Weak coupling limits and colliding punctures in class- $\mathcal{S}$ theories

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Class- $\mathcal{S}$ theories are four-dimensional $\mathcal{N} = 2$ supersymmetric field theories constructed by the reduction of a (2,0) six-dimensional theory on a punctured Riemann surface C (called the UV curve). A basic degeneration limit of the surface C is when several punctures are brought close to each other. As this happens, a long tube appears in C and a weakly coupled gauge group emerges. When the corresponding gauge coupling is turned off, we are left with two surfaces. This decoupling leads to a unique result. We explain how to fix the gauge group that becomes weakly coupled and the resulting two theories, in terms of the types of punctures on the surface C . Each of the resulting theories is given in terms of a class- $\mathcal{S}$ theory (plus possibly additional free hypermultiplets). Many other more complicated degeneration limits can be described by repeatedly using this specification. For questions about weak coupling limits in such theories, it is useful to consider the set of punctures that can appear at the end of a tube. These are the punctures that are created in the resulting surfaces when the gauge coupling is taken to be extremely weak. Not any puncture can be formed in a decoupling process, and this set is described. Each such puncture can be found diagrammatically in terms of the other punctures on C in a particular decoupling limit. The relation of the gauge group to the symmetry associated with this puncture is discussed. We discuss the relation of such degeneration limits of large N theories, through the AdS/CFT correspondence, to decoupled field theories on AdS_5 .

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I. INTRODUCTION

Consider the Coulomb branch of $\mathcal{N} = 2$ superconformal field theories in four dimensions. At low energies we have $U(1)^n$ gauge fields, which are part of $\mathcal{N} = 2$ vector multiplets. Denote the $\mathcal{N} = 1$ chiral superfield part of these vector multiplets by a_i . In the low energy effective action, the gauge coupling constants matrix is promoted to $\tau^{ij}(a)$ (a function of the a_i). The gauge coupling matrix is related to the $\mathcal{N} = 2$ prepotential F by $\tau^{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j}$, and we have the definition $a_D^i = \frac{\partial F}{\partial a_i}$. a_D plays the role of a in the electric-magnetic dual theory. Additionally, as can be observed from the low energy effective theory, the expectation values of a_i and a_D^i are the coefficients of the electric and magnetic charges under the i th $U(1)$ in the Bogomolnyi-Prasad-Sommerfield (BPS) bound.

The solution of the theory on the Coulomb branch (in the infrared) is described by the Seiberg-Witten (SW) curve [1,2]. A basis of nontrivial cycles and dual cycles is chosen,

and the integrals of the SW differential λ along them give a_i and a_D^i . The SW curve contains gauge invariant Coulomb branch parameters, and a_i , a_D^i are holomorphic functions of these parameters.

The SW curve can actually be realized geometrically [3]. Linear quivers of $SU(k_i)$ gauge groups with $\mathcal{N} = 2$ supersymmetry were constructed through the decoupling limit of configurations of NS5 and D4 branes in type-IIA superstring. These configurations have singularities, but when interpreted in M theory, could be described by a single M5 brane. For generic parameters, the M5 brane is smooth and contains no singularities. The description of the M5 brane includes the SW curve solution of the theory. Linear quivers with generic flavor groups of fundamental hypermultiplets of the $SU(k_i)$ gauge groups are constructed by including D6 branes as well. Circular quivers, as well as other classes of $\mathcal{N} = 2$ theories can also be obtained by brane constructions.

Following [4], consider a linear quiver of n $SU(N)$ gauge groups, with N fundamentals at each end of the quiver. For $N = 2$, when all the gauge groups but one are arbitrarily weakly coupled, we can take the coupling constant of that one gauge group to be very strong and apply the familiar $SL(2, \mathbb{Z})$ S duality of $SU(2)$ with four flavors. This leads to a generalized quiver (when expressed in terms of the weakly coupled gauge groups), differing from the linear quiver we began with. Similarly, for $N = 3$, we may apply the Argyres-Seiberg duality (mentioned below), and get another

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kind of generalized quiver. These generalized quivers are composed of elementary building blocks. The E_6 superconformal field theory (SCFT) is one of the building blocks for $N = 3$. The generalized quivers constructed starting from some fixed quiver gauge theory are weak coupling cusps of a single theory.

The linear quiver we started with can be constructed, as mentioned above, by taking N M5 branes that are intersected by other $n + 1$ transverse M5 branes. This configuration can be viewed as N M5 branes wrapping a Riemann sphere, in the presence of $n + 1$ punctures at some points on the Riemann sphere, as well as punctures at 0 and ∞ . In the circular quivers construction, the Riemann sphere is replaced by a torus T^2 (as a result of the topology of the space that the N M5 branes wrap).

Other kinds of linear quivers are described by including two punctures of a more general type. These punctures are local and thus suggest that we can have additional theories by combining different sets of the punctures. This more general sort of theory [4] is therefore defined by taking the A_{N-1} (2,0) six-dimensional theory on a Riemann surface C , with codimension 2 defect operators (a twisting is required to get $\mathcal{N} = 2$ in the four-dimensional macroscopic theory). There are (2,0) six-dimensional theories of A, D, and E types. This defines a four-dimensional theory by a simply laced Lie group and a punctured Riemann surface. These theories are known as class- $\mathcal{S}$. We restrict ourselves to the A_{N-1} theories.¹

A basic building block of the generalized quivers is the T_N theory, which can be identified by an A_{N-1} theory on a sphere with three full punctures (a review of the types of punctures is given below). It plays a similar role to that played by the E_6 SCFT in $N = 3$ (even though there is another kind of isolated SCFT that is identified more naturally with the generalization of the E_6 SCFT to general N). The T_N theory is an example of a class- $\mathcal{S}$ theory with no brane construction of the sort described above. There is a description [9] of it in terms of a web of 5 branes (and 7 branes) in type-IIB giving a five-dimensional theory, which when compactified on S^1 gives the T_N theory (as well as more general isolated SCFTs).

The SW curve for the A_{N-1} class- $\mathcal{S}$ theories can be written as an N sheeted branched covering of C . It has the canonical form

$$x^N + \sum_{k=2}^N \phi_k(z) x^{N-k} = 0, \quad (1.1)$$

where z parametrizes C with the SW differential being $\lambda = x dz$. The ϕ_k are more naturally used as k differentials

¹For a review, see [5]. We do not consider here twisted punctures of the A-type theories [6]. Generalizations of our analysis to include those, as well as the D, E-type theories, are left for the future; see [7,8].

$\phi_k dz^k$, having appropriate poles at the punctures. Usually the SW curve describes the infrared limit of the four-dimensional theory on the Coulomb branch. Here, as is manifest in the brane constructions mentioned above, the structure of the SW curve as a branched covering of C identifies the four-dimensional theory.

There are discrete holomorphic transformations keeping the SW curve with the covering structure above invariant, and are therefore symmetries of the full four-dimensional theory. Then, the S-duality invariant space of exactly marginal deformations is the complex structure moduli space of C , with punctures of the same kind being indistinguishable.

At various cusps of the moduli space of the punctured Riemann surface C , the surface degenerates, long tubes emerge, and some cycles shrink. In such cases, weakly coupled gauge groups emerge. A common situation where this happens is when some punctures on C are brought close to each other. At different degenerations of the same surface C , different gauge groups become weakly coupled. This provides us with S dualities between *a priori* different theories. An example is the Argyres-Seiberg duality [10], stating that the strongly coupled $SU(3)$ superconformal theory with six fundamental hypermultiplets is a weakly coupled $SU(2)$ theory, coupled to one fundamental hypermultiplet and the T_3 theory (which has global symmetry E_6 and no marginal deformations).

Let us write the form of the SW curve more explicitly for a few low genus surfaces C . On a sphere (genus $g = 0$) with punctures at $z_1, z_2, \dots, \phi_k$ of a massless theory is of the following form,

$$\phi_k = \frac{Q_k(z)}{(z - z_1)^{p_k^1} (z - z_2)^{p_k^2} \dots} dz^k, \quad (1.2)$$

where we label by p_k^i the pole structure corresponding to the puncture located at z_i . Since we do not have a pole at infinity, a change of variable $z = 1/w$ shows that the polynomial Q_k is of order at most $\sum_i p_k^i - 2k$. Therefore the number of Coulomb branch parameters ϕ_k gives rise to is $\sum_i p_k^i - 2k + 1$. (Note that we could choose to position one of the poles at infinity, with the same result.)

On a torus ($g = 1$), the general ϕ_k with the required pole structure (and no masses) is of the form

$$\begin{aligned} \phi_k &= A_k \frac{\theta(z - n_k^1) \dots \theta(z - n_k^{d_k})}{\theta(z - z_1)^{p_k^1} \theta(z - z_2)^{p_k^2} \dots} dz^k, \\ d_k &= \sum_i p_k^i, \quad \sum_i n_k^i = \sum_i p_k^i z_i, \end{aligned} \quad (1.3)$$

where the θ is a Jacobi theta function. Including A_k and the restriction on $\sum_i n_k^i$, we see that ϕ_k gives rise to $\sum_i p_k^i$ Coulomb branch parameters. In a general surface of genus g , the number of Coulomb branch parameters of dimension

k is the dimension of the space of k differentials and is given by

$$d_k = \sum_i p_k^i + (g-1)(2k-1). \quad (1.4)$$

The pole structure of regular (untwisted) punctures in a superconformal class- $\mathcal{S}$ A_{N-1} theory is restricted. A regular puncture P is described in terms of a Young diagram with N boxes in total. The pole structure of the puncture is fixed by the diagram as follows. For $k = 2, \dots, N$, ϕ_k has a pole of order p_k at the puncture, where p_k is given by $p_k = k - h(k, P)$, in which $h(k, P)$ is the row number of the k th box in the Young diagram (we label the rows starting with 1). The most common punctures are simple and full punctures. A simple puncture has a diagram with rows of width 2, 1, 1, ... and pole structure 1, 1, 1, ... ($p_k = 1$). A full puncture has a single row (of width N) and pole structure 1, 2, 3, ... ($p_k = k - 1$). Any Young diagram corresponds to a regular puncture, except for a single column diagram, referred to as a no puncture.

Since we commonly use punctures, let us introduce a convenient (but slightly subtle) notation. Punctures are denoted by uppercase roman letters, such as P . For each such puncture P we use the following notations:

- (i) P_i , the width of row number i .
- (ii) p_k , the pole structure at the value k .
- (iii) p , the number of boxes outside the first column [explicitly $p = \sum_i (P_i - 1)$].
- (iv) $h(k, P)$, the row number of box number k .

When we have several punctures, we add a superscript, as in P^i [and then we have as before P_j^i , p_k^i , p^i , and $h(k, P^i)$].

Each regular puncture has a global symmetry associated with it. Corresponding to this symmetry, mass deformations can be introduced. The symmetry associated with a regular puncture P is denoted by $G(P)$ and is given by

$$G(P) = S\left(\prod_i U(P_i - P_{i+1})\right) \quad (1.5)$$

[where $S(\dots)$ means removing the diagonal $U(1)$]. The product of the punctures' symmetries does not have to be the full symmetry of the theory. A method to find the full symmetry which can be used in some of the cases is by considering the mirror of the theory compactified to three dimensions; see [11] using [12,13].

Sections II–IV analyze decoupling limits in A_{N-1} class- $\mathcal{S}$ theories. These decoupling limits were already analyzed in the literature [6,11] by methods that allow studying any given case, but the systematic analysis of Secs. II–IV provides a direct answer for all cases, which can then be applied at once. This is especially useful in classification problems of decoupling limits. The direct method enables us to classify all possible cases corresponding to large N

decouplings, as described in Sec. V, and this would be difficult to do using the existing methods.

In Sec. II we describe the result of decoupling in a general surface. We imagine that several punctures are brought close to each other, resulting in a formation of a long tube and a weakly coupled gauge group (see for instance Fig. 2). In the extreme weak coupling limit, the Riemann surface is separated into two surfaces. We explain how, in a unique and simple manner, the result of this decoupling can be obtained. That is, what is the weakly coupled gauge group that arises, and what are the two theories corresponding to the two surfaces.

Then, in Sec. III we first identify the punctures that are created at the ends of a tube as a set that is useful to describe. The set of regular punctures that can appear in a decoupling process at the end of a tube is precisely the set of punctures L satisfying $L_1 \geq 2L_2$, excluding the simple punctures of $N > 2$. A simple diagrammatic method for finding these punctures in a given decoupling process is described. There are theories that are formed after a decoupling that can be described in terms of irregular punctures, as reviewed in Sec. II following [11,14]. The set of irregular punctures in $\mathcal{N} = 2$ SCFTs of the sort described there is classified in terms of Young diagrams.

In Sec. IV similar questions are asked from a different point of view, in which we consider what gauging of some regular puncture is possible. In other words, gauging a diagonal subgroup of the symmetry associated with a regular puncture and of a global symmetry from an additional theory, what are the possibilities for that additional theory that will result in a SCFT? The possible such gauge groups are listed. Additionally, we discuss the embedding of the gauged group in the symmetry associated with the puncture and a few implications of that.

The $\mathcal{N} = 2$ circular quiver theory is studied using holography in [15]. It turns out that in a certain limit and when N is large, it contains the $\mathcal{N} = (2, 0)$ six-dimensional A_{K-1} theory on $\text{AdS}_5 \times S^1$. In order to place the $(2, 0)$ theory on $\text{AdS}_5 \times S^1$, boundary conditions must be specified. Two sorts of boundary conditions in this context are discussed in [15] and are mentioned here. It is natural to check whether there are other class- $\mathcal{S}$ theories in which there is a decoupled field theory on AdS_5 . Additionally, for the $\mathcal{N} = 2$ theories in which this is the case, various additional boundary conditions might be possible, implemented in other $\mathcal{N} = 2$ theories analogous to the ones described in [15]. These issues are discussed in Sec. V and the conclusions of the other sections can be applied in this context.

II. WEAK COUPLING LIMITS OF A CLASS- $\mathcal{S}$ THEORY

As was mentioned in the previous section, a common situation in which the Riemann surface C of an A_{N-1} class- $\mathcal{S}$ theory degenerates is when several punctures are brought

close to each other. When C is a sphere, this is the only possibility for a degeneration. A long tube is then formed, and is associated with an emergent weakly coupled gauge group that we denote by G_T (see Fig. 2). In the extreme weak coupling limit we are left with two surfaces describing two theories. We describe what these theories are for the different possible surfaces C with a generic set of punctures.

A. Maximal gauge group along a tube

In this subsection, we examine the SW curve in the region of the tube when several punctures are brought close together, and get some preliminary information about the gauge group along the tube, G_T . This is a review of a discussion that was done in [4]. Later on we use the characterization of the punctures that are brought together as it appears here. The naive argument that is given is refined in the consequent subsections.

