Geometry of the theory space in the exact renormalization group formalism

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We consider the theory space as a manifold whose coordinates are given by the couplings appearing in the Wilson action. We discuss how to introduce connections on this theory space. A particularly intriguing connection can be defined directly from the solution of the exact renormalization group (ERG) equation. We advocate a geometric viewpoint that lets us define straightforwardly physically relevant quantities invariant under the changes of a renormalization scheme.

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I. INTRODUCTION

The theory space is a key ingredient of our modern understanding of quantum and statistical field theory. On very general grounds, one may define the theory space as the set of theories that are identified by the following common features: dimensionality, field content, and symmetries. The renormalization group (RG) brings further qualitative and quantitative information through the notion of relevant, irrelevant, and marginal directions. Indeed, the study of the RG flow of the couplings allows us to define the continuum limit of quantum field theories and to derive the scaling properties of the operators by studying the linearized RG flow around a fixed point [1].

In this work we study the possibility of considering the theory space as a manifold with geometric structures. In particular, we will show that it is possible to define connections on the theory space. The introduction of a connection is an important step as it allows us to study in a general way both local and global quantities defined over the theory space. We will pay particular attention to a connection stemming directly and nonperturbatively from the exact renormalization group (ERG) equation. By means of a connection, it is then straightforward to construct the quantities that are invariant under the changes of coordinates. A coordinate change can be identified as a change of schemes (choice of a cutoff function in the ERG framework). Scheme independence is important since physical observables such as critical exponents are scheme independent.

In different forms, a geometric viewpoint of the theory space has already been invoked in the past. In [2], the RG flow is identified as a one-parameter group of diffeomorphism generated by the beta functions as a vector field. A connection was also identified in the formulation of renormalization in coordinate space by requiring covariant transformation properties of the correlation functions [3,4]. Apart from the linearized behavior around the fixed point, little effort has been made to investigate seriously the information encoded in the RG flow beyond critical exponents. More recently, however, the transformation properties of RG flows at the second order around a fixed point have been considered in order to make contact with the operator product expansion (OPE) [5,6]. We will comment also on the relation between our result and the OPE.

The paper is organized as follows. In Sec. II we introduce the theory space as a manifold and explain its basic features. In Sec. III we consider the ERG equation and show that its solution implies the existence of a connection and define its curvature. In Sec. IV we consider a covariant expansion of the RG flow and comment on its possible applications. In Sec. V we generalize our consideration to the full (infinite-dimensional) theory space. We summarize our findings in Sec. VI.

II. THE THEORY SPACE AS A MANIFOLD

In the Wilsonian renormalization program, one is instructed to write in the action all possible terms compatible with the symmetries and the field content of the theory. Generally, this implies that one has to consider infinitely many terms in the Wilson action, and

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consequently introduce infinitely many couplings. Therefore, the theory space is, generally speaking, infinite dimensional. However, if we consider only theories that are defined in the continuum limit, the actual dimension of the space spanned by the theory is N, the number of relevant directions associated to the fixed point. In this work we will mainly consider this latter setting and take a field theory whose continuum limit is well defined. This permits us to work with a finite-dimensional manifold. Some considerations regarding the infinite-dimensional theory space will be given in Sec. V.

Let $g^i (i = 1, \ldots, N)$ be the N coupling constants parametrizing the theory.[1] We view the couplings g^i as coordinates of the theory space and view the latter as a manifold. A change of scheme, or cutoff function in the ERG case, results in a possibly very complicated redefinition of the couplings: $g'^i = g'^i(g)$. We view such a redefinition as a change of coordinates on the theory space. Note that schemes like minimal subtraction are not included straightforwardly in the functional RG equations, although it is known how to retain the former's quantities from the latter; see [7] and references therein. Physical quantities should not depend on the RG scheme employed. Hence, in the ERG framework, physical quantities should be independent from the chosen cutoff function, or, equivalently, from the specific coordinates employed.

