Chiral anomaly, Berry phase, and chiral kinetic theory from worldlines in quantum field theory

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In previous work, we outlined a worldline framework that can be used for systematic computations of the chiral magnetic effect (CME) in ultrarelativistic heavy-ion collisions. Towards this end, we first expressed the real part of the fermion determinant in the QCD effective action as a supersymmetric worldline action of spinning, colored, Grassmannian point particles in background gauge fields, with equations of motion that are covariant generalizations of the Bargmann-Michel-Telegdi and Wong equations. The chiral anomaly, in contrast, arises from the phase of the fermion determinant. Remarkably, the latter too can be expressed as a point particle worldline path integral, which can be employed to derive the anomalous axial vector current. We will show here how Berry's phase can be obtained in a consistent nonrelativistic adiabatic limit of the real part of the fermion determinant. Our work provides a general first principles demonstration that the topology of Berry's phase is distinct from that of the chiral anomaly confirming prior arguments by Fujikawa in specific contexts. This suggests that chiral kinetic treatments of the CME in heavy-ion collisions that include Berry's phase alone are incomplete. We outline the elements of a worldline covariant relativistic chiral kinetic theory that captures the physics of how the chiral current is modified by many-body scattering and topological fluctuations.

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I. INTRODUCTION

The possibility that topological sphaleron transitions can be identified in heavy-ion collision experiments has aroused great interest. Besides the information they provide about the nonperturbative real time dynamics of the QCD vacuum, sphaleron transitions are conjectured to play a role in electroweak baryogenesis [1,2]. A striking manifestation of the role of topology is the chiral magnetic effect (CME) where, as a consequence of the chiral anomaly, an induced current is generated in the direction of the external magnetic field [3,4]. Whether such an effect is seen in heavy-ion collisions is still unclear and is a focus of experimental research in the field [5,6]. We note that the CME has been observed in condensed matter systems [7].

For the CME to be large enough to be observed, it must be generated at the earliest times in the heavy-ion collision where the magnetic fields are very large initially, before dying off rapidly [8,9]. First principles weak coupling computations of sphaleron transitions [10] in the produced

nonequilibrium Glasma matter indicate that the sphaleron transition rate is significantly larger,¹ than the corresponding equilibrium rate [11]. Because the occupancies of gluons in the Glasma are large, classical-statistical simulations can be employed [12] to compute *ab initio*, in a sphaleron background, and in the presence of external magnetic fields [13,14], the development of a chiral magnetic, and accompanying “chiral separation,” wave [15,16].

Detailed thermalization scenarios however suggest that thermalization occurs at parametrically later times [17,18]. Because the gluon occupancy is of order unity or lower in this regime, kinetic theory provides the appropriate description of quark-gluon dynamics. The classical-statistical simulations, that are valid for the early time description of the CME, must therefore be matched to this framework. Likewise, at even later times, when the quark-gluon matter is strongly coupled, the kinetic description of chiral currents must be matched to anomalous hydrodynamics [19–22].

There has been a considerable amount of recent work on chiral kinetic theory, and of the role therein of the well known Berry phase [23] and of the chiral anomaly [24–42]. We will discuss here a first principles worldline approach developed by us [43] toward deriving a covariant kinetic theory in QCD (or even QED). In this worldline approach,

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¹The computations are for over-occupied SU(2) gauge fields in a fixed box. More realistic geometries and $N_c = 3$ will not modify this qualitative conclusion.

it is manifest that the topology of Berry's phase is distinct from that of the chiral anomaly. The two are often conflated, a point that has been made previously in the context of specific condensed matter systems by Fujikawa and collaborators [44–46]. Our work leaves no room for ambiguity. We will show that Berry's phase arises from the real part of the worldline effective action in a particular nonrelativistic adiabatic limit. In contrast, the chiral anomaly is obtained from the imaginary part of the worldline effective action. The proper identification of the two effects is not academic but is important for the treatment of topological fluctuations experienced by a chiral current traversing the hot and dense fireball created in heavy-ion collisions.

We further outline key elements in the worldline formalism necessary for a first principles derivation of a relativistically covariant kinetic theory. One such element is provided by the Euler-Lagrange equations of motion derived from the worldline action² for the spinning and colored (in the QCD case) Grassmannian fields [47–56]. Variation of the worldline action for spinning particles yields the well known Bargmann-Michel-Telegdi equations [47–51,57]. Further, by embedding the saddle point describing worldline trajectories in the Schwinger-Keldysh real time formulation [58,59], one obtains classical many-body phase space of fermions. In this generalized formulation of phase space, the internal spin degrees of freedom are anticommuting coordinates [55]. Employing a standard fluctuation analysis in the Schwinger-Keldysh framework, we are able make contact with the Boltzmann transport equation, where we identify the collision terms. While the dynamics of spinning particles can be formulated in terms of an extended phase space, the source of the anomaly however is found in the structure of worldline quantization itself. A self-consistent derivation of the detailed structure of the many-body kinetic theory for spinning particles, and of the topological fluctuations induced by the QCD axial anomaly, is out of the scope of this work and will be reported on in the near future [60].