Suppose then that we bring m regular punctures P^i together. Let them be positioned at $z = \alpha_i$, with $\alpha_i \propto w$ and $w \rightarrow 0$ being the limit of bringing them close to each other. As the surface could be of any genus, and there might be additional punctures except for P^i , the explicit expression for the entire curve is not straightforward. However, it will be enough to concentrate on the form of the curve in the region of small z . This can be written down, and the other details are essentially immaterial for this purpose. The way they do affect the analysis is explained. As described in the previous section, the SW differential is $\lambda = xdz$. The curve in the region of small z is approximated by

$$x^N + \sum_{k=2}^N \frac{Q_k(z)}{\prod_i (z - \alpha_i)^{p_k^i}} x^{N-k} = 0. \quad (2.1)$$

The polynomials Q_k are determined by the behavior of the Coulomb branch parameters as $w \rightarrow 0$. Now let us look at the behavior of the curve in the tube region, $|w| \ll |z| \ll 1$. Substitute $x = y/z$ to get

$$y^N + \sum_{k=2}^N \frac{Q_k(z)}{z \sum_i p_k^i - k} y^{N-k} = 0. \quad (2.2)$$

For some set of punctures P^i , we use the following notation:

$$\Delta_k \equiv \sum_i p_k^i - k. \quad (2.3)$$

We always start with $\Delta_2 = m - 2$. There are essentially two possible qualitative behaviors of Δ_k for any set of regular punctures. We first review the reasoning for that and then summarize the two types of behavior.

Recall $p_k = k - h(k, P)$, $h(k, P)$ being the row number of the k th box in the Young diagram. When k increases by

1, p_k can either increase by 1 if the box is not at the end of a row, or p_k can stay the same if we are at the last box of a row. If a diagram ends with a series of rows of width 1, call this region of the diagram the “tip” of the diagram.

Δ_k can decrease by 1, stay the same, or increase by 1, 2, ..., $m - 1$, as k increases by 1. It decreases by 1 only if we are at the end of a row in each diagram. It stays the same if we are at the end of a row in $m - 1$ of the diagrams. If Δ_k decreases by 1 twice consecutively, it means that we were twice at the end of a row consecutively (in all diagrams), and therefore we passed through a row of width 1. Since the row width is nonincreasing, we are at the tip of each diagram. All p_k^i stay the same, and Δ_k continues decreasing by 1. We can also note that if Δ_k decreases by 1 and then stays the same, or stays the same and then decreases by 1, it means that in $m - 1$ of the Young diagrams we were at the end of a row twice consecutively, and therefore we are at their tip. We stay there in these diagrams, and overall Δ_k continues either decreasing or staying the same (it does not increase).

What are the options we have? First look at $m = 2$ for which we start at $\Delta_2 = 0$. Suppose first that when going from $k = 2$ to $k = 3$ we increase at first. Then if we were to go to negative Δ_k , we need to decrease by 1 from $\Delta = 0$, and before that we either stay the same or decrease by 1. In both cases we do not increase anymore, and stay at negative Δ . Therefore in this case, once we cross to negative Δ we do not get back. Next suppose Δ stays the same going from $k = 2$ to $k = 3$. If we sometime later increase by 1, we are in the previous situation. Otherwise we stay the same and then decrease, and so again we stay only at negative Δ , and the behavior is essentially the same as before. Lastly, suppose we first decrease by 1 going from $k = 2$ to $k = 3$. This means that $k = 2$ is the end of the row for both diagrams and they are of width 2. We then have to increase by 1, decrease by 1, and so on, until one of the diagrams gets to its tip. So after one of the decreases, we do not increase anymore and stay at negative Δ . Qualitatively, we get two options, which are demonstrated in Fig. 1. In the first option, we might not get to a k where Δ_k becomes negative, and so it happens that $\Delta_k \geq 0$ for all k in this option.

For $m > 2$, $\Delta_2 = m - 2 > 0$, and we do not have the second behavior of the $m = 2$ analysis. To get to negative

FIG. 1. The behavior of Δ_k in the two cases SU and Sp .

Δ_k we must either stay the same and then decrease by 1, or decrease by 1 twice. In both cases we stay at negative Δ_k .

Let us call the case in which Δ_k behaves as in the left diagram of Fig. 1 the *SU* case, and when it behaves as in the right diagram, the *Sp* case. In the *SU* behavior, Δ_k is non-negative up to some k , and once it crosses the horizontal axis and becomes negative, it is nonincreasing (as mentioned, it happens that Δ_k does not get to that region where it is negative). In the *Sp* case, Δ_k is zigzagging between 0 and -1 until some stage where it does not increase anymore. It should be kept in mind that the set of punctures P^i is of *Sp* type exactly when $m = 2$ and the first row of each of the two punctures is of width 2. The rest are of the *SU* behavior. This is an immediate way to specify whether we are in the *SU* or *Sp* case.

Define T to be the last k such that $\Delta_k \geq 0$ for both the *SU* and *Sp* cases (and for any m). Note that N might be not large enough, in which case the left diagram of Fig. 1 may not reach $\Delta_k < 0$. The definition of T is useful in this situation too.

Now let us apply a naive argument for the gauge group along the tube that is refined later on. Start with the *SU* case. The behavior around the tube is governed by (2.2). Choose the Coulomb branch parameters, such that $Q_k(z)$ behaves as z^{Δ_k} for $k \leq T$ around the tube, in the limit $w \rightarrow 0$. As is discussed later, we might not have enough Coulomb branch parameters for that. The terms in (2.2) with $k > T$ are negligible. We get an algebraic equation for y with constant solutions. Recall that $\lambda = xdz = y \frac{dz}{z}$ is the SW differential. It follows that these constant solutions give the values of the integrals over the cycles surrounding the tube. These T values (which sum to 0) are vacuum expectation values of the scalars in the vector multiplet of an $SU(T)$ gauge group. This gives naively $SU(T)$ along the tube. In the *Sp* case (for $m = 2$), the odd k terms in (2.2) are negligible, and we get $USp(T)$ along the tube (the group of rank $T/2$). If we do not have enough Coulomb branch parameters, we get a smaller gauge group. In this sense, the gauge group we found given only P^i is the maximal possibility.

B. Decoupling on a sphere using the curve

The case where C is a sphere is important and instructive. Whenever several punctures on a sphere are close to each other, this is conformally the same as having the rest of the punctures be close to each other. Therefore for a sphere the situation is quite symmetric, in which we have two sets of punctures; see Fig. 2. We can think of the punctures on the right as being close to each other, or the punctures on the left being close. As was mentioned before, the curve degenerates into two spheres and a long tube connecting them that represents a weakly coupled gauge group G_T . In the simplest description, when the gauge coupling is turned off, we remain with two spheres representing two theories, and at the points where the tube ended before, two new

FIG. 2. Decoupling on a sphere.

punctures arise, denoted as L on the left sphere and R on the right one. The question we ask is, what are the resulting two theories and what is G_T in general?

In the analysis of the previous subsection, we considered only the side of the surface with the punctures brought close to each other. We did not have enough information to determine what the resulting G_T is. Indeed, it cannot be fixed uniquely by considering punctures on one side alone, as is discussed in Sec. II D.

In the case of a sphere, we can write simply the full curve. We perform an analysis that is more precise than that of the previous subsection, and at the end of this subsection we summarize the result that we get. The main question in such an analysis is how the Coulomb branch parameters should behave as a function of the scale of the punctures that are taken close to each other (as $w \rightarrow 0$ above), with the requirement that the curves that are left on both sides make sense. This question can also be rephrased as how the zeros of the meromorphic differentials ϕ_k should behave in the limit we take.

Let us then write the curve of the theory. Denote the set of punctures on the right by P^i , and associate to them $\Delta_k^R = \sum_i p_k^i - k$ as before; similarly for the left side we define $\Delta_k^L = \sum_i q_k^i - k$ with Q^i being the punctures on the left. The curve is of the form

$$\lambda^N + \phi_2 \lambda^{N-2} + \dots + \phi_N = 0, \quad \lambda = xdz$$

$$\phi_k = \frac{Q_k(z)}{\prod (z - \alpha_i)^{p_k^i} \prod (z - \beta_i)^{q_k^i}} dz^k. \quad (2.4)$$

Suppose we bring the punctures on the right side close together and to the point labeled $z = 0$. The position of these points is proportional to some w and we take $w \rightarrow 0$. The ϕ_k 's appearing in the curve in this limit $w \rightarrow 0$ are

$$\phi_k \sim \frac{Q_k(z)}{z^{\Delta_k^R + k}} dz^k = u_k \frac{(z - z_1^{(k)}) \dots (z - z_{n_k}^{(k)})}{z^{\Delta_k^R + k}} dz^k. \quad (2.5)$$

When we take $w \rightarrow 0$ we have to decide how the parameters behave in order to get a sensible curve that will remain on the lhs. The degrees of the poles in (2.4) fix the degree of the polynomial in the numerator of ϕ_k ,

$$n_k = \Delta_k^L + \Delta_k^R, \quad (2.6)$$

where n_k is defined in (2.5) (it is related to d_k from the previous section by $d_k = n_k + 1$).

For a set of punctures we defined T as the last k such that $\Delta_k \geq 0$. Here we have such T associated to the left punctures and the right punctures, T^L and T^R .

We start by assuming that both Δ_k^L and Δ_k^R are of the SU case. We cannot have simultaneously $T^R < N$ and $T^L < N$ because then $n_N < -1$. Then suppose $T^L = N$, $T^R = T$; we are in the situation depicted in Fig. 3.

The following discussion is done for Δ_k^R , but is used afterwards for Δ_k^L and Δ_k^R interchanged. Statements such as “if $\Delta_k^L \geq 0$ then...,” which seem redundant for Δ_k^L , are not redundant when we switch Δ_k^L by Δ_k^R . The following is then a general discussion.

Fix some k . If $\Delta_k^R \geq 0$ we ask if we can flatten Δ_k^R , by which we mean that we can have enough $z_i^{(k)}$'s in Q_k that we can take to 0 together with $w \rightarrow 0$, such that the power of $1/z$ in ϕ_k reduces from $\Delta_k^R + k$ to k . This can be done only when $n_k \geq \Delta_k^R$, which is $\Delta_k^L \geq 0$. So we say that we can flatten Δ_k^R , if $\Delta_k^L \geq 0$.

- (i) If we are not able to flatten Δ_k^R , we know we cannot be left with a pole of order $> k$ (since we do not have those in the superconformal theories we consider), and we are forced to take $u_k = 0$ [u_k defined in (2.5)].
- (ii) If we can flatten Δ_k^R , we are brought to $(A + Bz + Cz^2 + \dots)/z^k$ in ϕ_k . Note that $z = 0$ becomes the position of the created puncture L on the remaining lhs sphere.

FIG. 3. Both Δ_k^L and Δ_k^R are of the SU case, which means that up to some T , $\Delta_k \geq 0$ and after it $\Delta_k < 0$. Here we have $T^L = N$ (N is the last value k obtains), while T^R may be smaller or equal to N . The actual values in the plot are chosen arbitrarily and should not have to correspond to a realistic case, but are shown for demonstration purposes only.

The constants A in the various A/z^k of ϕ_k are exactly the Coulomb branch parameters of the gauge group along the tube. This is so because these terms give in the curve $x^N + \sum_{k=2}^T \frac{A_k}{z^k} x^{N-k} = 0$, which after a change of variables $x = y/z$ gives an algebraic equation for y : $y^N + \sum_{k=2}^T A_k y^{N-k} = 0$. As explained in the previous subsection, the solutions for this equation give the Coulomb branch parameters of the gauge group (denoted usually by a_i).

Since we are interested in the lhs sphere, we take the A 's to 0, which just amounts to taking the Coulomb branch parameters of the gauge group along the tube to 0. This does not affect $B, C, \dots$. Therefore we get that when $\Delta_k^R \geq 0$ can be flattened, $l_k = k - 1$.

- (iii) If $\Delta_k^R < 0$, we do not have to do anything and therefore $l_k = k + \Delta_k^R = \sum_i p_k^i$.²

Note that in the second case, it might be that we do not have the $B, C, \dots$ terms. When we flattened $\Delta_k^R \geq 0$, we had to use Δ_k^R number of the $z_i^{(k)}$'s in (2.5). There are no $B, C, \dots$ terms exactly when $n_k = \Delta_k^R$, or equivalently $\Delta_k^L = 0$. In this case, even though we can flatten Δ_k^R , we are left with A/z^k and then take $A \rightarrow 0$ to get the massless curve. Therefore we get $\phi_k^L = 0$ in the SW curve of the lhs sphere for these values of k , and l_k is not defined by the SW curve. However, the assignment of $l_k = k - 1$ above can still be used as is explained in a moment. The idea briefly is that if we have on the lhs sphere the punctures q_k^i and $l_k = k - 1$ then the sum $\sum_i q_k^i + l_k = \Delta_k^L + k + k - 1 = 2k - 1$ and there is no k differential with poles of total order $2k - 1$ on the sphere, forcing indeed $\phi_k^L = 0$.

Let us apply the above to the situation of Fig. 3. For $k \leq T$, we can flatten all the Δ_k^R since $\Delta_k^L \geq 0$, and therefore necessarily $l_k = k - 1$ there. For $k > T$, $l_k = \sum_i p_k^i$.

The same can be done in the other direction, by asking what is left on the right-hand sphere. For this we just interchange the roles of Δ_k^R and Δ_k^L . Δ_k^L is always non-negative, but we cannot flatten all the Δ_k^L : we see this by Δ_k^R . We can flatten only $k \leq T$. Therefore $r_k = k - 1$ for $k \leq T$, where r_k is the puncture created on the rhs sphere. For $k > T$, we cannot flatten Δ_k^L , so we get $\phi_k^R = 0$ in the SW curve of the rhs sphere. r_k is not defined by the SW curve for k in this range.

We had one Coulomb branch parameter (an A) for each $2 \leq k \leq T$. From the point of view of Δ_k^R this was so because $\Delta_k^R \geq 0$ for these k 's and all of them can be flattened because $\Delta_k^L \geq 0$. From the point of view of Δ_k^L this was so because even though $\Delta_k^L \geq 0$ for all k 's, only for

²We could think that a pole of lower order can be obtained as well. This is addressed in the discussion about surfaces of general genus (Appendix).

$2 \leq k \leq T$, Δ_k^L can be flattened because only there $\Delta_k^R \geq 0$. We then have along the tube $SU(T)$.

We have seen several cases in which we get $\phi_k = 0$ in the lhs or rhs spheres (and therefore we cannot get the pole structures of L or R). There are basically two ways to approach this. We start with the first approach by describing an additional meaning in which we can assign a value to l_k or r_k in such situations.

In general, define d_k associated to some theory to be the number of Coulomb branch parameters of dimension k . We had a formula (1.4) for d_k in terms of the pole structures for a nonzero ϕ_k . The theory we started from had $d_k = n_k + 1$ parameters of dimension k . For $k \leq T$ one parameter became the single Coulomb branch parameter of dimension k for the gauge group along the tube (the one that was denoted by A). As we said before, after flattening Δ_k^R , we are left with $(A + Bz + \dots + Dz^{n_k - \Delta_k^R})/z^k$ and take $A \rightarrow 0$. This leaves $n_k - \Delta_k^R = \Delta_k^L$ parameters on the lhs sphere. Therefore $d_k^L = \Delta_k^L$ and similarly $d_k^R = \Delta_k^R$. As expected, the number of parameters is conserved, $d_k = d_k^L + d_k^R + d_k^{\text{tube}} = \Delta_k^L + \Delta_k^R + 1$. For $k > T$, no parameters go to the tube. One of the remaining curves' ϕ_k is 0 as we saw: in Fig. 3 it was ϕ_k^R , so $d_k^R = 0$. ϕ_k^L was just inherited from ϕ_k , and therefore $d_k^L = d_k$. So again the number of parameters is conserved, $d_k = d_k^L + d_k^R + d_k^{\text{tube}}$.