The RG flow is expressed by the beta functions, which constitute a vector field over the theory space. More precisely, a RG trajectory is described by the beta functions

$$\beta^i = \frac{dg^i}{dt} \quad (i = 1, \ldots, N) \tag{1}$$

that enjoy the transformation properties of a vector under a coordinate change. (We define the "RG-time" t by $t \equiv -\log\frac{\Lambda}{\mu}$, where Λ is the cutoff scale introduced in Sec. III.)

As we already said, physical quantities must be independent of the RG scheme used to compute them. Translated into a geometric language, this means that physical quantities must be invariant under any change of coordinates. An example of such a coordinate invariant quantity is the critical exponents. Let us consider

$$\begin{aligned}\frac{\partial\beta^i}{\partial g^j} &= \frac{\partial}{\partial g^j}\sum_{k=1}^{N}\left(\frac{\partial g^i}{\partial g'^k}\beta'^k\right)\\ &= \sum_{k,l=1}^{N}\left(\frac{\partial g'^l}{\partial g^j}\frac{\partial^2 g^i}{\partial g'^l\partial g'^k}\beta'^k + \frac{\partial g'^l}{\partial g^j}\frac{\partial\beta'^k}{\partial g'^l}\frac{\partial g^i}{\partial g'^k}\right).\end{aligned} \tag{2}$$

It is clear that at a fixed point g^* the first term in (2) vanishes. The critical exponents are defined as the eigenvalues of the matrix $\partial_j\beta^i$ at the fixed point. Since the eigenvalues are independent of the basis used to compute them, we see that the matrices $\partial_j\beta^i$ and $\partial'_j\beta'^i$ possess the same spectrum and hence yield the same critical exponents. For later purposes, let us denote the eigendecomposition of the linearized RG flow at the fixed point as follows:

$$\left.\frac{\partial\beta^i}{\partial g^j}\right|_{g=g^*} = \sum_{m,n=1}^{N} A^i_m Y^m_n (A^{-1})^n_j, \tag{3}$$

where Y is the eigenvalue matrix, and A is the eigenvector matrix. It is straightforward to check that $A^i_j = \sum_{k=1}^{N}\frac{\partial g^i}{\partial g'^k}A'^k_j$.

We note that the coordinate independence of the critical exponents relies crucially on the vanishing of the inhomogeneous term in (2) at the fixed point, so that the matrix of the linearized RG flow transforms covariantly under a coordinate transformation at the fixed point. It is clear, however, that no such simplification occurs when taking further derivatives of the beta function. To obviate such difficulties, instead of employing partial derivatives, it is natural to employ covariant derivatives that allow us to write down covariant quantities directly. It is the purpose of this work to show that such a geometric structure, namely, a connection on the tangent space, can naturally be introduced from the ERG flow equation.

III. A CONNECTION FROM THE ERGE

Let $S[\phi]$ be a bare action with an ultraviolet (UV) cutoff incorporated. Following [8], we introduce $W_\Lambda[J]$, the generating functional of connected Green functions with an infrared (IR) cutoff Λ, by

$$e^{W_\Lambda[J]} \equiv \int \mathcal{D}\phi\, e^{-S[\phi]-\Delta S_\Lambda+\int d^dxJ\phi}, \tag{4}$$

where

$$\Delta S_\Lambda = \frac{1}{2}\int d^dx\phi(x)R_\Lambda(-\partial^2)\phi(x)$$

is an IR regulator. The kernel $R_\Lambda(-\partial^2)$ suppresses the integration over the modes with momenta lower than the scale Λ in (4). If we denote the Fourier transform of R_Λ by the same symbol $R_\Lambda(p)$, it approaches a positive constant of order Λ^2 as $p^2 \to 0$ and vanishes at large momentum.