II. REVIEW OF THE WORLDLINE FORMALISM IN QED

We will begin by briefly reviewing aspects of the worldline formalism necessary for the construction of a chiral kinetic theory. These were discussed by us at greater length in [43]. For simplicity, we will consider QED; the extension to QCD is straightforward. The real part of the fermion determinant can be expressed exactly in terms of the supersymmetric quantum mechanical path integral for point particle worldlines [58,61–67], where the internal

spin and color degrees (in QCD) of freedom are expressed in terms of Grassmann variables [55,56].

We begin with the Euclidean action for massless fermion fields in the background of a vector (A) and an auxiliary Abelian axial-vector (B) field³

$$S_F[A, B] = \int d^4x \bar{\psi} (i\partial + \not{A} + \gamma_5 \not{B}) \psi, \quad (1)$$

and allow the fermion fields to transform under internal (gauge) symmetry. Performing the path integral over the fermion fields, one obtains the effective action

$$-W[A, B] = \log \det(\theta) \quad \theta = i\partial + \not{A} + \gamma_5 \not{B}. \quad (2)$$

The fermion determinant has a relative phase that is well known to be related to the physics of the chiral anomaly [68–71]. We will here adapt a trick due to D'Hoker and Gagné [72,73], which provides an alternative derivation to those often discussed in the literature. One begins by formally splitting Eq. (2) into real and imaginary parts,

$$W[A, B] = W_{\mathbb{R}}[A, B] + iW_{\mathbb{I}}[A, B]. \quad (3)$$

The real part of the effective action can be expressed as

$$W_{\mathbb{R}} = -\frac{1}{2} \log \det(\theta^\dagger \theta). \quad (4)$$

As shown in [72,73], this can be rewritten as

$$W_{\mathbb{R}} = -\frac{1}{8} \log \det(\tilde{\Sigma}^2) = -\frac{1}{8} \text{Tr} \log(\tilde{\Sigma}^2). \quad (5)$$

Here $\tilde{\Sigma}^2$ is a sixteen dimensional matrix given by

$$\tilde{\Sigma}^2 = (p - \not{A})^2 \mathbb{I}_8 + \frac{i}{2} \Gamma_\mu \Gamma_\nu F_{\mu\nu} [A], \quad (6)$$

with $\mathbb{I}_8$ the 8-dimensional identity matrix and

$$\begin{aligned} \Gamma_\mu &= \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, & \Gamma_5 &= \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \\ \Gamma_6 &= \begin{pmatrix} 0 & i\mathbb{I}_4 \\ -i\mathbb{I}_4 & 0 \end{pmatrix}, \end{aligned} \quad (7)$$

are 8×8 dimensional gamma matrices. We further define a Γ_7 matrix anticommuting with all other elements of the algebra,

$$\Gamma_7 = -i \prod_{A=1}^6 \Gamma_A = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}. \quad (8)$$

The gauge fields in Eq. (5) that appear explicitly and in the field-strength tensor $F_{\mu\nu} [A]$ can be split into left-right chiral structures with the 2×2 dimensional matrix form,⁴

²Prior work in this context was motivated from either symmetry considerations [47–51] or Schwinger's proper time representation of the propagator [52–54].

³Couplings are absorbed into the definitions of the fields.

⁴The 8×8 Gamma matrices in $\tilde{\Sigma}^2$ should be understood as multiplying the diagonal terms of the 2×2 matrices formed by the gauge fields.

$$\mathcal{A} = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}. \quad (9)$$

$\tilde{\Sigma}^2$ admits a manifestly positive-definite heat kernel regularization. Therefore, using Schwinger's proper time scheme, the real part of the effective action can be rewritten as

$$W_{\mathbb{R}} = \frac{1}{8} \int_0^\infty \frac{dT}{T} \text{Tr}_{16} e^{-\frac{\mathcal{E}}{2} T \tilde{\Sigma}^2}, \quad (10)$$

where $\mathcal{E}$ is the einbein, to be discussed further later.

This 16-dimensional representation of $\tilde{\Sigma}^2$ is useful because it is conveniently cast into a path integral in terms of Grassmanian variables. These variables are eigenvalues of coherent states of creation/annihilation operators that generate finite dimensional representations of the internal symmetries of the theory [55,56]. One obtains after some algebra,

$$W_{\mathbb{R}} = \frac{1}{8} \int_0^\infty \frac{dT}{T} \mathcal{N}(T) \int_P \mathcal{D}x \int_{AP} \mathcal{D}\psi \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \mathcal{J}(\lambda^\dagger \lambda) \times \left(e^{-\int_0^T d\tau \mathcal{L}_L(\tau)} + e^{-\int_0^T d\tau \mathcal{L}_R(\tau)} \right). \quad (11)$$