Now take some k for which we assume that $\Delta_k^R = n_k > 0$ (and then $\Delta_k^L = 0$), corresponding to the first situation of a vanishing ϕ_k that we encountered. According to the discussion above, ϕ_k^R is nonzero, but $\phi_k^L = d_k^L = 0$. Now suppose, by definition, that we want to maintain the equation

$$d_k = \sum p_k^i - 2k + 1 \quad (p_k^i \text{ here denote general punctures}) \quad (2.7)$$

for the lhs sphere, even though for those k 's $\phi_k^L = 0$ and l_k is not defined by the SW curve. This implies that $l_k = d_k^L - \sum q_k^i + 2k - 1 = k - 1 - \Delta_k^L = k - 1$. In this sense, we get again $l_k = k - 1$. For k 's satisfying $\Delta_k^L = n_k$ we get by the same reasoning that even though r_k is not defined by the SW curve, to preserve (2.7) we still can define $r_k = k - 1$ as we did before naively.

For $k > T$ we said that r_k is not defined by the SW curve. Using the definition we just had, we can similarly define r_k there. $d_k^R = 0$, so $r_k = d_k^R - \sum p_k^i + 2k - 1 = 2k - 1 - \sum p_k^i$. These r_k 's in the range $k > T$ satisfy $r_k \geq k$. Punctures having a pole structure greater than $k - 1$ are called irregular punctures.³ Irregular punctures and this approach

of using the graded dimension of the Coulomb branch are discussed in [11].

Note that we could be in the situation in which we started from a curve on C in which $\phi_k = 0$ for some k . The pole structures of the punctures on C (p_k^i and q_k^i) are not defined by the massless curve, but are given as part of the definition through a Riemann surface with codimension 2 defect operators. We can do the same procedure: the resulting curves have of course $\phi_k^L = \phi_k^R = 0$ for the k 's with $\phi_k = 0$, and then $d_k^L = d_k^R = 0$ again fix the l_k and r_k from q_k^i and p_k^i .

We now complete the discussion of all the possible decouplings of a sphere into two spheres in the current approach, and after this we describe another equivalent description in which we do not use irregular punctures. If both Δ_k^L and Δ_k^R are of the SU case, as was discussed $T^L < N$ together with $T^R < N$ is not possible because it implies $d_N < 0$. Therefore if both Δ_k are of the SU case, we have one with $T = N$, say $T^L = N$ and the other $T^R = T \leq N$. Both Δ_k^L , Δ_k^R cannot be of the Sp case because they give once again some $d_k < 0$.⁴ If one of the Δ_k 's is of the Sp type, then the other must be an SU . If the Sp have $T < N$, the SU must have $T = N$ (otherwise $d_N < 0$). If on the other hand the Sp have $T = N$, the SU can have $T = N$ or $T < N$. If it is $T < N$, it actually must be $T = N - 1$ because otherwise $d_{N-1} < 0$. The following three cases then cover all the situations:

- (i) Left: SU , $T^L = N$. Right: SU , $T^R = T \leq N$.
 - (ii) Left: SU , $T^L = N$. Right: Sp , $T^R = T \leq N$.
 - (iii) Left: Sp , $T^L = N$. Right: SU , $T^R = T = N - 1$.
- Possible only for even N .

(We have chosen what is left and what is right for convenience.)

We described how l_k and r_k are fixed in general. Let us summarize it for instance for l_k . r_k is obtained in the same way by switching left and right. Consider Δ_k^R and suppose $\Delta_k^R \geq 0$. If $\Delta_k^L > 0$, Δ_k^R can be flattened and we are left with a nonzero ϕ_k^L . In this case $l_k = k - 1$. If $\Delta_k^L = 0$ we can flatten Δ_k^R but are left with $\phi_k^L = 0$ when taking $A = 0$. If $\Delta_k^L < 0$ we cannot flatten Δ_k^R , and must take $u_k = 0$ to get $\phi_k^L = 0$. Therefore for $\Delta_k^L \leq 0$, $\phi_k^L = 0$, or equivalently $d_k^L = 0$. We now use (2.7) to fix $l_k = d_k^L - \sum_i q_k^i + 2k - 1 = k - 1 - \Delta_k^L$. In the case where $\Delta_k^R < 0$ we do not need to do anything and just have $l_k = \sum_i p_k^i$.

These considerations assume $\phi_k \neq 0$, but hold also in the case that $\phi_k = 0$. To see this, note first that clearly $d_k^L = d_k^R = 0$ for such k . If $\Delta_k^R \geq 0$ then $\Delta_k^L = n_k - \Delta_k^R = -1 - \Delta_k^R < 0$. l_k is fixed by $l_k = d_k^L - \sum_i q_k^i + 2k - 1 = k - 1 - \Delta_k^L$ as obtained in the previous paragraph. If

³In the context of class- $\mathcal{S}$ theories, the term irregular puncture has an additional meaning as, e.g., in [16], which differs from the one used here.

⁴ $\Delta_k < 0$ for all $k > 2$ satisfies both the SU and Sp behaviors, and we consider it to be of the SU type.

$\Delta_k^R < 0$, again $\Delta_k^L = -1 - \Delta_k^R$. Still $l_k = -\sum_i q_k^i + 2k - 1 = k - 1 - \Delta_k^L = k + \Delta_k^R = \sum_i p_k^i$ as before.

To summarize,

$$l_k = \begin{cases} k-1 & \Delta_k^R \geq 0, \Delta_k^L \geq 0 \\ k-1 - \Delta_k^L & \Delta_k^R \geq 0, \Delta_k^L < 0 \\ \sum_i p_k^i & \Delta_k^R < 0 \end{cases}. \quad (2.8)$$

This equation and the one corresponding to switching left and right fix l_k and r_k in the three cases mentioned above.

Every given situation falls into one of these three cases, which are shown in Figs. 4–6. The plots are for illustration, and the essential behavior is indicated above the plots. As we saw, there is one Coulomb branch parameter of dimension k in the gauge group along the tube exactly for $\Delta_k^L, \Delta_k^R \geq 0$. This gives us the gauge group along the tube in the three cases: $SU(T)$, $USp(T)$, and $USp(N-2)$. After the decoupling the resulting two theories are those defined by a sphere with the punctures P^i and R for the rhs theory, and Q^i and L for the lhs theory. L and R are indicated in the appropriate figure.

The irregular punctures are R in the first case if $T < N$, R in the second case, and both L and R in the third case. We have two irregular punctures at both ends of the tube only for even N , in which case we get $USp(N-2)$ along the tube. The only case in which both punctures at the ends of the tube are regular is Fig. 4 with $T = N$, in which both punctures are full punctures ($l_k = r_k = k-1$ for all k) and the gauge group along the tube is $SU(N)$.

We can describe the result of the decoupling using only the familiar regular punctures. In all of the resulting spheres that we have found having an irregular puncture, the ϕ_k are 0 starting from some k .⁵ There are then a set of branches of the curve that are decoupled from the rest of it. This is the same as the following:

- (i) A usual $A_{N'-1}$ theory with lower $N' < N$ having only regular punctures, with the curve obtained by canceling a common factor of x in (1.1) from the curve with the vanishing ϕ_k 's. If all $\phi_k = 0$ for $k \geq 2$ we have only the second ingredient below. The punctures are obtained by truncating the pole structures to $2 \leq k \leq N'$.
- (ii) Plus possibly additional free hypermultiplets.

The number of additional free hypermultiplets in such a theory can be found as follows. We started from some theory C and after a decoupling limit, had a theory C_1 , a theory C_2 which is a theory of regular punctures, and n_h additional free hypermultiplets. Each of C , C_1 , and C_2 is a theory of regular punctures. For each theory, a number of effective hypermultiplets and vector multiplets was defined, by a relation to the a , c anomalies [17].

⁵This might also happen in theories with regular punctures only.

FIG. 4. First case. We get in the tube $SU(T)$. If $T = N$ both l_k and r_k are full punctures.

Subtracting the number of effective hypers of C_1 and C_2 from C we find n_h . We do not quote this simple algebraic calculation, but merely give the result.

In the Sp behavior, the resulting curve on the Sp side is trivial ($x^N = 0$), and the theory there is then only a set of free hypermultiplets.

The missing ingredient is an expression for N' in the SU case. We use results that are obtained later on in order to express the different quantities using the diagrams of the punctures. According to what we saw, in the resulting theories of interest, the last k for which $\phi_k \neq 0$ and afterwards all $\phi_k = 0$ is the last k such that $\Delta_k > 0$. Using the discussion in Sec. II A and Eq. (4.4), if there exists a last k such that $\Delta_k > 0$ then it is given by $\sum_{i=1}^{a-1} P_i^1$ [we use the conventions and notations from the text around (4.4) and write these below when we summarize the result⁶].

If there is no $\Delta_k > 0$, the resulting theory is a theory of free hypermultiplets ($N' = 0$). By following the lines of the analysis of the behavior in the SU case, we can construct quite easily the diagrams that result in all $\Delta_k \leq 0$ by following the behavior of Δ_k as we append each box. Since $\Delta_2 > 0$ if $m > 2$, this can happen only with two punctures P^1, P^2 that are brought together to form the decoupling sphere in which we are interested. By doing this exercise, we find that there is no $\Delta_k > 0$ in three basic cases (which are not distinct in the way they are written). In the first option we have a simple puncture and some other puncture P^1 . In this case $\alpha = 1$ and the resulting theory is only a theory of $P_1^1(P_1^1 - P_2^1)$ free hypermultiplets. The second option is when we have $P_1^2 = 2, P_1^1 = 3, P_2^1 \leq 2$. Finally, we can have any $P_1^1 \leq 3$ and P^2 of rows 2, 2, 1, 1, ...

For these cases we just need the number of free hypers. In general, as was explained above, the number of effective

⁶In these conventions we choose P^1 to be of largest p . Note that there might be several punctures with the largest p . In all the calculations [such as that of T in (4.4)] it does not matter which one we choose, except for calculating N' in some cases where we have $m = 2$ punctures. In these cases, if only one of them has $P_\alpha > 1$ we choose it as P^1 (and otherwise it does not matter which one is chosen).

FIG. 5. Second case. We get in the tube $USp(T)$. FIG. 6. Third case. Possible only for even N . In the tube we get $USp(N-2)$.

hypers minus the number of effective vector multiplets can be calculated for the decoupled rhs theory in the first general case (Fig. 4). The diagrammatic description that is given later on is used in this calculation. The number that is found is (still ordering $p^1 \geq p^i$)

$$\begin{aligned} n_\Delta &= n_h - n_v = -1 + \frac{1}{2} P_{1,\alpha}^1 (P_{1,\alpha}^1 - P_{\alpha+1}^1) \\ &\quad + \frac{1}{2} \sum_{i=1}^{\alpha} P_{1,i}^1 (P_i^1 - P_{i+1}^1) + \sum_{j \geq 2} n_\Delta(P^j) \\ n_\Delta(P) &= -\frac{N}{2} + \sum_i \frac{1}{2} P_{1,i} (P_i - P_{i+1}) \geq 0, \\ P_{1,i} &\equiv \sum_{j=1}^i P_j. \end{aligned} \quad (2.9)$$

In these cases in which there is no $\Delta_k > 0$ we find that the total number of free hypers is

$$\begin{aligned} n_h &= -1 + \frac{1}{2} P_{1,\alpha}^1 (P_{1,\alpha}^1 - P_{\alpha+1}^1) + \frac{1}{2} \sum_{i=1}^{\alpha} P_{1,i}^1 (P_i^1 - P_{i+1}^1) \\ &\quad + \sum_{j \geq 2} n_\Delta(P^j). \end{aligned} \quad (2.10)$$

Let us summarize how the result of decoupling can be found very easily, in terms of usual theories with regular punctures. We are given a sphere C with a set of punctures P^i on the right and a set of punctures Q^i on the left. We assume that they are ordered such that $p^1 \geq p^i$ for any i (see the previous section for the definition of p), and the same for Q^1 relative to Q^i .⁷ Up to a possible renaming of what we call left and right, any given configuration should fall into one of the following cases, where by “ P^i are of Sp type” we simply mean that we have two punctures with

⁷For choosing P^1 among several punctures with the largest p , see the previous footnote.

their first row being of width 2 and all the rest are referred to as SU type.

- (i) P^i and Q^i are of SU type, and $\sum_i q^i \geq N$.

In that case, we have along the tube $SU(T)$ with $T = \sum_{i=1}^{\alpha} P_i^1$ where we denote $\alpha = \sum_{j \geq 2} p^j$. The resulting theory on the lhs is given by an A_{N-1} theory with the punctures Q^i and additionally $l_k = \min(k-1, \sum_i p_k^i)$. We show in Sec. III that the diagram of L is the diagram of P^1 with the $\alpha + 1$ first rows merged to a single one.

The theory on the rhs is a theory in $A_{N'-1}$ where $N' = \sum_{i=1}^{\alpha-1} P_i^1$, with the punctures P^i (truncated to $k = 2, \dots, N'$) and an additional full puncture instead of the tube, plus $(N' + P_{\alpha}^1)(P_{\alpha}^1 - P_{\alpha+1}^1)$ free hypers.

If there are only two P^i punctures such that one of them is a simple puncture, or one of them has $P_1 = 2$ and the other $P_1 = 3, P_2 \leq 2$, or one of them has $P_1 \leq 3$ and the other is of rows $2, 2, 1, 1, \dots$, then the rhs is a free theory with the number of hypers given by (2.10).

- (ii) Q^i are of SU type while P^i are of Sp type, and $\sum_i q^i \geq N$.

Along the tube we get $USp(T)$ where now $\alpha = P^2$ since there are only two punctures on the rhs. The resulting theory on the lhs is described just as in the first case. The theory on the rhs is just $2\alpha(2 - P_{\alpha+1}^1)$ free hypers.

- (iii) The third case is quite special. It occurs only for even N . On the lhs there are two punctures with all rows being of width 2. On the rhs P^i are of the SU type with $\sum_i p^i = N - 1$.

On the tube we have $USp(N - 2)$. The lhs theory is a free theory of $2N$ hypers. The theory on the rhs is as described in the first case.

What is needed above are only the rows structure of the Young diagrams of the punctures, and this specification is therefore very easy to apply.⁸

C. $g \geq 1$ surfaces

Suppose we have a $g \geq 1$ surface with regular punctures, and we bring several of them close together. We would like to get the resulting theories and tube as was done for the sphere. The scenario we consider is depicted in Fig. 7.

In the case of the sphere, the situation between the two sets of punctures was symmetric. We could learn on each side by bringing the punctures on the other side close to each other. In a $g \geq 1$ surface the situation is not symmetric. We bring punctures say on the right close to each other. A sphere bubbles off and a long tube is formed, connecting the sphere and the remaining surface (as in Fig. 7). An

FIG. 7. Decoupling in a $g \geq 1$ surface. The punctures L and R are shown but do not appear until the complete decoupling of the tube.

additional complication is that on the sphere we could use the full curve, which is more involved for a general surface. We should avoid writing its full form, and instead concentrate on the region of the punctures that are brought together, and be able to get both of the resulting theories.