The Λ-dependence of W_Λ, derived in [8], is given by

$$\begin{aligned}-\Lambda\frac{\partial W_\Lambda[J]}{\partial\Lambda} = \int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{2}\bigg\{&\frac{\delta W_\Lambda[J]}{\delta J(-p)}\frac{\delta W_\Lambda[J]}{\delta J(p)}\\ &+\frac{\delta^2 W_\Lambda[J]}{\delta J(-p)\delta J(p)}\bigg\}.\end{aligned} \tag{5}$$

Here, we wish to consider instead a generalized equation with a positive anomalous dimension $\eta/2$ for the scalar field [9]:

[1]Throughout this work the couplings g^i are taken to be dimensionless as all dimensionful quantities have been rescaled in units of the cutoff.

$$-\Lambda\frac{\partial W_\Lambda[J]}{\partial\Lambda}=\frac{\eta}{2}\int_p J(p)\frac{\delta W_\Lambda[J]}{\delta J(p)}+\int_p\left(\Lambda\frac{\partial}{\partial\Lambda}-\eta\right)R_\Lambda(p)\cdot\frac{1}{2}\left\{\frac{\delta W_\Lambda[J]}{\delta J(-p)}\frac{\delta W_\Lambda[J]}{\delta J(p)}+\frac{\delta^2 W_\Lambda[J]}{\delta J(-p)\delta J(p)}\right\}. \quad (6)$$

In the dimensionful convention adopted here, the N parameters of the theory, say $G^i(i=1,\ldots,N)$, do not run as Λ changes. To obtain the running parameters of Sec. II, we introduce $\bar{g}^i(t;G)(i=1,\ldots,N)$ as the solution of

$$\frac{\partial}{\partial t}\bar{g}^i(t;G)=\beta^i(\bar{g}), \quad (7)$$

satisfying the initial condition

$$\bar{g}^i(0;G)=G^i. \quad (8)$$

We then define

$$g^i\equiv\bar{g}^i\left(-\ln\frac{\Lambda}{\mu};G\right), \quad (9)$$

where μ is a reference scale, such that

$$\lim_{\Lambda\to\infty}g^i=g_*^i, \quad (10)$$

where g_* denotes the fixed point. These g's are the parameters discussed in Sec. II, and they parametrize the theory in the dimensionless convention.

To switch to the dimensionless convention we divide all physical quantities by appropriate powers of Λ to make them dimensionless. We define

$$\bar{J}(p)\equiv\Lambda^{\frac{d-2}{2}}J(p\Lambda) \quad (11)$$

which is a dimensionless field with dimensionless momentum. We then define

$$W(g)[\bar{J}]\equiv W_\Lambda(G)[J], \quad (12)$$

where g's are related to G's via (9). All the Λ-dependence of the original functional has been incorporated into g's and $\bar{J}$. We wish to emphasize that we consider only theories in the continuum limit. The Wilson action and the functional W have an infinite number of terms, but they are related so that these functionals depend only on a finite number of couplings. In Appendix C, we give an explicit but perturbative construction of a continuum limit. The continuum limit in the ERG framework has been discussed in detail in Ref. [10].

For fixed G's, we have

$$-\Lambda\frac{\partial}{\partial\Lambda}g^i|_G=\beta^i(g), \quad (13)$$

and for fixed J, (11) gives

$$-\Lambda\frac{\partial}{\partial\Lambda}\bar{J}(p)|_J=\left(\frac{d-2}{2}+p\cdot\partial\right)\bar{J}(p). \quad (14)$$

Thus, we obtain

$$-\Lambda\frac{\partial}{\partial\Lambda}W_\Lambda(G)[J]=\sum_{i=1}^N\beta^i(g)\frac{\partial}{\partial g^i}W(g)[\bar{J}]+\int_p\left(\frac{d-2}{2}+p\cdot\partial\right)\bar{J}(p)\frac{\delta}{\delta\bar{J}(p)}W(g)[\bar{J}]. \quad (15)$$

Hence, (6) implies that $W(g)[\bar{J}]$ obeys the ERG differential equation

$$\sum_{i=1}^N\beta^i(g)\frac{\partial}{\partial g^i}W(g)[\bar{J}]=\int_p\left(\frac{d-2+\eta}{2}+p\cdot\partial\right)\bar{J}(p)\cdot\frac{\delta W(g)[\bar{J}]}{\delta\bar{J}(p)}+\int_p(2-\eta-p\cdot\partial)R(p)\frac{1}{2}\left\{\frac{\delta W(g)}{\delta\bar{J}(p)}\frac{\delta W(g)}{\delta\bar{J}(-p)}+\frac{\delta^2W(g)}{\delta\bar{J}(p)\delta\bar{J}(-p)}\right\}, \quad (16)$$

where $R(p)$ is related to $R_\Lambda(p)$ of Sec. II by

$$R_\Lambda(p)=\Lambda^2R(p/\Lambda). \quad (17)$$

From now on we work only in the dimensionless convention, and we omit the bar above J.

