solved the diagonalized RG equation. The same method was also applied to the evolution of the jet function. Combining the evolution of each part together, we obtained the RG-improved $B \rightarrow \pi$ form factors.

On the numerical side, we checked behaviors of the four evolution kernels (U_1 , U_2 , U_j , and U_ρ) and illustrated cancellation effects among the kernels. We examined the factorization-scale dependence of the RG-improved form factors and compared our predictions with previous results. We extrapolated the q^2 dependence of the form factors to the whole physical region using the z -series expansion. Then, we compared values of the form factors in this work with that in the lattice QCD and the pion LCSR. Phenomenologically, we extracted the CKM matrix element $|V_{ub}|$ and analyzed the normalized differential q^2 dependence of $B \rightarrow \pi\mu\nu_\mu$. The B -meson form factors have many other phenomenological applications, such as the fact that the tensor form factor can give important contributions to flavor changing neutral current processes $B \rightarrow (\pi, K)l^+l^-$. Of course, a complete study of phenomenological applications are far more complicated, and we left it for the future work. This work supplements the framework proposed in Ref. [16] and can be applied to various transition processes.

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APPENDIX A: JET FUNCTION IN THE DUAL SPACE

The jet function in the dual space is defined by

$$\begin{aligned} j^{(-)}\left(\frac{\mu_{hc}^2}{n \cdot p \hat{\omega}'}, \frac{\hat{\omega}'}{\bar{n} \cdot p}\right) &= \int_0^\infty \frac{d\omega}{\omega - \bar{n} \cdot p} J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) J^{(-)}\left(\frac{\mu^2}{n \cdot p \omega}, \frac{\omega}{\bar{n} \cdot p}\right) \\ &= \int_0^\infty \frac{d\eta}{1+\eta} J_0\left(2\sqrt{\frac{\eta}{\eta'}}\right) J^{(-)}(\eta, \mu), \end{aligned} \quad (\text{A1})$$

where

$$J^{(-)}(\eta, \mu) = 1 + \frac{\alpha_s C_F}{4\pi} \left[\ln^2 \frac{\mu^2}{-p^2} - 2 \ln(1+\eta) \ln \frac{\mu^2}{-p^2} + \ln^2(1+\eta) - \frac{\eta-2}{\eta} \ln(1+\eta) - \frac{\pi^2}{6} - 1 \right]. \quad (\text{A2})$$

Using the formula

$$\int_0^\infty \frac{x dx}{(x^2 + k^2)^{1-\lambda}} J_0(ax) = \frac{1}{\Gamma(1-\lambda)} \left(\frac{2k}{a}\right)^\lambda K_\lambda(ka), \quad (\text{A3})$$

which is valid for $\lambda < 3/4$, and performing a derivative with respect to λ and taking the limit $\lambda \rightarrow 0$, we can get

$$\begin{aligned} \int_0^\infty \frac{d\eta}{1+\eta} J_0\left(2\sqrt{\frac{\eta}{\eta'}}\right) &= 2K_0\left(2\sqrt{\frac{1}{\eta'}}\right), \\ \int_0^\infty \frac{d\eta}{1+\eta} J_0\left(2\sqrt{\frac{\eta}{\eta'}}\right) \ln(1+\eta) &= (\ln \eta' - 2\gamma_E) K_0\left(2\sqrt{\frac{1}{\eta'}}\right), \\ \int_0^\infty \frac{d\eta}{1+\eta} J_0\left(2\sqrt{\frac{\eta}{\eta'}}\right) \ln^2(1+\eta) &= \left[\frac{1}{2} \ln^2 \eta' - 2\gamma_E \ln \eta' + 2\gamma_E^2 - 2\psi'(1)\right] K_0\left(2\sqrt{\frac{1}{\eta'}}\right) + 2K_0^{(2,0)}\left(2\sqrt{\frac{1}{\eta'}}\right). \end{aligned} \quad (\text{A4})$$

Another useful equation is

$$\int_0^\infty \frac{dx}{x} J_0(bx) \ln(1+x^2) dx = 2 \int_b^\infty \frac{K_0(\beta)}{\beta} d\beta. \quad (\text{A5})$$

Taking the advantage of Eqs. (A4) and (A5), one can obtain Eq. (40).