This analysis appears in the Appendix. The answer is that any situation is either the first or the second scenario described for the sphere (without the third one). The gauge groups along the tube and the resulting two theories are the ones described there. In the convention used, the rhs theory is a sphere, while the lhs is of the same genus as that of C .

D. Do the decoupling punctures fix the tube?

In this section, we saw that given the punctures on any surface, when some of them are brought together, the resulting tube and theories on both sides of it are determined completely. We gave the gauge group along the tube and the pole structures of the created punctures for all the cases (as well as any needed additional information such as the $N' \leq N$ in which a resulting theory is defined and the number of additional free hypermultiplets in the corresponding description).

We could hope naively that if for instance we bring the punctures on the right side close together, the pole structures of these punctures alone may be sufficient to fix the additional data required to specify the decoupling result (that is, the gauge group along the tube, the punctures L and R , and the $N' \leq N$ and additional number of hypers when needed). From what we saw, it does not have to be the case. An example is shown in Fig. 8. There, two full punctures are brought together, but give different tubes.

We may ask, when does one side of the decoupling determine the gauge group along the tube and the additional data mentioned above (all of these together are called shortly the tube below). For the sphere, we can get the answer by inspecting the three cases from Sec. II B that cover all the scenarios. Suppose we are given a set of punctures that are brought together, and calculate Δ_k . Begin with the case in which they are of the SU case. If T of that Δ_k equals N , we might be for instance on the left side of the first case, with different possibilities for the right side. In

⁸Note that we assume that the sphere we began with is a legitimate theory, since not any collection of punctures on the Riemann sphere is an acceptable theory.

FIG. 8. Several options for decoupling of the same right sphere (see [11,14] for many examples).

the first case, the tube is determined by the right side as can be seen by the results there, and hence we cannot determine the tube uniquely. If $T = N - 1$ we might still be either in the first or the third case, and the tube is not determined. If $T < N - 1$ we are necessarily on the right side of the first case, and it determines the tube uniquely. Now suppose that the Δ_k we obtained is of the Sp type. If $T = N$ even, we might be either in the situation of the second case or of the third one, which gives different tubes (in particular, different gauge groups). If $T < N$, we are necessarily in the second case, the right side of which fixes the tube.

To conclude, a set of punctures P^i brought together determines the information needed to specify the result of the decoupling, when they are of the SU type with $T < N - 1$ or the Sp type with $T < N$. As is explained later on, T can be expressed in terms of the Young diagrams of P^i through (4.4) where $p^1 \geq p^i$ is the puncture with the largest p and $\alpha = \sum_{i \geq 2} p^i$.

For a $g \geq 1$ surface, we have only the first two cases and the punctures that are brought together correspond to the right-hand sides there. Therefore the punctures that are brought together determine the tube completely.

E. Relation to previous works

Since this section attempts to answer questions that were already raised in other papers, we mention the most relevant papers in this context, and explain the differences between the various approaches.

One of the most relevant references to this work is [11], as it addresses systematically degenerations in class- S theories of type A_{N-1} . The idea there is that since three-punctured spheres and tubes are the basic building blocks of any punctured Riemann surface, knowing these ingredients allows one to answer the question of the result of the degeneration. In order to do that, at every given finite value of N , various consistency conditions (many relying on the graded Coulomb branch dimension) and analysis of the collision of punctures are used in order to classify all the possible three-punctured spheres and cylinders. Once this is done for that value of N , the degeneration of an A_{N-1} theory can be found by assembling the three-punctured spheres and cylinders in order to form the surface. Punctures with nonstandard pole structures arise as solutions to the consistency conditions, which are the irregular punctures.

In the current section what was done is to perform a full analysis of the SW curve, and to use only this in order to fix

the final result of the decoupling. It is not necessary then to classify all the three-punctured spheres and cylinders for every N (as the number of these increases significantly with N). The analysis of the SW curve appearing in this section was performed only once, and there is no need to repeat it for each given case: rather than giving an algorithm to find the result of the decoupling, this section gives the final result straight away. For a degenerating Riemann surface (of any genus), the emergent weakly coupled gauge group (along the cylinder) and the resulting theories after the complete decoupling are given (for any N), in terms of the punctures on the degenerating surface. The resulting theories are expressed using usual class- S theories (with only regular punctures) plus possibly some free hypermultiplets.

Following [11], the same question is discussed in terms of irregular punctures, and the result is given also in terms of those. It is also instructive to mention further a couple of points that are relevant and complementary to [11]. Even though the properties of irrelevant punctures are given in [11], not any pole structure satisfying them is an honest irregular puncture. It should be checked that it indeed occurs in a three-punctured sphere. Therefore, there is no classification of all the irregular punctures *a priori*, other than going over the construction for any particular N . In Sec. III D we give a simple classification of the irregular punctures, similar to the usual regular punctures. Another point is that as mentioned in [11], a cylinder connecting two irregular punctures occurs rarely. In fact, we saw exactly when it happens, which is the third case, Fig. 6 [giving always $USp(N-2)$ along the tube].

Another relevant reference that should be mentioned is [6]. The analysis of the SW curve that appears in Sec. II A appeared already in Sec. 3 of [6].⁹ Stated in notations similar to those that were used there, when several punctures P_i are brought close to each other, a sphere X bubbles off, a tube with gauge group H is formed, and a puncture Q is created on the other side of the tube.

Now suppose that we have a sphere with two sets of punctures on each side, and we would like to apply Sec. II A for each of the two sides. This gives us X, H, Q from one side and X', H', Q' from the other. However, as was mentioned in Sec. II A this is only a naive analysis and is not guaranteed to give the correct result in general (it does hold for instance when we consider punctures brought

⁹It is applied there to theories beyond the A_{N-1} which we consider here, and also in Sec. 2.2 of [18] and Sec. 2.4.5 of [8].

together when at the other side of the tube there is a $g \geq 1$ surface). That is, H, H' are only the maximal gauge groups as was explained, and also it is not guaranteed that at the ends of the final tube we get Q, Q' . In order to fix this, a possible approach which was taken in [6] is to consider the theories X, X' connected by the tubes H, H' to a sphere containing both Q, Q' . In other words, using the naive analysis on each side independently gives two results that need to be sewn, and this is achieved by considering the two-punctured sphere with Q, Q' . This can then be analyzed by using the description of this two-punctured sphere as a nonlinear sigma model following [19]. In this description the product $H \times H'$ should be Higgsed to give the actual final gauge group G_T that emerges along the tube. However, in addition to this, it is important that the resulting theories after the decoupling might themselves differ from X, X' obtained by the naive analysis. This needs to be checked, and is accounted in the description of [19] by Higgsing X, X' to some other theories due to the F- and D-term constraints. As before, this gives another procedure that eventually should agree with the result given at the end of Sec. II B, and in Sec. II C for $g \geq 1$ surfaces.

III. DIAGRAMMATIC DECOUPLING

A. Punctures appearing at the end of a tube

It turns out to be instructive to distinguish the class of punctures that can be created when a tube decouples in a weak coupling limit.

According to what we saw in the discussion of the sphere and $g \geq 1$ surfaces, the three cases mentioned in Sec. II B exhaust all the possibilities. By examining these options, the (regular) punctures that appear at the end of a tube are exactly those that are given by the formula

$$l_k = \min\left(k - 1, \sum_{i=1}^m p_k^i\right), \quad (3.1)$$

where P^i are the (regular) punctures that are brought together on the other side of the tube.

Any puncture that can be expressed as in (3.1) for some set of m punctures can also be obtained from only two punctures. A short way to see this is to note the identity

$$\begin{aligned} \min(p_k^1 + \dots + p_k^m, k - 1) \\ = \min(k - 1, p_k^1 + \min(k - 1, p_k^2 + \dots + \min(k - 1, p_k^{m-1} + p_k^m))) \end{aligned} \quad (3.2)$$

for regular punctures $P^1, \dots, P^m$. Let us just write a particular case of this formula, so that the form of the formula is clear: (3.2) for $m = 3$ is

$$\begin{aligned} \min(k - 1, p_k^1 + p_k^2 + p_k^3) \\ = \min(k - 1, p_k^1 + \min(k - 1, p_k^2 + p_k^3)). \end{aligned} \quad (3.3)$$

Let us prove (3.2). Denote the term added to each p_k^i by $\bar{p}_k^i$. Look at $p_k^1 + \bar{p}_k^1$. When $p_k^1 + \bar{p}_k^1 \leq k - 1$ necessarily we do not choose in the next minimum the $k - 1$. So in this region, $p_k^1 + \bar{p}_k^1 = p_k^1 + p_k^2 + \min(k - 1, p_k^3 + \bar{p}_k^3)$. Continuing with the same reasoning, we get that when $p_k^1 + \bar{p}_k^1 \leq k - 1$, $p_k^1 + \bar{p}_k^1 = p_k^1 + \dots + p_k^m$.

By repeated use of the rhs of (3.1) being regular, $\bar{p}_k^1, \dots, \bar{p}_k^{m-2}$ are regular punctures. Apply now the analysis of Sec. II A to p_k^1 and $\bar{p}_k^1$. For $k > T$, $p_k^1 + \bar{p}_k^1 \leq k - 1$ and we saw what happens for these values of k . For $k \leq T$, $p_k^1 + \bar{p}_k^1 \geq k - 1$. The behavior of $p_k^1 + \dots + p_k^m$ is of the SU type only. It implies that $p_k^1 + \dots + p_k^m \geq k - 1$. For both of the ranges of k we get

$$\min(k - 1, p_k^1 + \bar{p}_k^1) = \min(k - 1, p_k^1 + \dots + p_k^m) \quad (3.4)$$

giving (3.2) indeed. This shows, in particular, that the punctures obtained by the rhs of (3.4) can also be achieved using the lhs.

Call the set of all (regular) punctures l_k that can be written as $l_k = \min(k - 1, p_k + p'_k)$ where p_k and p'_k are regular punctures, “primary regular punctures” (PRPs). The regular punctures that can appear at the end of a tube are exactly the PRPs. We give a simple classification of the PRPs now.

B. Classification of PRPs

Let us give a simple characterization of the possible PRPs in the language of the corresponding Young diagram. We saw that any PRP can be obtained by colliding two punctures P and P' . We defined T to be the last k for which $\Delta_k = p_k + p'_k - k \geq 0$. Let us show that for a PRP $L_1 - L_2 \geq T$ and $T \geq L_2$. Assume meanwhile that $T < N$. After $k = T$ we saw that Δ_k does not increase anymore, and therefore $k = T + 1$ is already at the tip of at least one of the diagrams, say P' . At $k = T + 1$ we must have $\Delta_k = -1$, that is $p_k + p'_k = k - 1$, and we get $l_{T+1} = T$. $k = T$ is the end of a row in both P and P' (because afterwards Δ_k decreases by 1). Denote the width of the row coming right after $k = T$ in P by P_i . After $k = T$, p'_k does not change anymore (stuck at the tip), so Δ_k stays -1 as long as we increase p_k , that is, P_i more times. We get therefore that $L_1 = T + P_i$. Afterwards, l_k becomes $k - 2$, and stays so as long as Δ_k stays the same, which happens for P_{i+1} more steps. Then $L_2 = P_{i+1}$. We get that $L_1 - L_2 = T + P_i - P_{i+1} \geq T$, and $L_2 = P_{i+1} \leq P_1 \leq T$ (the last inequality holds because as long as $k \leq P_1$, $p_k = k - 1$ and $\Delta_k = p_k + p'_k - k \geq p'_k - 1 \geq 0$ so $T \geq P_1$).

We assumed that $T < N$. If this does not happen, then $T = N$. In both the SU and the Sp cases, this means that $p_k + p'_k \geq k - 1$ for all $2 \leq k \leq N$. By the PRP

formula (3.1), $l_k = k - 1$ for all $2 \leq k \leq N$. Remembering that $l_k = k - h(k, L)$ [$h(k, L)$ being the row number of the box number k in the Young diagram], it implies that $L_1 = N$ and $L_i = 0$, $i \geq 2$.

We obtained that anyway, for a PRP,

$$L_1 - L_2 \geq T \quad \text{and} \quad T \geq L_2. \quad (3.5)$$

This implies that $L_1 - L_2 \geq L_2$, or $L_1 \geq 2L_2$ is a necessary condition for a PRP.

Now we claim that it is almost sufficient. Given $L_1 \geq 2L_2$ (except for a simple puncture when $N > 2$), it is a PRP.

Define p_k through its Young diagram. Take it to have rows with the number of boxes being $L_1 - L_2, L_2, L_2, L_3, L_4$ and so on. Choose p'_k to be a simple puncture. The values of $p_k + p'_k$ for k 's in the corresponding rows of p_k are k in first, $k - 1$ in second, $k - 2$ in third, and so on (recall $p'_k = 1$ for a simple puncture for all k , and p_k is k minus the height of the k th box in the diagram). $\min(k - 1, p_k + p'_k)$ is $k - 1$ for the first $L_1 - L_2 + L_2 = L_1$ values of k , $k - 2$ for the next L_2 values, $k - 3$ for the next L_3 values, and so on. We therefore constructed L (rows of width $L_1, L_2, L_3, \dots$) using $\min(p_k + p'_k, k - 1)$. We required that L is not a simple puncture if $N > 2$, because otherwise our construction of p_k gives a no puncture (all rows have a single box). A simple puncture in $N > 2$ is not a PRP, because for instance we saw the requirement $L_1 - L_2 \geq T \geq 2$, which is not satisfied by a simple puncture. When $N = 2$ a simple puncture is also a full puncture, having $L_2 = 0$.

To summarize, PRPs are the regular punctures having $L_1 \geq 2L_2$, not including the simple punctures of $N > 2$.

C. Diagrammatic construction of the decoupling

A (regular) puncture that can appear at the end of a tube can be obtained by the decoupling of some punctures P^i , and is given by (3.1). This relation can be described diagrammatically in a simple way. We show now how the Young diagram of L is found easily from those of the P^i .

Recall that a Young diagram of a puncture can end with consecutive rows of width 1, and we called that region of the diagram the tip of the diagram. We also denoted the row number of box number k in the diagram of the puncture P by $h(k, P)$. Note that since $p_k = k - h(k, P)$, p_k equals the number of boxes until box number k that are not in the first column.

For a general puncture P we have the following restrictions. At $k = \min(2p, N)$ we at least reach the last box of the diagram without the tip, and hence after this k , p_k does not change anymore (Fig. 9 might be useful in this discussion),

$$p_k = p \quad \text{for } k \geq \min(2p, N). \quad (3.6)$$

FIG. 9. Demonstration of the tip of a diagram for a puncture P and the value p .