Some details of this derivation are given in [43]. The point particle Lagrangian for left/right chiralities is

$$\mathcal{L}_{L/R}(\tau) = \frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2} \psi_a \dot{\psi}_a + \lambda^\dagger \dot{\lambda} - \lambda^\dagger [i\dot{x}_\mu (A \pm B)_\mu - \frac{i\mathcal{E}}{2} \psi_\mu \psi_\nu F_{\mu\nu} [A \pm B]] \lambda, \quad (12)$$

with the normalization $\mathcal{N}(T) = \int \mathcal{D}p e^{-\frac{\mathcal{E}}{2} \int_0^T d\tau p^2(\tau)}$. Here $\psi_a = \sqrt{2} \langle \psi | \Gamma_a | \psi \rangle$, with $a = 1, \dots, 6$ are Grassmann variables, which are defined over a real vector space, where $|\psi\rangle$ represents a coherent state basis of the Clifford algebra. Likewise, $\lambda^\dagger$ and λ are independent Grassmanian eigenvalues respectively of creation and annihilation fermion operators that generate finite dimensional representation of $SU(N_c)$, where N_c is the number of colors. The factor $\mathcal{J}(\lambda^\dagger \lambda) = (\frac{\pi}{T})^{N_c} \sum_\phi \exp[i\phi(\lambda^\dagger \lambda + N_c/2 - 1)]$ is required to project intermediate states in the path integral on to coherent states with unit occupancy [73]. The labels P and AP denote periodic and antiperiodic boundary conditions for the configuration space and Grassmanian variables respectively. Henceforth, the Abelian reduction of Eq. (11) will be sufficient for our purposes; the extension of our discussion to color degrees of freedom is straightforward [47–49].

Varying the real part of the effective action with respect to the vector gauge field gives the vector current,

$$\langle j_\mu^V(y) \rangle = \frac{\delta \Gamma_{\mathbb{R}}}{\delta A_\mu(y)} = -\frac{i}{8} \int_0^\infty \frac{dT}{T} \mathcal{N} \int_P \mathcal{D}x \int_{AP} \mathcal{D}\psi j_\mu^{V,cl} \times \left(e^{-\int_0^T d\tau \mathcal{L}_L(\tau)} + e^{-\int_0^T d\tau \mathcal{L}_R(\tau)} \right), \quad (13)$$

with $j_\mu^{V,cl}(y) = \int_0^T d\tau [\mathcal{E} \psi_\nu \psi_\mu \partial_\nu - \dot{x}_\mu] \delta^4(x(\tau) - y)$. This satisfies both $\partial_\mu j_\mu^{V,cl} = 0$ and $\partial_\mu \langle j_\mu^V \rangle = 0$.

The imaginary relative phase in the effective action can be written as

$$W_{\mathbb{I}} = -\frac{1}{2} \arg \det[\Omega], \quad \Omega = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}, \quad (14)$$

where θ is given in Eq. (2) and the matrix Ω is

$$\Omega = \Gamma_\mu (p_\mu - A_\mu) - i\Gamma_7 \Gamma_\mu \Gamma_5 \Gamma_6 B_\mu. \quad (15)$$

The D'Hoker and Gagné [72,73] trick consists of introducing a parameter that regulates chiral symmetry breaking—distinct from those employed previously [71]—to write $W_{\mathbb{I}}$ as

$$W_{\mathbb{I}} = \frac{i\mathcal{E}}{64} \int_{-1}^1 d\alpha \int_0^\infty dT \text{Tr} \{ \hat{M} e^{-\frac{\mathcal{E}}{2} T \tilde{\Sigma}_{(\alpha)}^2} \}, \quad (16)$$

with a trace insertion

$$\hat{M} = \Gamma_7 (2\Gamma_5 \Gamma_6 [\partial_\mu, B_\mu] + [\Gamma_\mu, \Gamma_\nu] \{ \partial_\mu, B_\nu \} \Gamma_5 \Gamma_6) \mathbb{I}_2, \quad (17)$$

that is linear in the axial-vector field and diagonal in the two dimensional field representation space introduced in Eq. (9). $\tilde{\Sigma}_{(\alpha)}^2$ is identical to the expression in Eq. (6), with $B \rightarrow \alpha B$, where α breaks chiral symmetry explicitly for $\alpha \neq \pm 1$.

This form of the phase of the fermion determinant is useful because it has a heat-kernel structure that can be computed, in a manner identical to the real part, using Grassmanian path integrals [55,56]. The path integral representation of $W_{\mathbb{I}}$ is given in [43]; it is further shown explicitly there that this representation gives the well-known anomaly relation

$$\begin{aligned} \partial_\mu \langle j_\mu^5(y) \rangle &\equiv \partial_\mu \left. \frac{i\delta W_{\mathbb{I}}}{\delta B_\mu(y)} \right|_{B=0} \\ &= -\frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(y) F_{\rho\sigma}(y). \end{aligned} \quad (18)$$