For our purposes, it is useful to think of W as a function of the couplings, $W=W(g)$, which is a scalar on the theory space, $W(g)=W'(g')$. By taking a derivative with respect to g^i, we obtain a zero momentum operator

$$\mathcal{O}_i\equiv\frac{\partial W(g)}{\partial g^i} \quad (18)$$

that has covariant transformation properties,

$$\mathcal{O}_i=\frac{\partial g'^j}{\partial g^i}\mathcal{O}'_j, \quad (19)$$

where we have adopted the Einstein convention for repeated indices.

In full analogy we can define the products of the operators $\mathcal{O}_i$ as follows:

$$[\mathcal{O}_{i_1}\cdots\mathcal{O}_{i_n}]\equiv e^{-W(g)}\frac{\partial}{\partial g^{i_1}}\cdots\frac{\partial}{\partial g^{i_n}}e^{W(g)}. \quad (20)$$

For the case of $[\mathcal{O}_{i_1}\mathcal{O}_{i_2}]$ we have

$$[\mathcal{O}_{i_1}\mathcal{O}_{i_2}] \equiv \frac{\partial W}{\partial g^{i_1}}\frac{\partial W}{\partial g^{i_2}} + \frac{\partial^2 W}{\partial g^{i_1}\partial g^{i_2}}. \tag{21}$$

Clearly $[\mathcal{O}_{i_1}\mathcal{O}_{i_2}]$ is not a covariant quantity. This is because the "connected term"

$$\mathcal{P}_{ij} \equiv \frac{\partial^2 W}{\partial g^i \partial g^j} \tag{22}$$

is not covariant. Furthermore, $[\mathcal{O}_{i_1}\mathcal{O}_{i_2}]$ is related to the product of two (zero momentum) operators, and $\mathcal{P}_{ij}$ is related to the short distance singularities of this product. Thus, one expects $\mathcal{P}_{ij}$ to be related to the OPE's singularities. The precise relation is hindered by the fact that we are considering zero momentum operators (i.e., operators integrated over space). (A detailed discussion regarding $[\mathcal{O}_{i_1}\mathcal{O}_{i_2}]$ and $\mathcal{P}_{ij}$ in the general case of momentum-dependent operators can be found in [11].)

Now we consider the flow equation for the operators $\mathcal{O}_i$ and their products. The flow of the operator $\mathcal{O}_i$ can be directly obtained from (16) by taking a derivative with respect to g^i:

$$\frac{\partial \beta^k}{\partial g^i}\mathcal{O}_k + \left(\beta \cdot \frac{\partial}{\partial g}\right)\mathcal{O}_i = \mathcal{D}\mathcal{O}_i \tag{23}$$

(please recall the Einstein convention for the repeated k), where we define

$$\begin{aligned}\mathcal{D} \equiv \int_p \Bigg[&\left(\left(\frac{d-2+\eta}{2} + p\cdot\partial_p\right) J(p) \cdot \frac{\delta}{\delta J(p)}\right. \\ &+ (2-\eta - p\cdot\partial)R(p) \\ &\left.\cdot \left\{\frac{\delta W(g)}{\delta J(-p)}\frac{\delta}{\delta J(p)} + \frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}\right\}\right)\Bigg]. \end{aligned} \tag{24}$$

In deriving (23) we assume that the anomalous dimension η is independent of g's. This is actually true only near the fixed point. The extension to a g-dependent anomalous dimension is given in Appendix A.