What about the region $k \leq \min(2p, N)$? If $P_1 = 2$ then $p_k = \lfloor \frac{k}{2} \rfloor$ for these k 's. If $P_1 > 2$, at any box in this range at least half of the boxes lay outside the first column, and so $p_k \geq \frac{k}{2}$. Together,

$$\begin{aligned} p_k &= \lfloor \frac{k}{2} \rfloor & \text{if } P_1 = 2, \\ p_k &\geq \frac{k}{2} & \text{if } P_1 > 2, \end{aligned} \quad \text{for } 2 \leq k \leq \min(2p, N). \quad (3.7)$$

Arrange the punctures P^i such that $p^1 \geq p^2 \geq p^3 \geq \dots$. Note that for $k \leq \min(2p^2, N)$ we have $\sum_i p_k^i \geq k - 1$ by (3.7). Define $\alpha = \sum_{i \geq 2} p^i$. Suppose that $2p^2 \leq N$. Since $2p^2 \leq 2p^1$, we have $p_{k=2p^2}^1 \geq \lfloor \frac{k}{2} \rfloor = p^2$ and the row number of the corresponding box is $h(2p^2, P^1) = 2p^2 - p_{2p^2}^1 \leq p^2 \leq \alpha$. For $k \geq \min(2p^2, N)$, using (3.6), $\sum_{i \geq 2} p_k^i = \alpha$ is constant. In this range, $l_k = \min(k - 1, \sum_i p_k^i) = \min(k - 1, \alpha + k - h(k, P^1))$. We see that until row number $\alpha + 1$ in P^1 [which includes the range $k \leq \min(2p^2, N)$ indeed since $h(2p^2, P^1) \leq \alpha$] $l_k = k - 1$, that is we are in the first row of L . For k in row number $\alpha + n$ in P^1 (with $n > 1$), $l_k = k - n$ so we are in the n th row of L . In other words, L is just P^1 with the $\alpha + 1$ first rows combined in a single row.

If on the other hand $2p^2 > N$ then at $k = N$ we have $p_N^1 \geq \lfloor \frac{N}{2} \rfloor = N - \lceil \frac{N}{2} \rceil \geq N - p^2$, or $h(N, P^1) \leq p^2$. We saw that for $k \leq \min(2p^2, N)$ we have $\sum_i p^i \geq k - 1$ and so for all k , $l_k = k - 1$, which is a full puncture. This is still described by combining the first $\alpha + 1$ rows of P^1 in the sense that $\alpha + 1 \geq p^2 \geq h(N, P^1)$; that is, there are more rows to combine than we have in P^1 .

To summarize, arranging the punctures such that $p^1 \geq p^i$ for all i and defining $\alpha = \sum_{i \geq 2} p^i$, the Young diagram of L is obtained by combining the first $\alpha + 1$ rows of P^1 to a single row. This is demonstrated in Fig. 10. If the number of rows in P^1 is less than $\alpha + 1$, we just combine all of them to a single row, and L is a full puncture.

D. Irregular punctures

We have given a simple result for the process of decoupling, which requires the use of the familiar regular punctures only. The same applies to the rest of the sections. However, if we would like to describe the resulting theories in the same A_{N-1} theories (that is, the same N), we have to

FIG. 10. The diagrammatic method.

use irregular punctures.¹⁰ In this section we pause to discuss them. These irregular punctures are discussed in [11,14]. *A priori* it is not possible to say what is the set of all irregular punctures for N general (for that, it is necessary to check in what three-punctured spheres they might appear). With the tools we gained, we can now classify all the irregular punctures. We introduce Young diagrams analogous to those of regular punctures. The forms of these diagrams that correspond to irregular punctures are described.

By considering Figs. 4–6 we see that all the irregular punctures can be obtained in a decoupling process on the rhs of Figs. 4 or 5 (these include the irregular punctures of

Fig. 6). Therefore, a general irregular puncture can be realized in the setup of Fig. 11 in which L is a regular puncture (a PRP). Necessarily the puncture R is either an irregular puncture or a full puncture. It is then convenient to refer to a puncture that is either a full puncture or an irregular puncture as a PIP. We study PIPs in this setup of Fig. 11.

From Figs. 4 and 5, there are two types of irregular punctures, depending on whether the punctures P^i are of the SU type or the Sp type. We call the corresponding irregular punctures SU and Sp irregular punctures. Their pole structure is given by

$$SU \text{ irregular puncture: } p_k = k - 1 \quad \text{for } k \leq T < N \quad \text{and} \\ p_k = 2k - 1 - \sum_i p_k^i \geq k \quad \text{for } k > T, \quad \text{with some set of regular punctures } p_k^i, \quad (3.8)$$

Sp irregular puncture: p_k oscillates $k-1, k, k-1, k, \dots, k-1$ up to some even T and

$$p_k = 2k - 1 - \sum_i p_k^i \geq k \quad \text{for } k > T, \quad \text{with some set of regular punctures } p_k^i. \quad (3.9)$$

In Fig. 11, while L is given by (3.1), R can be expressed as (see Figs. 4 and 5)

$$r_k = \begin{cases} k-1 & \Delta_k \geq 0 \\ 2k-1 - \sum_i p_k^i & \Delta_k < 0 \end{cases}$$

$$= \max\left(k-1, 2k-1 - \sum_i p_k^i\right). \quad (3.10)$$

¹⁰To avoid confusion, we note that there is another sense in which the term irregular puncture is used; see footnote 3.

In Sec. III C we gave several arguments that when combined with (3.1) resulted in the diagrammatic description. We now apply the results there and use (3.10). First, consider the case where P^i have the SU behavior. In that case, we can refine the inequality in the range $k \leq \min(2p^2, N)$ to $\sum_i p_k^i \geq k$ by (3.7) (since either $m > 2$ or at least one of the P^i have $P_1^i > 2$). For $k \geq \min(2p^2, N)$, $\sum_{i \geq 2} p_k^i = \alpha$. Substituting in (3.10), in this range, $r_k = \max(k - 1, k - 1 - \alpha + h(k, P^1))$. This equation actually holds for all k since for $k \leq \min(2p^2, N)$ we have $k - 1 - \alpha + h(k, P^1) \leq 2k - 1 - \sum_i p_k^i \leq k - 1$. Therefore, until row number α in P^1 , $r_k = k - 1$, and for a k in row number $\alpha + n$ in P^1 , $r_k = k - 1 + n$.

FIG. 11. Bringing several punctures together.

We introduce Young diagrams for SU PIPs (that is, an SU irregular or a full puncture) as on the left side of Fig. 12. These are usual Young diagrams, colored in red to distinguish them. The box number k in the n th row gives $r_k = k - 2 + n$. We see by the discussion above that the puncture R of SU PIP type is obtained diagrammatically from the P^i 's in a similar way to the regular L . Arranging $p^1 \geq p^i$ for all i , the diagram of R is that of P^1 with the α first rows combined to a single row, $\alpha = \sum_{i \geq 2} p^i$. In a moment we classify the diagrams that are obtainable.

Before that, let us address the remaining Sp case. We have only P^1 and P^2 , $P_1^i = 2$ for both, and $\alpha = p^2$ assuming $p^1 \geq p^2$. Necessarily $2p^2 \leq N$ since $P_i^2 \leq 2$. In the range $k \leq 2p^2$, by (3.7), $\sum_i p_k^i = 2\lfloor \frac{k}{2} \rfloor$. Using (3.10), $r_k = \max(k - 1, 2\lfloor \frac{k}{2} \rfloor - 1) = 2\lfloor \frac{k}{2} \rfloor - 1$. For $k \geq 2p^2$, $p_k^2 = p^2$ and $r_k = \max(k - 1, k - 1 - \alpha + h(k, P^1))$. Thus again if k is in row number $\alpha + n$ in P^1 , $r_k = k - 1 + n$. Note that $p_{k=2p^2}^1 = \lfloor \frac{k}{2} \rfloor = p^2$ and $h(2p^2, P^1) = 2p^2 - p_{2p^2}^1 = p^2 = \alpha$.

We see that for the α first rows of P^1 , $r_k = 2\lfloor \frac{k}{2} \rfloor - 1$, while for the $(\alpha + n)$ th row ($n > 0$), $r_k = k - 1 + n$. We represent Sp PIPs (an Sp irregular puncture or a full puncture) by a Young diagram, colored in blue, where a k in the first row is associated with $r_k = 2\lfloor \frac{k}{2} \rfloor - 1$ and for the n th row ($n > 1$), $r_k = k - 2 + n$; see the right side of Fig. 12. Diagrammatically, we get R by just combining the first $\alpha = p^2$ rows of P^1 to a single row (where again $p^1 \geq p^2$). Since $R_1 = 2\alpha$, the number of boxes in the first row of an Sp PIP diagram is always even.

We are now in a position to classify the PIPs and the irregular punctures. Any PIP is described by a Young diagram from Fig. 12. The question is what subset of these diagrams give precisely all the PIPs. Note that the two types

FIG. 12. Diagrams for PIP (irregular and full) punctures. SU PIPs are marked in red (left), and Sp PIPs are marked in blue (right).

of diagrams have an overlap that is given by the Sp diagrams of $R_1 = 2$. These diagrams are realized precisely when $m = 2$, $P_1^1 = 2$ and P^2 is a simple puncture. To avoid this redundancy, even though this set of P^i is of Sp type, we group it with the SU PIPs. First, we claim that any SU PIP diagram is obtainable (except for $R_1 = 1$ as usual). The reason is that we can simply take P^1 to be a general regular puncture and P^2 to be a simple puncture having $p^2 = 1$, and then by the diagrammatic description R has the same diagram as P^1 . The SU PIP diagrams are the same as the regular puncture diagrams (which are all the Young diagrams with more than one column). Next consider Sp PIPs. We have P^1 with p^1 rows of length 2 and the rest contain a single box, and P^2 with $p^2 \leq p^1$ rows of two boxes and the rest with a single one. By the diagrammatic rules, R has $R_1 = 2p^2$, then $p^1 - p^2$ rows of length 2, and the rest with a single box. Thus all the Sp PIPs are such that R_1 is even and $R_i \leq 2$ for $i > 1$. To avoid the redundancy above we restrict to $R_1 > 2$.

To recover the irregular punctures we just need to throw away the full punctures, which are the SU PIPs with $R_2 = 0$. Summarizing, the irregular punctures are described by all the SU Young diagrams with at least two rows and columns and by the Sp Young diagrams with $R_1 > 2$ even and $R_i \leq 2$ for $i > 1$. The Sp irregular punctures are described by just two numbers, $p^2 > 1$ and $p^1 - p^2 \geq 0$.

IV. GAUGING A GIVEN THEORY

In the discussion of the decoupling limits we considered, almost always¹¹ the Riemann surface in an A_{N-1} theory degenerated to a surface with the same genus and same N , with an additional regular puncture L by which it was connected through a tube to a sphere. This situation is demonstrated in Fig. 11. We classified what the regular punctures are (the possible L 's) that appear at the end of tubes (which we called PRPs). In this section we want to look at this situation from the other direction. Given a PRP L , we know it can be gauged, that is, connected through a tube to a sphere with punctures P^i . This amounts to gauging a diagonal global symmetry group of the symmetry of L and of the additional sphere. We ask then in Sec. IV A what the possibilities are for this sphere given a PRP L . In Sec. IV B we consider what subgroups of the global symmetry associated with the puncture L can be gauged (that is, what gauge group G_T we can have along a tube). In Sec. IV C we describe how G_T is embedded in the symmetry of L and what restrictions that were obtained for class-S theories of A type (without twisted punctures) are still valid (from field theory considerations) when we replace the additional sphere by a generic theory. A bound

¹¹Recall that the exception for that is the third case mentioned at the end of Sec. II B, which occurs only for even N and gives just $USp(N-2)$.

FIG. 13. Constructing A and B .

relating the symmetry of G to the symmetries of the punctures P^i is given in Sec. IV D.

A. Gauging a puncture

Given a PRP L , we show here how we can find all the possibilities for P^i in Fig. 11. For this purpose, the diagrammatic description in Sec. III C (which applies here) is very useful. We restrict ourselves to L , which is not a full puncture, and at the end we address the case of a full puncture. In the discussion in Sec. III C we saw that for $k \geq 2p^2$ (in the ordering of the punctures used there) all p_k^i do not change with k for all $i \geq 2$ (we have used $2p^2 < N$ since L is not a full puncture), and that $k = 2p^2$ is in the first α rows of P^1 . L is obtained by combining the first $\alpha + 1$ rows of P^1 . It follows that after the α th row of P^1 , all other P^i with $i \geq 2$ are at their tip.

Any PRP can be gauged by two punctures as we saw, and therefore we start by addressing the question for $m = 2$. Let us denote them instead of P^1 and P^2 , by A and B . We order them as in the diagrammatic description with $a \geq b$. We saw that if we restrict their diagrams to the first L_1 boxes, we cover exactly the first $b + 1$ rows of A , and in B we are necessarily at the tip already. Additionally, since $A_{b+2} = L_2$, we must have $A_{b+1} \geq L_2$. The first L_1 boxes of A necessarily belong to the following class of diagrams,

- (i) $A(L_1, L_2, n) =$ diagrams of L_1 boxes, n rows, and $A_n \geq L_2$,

with $n = b + 1$. The full A diagram is given by removing the first row of L and placing the resulting diagram on top of a diagram in $A(L_1, L_2, b + 1)$. The first L_1 boxes of B give a diagram in the class

- (ii) $B(L_1, n) =$ diagrams of L_1 boxes, n rows, and $B_n = 1$.

The full diagram of B is given by extending the tip of this diagram.

Given any diagram in $A(L_1, L_2, n_A)$ and a diagram in $B(L_1, n_B)$, by extending them as above, we get in their collision precisely L if $n_A = b + 1$ while $b = L_1 - n_B$, that is, if

$$n_A + n_B = L_1 + 1. \quad (4.1)$$

For an example, see Fig. 13. Note that $n_A \leq L_1/L_2$ and so if $L_2 > 1$, then $n_B > n_A$ and $A(L_1, L_2, n_A) > B(L_1, n_B)$ (we sometimes denote $A > B$ meaning $a > b$ for the punctures A, B). If $L_2 = 1$ there is an overlap between the two classes of punctures. To avoid some of this redundancy we can restrict $n_A \leq \frac{L_1+1}{2}$, which ensures that still $a \geq b$.

To encapsulate this part, we do the following to find all the possibilities with $m = 2$. For every $2 \leq n_A \leq \lfloor \min(\frac{L_1}{L_2}, \frac{L_1+1}{2}) \rfloor$ (the upper bound is $\lfloor \frac{L_1}{L_2} \rfloor$ unless $L_2 = 1$), we find all the diagrams in the class $A(L_1, L_2, n_A)$ (there is at least one diagram for each such n_A). Each such diagram, completed by appending the rows of L starting from the second one, is the first puncture. For each of them, we look for all the diagrams of $B(L_1, L_1 + 1 - n_A)$ type, in which we complete the tip to get N boxes in total (there is at least one such diagram for every n_A above). All the pairs of diagrams that were obtained are the $m = 2$ solutions that can appear in a gauging of L . If $L_2 = 1$ we might get the same configuration more than once; for $L_2 > 1$ we get each possibility exactly once.

The extension to a general number of punctures m is given in a similar manner. Ordering the punctures as in the diagrammatic description $P^1 \geq P^2 \geq \dots$, the diagrams corresponding to the first L_1 boxes are of $A(L_1, L_2, n_1)$ class for P^1 and $B(L_1, n_i)$ for $P^i, i = 2, \dots, m$. The ordering of the punctures means that $n_1 \leq n_2 \leq \dots$. To get L in the collision of the P^i we need that $n_1 = \alpha + 1 = \sum_{i \geq 2} P^i + 1 = \sum_{i \geq 2} (L_1 - n_i) + 1$, that is,

$$\sum_{i=1}^m n_i = (m-1)L_1 + 1. \quad (4.2)$$

For every diagram in $B(L_1, n)$, $n \leq L_1 - 1$ (since a single column diagram is a no puncture); thus $n_i \leq L_1 - 1$ for $i \geq 2$ and from (4.2) $m \leq n_1$. As before, $n_1 \leq L_1/L_2$ for $L_2 > 1$ and for $L_2 = 1$, $n_1 \leq L_1 - 1$ again not to have a no puncture. We have found

$$2 \leq m \leq \frac{L_1}{L_2} - \delta_{L_2,1}. \quad (4.3)$$

Every m in that range is indeed obtainable: take $m - 1$ simple punctures, $n_1 = m$ with the first row of length $L_1 - (m - 1)L_2$ and the rest of length L_2 ; all of these punctures are legitimate regular punctures, whose collision gives L .