III. BERRY'S PHASE FROM THE REAL PART OF THE QED EFFECTIVE ACTION

From the discussion above, it is clear that the chiral anomaly arises from the imaginary part of the effective action $W_{\mathbb{I}}$. We will now show instead that Berry's phase arises from taking a nonrelativistic and adiabatic limit of the real part of the effective action $W_{\mathbb{R}}$. Our starting point is the worldline Lagrangian in Eq. (12) continued to Minkowskian metric ($g = \text{diag}[-, +, +, +]$). We proceed by introducing Lagrange multipliers in Eq. (12) to (i) impose the mass-shell constraint and (ii) to project out unphysical spin degrees of freedom for both massless and massive point particles. After imposing all constraints,⁵

⁵The integration over the parameter T represents the diffeomorphism invariance of the worldline action—it can either be performed explicitly [74,75] or fixed equivalently for a choice of the einbein that respects the constrained Hamiltonian dynamics of massless and massive spinning particles. Since we are interested in real-time dynamics, we will adopt the latter approach.

and eliminating thereby unphysical degrees of freedom,⁶ the Lagrangian can be written as $S = \int d\tau \mathcal{L}$ (setting $\tau = ct\sqrt{1 - \mathbf{v}^2/c^2}$, with $x^\mu = (ct, \vec{x})$, defining $z = \sqrt{-\dot{\mathbf{x}}^2}$ and putting the auxiliary field $B = 0$),

$$\mathcal{L} = -\frac{m_R c z}{2} \left(1 + \frac{m^2}{m_R^2}\right) + \frac{i}{2}(\boldsymbol{\psi} \dot{\boldsymbol{\psi}} - \psi_0 \dot{\psi}_0) + \frac{\dot{x}_\mu A^\mu(x)}{c} - \frac{iz}{m_R c} \psi^0 F_{0i} \psi^i - \frac{iz}{2m_R c} \psi^i F_{ij} \psi^j, \quad (19)$$

where $m_R^2 = m^2 + i\psi^\mu F_{\mu\nu} \psi^\nu / c^2$ is the effective mass for the spinning worldline. The spin three-vector can be defined as $S^i = -\frac{i}{2} \epsilon^{ijk} \psi^j \psi^k$, where ϵ^{ijk} is the Levi-Civita symbol. Likewise, the magnetic field $B^i = \frac{1}{2} \epsilon^{ijk} F^{jk}$ and the electric field $E^i = F^{0i}$. The Euler-Lagrange equations of motion are [43]

$$-\frac{\bar{m}}{Y} \frac{d^2 x^\mu}{du^2} + \frac{iY}{2\bar{m}} \psi^\alpha F_{\alpha\beta} \psi^\beta + F^{\mu\nu} \frac{dx_\nu}{du} = 0, \quad \frac{d\psi_\mu}{du} - \frac{Y}{\bar{m}} F_{\mu\nu} \psi^\nu = 0, \quad (20)$$

which represent the covariant form of the Bargmann-Michel-Telegdi [47–51,57] and Wong's equations [76]. We have abbreviated $u \equiv \tau/T$, $Y = \sqrt{-\int_0^1 du (dx/du)^2}$ and $\bar{m} = \sqrt{m^2 + i \int_0^1 du \psi^\mu F_{\mu\nu} \psi^\nu}$. The last two terms in Eq. (19) are respectively,

$$-i\psi^0 F_{0i} \psi^i = \frac{\mathbf{S} \cdot (\mathbf{P} \times \mathbf{E})}{cP^0}; \quad -\frac{i}{2} \psi^i F_{ij} \psi^j = \mathbf{S} \cdot \mathbf{B}. \quad (21)$$

Here $P^\mu \equiv p^\mu - A^\mu$, where p^μ is the canonical momentum. To take the nonrelativistic limit, we expand the effective mass as $m_R \approx m(1 + X)$ with $X = -(\mathbf{S} \cdot (\mathbf{P} \times \mathbf{E})/[cP^0] + \mathbf{S} \cdot \mathbf{B})/(2m^2 c^2)$. Observing that $X \propto (v/c)^2$, where $\vec{v}$ is the three-velocity of the spinning worldline, one can expand Eq. (19) to obtain

$$\mathcal{L}_{\text{NR}} = -mc^2 + \frac{1}{2}mv^2 + \frac{i}{2}(\boldsymbol{\psi} \dot{\boldsymbol{\psi}} - \psi_0 \dot{\psi}_0) - A^0 + \frac{\mathbf{v}}{c} \cdot \mathbf{A} + \frac{\mathbf{S} \cdot ([\mathbf{v}/c - \mathbf{A}/(mc^2)] \times \mathbf{E})}{mc} + \frac{\mathbf{S} \cdot \mathbf{B}}{m} + O\left(\frac{v^3}{c^3}\right). \quad (22)$$

Compactly expressing the nonrelativistic action as $S = \int dt (\mathbf{p} \cdot \dot{\mathbf{x}} + \frac{i}{2} \boldsymbol{\psi} \cdot \dot{\boldsymbol{\psi}} - H)$, the corresponding Hamiltonian is⁷

⁶A detailed discussion of the constrained Hamilton dynamics of worldlines can be found in [43].