APPENDIX B: DISPERSION INTEGRALS

To obtain final expressions of the form factors, we need to extrapolate $\vec{n} \cdot \vec{p}$ to the physical region. For the consistency of our derivation, we must have

$$\int_0^\infty \frac{d\omega}{\omega - \Omega - i\epsilon} J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) = 2K_0\left(-2i\sqrt{\frac{\Omega}{\omega'}}\right). \quad (\text{B1})$$

The above equation indicates that the branch cut of the root and logarithmic function are along the negative real axis, and $-\Omega - i\epsilon = \Omega e^{-i\pi}$. The following equation can be derived from Eq. (A3):

$$\int_0^\infty \frac{d\omega}{\omega - \Omega - i\epsilon} \left(1 - \frac{\omega}{\Omega}\right)^\lambda J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) = i\pi e^{i\lambda\pi} \left(\frac{\omega'}{\Omega}\right)^{\lambda/2} \frac{1}{\Gamma(1-\lambda)} H_\lambda^{(1)}\left(2\sqrt{\frac{\Omega}{\omega'}}\right). \quad (\text{B2})$$

From this equation, we obtain the following useful results:

$$\begin{aligned} \int_0^\infty \frac{d\omega}{\omega - \Omega - i\epsilon} \ln\left(1 - \frac{\omega}{\Omega}\right) J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) &= -\frac{\pi^2}{2} J_0\left(2\sqrt{\frac{\Omega}{\omega'}}\right) - \frac{\pi}{2} \ln \frac{\omega'}{\Omega} N_0\left(2\sqrt{\frac{\Omega}{\omega'}}\right) \\ &\quad - i \left[\frac{\pi^2}{2} N_0\left(2\sqrt{\frac{\Omega}{\omega'}}\right) - \frac{\pi}{2} \ln \frac{\omega'}{\Omega} J_0\left(2\sqrt{\frac{\Omega}{\omega'}}\right) \right], \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \int_0^\infty \frac{d\omega}{\omega - \Omega - i\epsilon} \ln^2\left(1 - \frac{\omega}{\Omega}\right) J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) &= \frac{i\pi}{2} \left(\frac{1}{2} \ln^2 \frac{\omega'}{\Omega} + i\pi \ln \frac{\omega'}{\Omega} - \frac{\pi^2}{3} \right) H_0^{(1)}\left(2\sqrt{\frac{\Omega}{\omega'}}\right) \\ &\quad + i\pi J_0^{(2,0)}\left(2\sqrt{\frac{\Omega}{\omega'}}\right) - \pi N_0^{(2,0)}\left(2\sqrt{\frac{\Omega}{\omega'}}\right). \end{aligned} \quad (\text{B4})$$

Following a similar way, another useful result is also obtained:

$$\int_0^\infty \frac{d\omega}{\omega} \ln\left(1 - \frac{\omega}{\Omega}\right) J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) = 2i\pi \int_{2\sqrt{\frac{\Omega}{\omega}}}^\infty \frac{d\beta}{\beta} H_0^{(1)}(\beta). \quad (\text{B5})$$

Taking the imaginary part of the above equation, we have

$$\begin{aligned} \text{Im} \int_0^\infty \frac{d\omega}{\omega} \ln\left(1 - \frac{\omega}{\Omega}\right) J_0\left(2\sqrt{\frac{\omega}{\omega'}}\right) \\ = 2\pi \left[-\gamma_E + \frac{\Omega}{\omega'} {}_2F_3\left(1, 1; 2, 2, 2; -\frac{\Omega}{\omega'}\right) - \ln \frac{\Omega}{\omega'} \right]. \quad (\text{B6}) \end{aligned}$$

Having all of the above equations in hand, we get final results in Eqs. (43) and (44).

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