Note that the condition (4.2) ensures that the collision of the punctures P^i gives L . If $L_2 > 1$ then necessarily $n_1 \leq n_i$ for $i \geq 2$ [since otherwise $\sum n_i \leq 2\frac{L_1}{L_2} + (m - 2)(L_1 - 1) \leq L_1 + (m - 2)(L_1 - 1) = (m - 1)L_1 - (m - 2) < (m - 1)L_1 + 1$, which is impossible]. Thus indeed $p^1 \geq p^i$ for $i \geq 2$ and the collision gives the combination of the first L_1 boxes of P^1 giving L . If $L_2 = 1$ then the extension of all the diagrams is with rows of a single box and it does not matter what the biggest puncture is (the combination of the first L_1 boxes of all of them gives an L-shaped diagram).

To get all the possible P^i for every m in (4.3) we do as before the following. We find all sets of $m - 1$ punctures of $B(L_1, n_i)$ class with $n_2 \leq n_3 \leq \dots \leq n_m \leq L_1 - 1$ such that $\sum_{i \geq 2} n_i \geq (m - 2)L_1 + 2$ (these ensure $1 \leq m \leq n_1 \leq L_1 - 1$). For each of them we look for all $A(L_1, L_2, n_1 = (m - 1)L_1 + 1 - \sum_{i \geq 2} n_i)$ punctures. Extending the tip of the $B(L_1, n_i)$ punctures to get N boxes in total, and appending to the $A(L_1, L_2, n_1)$ punctures the diagram of L without the first row, we get all the possibilities for the m punctures P^i . Note that again if $L_2 = 1$ we might get the same configurations more than once.

Finally we comment on the case where L is a full puncture. From Fig. 1 and (3.1) L is a full puncture exactly when $\sum_i p^i = \sum_i p_N^i \geq N - 1$. We can get a full puncture with any number $m \geq 2$ of punctures P^i . All the possibilities of such punctures with $\sum_i p^i \geq N - 1$ result in a full puncture.

B. The possible gauge groups G_T

We have found what P^i can appear for a given PRP L in Fig. 11. Now we look for all the possible G_T .

We have seen in the diagrammatic method that L is given by combining the first $\alpha + 1 = \sum_{i \geq 2} p^i + 1$ rows of P^1 (where $p^1 \geq p^i$ for all i). For the purpose of this subsection, it is more convenient to abuse notation and by α denote $\min(\sum_{i \geq 2} p^i, h(N, P^1))$ [where $h(N, P^1)$ is the number of rows in P^1] since there can be fewer rows in P^1 than α , in which case $P_{\alpha+1}^1 = P_{\alpha+2}^1 = \dots = 0$ (note that this redefinition is not essential and the original definition of α can be used in the formulas below). The diagrammatic rule is still the same with this definition of α . With these conventions, by considering Sec. II A (in particular, Fig. 1) and (3.1) (or alternatively from Sec. III D using the last k for which $r_k = k - 1$) one can see that the T associated with the P^i is

$$T = \sum_{i=1}^{\alpha} P_i^1 \quad (4.4)$$

[for both the SU and the Sp case; in the Sp case $p^2 \leq h(N, P^1)$ always]. We have seen in Sec. II that if the P^i are of the SU behavior, then $G_T = SU(T)$ while if they are of the Sp behavior, $G_T = USp(T)$.

From what we have seen just now, $L_1 = T + P_{\alpha+1}^1$, $L_2 = P_{\alpha+2}^1, \dots$. Therefore $T = L_1 - P_{\alpha+1}^1$, and we can bound T from both sides. From the usual structure of Young diagrams we have $P_{\alpha+1}^1 \geq L_2$ and so $T \leq L_1 - L_2$. Similarly, $L_1 - P_{\alpha+1}^1 \geq P_{\alpha}^1 \geq P_{\alpha+1}^1$, or $P_{\alpha+1}^1 \leq L_1/2$ and $T \geq L_1/2$. Additionally, always $T \geq 2$. Combining the bounds

$$\max\left(2, \frac{L_1}{2}\right) \leq T \leq L_1 - L_2. \quad (4.5)$$

Note that as we see later on, the bound $T \geq \frac{L_1}{2}$ also follows from demanding that the theory on the other side of the tube (not the one with L) is unitary.

First we claim that precisely all the $SU(T)$ with T in (4.5) are the options for $G_T = SU(T)$. Indeed any integer T is obtained in the following configuration. Take $m = 2$ punctures, with P^2 a simple puncture, and P^1 with rows of length $T, L_1 - T, L_2, L_3, \dots$. By (4.5) both punctures are legal, and they give $G_T = SU(T)$ and L in their collision.

In the Sp case, we have $m = 2$ and $P_1^1 = P_2^1 = 2$. Therefore necessarily $L_i \leq 2$ for $i \geq 2$. A puncture P with $P_1 = 2$ is fixed by p : it has p rows of width 2 and the rest with width 1. Suppose $L_2 = 2$; then necessarily $P_i^1 = 2$ for all $i \leq L_1/2$, L_1 is even, and $p^2 = \frac{L_1}{2} - 1$. In this case $G_T = USp(L_1 - 2)$. If $L_2 = 1$, then $P_{\alpha+1}^1$ is 1 or 2 and $T = L_1 - P_{\alpha+1}^1 \geq L_1 - 2$ and $T \leq L_1 - 1$ from (4.5). So if $L_2 = 1$ and L_1 is odd we have $G_T = USp(L_1 - 1)$ and if L_1 is even, then $G_T = USp(L_1 - 2)$. Both G_T are obtained with a single P^1, P^2 configuration. The case $L_2 = 0$ is completed in the same way, with the result below.

To summarize, for a given PRP L , the possible G_T are exactly

- (i) $SU(T)$ with T in (4.5).
- (ii) If $L_2 = 2$ and L_1 even, $USp(L_1 - 2)$.
- (iii) If $L_2 = 1$ then $USp(L_1 - 2)$ for even $L_1 \geq 4$ and $USp(L_1 - 1)$ for odd L_1 .
- (iv) If $L_2 = 0$ and L_1 even, can have $USp(L_1)$ and $USp(L_1 - 2)$ (the latter for $L_1 \geq 4$).
- (v) If $L_2 = 0$ and L_1 odd, $USp(L_1 - 1)$.

The Sp groups are obtained with a single possibility for P^1, P^2 , and when $T = 2$ the G_T is just $SU(2)$.

C. The embedding of G_T in a PRP

The bound $T \leq L_1 - L_2$ in (4.5) implies that G_T comes from gauging a subgroup of the $U(L_1 - L_2)$ factor only in the symmetry (1.5) associated with L . As a verification of this, we expect to be left with a $\prod_{i \geq 2} U(L_i - L_{i+1})$ global symmetry. Indeed, after the gauging, the bigger puncture

FIG. 14. The picture of gauging a diagonal flavor symmetry.

P^1 contains a symmetry of $U(P_{\alpha+2}^1 - P_{\alpha+3}^1) \times U(P_{\alpha+3}^1 - P_{\alpha+4}^1) \times \dots = U(L_2 - L_3) \times U(L_3 - L_4) \times \dots$ [possibly without a $U(1)$ factor that appears in another P^i] as we saw in the diagrammatic description (see Fig. 10). We show in this subsection how the possible G_T 's described above are embedded in this $U(L_1 - L_2)$. It is convenient to refer to the theories we consider here, which are from the A_{N-1} series of class- $\mathcal{S}$ without twisted punctures, as class- $\mathcal{S}_A$. We discuss the possibility of getting additional G_T 's if we gauge a diagonal subgroup of the symmetry of L and a nonclass- $\mathcal{S}_A$ theory (replacing the sphere containing the P^i). We see that many of the restrictions on G_T are purely field theoretic, valid for any $\mathcal{N} = 2$ SCFT, not necessarily from class- $\mathcal{S}_A$.

Any gauge group G_T along some tube in a superconformal theory should have a vanishing beta function. Before forming the tube, we had two theories described by two Riemann surfaces (or the tube can have two ends on the same surface). This is depicted in Fig. 14. This amounts to gauging some diagonal group from the flavor symmetries of the two sides. The contribution of each side to the beta function is proportional to the central charge of the flavor symmetry currents that we gauge. The total contribution of the matter should cancel the contribution from the vector multiplets. The vanishing beta function equation is then

$$2T(\text{adj}) = k^L + k^R, \quad (4.6)$$

where k^L and k^R are the contributions from the two sides of the tube. The vector multiplets are in the adjoint representation and therefore contribute $2T(\text{adj})$. The factor of 2 is a matter of normalization of this equation. We work in the normalization in which $T(\text{adj}) = 2K$ and $T(\text{fund}) = 1$ for $SU(K)$.

1. $G_T = SU(T)$

The central charge of the flavor symmetry of some puncture is determined by the puncture and the subgroup we gauge. We are still in the situation of Fig. 11, and suppose $G_T = SU(T)$, T being in the range (4.5). We can replace the right side of the tube by any other theory, and the contribution of L to the beta function of G_T does not change as long as we have the regular puncture L on the left

FIG. 15. Replaced sphere giving L and $G_T = SU(T)$. We used a shorthand notation for a diagram: a diagram with a label n in a box means that in the appropriate row we have n boxes. The 2, 1, 1, ... diagram is just a simple puncture.

side of the tube and G_T is the same. We choose the sphere shown in Fig. 15 for that. Note that the rightmost diagram is valid, $T \geq L_1 - T$ and $L_1 - T \geq L_2$. Using the analysis in Sec. II B or the diagrammatic method, it is seen immediately that we indeed get L and $G_T = SU(T)$.

We can get the contribution of L since the configuration of Fig. 15 appears in the linear quiver shown in Fig. 16. The rightmost sphere is just a free theory of $2T - L_1$ fundamentals of $SU(T)$, while the contribution of L to $SU(T)$ is the same as of L_1 fundamentals (the sphere to which it belongs gives a bifundamental hypermultiplet). If we have hypermultiplets in the representation $\oplus_i r_i$ of some group G , then $k_G = \sum_i 2T(r_i)$. Therefore

$$k_{SU(T)}^L = 2L_1 T(\text{fund}) = 2L_1. \quad (4.7)$$

We saw that any gauging of L must come from some subgroup of $U(L_1 - L_2)$. In general, if the central charge of a current of flavor symmetry U (U being a simple group) is k_U , then for a simple subgroup $W \subset U$, $k_W = x k_U$, where x is the embedding index of W in U . In our case, we found that $k_{SU(T)}^L = k_{SU(L_1-L_2)}^L$. This means that the embedding index is 1, and the embedding is the trivial embedding.

We can again check if this is consistent. After gauging $SU(T)$ from the $SU(L_1 - L_2)$ part of the symmetry of the puncture L , we expect that if the embedding is trivial, there will be a leftover $SU(L_1 - L_2 - T)$ symmetry. This is indeed the case, as the bigger puncture in the diagrammatic picture of gauging has an $SU(P_{\alpha+1}^1 - P_{\alpha+2}^1) = SU(L_1 - T - L_2)$ factor in its symmetry (as we saw in the previous subsection, with α defined as there); see Fig. 10.

We could ask whether in gauging L with a nonclass- $\mathcal{S}_A$ theory, we could obtain a gauging that is not possible in class- $\mathcal{S}_A$, that is, some other gauged subgroup or a different embedding (a nontrivial embedding). As the other theory is not from class- $\mathcal{S}_A$, there is no longer a reason to restrict L to be a PRP (which was the only class of punctures that could be gauged in class- $\mathcal{S}_A$), and therefore we consider any regular puncture L . Suppose we could gauge an $SU(T)$

FIG. 16. A linear quiver in which the same L and G_T appear. A small disk represents a simple puncture.

FIG. 17. A tail of a general linear quiver. In the top we see the Riemann surface with a puncture L and additional simple punctures. Below the corresponding linear quiver is displayed.

subgroup of $SU(L_1 - L_2)$ with an embedding index x , and at the other end of the tube we could have any other theory (not only of class- $\mathcal{S}_A$). Then the contribution of the rhs theory to the beta function is

$$k^R = 4T - k^L = 4T - 2L_1x. \quad (4.8)$$

We assume that the theory at the other end of the tube is unitary. The condition $k^R \geq 0$ means $T \geq L_1/2$. Therefore, we cannot gauge in general an $SU(T) \subset SU(L_1 - L_2)$ that does not appear in class- $\mathcal{S}_A$.

Since the Dynkin embedding index is an integer, for a nontrivial embedding $x \geq 2$, implying $k^R \leq 4(T - L_1) \leq -4L_2$. To be non-negative we must have $L_2 = 0$ and $T = L_1 = N$ for which $x = 1$. We conclude that the only possible gauging of $SU(T) \subset SU(L_1 - L_2)$ is by the trivial embedding.

Let us consider gauging a subgroup of the other $SU(L_i - L_{i+1})$, $i > 1$, symmetries, which was not possible

in class- $\mathcal{S}_A$. For any L we can form the linear quiver presented at the bottom of Fig. 17. The corresponding curve is the one at the top of Fig. 17. If we turn off the gauge coupling of the $SU(\sum_{j=1}^i L_j)$ gauge group, we find that $k_{SU(L_i - L_{i+1})} = 2 \sum_{j=1}^i L_j$. By the supersymmetry, this central charge is also an anomaly coefficient and is independent of the exactly marginal couplings. $k^R \geq 0$ together with $T \leq L_i - L_{i+1}$ is possible only if N is even and $T = L_1 = L_2 = N/2$ (so that $L_3 = 0$), giving $k^R = 0$. There is then a single possibility that was absent in class- $\mathcal{S}_A$ in which we gauge the $SU(L_2 - L_3) = SU(N/2)$ of a (non-PRP) puncture with $L_1 = L_2 = N/2$ for even N . This possibility actually appears in the presence of twisted punctures, realizing D -shaped conformal quivers (of special unitary gauge groups); see [6].

2. $G_T = USp(T)$

The only other possibility for G_T in class- $\mathcal{S}_A$ is $USp(T)$. We saw in the previous subsection the cases giving $USp(T)$. Let us see what are the possible embeddings given that we cannot have $k^R < 0$. For $USp(2r)$, $2T(\text{adj}) = 4(r+1)$. The options giving $USp(2r)$ are

- (i) $L_1 = 2r + 2$, $L_2 = 0$, 1 , 2 . In these cases, $k^R = 4(r+1) - xk_{SU(L_1 - L_2)} = 4(r+1) - 2(2r+2)x$. Only $x = 1$ is possible, and it gives $k^R = 0$.
- (ii) $L_1 = 2r + 1$, $L_2 = 0$, 1 . $k^R = 4(r+1) - 2x(2r+1)$. For $x = 1$ we get $k^R = 2$. If $x \geq 2$, $k^R \leq -4r < 0$, so it is not possible.
- (iii) $L_1 = 2r$, $L_2 = 0$. $k^R = 4(r+1) - 4rx$. For $x = 1$ we get $k^R = 4$. If $x \geq 2$, $k^R \leq 4(1-r) < 0$, which is not possible again.