⁷The variable ψ_0 is not dynamical and we will drop it henceforth.

$$H \equiv mc^2 + \frac{(\mathbf{p} - \frac{\mathbf{A}}{c})^2}{2m} + A^0(x) - \frac{\mathbf{S} \cdot ([\mathbf{v}/c - \mathbf{A}/(mc^2)] \times \mathbf{E})}{2mc} - \frac{\mathbf{B} \cdot \mathbf{S}}{m}. \quad (23)$$

Expressed in this form, the nonrelativistic point particle Hamiltonian is familiar [77]; the penultimate term is the spin-orbit interaction energy from Thomas precession, while the final term is the Larmor interaction energy. In atomic physics their combined effect is of course to reduce the Larmor energy by the famous “Thomas 1/2”.

In the following, we will show how the system described by Eq. (23) contains, in an adiabatic approximation, a Berry phase; in this limit, it has the monopole form postulated in [24–26,28]. To recover the expressions in [24,25] we requantize the spin, by promoting the spin (phase-space) variables ψ to the Hilbert space operators $\psi_i \rightarrow \sqrt{\frac{\hbar}{2}} \sigma_i \equiv \hat{\psi}_i$ and $S_i \rightarrow \frac{\hbar}{2} \sigma_i \equiv \hat{S}_i$, where σ are the Pauli matrices and hats indicate operators. Further, to describe the finite phase space of Grassmannian variables ψ , we define the two dimensional Hilbert space for a spin-1/2 particle at every point in phase space $(\mathbf{p}, \mathbf{x})$ by the eigenstates $|\psi^\pm\rangle = |\psi^\pm(\mathbf{p})\rangle$. Defining $\mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|} \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, one has two choices

$$|\psi_+^{(1)}(\mathbf{p})\rangle = \frac{N}{2} (1 + \mathbf{n} \cdot \boldsymbol{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad (24)$$

$$|\psi_+^{(2)}(\mathbf{p})\rangle = \frac{N}{2} (1 + \mathbf{n} \cdot \boldsymbol{\sigma}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad (25)$$

for the “spin up” + basis vectors (where N is a normalization factor) and similarly for the “spin down” basis vectors. (See also [78].) The two choices of basis vectors are not defined globally for all $\mathbf{p}$ with Eq. (24) [Eq. (25)] ill defined for the south (north) pole for $\theta = \pi(0)$. One set can however be used for the northern hemisphere and the other for the southern one, and are related as $|\psi_+^{(1)}(\mathbf{p})\rangle = e^{i\phi} |\psi_+^{(2)}(\mathbf{p})\rangle$ [78].

These basis states allow us to derive a path integral formulation in the adiabatic limit of the theory defined by Eq. (23). The transition amplitude for the Hamiltonian operator corresponding to Eq. (23) from an initial state $|\psi^+(\mathbf{p}_i)\rangle$ at time t_i to the state with momentum $\mathbf{p}_f$ at finite time t_f is

$$T(\mathbf{p}_f, \mathbf{p}_i, +) \equiv \langle \mathbf{p}_f, \psi^+(\mathbf{p}_f) | e^{-i\hat{H}(t_f - t_i)} | \mathbf{p}_i, \psi^+(\mathbf{p}_i) \rangle. \quad (26)$$

The construction of the path integral for this amplitude requires insertions of complete sets of intermediate states satisfying $\mathbb{1} = \int d^3 x_k |\mathbf{x}_k\rangle \langle \mathbf{x}_k| = \int d^3 p_k |\mathbf{p}_k\rangle \langle \mathbf{p}_k|$, as well as one for the two dimensional spin-Hilbert space: $\mathbb{1}_2 = |\psi^+\rangle \langle \psi^+| + |\psi^-\rangle \langle \psi^-|$. The adiabatic approximation corresponds to $\frac{\mathbf{B} \cdot \mathbf{S}}{2m} \approx 0$. Therefore in this limit we can

neglect the second term $|\psi^-\rangle\langle\psi^-|$, thereby constraining the dynamical spin degrees of freedom.

The transition matrix element can thus be written as

$$T(\mathbf{p}_f, \mathbf{p}_i, +) = \int \left(\prod_{k=1}^{N-1} d^3 p_k \right) \left(\prod_{l=1}^N d^3 x_l \right) \times \prod_{j=1}^N \frac{1}{(2\pi)^3} e^{-i\mathbf{x}_j \cdot (\mathbf{p}_j - \mathbf{p}_{j-1}) - iH_j \Delta} \langle \psi^+(\mathbf{p}_j) | \psi^+(\mathbf{p}_{j-1}) \rangle, \quad (27)$$

where $\Delta \equiv (t_f - t_i)/N$ and H_j is Eq. (23) evaluated at $(\mathbf{x}_j, \mathbf{p}_j)$. Taylor expanding $|\psi^+(\mathbf{p}_{j-1})\rangle = \{1 + [\mathbf{p}_j - \mathbf{p}_{j-1}] \cdot \nabla_{\mathbf{p}}\} |\psi^+(\mathbf{p}_j)\rangle + \dots$, it is straightforward to show in the continuum limit that one obtains Berry's phase,