For the $USp(2r)$ case, again only the trivial embedding in $SU(L_1 - L_2)$ is possible. We already saw that there is indeed a leftover $SU(L_1 - L_2 - T)$ symmetry after the gauging.

Let us ask once again about gauging L with a nonclass- $\mathcal{S}_A$ theory. As before, only an embedding with index $x = 1$ is possible, even when the other theory that we gauge is not necessarily from class- $\mathcal{S}_A$. The reason for this is the following. An embedding of $USp(2r) \subset SU(L_1 - L_2)$ implies $L_1 \geq 2r + L_2$. Again we use $k^R = 4(r+1) - 2xL_1 \leq 4(r+1) - 2x(2r + L_2)$. An embedding with $x \geq 2$ would imply $k^R < 0$ [using $r > 1$ since otherwise the algebra is the same as that of $SU(2)$]. We saw that in class- $\mathcal{S}_A$ we cannot gauge some $USp(2r)$ of the flavor symmetry of a puncture with $L_2 \geq 3$, and this still holds even when the other theory we couple to the gauge group is not from class- $\mathcal{S}_A$ by the same bound.

3. $G_T = SO(T)$

In the theories discussed here, it was not possible to have an $SO(n)$ group along the tube. Let us see that it is still impossible even if we would like to gauge a diagonal subgroup of the symmetry of any regular puncture L , and of some other theory that is not necessarily of class- $\mathcal{S}_A$. $2T(\text{adj}) = 4(n-2)$ for $SO(n)$. Additionally, in any embedding $SO(n) \subset SU(L_1 - L_2)$, $x \geq 2$ necessarily [since the minimal index of a representation is 2 in $SO(n)$]. The exceptions for that are the low-rank cases in which the two algebras are just the same, but we are not interested clearly in these cases. In all the other cases, $x \geq 2$ and $L_1 - L_2 \geq n$. We obtain then $k^R = 4(n-2) - 2xL_1 \leq 4(n-2) - 4n < 0$.

D. A bound on the rank of the symmetry of a PRP

Given $m \geq 2$ regular punctures $P^1, \dots, P^m$, the regular puncture L defined by (3.1) satisfies

- (i) $L_1 \geq \min(\sum_{i=1}^m P_1^i - m + 1, N)$ or equivalently

$$rkG(L) \geq \min\left(\sum_{i=1}^m rkG(P^i), N - 1\right) \quad (4.9)$$

[where $G(P)$ is the global symmetry of the puncture P].

- (ii) If in addition at least one $P_2^i \geq 2$, then $L_1 \geq \min(\sum_{i=1}^m P_1^i - m + 2, N)$ or equivalently

$$rkG(L) \geq \min\left(\sum_{i=1}^m rkG(P^i) + 1, N - 1\right). \quad (4.10)$$

These follow easily from Eq. (3.1). In any puncture $P_1 \geq 2$, because if it was 1 it was a no puncture. Define $k_0 = \min(\sum_i P_1^i - m + 1, N)$. If $k_0 = \sum_i P_1^i - m + 1$, then for any i , $k_0 = P_1^i + \sum_{j \neq i} P_1^j - m + 1 \geq P_1^i + 2(m-1) - m + 1 \geq P_1^i$ and therefore $p_{k_0}^i \geq P_1^i - 1$. This is clearly true also if $k_0 = N$. It follows that $\sum_i p_{k_0}^i \geq \sum_i P_1^i - m \geq k_0 - 1$, and then by the definition of l_k ,

$l_{k_0} = k_0 - 1$. This means that $L_1 \geq k_0$, establishing (4.9) [since $rkG(P) = P_1 - 1$].

For the second statement, define $k'_0 = \min(\sum_i P_1^i - m + 2, N)$. It is not smaller than the previously defined k_0 and therefore still $p_{k'_0}^i \geq P_1^i - 1$ for every i . For the particular i for which $P_2^i \geq 2$, if $k'_0 = \sum_j P_1^j - m + 2$ then $k'_0 = P_1^i + \sum_{j \neq i} P_1^j - m + 2 \geq P_1^i + 2(m-1) - m + 2 = P_1^i + m \geq P_1^i + 2$. Since $P_2^i \geq 2$, it means that $p_{k'_0}^i \geq P_1^i$. If $k'_0 = N$ this clearly still holds. Together, these give $\sum_i p_{k'_0}^i \geq \sum_i P_1^i - m + 1 \geq k'_0 - 1$, and once again $L_1 \geq k'_0$.

V. LARGE N AND FIELD THEORIES ON AdS_5

Certain theories of the sort we have used are relevant in the large N limit for describing field theories on AdS_5 with various boundary conditions. In this section we address this connection and apply the tools from the previous sections.

In [15] the AdS/CFT correspondence between type-IIB string theory on $\text{AdS}_5 \times S^5/\mathbb{Z}_K$ and the four-dimensional $\mathcal{N} = 2$ theory with a circular quiver $SU(N)^K$ [20] was considered. The four-dimensional side is the theory where the Riemann surface C is a torus with K simple punctures. The singular limit where the integrals of the NS-NS and R-R 2-forms on the 2-cycles of the orbifold vanish is particularly interesting. This corresponds in the four-dimensional side to bringing the simple punctures close to each other. In the large N limit, we get the six-dimensional (2,0) A_{K-1} theory on $\text{AdS}_5 \times S^1$ decoupled from gravity.

In the four-dimensional picture, this situation is a simple instance of the degeneration limits we considered in the previous sections. As the simple punctures are taken close, a sphere bubbles off, connected by a tube to the remaining torus on which a single puncture is created if the tube is completely decoupled. This puncture is an L-shaped puncture with rows of width $K+1, 1, 1, \dots$ and pole structure $1, 2, \dots, K, K, \dots$. The gauge group that becomes weakly coupled is an $SU(K)$. As we explained in Sec. II, the curve on the decoupling sphere terminates within a number of terms of the order of K . The sphere and the tube are independent of N .

In this limit where we have an A_{K-1} singularity in the type-IIB theory, we expect to get an $SU(K)$ gauge symmetry in the five-dimensional theory on AdS_5 and an $SU(K)$ global symmetry in the corresponding four-dimensional theory. This does not happen and the global symmetry in our four-dimensional theory is $U(1)^K$. This conflict is avoided as described in [15] by having the $SU(K)$ tube on the boundary gauging both the decoupling sphere with simple punctures sitting on the boundary as well, and the five-dimensional theory on AdS_5 in the bulk. The four-dimensional $SU(K)$ symmetry is thus gauged and not global, and the global $U(1)^K$ symmetry arises from the global symmetry of the theory on the decoupling sphere.

FIG. 18. General form of four-dimensional theories that might give decoupled theories on AdS_5 .

Having these theories on the boundary provides a specific choice of boundary conditions for the (2,0) theory on $\text{AdS}_5 \times S^1$, which is realized in the type-IIB construction. We could give the (2,0) theory on $\text{AdS}_5 \times S^1$ alternative boundary conditions in which we do not have the ingredients just mentioned on the boundary. In this case the corresponding four-dimensional theory is a torus with the L-shaped puncture, having now $SU(K)$ in its global symmetry. It is not known how to realize these alternative boundary conditions in type-IIB string theory. In principle, an M-theory dual can be found along the lines of [17], but it is not reliable due to high curvatures.

We would like to see by how much this picture can be generalized. The configurations we need to examine appear in Fig. 18. The $Q(N, G)$ ingredient is the analog of the torus with the L-shaped puncture. It is now some surface including a puncture with global symmetry G (as well as additional punctures possibly). The tube, gauging some subgroup of G , together with the $P(G)$ theory may provide different boundary conditions in case we have a decoupled theory on AdS_5 . The tube and the $P(G)$ theories are finite in the large N limit.

In order for the tube and $P(G)$ to be independent of N , we must be either in the first or the second general cases, which were described at the end of Sec. II B. In particular, as was discussed in Sec. II C, $P(G)$ must be defined on a sphere. We are once again in the situation we inspected a lot, which appears in Fig. 11 with the puncture of symmetry G being L .

As was mentioned above, a torus with an L-shaped puncture contains a decoupled (2,0) theory on $\text{AdS}_5 \times S^1$ [15]. We first try to see in what theories $Q(N, G)$ we have a decoupled field theory on AdS_5 . We give a partial answer for this, not covering all the possible situations. For the cases where this does happen, we assume that different choices of the gauged subgroup $G_T \subset G$ and of $P(G)$ provide different boundary conditions for the decoupled theory. Under this assumption we can enumerate what those possible boundary conditions are.

A. Large N

The first immediate generalization of an L-shaped puncture is a puncture with a tip (all rows are of width

1 starting from some row number). When N is large, we say that a puncture has a “long tip” if p is independent of N . All punctures of this sort have a finite large N limit for their global symmetry. That is, only for these punctures the global symmetry (1.5) does not change as we append a box (or any number of boxes) starting from some point, and is independent of N . As is explained in a moment, if all the punctures P^i on $P(G)$ before the decoupling of the tube have a long tip in the large N limit, after the decoupling the $P(G)$ side represents a theory that is independent of N .

By the requirement that after decoupling $P(G)$ is independent of N we are led to consider additional punctures, namely, punctures of “finite P_1 ,” which means that the first row of the Young diagram has a number of boxes independent of N . We now explain that except for exceptions of a certain type, the theory $P(G)$ after its decoupling is independent of N when all punctures except one have a long tip, and the remaining puncture is of finite P_1 . Therefore the punctures P^i we need to consider are of finite P_1 (which includes punctures with a long tip in particular).

The explanation of the statement above is the following. As mentioned before, the $P(G)$ theory corresponds to the rhs in the first or the second general cases, which were described at the end of Sec. II B. We use the (by now) usual convention of $p^1 \geq p^i$ for $i > 1$ and $\alpha = \sum_{j \geq 2} p^j$. By the description of the second case in which we have only $m = 2$, we basically need α to be independent of N for $P(G)$ to be independent of N and so P^2 has a long tip and P^1 automatically has finite P_1^1 (it equals 2). There is an exception for that, in which we have $p^2 = \alpha$, which is large, and $P_{\alpha+1}^1 = 2$. In this case the decoupled theory is an empty theory. Next, consider the first general case. Assume first that there is a $\Delta_k > 0$ (the Δ_k here corresponds to the P^i punctures). To remain with a finite theory we must have a finite N' (see the discussion regarding N' in Sec. II B), which means that P_1^1 and α are necessarily independent of N and so P^i for $i > 1$ have a long tip. The number of additional free hypers in the resulting decoupled theory is also finite. Finally consider what happens if there is no $\Delta_k > 0$. As mentioned in Sec. II B, the decoupling theory is a theory of free hypers. For the number of hypers [given by (2.10)] to be independent of N we need, in principle, that again P_1^1 and α are finite and so again P^1 is of finite P_1^1 and the P^j for $j > 1$ have long tips. The exception for this is when $m = 2$, P^2 is a simple puncture, and $P_1^1 = P_2^1$, which is large. In this case the decoupling theory is empty as in the previous exception. To summarize, we have found that $P(G)$ is independent of N after decoupling exactly when P^1 is of finite P_1^1 and the other P^i have long tips, with the exceptions mentioned above in which the decoupled theory is empty.

Note that we cannot have $Q(N, G)$ a sphere with only a single puncture of symmetry G in Fig. 18, because the

number of Coulomb branch parameters of dimension k in the sphere composed of $Q(N, G)$, $P(G)$ and the tube is $d_k = \Delta_k - k + 1$, and for P^i giving $P(G)$ finite, Δ_k and d_k become negative. On a torus, and higher genus surfaces, the Coulomb branch graded dimension (1.4) is always non-negative, $d_k \geq 0$, and we do not have this restriction.

B. Decoupled field theories in the large N limit

Consider a genus $g > 1$ surface C having punctures with a long tip. We would like to see if analogously to the case of a torus with an L-shaped puncture, there is in the large N limit a field theory that is decoupled from gravity. Gaiotto and Maldacena constructed the gravity duals of such theories [17]. These are given by M theory on a background of the form $\text{AdS}_5 \times \mathcal{M}_6$ with $\mathcal{M}_6$ being compact. The M-theory spacetime is smooth for $g > 1$, and in the large N limit adding to it punctures of the sort we consider is a small local deformation of the background.

The background is found by solving a Toda equation with boundary conditions specified by the different punctures. Around each puncture, the Toda equation is analogous to three-dimensional electrostatics with a cylindrical axial symmetry. Every puncture is associated with a line charge density profile denoted by $\lambda(\eta)$ which is piecewise linear. The slopes are integer and change at integer values of η . Specifically, for a given puncture P , the slopes are $P_1, P_2, \dots$ and they change at $\eta = 1, 2, \dots$. Whenever $P_i - P_{i+1} \geq 2$, we have corresponding to a slope change, in the appropriate position in spacetime, an A_{k-1} singularity, with $k = P_i - P_{i+1}$. Each A_{k-1} singularity gives a non-Abelian gauge theory on AdS_5 that is associated with an appropriate global symmetry in the four-dimensional $\mathcal{N} = 2$ theory. The radius of the AdS_5 in Planck units is $R_{\text{AdS}_5}/l_P \sim \lambda(\eta_k)^{1/3}$ [see Eq. (3.22) in [17]] where $\lambda(\eta_k)$ is the value of λ where the slope changes, and the five-dimensional gauge coupling is $R_{\text{AdS}_5}/g_5^2 \sim \lambda(\eta_k)$. To get a nontrivial theory decoupled from gravity we need to have a finite nonzero value of R_{AdS_5}/g_5^2 while $l_P \rightarrow 0$ (compared to R_{AdS_5} and g_5^2). Therefore we see that we do not have a nontrivial field theory decoupled from gravity.

We do not provide indication for or against the existence of decoupled field theories on AdS_5 in the large N limit in the rest of the cases. In $g > 1$ surfaces having only punctures of the type we considered there are no such decoupled theories, but if there are other punctures, different arguments should be used.¹² In a $g = 1$ surface with an L-shaped puncture there is a decoupled theory, but for other punctures, as well as for the sphere, the situation should be investigated further.

¹²For any theory having an M-theory dual that is given by a weakly curved spacetime with A_{k-1} singularities, there are no such decoupled field theories on AdS_5 , as was argued.

Let us assume that in the cases where we do have a decoupled field theory, the various choices for $P(G)$ in Fig. 18 with the appropriate tube amount to different boundary conditions for the decoupled theory on AdS_5 . As was mentioned before, $P(G)$ is defined on the sphere and we are in the situation of Fig. 11 with L having symmetry G . We can use all the conclusions developed so far.

Under this assumption, Sec. IV A essentially is a classification of all the possible boundary conditions of this sort. In particular, this can be applied to the torus with a single L-shaped puncture where we know that we have a decoupled field theory.

If we do not restrict to boundary conditions that can be implemented in class- $\mathcal{S}$, the possible options for gauging some subgroup of G depend only on the central charge of this subgroup, which is related to the central charge of G as reviewed above. The options for possible boundary conditions are thus determined by the central charge of G , which is related to the gauge coupling of G in the bulk in units of the AdS radius.

C. Nonsingular weakly curved gravitational duals

Having or not having a decoupled field theory on AdS_5 , many of the four-dimensional theories do not have a dual string or M theory on a nonsingular background that is weakly curved. The symmetry of a four-dimensional theory having such a dual should be a product of $U(1)$'s. In such a case, all the punctures in $P(G)$ in Fig. 18 must have $P_i - P_{i+1} = 0, 1$. Using the diagrammatic method, the resulting puncture with symmetry G necessarily has $L_i - L_{i+1} = 0, 1$ for $i > 1$ but not for $i = 1$; see Fig. 19. Conversely, not every puncture of the form shown in Fig. 19 can be gauged such that at the other side of the tube there is a sphere with punctures having symmetries $U(1)^{n_i}$ only. This is easily seen by considering Fig. 10, since it is not always possible to partition L_1 into $P_1^1, P_2^1, \dots, P_{\alpha+1}^1 \geq L_2$ with $P_i - P_{i+1} \leq 1$. Only surfaces with punctures for which this is possible might have boundary conditions such that the resulting theory has a gravitational dual that is weakly curved and nonsingular.

FIG. 19. A PRP obtained by decoupling punctures with symmetries $U(1)^{n_i}$. It has $L_i - L_{i+1} \leq 1$ for $i > 1$.

These boundary conditions are a generalization of Fig. 18 in which each such puncture is connected through a tube to a sphere with punctures having a symmetry $U(1)^{n_i}$.

Concentrating on a single puncture as in Fig. 18, note that the case from the beginning of this section in which we have only simple punctures on $P(G)$ and L is an L-shaped puncture is special in the following sense. Assuming that L is not a full puncture, it is the only case in which the rank of the gauge symmetry G in the bulk is equal to the rank of the introduced Abelian global symmetry group arising after gauging G in a particular choice of boundary conditions. This means in the context of Fig. 18 that for L which is not a full puncture being replaced by the punctures P^i of symmetry $U(1)^{n_i}$ each, the case of simple punctures giving an L-shaped puncture is the only one in which $rkG(L) = \sum_i rkG(P^i)$. This is a simple consequence of Sec. IV D. Since L is not a full puncture, $rkG(L) \leq N - 2$ and we can choose the first argument of the minimum in both (4.9) and (4.10). It then follows that all $P^i_2 \leq 1$, and to have symmetries of $U(1)^{n_i}$ all the P^i must be simple punctures [and from (3.1), the resulting l_k is $(1, 2, \dots, m, m, \dots)$, an L-shaped puncture].

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APPENDIX: DECOUPLING ON GENERAL SURFACES THROUGH THE CURVE

In this appendix we generalize the analysis of the curve in a decoupling that was done for a sphere in Sec. II B to any surface. The result is used in Sec. II C.

The ϕ_k appearing in the SW curve are no longer of the form (2.4). Despite this, we are now interested in the region of small z , which captures the punctures on the right and the tube (see Fig. 7). The local behavior of ϕ_k in this region is independent of what happens far away, as well as of the genus of the surface. The poles and the zeros in this region fix it. If we bring n'_k of the zeros of ϕ_k to 0 while taking $w \rightarrow 0$, then we have

$$\phi_k \approx u_k \frac{\prod_1^{n'_k} (z - z_i^{(k)})}{\prod (z - \alpha_i)^{p_i}} dz^k \quad (\text{A1})$$

with $n'_k \leq n_k$ and

$$n_k = \Delta_k^L + \Delta_k^R + g(2k - 1). \quad (\text{A2})$$

For $\Delta_k^R \geq 0$ we ask, as for the sphere, whether we can flatten Δ_k^R . When $g \geq 1$ it is always possible to do that: $n_k - \Delta_k^R \geq \sum_i q_k^i + k - 1 \geq 1$. We argued that on the sphere, if $n_k - \Delta_k^R < 0$ (and then could not flatten Δ_k^R , and had to take $u_k \rightarrow 0$ as $w \rightarrow 0$) or if $n_k - \Delta_k^R = 0$, we got $\phi_k^L = 0$, which fixed l_k through (2.7). It is useful to reproduce this case in the language that is used below.

The case $\Delta_k^R \geq 0$ and $n_k - \Delta_k^R \leq 0$ is then possible only for the sphere. We have $\Delta_k^R \geq 0$ and $\Delta_k^L \leq 0$, and the curve for this case is (2.4). In the region of small z , it is approximated by

$$\phi_k \approx \frac{a_0 + a_1 z + \dots + a_{n_k} z^{n_k}}{\prod (z - \alpha_i)^{p_i}} dz^k. \quad (\text{A3})$$

For $w \ll |z| \ll 1$,

$$\phi_k \approx \frac{a_0 + a_1 z + \dots + a_{n_k} z^{n_k}}{z^{\Delta_k^R + k}} dz^k. \quad (\text{A4})$$

In the limit $w \rightarrow 0$, this region becomes the region of l_k . For this we need to decouple the tube completely. For $\Delta_k^L = 0$, $a_{n_k = \Delta_k^R}$ becomes a Coulomb branch parameter for the gauge group along the tube. To decouple the tube, we take $a_{\Delta_k^R} = 0$ (not just approach 0 as $w \rightarrow 0$, but equal to 0). For $\Delta_k^L < 0$ we do not have a parameter for the gauge group for this k . In both cases we are left with possible nonzero a_i for $i \leq \min(n_k, \Delta_k^R - 1) = n'_k$. We must take all the remaining $a_i \rightarrow 0$ as $w \rightarrow 0$, because otherwise we would get $l_k > k$. This fixes $\phi_k^L = 0$ for this k .

To recover r_k we do the following. To describe the decoupling rhs sphere we change variables $z = yw$, so

$$\phi_k \approx w^{-\Delta_k^R} \cdot \frac{a_0 + a_1 yw + \dots + a_{n'_k} y^{n'_k} w^{n'_k}}{\prod (y - \alpha'_i)^{p_i}} dy^k \quad (\text{A5})$$

where $\alpha_i = w\alpha'_i$. ϕ_k in the y coordinate should be finite to describe the rhs sphere (we have an equation for $\lambda = xdz$ that should give finite cycles; the integration in y should be finite). Therefore $a_i = w^{\Delta_k^R - i} a'_i$ as $w \rightarrow 0$ (they indeed go to 0 as we demanded since $\Delta_k^R - n'_k > 0$). The resulting curve of the decoupling sphere is

$$\phi_k = \frac{a'_0 + a'_1 y + \dots + a'_{n'_k} y^{n'_k}}{\prod (y - \alpha'_i)^{p_i}} dy^k. \quad (\text{A6})$$

The pole order at infinity is obtained as usual by a change of variable $y = 1/y'$ with $y' \rightarrow 0$ as $y \rightarrow \infty$. This gives the pole structure of r_k : $r_k = k - \Delta_k^R + n'_k$. For $\Delta_k^L < 0$ this is $r_k = k - \Delta_k^R + n_k = k + \Delta_k^L = \sum_i q_k^i$. This is what we are familiar with, when looking from the lhs point of view having $\Delta_k^L < 0$ where we said we do not need to do anything, giving $r_k = \sum_i q_k^i$. For $\Delta_k^L = 0$, $r_k = k - 1$, again as we are familiar with. Note, as stated before, if $\Delta_k^R = 0$, then in our case either $\phi_k \approx a_0/z^k \cdot dz^k$ or $\phi_k = 0$ to begin with. In both cases, turning off the tube, $\phi_k^R = 0$ fixing r_k from d_k^R , $r_k = k - 1$.

Now return to surfaces of general genus. Suppose that $\Delta_k^R \geq 0$ and $n_k - \Delta_k^R \geq 1$. As mentioned, the first condition implies the second in $g \geq 1$ surfaces. Meanwhile assume $\Delta_k^R > 0$. In the region of small z , where the right punctures approach each other, we have the curve behavior for any surface (A1),

$$\begin{aligned} \phi_k &\approx \frac{a_0 + a_1 z + \dots + a_{n'_k} z^{n'_k}}{\prod (z - \alpha_i)^{p_i}} dz^k \\ &= w^{-\Delta_k^R} \cdot \frac{a_0 + a_1 w y + \dots + a_{n'_k} (w y)^{n'_k}}{\prod (y - \alpha'_i)^{p_i}} dy^k. \end{aligned} \quad (\text{A7})$$

Again ϕ_k in the y coordinate should be finite since it describes the rhs sphere. It means we should take $a_i = w^{\Delta_k^R - i} a'_i$ as $w \rightarrow 0$.

The leading behavior of ϕ_k in the region where $w \ll |z|$ and $|z|$ much smaller than the modulus of other poles and zeros, is (using $z \gg z_i^{(k)}, \alpha_i$)

$$\phi_k \approx \frac{a_{n'_k} z^{n'_k} + \dots}{z^{\Delta_k^R + k}} dz^k. \quad (\text{A8})$$

The higher order in z terms in “...” are finite in the decoupling limit in general (they can go to 0, but this corresponds to further taking zeros of ϕ_k in the rest of the surface to 0, but still much larger than w). There is at least one such term for the choice of n'_k we do. This form fixed for us l_k . Since we cannot have $l_k > k - 1$ we take $n'_k = \Delta_k^R - 1$. $a_{n'_k} \rightarrow 0$ necessarily as we saw, so there is no $1/z^{k+1}$ term in l_k . The Coulomb branch parameter of the gauge group along the tube is taken to 0. We therefore get $l_k = k - 1$ (the value of n'_k we have taken, satisfying $n_k - n'_k > 0$, ensures we have higher orders in z , so $\phi_k^L \neq 0$). For r_k we use the curve on the right sphere,

$$\phi_k = \frac{a'_0 + a'_1 y + \dots + a'_{\Delta_k^R - 1} y^{\Delta_k^R - 1}}{\prod (y - \alpha'_i)^{p_i}} dy^k \quad (\text{A9})$$

(an index of k is omitted from a_i and a'_i for clarity of the equations, but it should be kept in mind that these a 's are different for different k 's). Note $\frac{a_i}{a_{n'_k}} \sim w^{n'_k - i}$ as indeed

expected since the zeros in $a_{n'_k} \prod (z - z_i^{(k)})$ are proportional to w . r_k can be read from the pole at infinity and is $r_k = k - \Delta_k^R + \Delta_k^R - 1 = k - 1$.

Note that we have taken $n'_k = \Delta_k^R - 1$. We could not take smaller n'_k because it would give $l_k > k - 1$ as mentioned, but we could ask why not take $n'_k > \Delta_k^R$ (note $n'_k = \Delta_k^R$ is the same as what we did before since we take its coefficients to be 0, not affecting either curve). The reason is that in the last equation we would get in the numerator terms up to $y^{n'_k}$ giving $r_k = k - \Delta_k^R + n'_k > k$. This justifies not flattening further.

Regarding $\Delta_k^R = 0$, we must take $n'_k = 0$ for the same reason. We still have $l_k = k - 1$. In the SW curve of the rhs sphere we only have a'_0 in the numerator. We should take it to 0 in order to decouple the tube, giving $\phi_k^R = 0$ as we already know. This fixes $r_k = d_k^R - \Delta_k^R + k - 1 = k - 1$.

Lastly, in $g \geq 1$ surfaces we need also to consider k 's for which $\Delta_k^R < 0$ and $n_k - \Delta_k^R \geq 1$. We said we do not need to bring any $z_i^{(k)}$'s to 0, that is, $n'_k = 0$. The curve in the region $w \ll |z| \ll 1$ is then

$$\phi_k \approx a_0 \frac{1}{z^{\sum_i p_i}} dz^k. \quad (\text{A10})$$

There are no Coulomb branch parameters for the gauge group along the tube. This gives $l_k = \sum_i p_k^i$. In the region of the decoupling sphere we have

$$\phi_k \approx \frac{a_0}{\prod (z - \alpha_i)^{p_i}} dz^k = w^{k - \sum_i p_i} \frac{a_0}{\prod (y - \alpha'_i)^{p_i}} dy^k \xrightarrow{w \rightarrow 0} 0 \quad (\text{A11})$$

since $\sum_i p_k^i < k$ and a_0 cannot diverge because of (A10). For the rhs curve we get $\phi_k^R = 0$ fixing $r_k = d_k^R - \sum_i p_k^i + 2k - 1 = 2k - 1 - \sum_i p_k^i$.

We could as before wonder what happens if we were taking $n'_k > 0$ $z_i^{(k)}$'s to 0, giving $l_k = \sum_i p_k^i - n'_k < \sum_i p_k^i$. Again, this is the question of why not flatten further in our previous considerations. The rhs curve would then be (A7). The parameters in the resulting rhs curve should go to 0; otherwise we would get $r_k > k$. Then $\phi_k^R = 0$ and still $r_k = 2k - 1 - \sum_i p_k^i$, but now $l_k < \sum_i p_k^i$. It does not give a new decoupling from the following reason. We took n'_k parameters to 0 as $w \rightarrow 0$ but we do not see them in the rhs SW curve. It seems as if the number of parameters is not conserved. To understand what happens, return to the first situation we analyzed—the sphere with $\Delta_k^R > 0$ and $\Delta_k^L < 0$. This is just switching left and right of this situation, on the sphere. The curve was approximated by (A3) for small z there, corresponding to nonsmall z here. Write $a_0 + a_1 z + \dots + a_{n_k} z^{n_k} = a_{n_k} \prod_{i=1}^{n_k} (z - z_i)$. If all the zeros z_i behave as $z_i \sim w^1$ and $a_{n_k} \sim w^{\Delta_k^R - n_k}$, we get $a_i \sim a_{n_k} z_i^{n_k - i} \sim w^{\Delta_k^R - i}$ exactly as we obtained there. Now

suppose we want to lower the degree of the polynomial in the numerator of (A6), which is exactly what we are doing here (fewer parameters in the $\Delta > 0$ side). Let us lower it by 1 for instance. For this we need $a'_{n_k} \rightarrow 0$, or $a_{n_k} \sim w^{\Delta_k^R - n_k + 1}$ (or $w^{\Delta_k^R - n_k + s}$ with $s > 0$). To preserve a'_{n_k-1} finite, we need a_{n_k-1} to still behave as $w^{\Delta_k^R - n_k + 1}$. This requires that at least one z_i behaves as w^0 (or $w^{1-s} \gg w$). We get a z_i that stays in the lhs of the analysis there, which corresponds to the rhs in our current analysis (having $\phi_k = 0$). We conclude that getting $l_k < \sum_i p_k^i$ in our case is just like taking a'_i 's to 0 in (A6), which is a degenerate

case of the general $l_k = \sum_i p_k^i$ situation; we just take some Coulomb branch parameters to 0 there.

To conclude, for $g \geq 1$ surfaces, we obtained that for $\Delta_k^R \geq 0$, $l_k = r_k = k - 1$, while for $\Delta_k^R < 0$, $l_k = \sum_i p_k^i$ and $r_k = 2k - 1 - \sum_i p_k^i$. There was a single tube Coulomb branch parameter of dimension k for $\Delta_k^R \geq 0$. The punctures that decouple alone fix the gauge group along the tube and the two punctures that are created. The two possibilities for Δ_k^R of the right side of Fig. 7 are just as the ones on the right of Figs. 4 and 5. The resulting gauge group, r_k , and l_k are just as there.

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