$$\prod_{j=1}^N \langle \psi^+(\mathbf{p}_j) | \psi^+(\mathbf{p}_{j-1}) \rangle \rightarrow \exp \left(i \int dt \dot{\mathbf{p}} \cdot \mathcal{A}(\mathbf{p}) \right), \quad (28)$$

where $\mathcal{A}(\mathbf{p}) \equiv -i \langle \psi^+(\mathbf{p}) | \nabla_{\mathbf{p}} | \psi^+(\mathbf{p}) \rangle$ is the Berry connection. The final expression for the path integral is

$$T(\mathbf{p}_f, \mathbf{p}_i, +) = \int \mathcal{D}x \mathcal{D}p \exp \left(i \int dt [\dot{\mathbf{x}} \cdot \mathbf{p} - \tilde{H}] \right), \quad (29)$$

with $\tilde{H} = mc^2 + \frac{(\mathbf{p} - \mathbf{A}/c)^2}{2m} + A^0(x) - \dot{\mathbf{p}} \cdot \mathcal{A}(\mathbf{p})$. Eq. (29) is closely related to a similar formulation in [24–26, 28].

We note a few crucial points in our derivation and interpretation for systems where the chemical potential is much smaller than the rest mass. First, as we showed, the structure of Berry's phase is restricted to the nonrelativistic adiabatic limit where the Larmor interaction energy is much smaller than the rest energy. It is ill-defined in the massless case albeit the spin basis vectors in Eq. (24) look similar to Weyl spinors. An exception holds for massless systems with a large chemical potential; the latter in that case takes over the role of the mass [43].

Our derivation further makes it clear that the topological structure of Berry's phase [79] is distinct from that of the chiral anomaly. The former arises from the real part of the QED/QCD effective action while the latter can be traced to its relative phase. This point is also stressed in Refs. [44–46] where it was argued that the Berry phase in Eq. (29) should not be confused with a Wess-Zumino term. In these papers, specific examples in condensed matter systems are worked out. In our work, the distinction between the two phenomena is demonstrated directly from the QED effective action.

IV. CHIRAL KINETIC THEORY

In the previous section, we illustrated the appearance of a Berry phase in the adiabatic limit of QED and QCD. In contrast, the chiral anomaly is independent of any approximations and arises from the fundamental structure of the

theory, embodied by fermionic zero modes in the worldline formulation. Since the CME dynamics in heavy-ion collisions is relativistic, and far from adiabatic, kinetic theory constructions that explicitly incorporate the Berry phase along the lines of Eq. (29) are insufficient for this case.⁸ Since the mass of the up and down quarks is much smaller than the temperature, we will consider here the formulation of a kinetic theory exclusively for chiral fermions. Finite mass will be treated separately in later work.

We will begin by expressing the many-body Schwinger-Keldysh (SK) path integral in the language of worldlines as [58],

$$Z = \int [d\xi] \exp(-G[\xi]) \int_{\mathcal{C}} [dA] \exp(iS_{\text{eff}}), \quad (30)$$

where

$$S_{\text{eff}}[A, \xi] = -\frac{1}{4} \int_{\mathcal{C}} d^4 x F_{\mu\nu} F^{\mu\nu} + \Gamma[A, \xi]. \quad (31)$$

Here $\xi \equiv \xi(x_{\text{init}}, x'_{\text{init}}, \psi_{\text{init}}, \psi'_{\text{init}}; A_{\text{init}})$ represents the initial conditions on the Keldysh contour, which we will assume are distributed probabilistically as $\exp(-G[\xi])$, where G is a positive-definite weight functional. Equation (30) and Eq. (31) were introduced in [58] for spinless colored particles; it was shown there that if one takes the saddle point of Γ , the dynamics of the matter fields is determined by a Liouville equation. For the relativistic fermions in our QED example, we will add extend phase space by adding spin degrees of freedom: $(x^\mu, p^\mu) \rightarrow (x^\mu, p^\mu, \psi^\mu)$.⁹ The equations of motion are derived from the chiral many-body Hamiltonian,¹⁰

$$H = \frac{\epsilon}{2} [P^2 + i\psi^\mu F_{\mu\nu}(x) \psi^\nu] + \frac{i}{2} c_+ \chi_+ - \frac{i}{2} c_- \chi_-, \quad (32)$$

where

$$c_\pm \equiv \frac{1}{2} \left(\pm P_\mu \psi^\mu + \frac{i}{3} \epsilon^{\mu\nu\alpha\beta} P_\mu \psi_\nu \psi_\alpha \psi_\beta \right). \quad (33)$$

Here Eq. (32) is the constrained relativistic Hamiltonian corresponding to Eq. (12), whereby Eq. (33) enforces the Weyl-constraints and $\chi_\pm$ are anticommuting Lagrange multipliers [84].¹¹ The equations of motion corresponding to this Hamiltonian cannot be obtained by putting $m = 0$ in Eq. (20), and instead are expressed as,

$$\dot{x}^\mu = \epsilon P^\mu + \frac{i}{2} \psi^\mu \tilde{\chi} - \frac{1}{6} \epsilon^{\mu\nu\alpha\beta} P_\nu \psi_\alpha \psi_\beta \tilde{\chi}, \quad (34)$$

⁸This may also be the case for topological effects in the transport of chiral fermions in an astrophysical context, if the chemical potential is much smaller than the mass [80–83].

⁹ ψ_5 is not dynamical in the chiral limit.

¹⁰Because we are describing the classical phase space of N particles, a sum over particle labels is always implied but omitted in our notation.

¹¹Some relevant details are presented in Supplemental Material [85].

$$\dot{\psi}^\mu = \varepsilon F^{\mu\alpha} \psi_\alpha + \frac{P^\mu}{2} \bar{\chi} + \frac{i}{4} \varepsilon^{\mu\nu\alpha\beta} P_\beta \psi_\nu \psi_\alpha \tilde{\chi}, \quad (35)$$

$$\begin{aligned} \dot{P}^\mu &= \varepsilon P_\alpha F^{\mu\alpha} - \frac{i\varepsilon}{2} \psi^\alpha \partial^\mu F_{\alpha\beta} \psi^\beta + \frac{i}{2} F^{\mu\alpha} \psi_\alpha \tilde{\chi} \\ &\quad - \frac{1}{12} \varepsilon_{\alpha\beta\lambda\sigma} F^{\mu\alpha} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi}, \end{aligned} \quad (36)$$

with $\bar{\chi} \equiv (\chi_+ + \chi_-)/2$ and $\tilde{\chi} \equiv \chi_+ - \chi_-$.

With these equations of motion in hand, Liouville's equation for chiral Dirac fermions reads as,

$$\begin{aligned} \{f, H\} &= f \left(\frac{\tilde{\partial}}{\partial x^\mu} \left[\varepsilon P^\mu + \frac{i}{2} \psi^\mu \bar{\chi} - \frac{1}{6} \varepsilon^{\mu\nu\alpha\beta} \psi_\nu \psi_\alpha \psi_\beta \tilde{\chi} \right] \right. \\ &\quad + \frac{\tilde{\partial}}{\partial P^\mu} \left[\varepsilon P_\alpha F^{\mu\alpha} - \frac{i\varepsilon}{2} \psi^\alpha \partial^\mu F_{\alpha\beta} \psi^\beta + \frac{i}{2} F^{\mu\alpha} \psi_\alpha \tilde{\chi} \right. \\ &\quad \left. \left. - \frac{1}{12} \varepsilon_{\alpha\beta\lambda\sigma} F^{\mu\alpha} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi} \right] + \frac{\tilde{\partial}}{\partial \psi^\mu} \left[\varepsilon F^{\mu\alpha} \psi_\alpha \right. \right. \\ &\quad \left. \left. + \frac{P^\mu}{2} \bar{\chi} + \frac{i}{4} \varepsilon^{\mu\nu\alpha\beta} P_\beta \psi_\nu \psi_\alpha \tilde{\chi} \right] \right) = 0, \end{aligned} \quad (37)$$

where f is the phase-space distribution function given by

$$\begin{aligned} f(x, p, \psi) &\equiv 4i\varepsilon \delta^{(4)}(x^\mu - x^\mu(\tau)) \delta^{(4)}(P^\mu - P^\mu(\tau)) \\ &\quad \times \delta^{(4)}(\psi^\mu - \psi^\mu(\tau)). \end{aligned} \quad (38)$$

The properties of this phase-space distribution are discussed further in Supplemental Material [85]. In Eq. (38), $x^\mu(\tau)$, $P^\mu(\tau)$ and $\psi^\mu(\tau)$ are the solutions to Eqs. (34)–(36) in the background of A , with the initial conditions for their respective temporal evolution specified by the initial density matrix of the SK path integral in Eq. (30). Since these initial conditions are probabilistic, we can split $f(x, p, \psi)$ into average and fluctuating parts, $f \equiv \bar{f} + \delta f$. Expressed in this way, $\langle f \rangle = \bar{f}$ coincides with the Boltzmann one-particle distribution function [58,86]. Expressed in terms of mean and fluctuating components, the ensemble-averaged Liouville equation reads

$$\begin{aligned} \bar{f} &\left(\frac{\tilde{\partial}}{\partial x^\mu} \left[\varepsilon P^\mu + \frac{i}{2} \psi^\mu \bar{\chi} - \frac{\varepsilon^{\mu\nu\alpha\beta}}{6} \psi_\nu \psi_\alpha \psi_\beta \tilde{\chi} \right] \right. \\ &\quad + \frac{\tilde{\partial}}{\partial P^\mu} \left[\varepsilon \bar{F}^{\mu\alpha} \psi_\alpha + \frac{P^\mu}{2} \bar{\chi} + \frac{i}{4} \varepsilon^{\mu\nu\alpha\beta} P_\beta \psi_\nu \psi_\alpha \tilde{\chi} \right] \\ &\quad + \frac{\tilde{\partial}}{\partial \psi^\mu} \left[\varepsilon \bar{F}^{\mu\alpha} P_\alpha - \frac{i\varepsilon}{2} \psi^\alpha \partial^\mu \bar{F}_{\alpha\beta} \psi^\beta + \frac{i}{2} \bar{F}^{\mu\alpha} \psi_\alpha \tilde{\chi} \right. \\ &\quad \left. \left. - \frac{\varepsilon_{\alpha\beta\lambda\sigma}}{12} \bar{F}^{\mu\alpha} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi} \right] \right) = C[\delta f, \delta F], \end{aligned} \quad (39)$$

where the fluctuation induced collision term is given by

$$\begin{aligned} C[\delta f, \delta F] &\equiv -\varepsilon \left\langle \delta f \frac{\tilde{\partial}}{\partial \psi^\mu} \delta F^{\mu\nu} \right\rangle \psi_\nu + \frac{i\varepsilon}{2} \left\langle \delta f \frac{\tilde{\partial}}{\partial P^\mu} \partial^\mu \delta F_{\alpha\beta} \right\rangle \psi^\alpha \psi^\beta \\ &\quad - \left\langle \delta f \frac{\tilde{\partial}}{\partial P^\mu} \delta F^{\mu\alpha} \right\rangle \left(\varepsilon P_\alpha + \frac{i}{2} \psi_\alpha \tilde{\chi} - \frac{1}{12} \varepsilon_{\alpha\beta\lambda\sigma} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi} \right). \end{aligned} \quad (40)$$

A similar expression is found for δf and equations for higher correlation functions can be derived. The result is an infinite tower of coupled equations, analogous to the Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy [87–89]. In a dilute limit, known as the second moment approximation, one can expand to linear order in the fluctuations,

$$\begin{aligned} \delta f &\left(\frac{\tilde{\partial}}{\partial x^\mu} \left[\varepsilon P^\mu + \frac{i}{2} \psi^\mu \bar{\chi} - \frac{1}{6} \varepsilon^{\mu\nu\alpha\beta} \psi_\nu \psi_\alpha \psi_\beta \tilde{\chi} \right] + \frac{\tilde{\partial}}{\partial P^\mu} \left[\varepsilon \bar{F}^{\mu\alpha} P_\alpha \right. \right. \\ &\quad \left. \left. - \frac{i\varepsilon}{2} \psi^\alpha \partial^\mu \bar{F}_{\alpha\beta} \psi^\beta + \frac{i}{2} \bar{F}^{\mu\alpha} \psi_\alpha \tilde{\chi} - \frac{1}{12} \varepsilon_{\alpha\beta\lambda\sigma} \bar{F}^{\mu\alpha} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi} \right] \right. \\ &\quad \left. + \frac{\tilde{\partial}}{\partial \psi^\mu} \left[\varepsilon \bar{F}^{\mu\alpha} \psi_\alpha + \frac{P^\mu}{2} \bar{\chi} + \frac{i}{4} \varepsilon^{\mu\nu\alpha\beta} P_\beta \psi_\nu \psi_\alpha \tilde{\chi} \right] \right) = K[\delta F], \end{aligned} \quad (41)$$

where

$$\begin{aligned} K[\delta F] &\equiv -\bar{f} \left(\frac{\tilde{\partial}}{\partial P^\mu} \left[\varepsilon \delta F^{\mu\alpha} P_\alpha - \frac{i\varepsilon}{2} \psi^\alpha \partial^\mu \delta F_{\alpha\beta} \psi^\beta \right. \right. \\ &\quad \left. \left. + \frac{i}{2} \delta F^{\mu\alpha} \psi_\alpha \tilde{\chi} - \frac{\varepsilon_{\alpha\beta\lambda\sigma}}{12} \delta F^{\mu\alpha} \psi^\beta \psi^\lambda \psi^\sigma \tilde{\chi} \right] \right. \\ &\quad \left. + \frac{\tilde{\partial}}{\partial \psi^\mu} [\varepsilon \delta F^{\mu\alpha} \psi_\alpha] \right). \end{aligned} \quad (42)$$

The equations for δf and $\bar{f}$ are supplemented by Maxwell's (Yang-Mills) equations, analogously split into background and fluctuating parts. Note that while ε and $\chi_\pm$ appear in the equations, physical observables should not depend on these gauge parameters. They can be fixed, with suitable choices, as discussed in Supplemental Material [85].

The resulting set of equations can be solved in closed form under systematic approximations for specific examples [89]. One nontrivial example is that of the real time formulation [59] of a pseudoclassical kinetic theory from the worldline action for spinless colored particles [58]. It was shown there that the analogous set of coupled equations is equivalent to the non-Abelian Boltzmann–Langevin Bödeker kinetic theory of hot QCD [86,90]. A novel element here is the systematic treatment of the chiral anomaly and the role of the resulting real time sphaleron transitions in the spacetime evolution of the chiral magnetic current. The explicit derivation of this “anomalous Bödeker theory” [91,92], including as well the matching of chiral kinetic theory to classical-statistical simulations at early times [12–14], and anomalous hydrodynamics [19–22] at late times, will be presented in followup work [60].

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