By taking a further derivative of the flow Eq. (16) with respect to g^j, we deduce the flow equation for $\mathcal{P}_{ij}$. This can be written as

$$\begin{aligned}&\frac{\partial^2 \beta^k}{\partial g^i \partial g^j}\mathcal{O}_k + \frac{\partial \beta^k}{\partial g^j}\mathcal{P}_{ki} + \frac{\partial \beta^k}{\partial g^i}\mathcal{P}_{kj} + \left(\beta^k \frac{\partial}{\partial g^k} - \mathcal{D}\right)\mathcal{P}_{ij} \\ &= \int_p ((2-\eta)R(p^2) - p\cdot\partial_p R(p^2))\frac{\delta \mathcal{O}_i}{\delta J(p)}\frac{\delta \mathcal{O}_j}{\delta J(-p)}. \end{aligned} \tag{25}$$

It is interesting to observe that the rhs of (25) is covariant since it is determined by the product of the covariant operators $\mathcal{O}_i$ and $\mathcal{O}_j$. It follows also that the lhs of (25) must be covariant, too.

In order to investigate the covariance of the lhs of (25), let us consider the transformation properties of $\mathcal{P}_{ij}$:

$$\mathcal{P}'_{ij} = \frac{\partial g^k}{\partial g'^i}\frac{\partial g^l}{\partial g'^j}\mathcal{P}_{kl} + \frac{\partial^2 g^k}{\partial g'^i \partial g'^j}\mathcal{O}_k. \tag{26}$$

$\mathcal{P}_{ij}$ is not covariant. Hence, the product $[\mathcal{O}_i\mathcal{O}_j]$ is not covariant as was already pointed out. Now we expand $\mathcal{P}_{ij}$ in terms of a basis of composite operators:

$$\mathcal{P}_{ij} = \sum_{k=1}^{N}\Gamma_{i\,j}^{\,k}\mathcal{O}_k + \sum_{a=N+1}^{\infty}\Gamma_{i\,j}^{\,a}\mathcal{O}_a, \tag{27}$$

where the operators $\mathcal{O}_k$ with $k \in [1, N]$ are the relevant operators conjugate to the couplings g^k, whereas the operators $\mathcal{O}_a$ with $a \in [N+1, \infty)$ are irrelevant operators. By inserting the expansion (27) into (26), we deduce the transformation properties of the terms appearing in (27). More precisely, we find that

$$\Gamma'^{\,k}_{i\ j} = \frac{\partial g'^k}{\partial g^n}\frac{\partial g^l}{\partial g'^i}\frac{\partial g^m}{\partial g'^j}\Gamma_{l\,m}^{\,n} + \frac{\partial g'^k}{\partial g^l}\frac{\partial^2 g^l}{\partial g'^i \partial g'^j}, \tag{28}$$

for $(i, j, k) \in [1, N]$ so that $\Gamma_i{}^k{}_j$ transforms as a connection in the theory space. Moreover, we deduce that the second term in (27) transforms as a tensor:

$$\sum_{a=N+1}^{\infty}\Gamma'^{\,a}_{i\ j}\mathcal{O}'_a = \frac{\partial g^k}{\partial g'^i}\frac{\partial g^l}{\partial g'^j}\sum_{a=N+1}^{\infty}\Gamma_{k\,l}^{\,a}\mathcal{O}_a. \tag{29}$$

Equation (27), together with the transformation properties (28) and (29), is one of the main results of this section. Indeed, our findings entail that, by solving the flow equation, we can determine a connection over theory space by considering the expansion of $\mathcal{P}_{ij}$ in (27). Note also that, by definition, this connection is torsionless, i.e., symmetric in the lower indices.

It is now natural to come back to Eq. (25) and consider its lhs in view of the expansion (27) and the new connection. To do so, we also expand the rhs of (25):

$$\begin{aligned}&\int_p ((2-\eta)R(p^2) - p\cdot\partial_p R(p^2))\frac{\delta \mathcal{O}_i}{\delta J(p)}\frac{\delta \mathcal{O}_j}{\delta J(-p)} \\ &\quad = d_{ij}^k \mathcal{O}_k + \cdots, \end{aligned} \tag{30}$$

where the dots are contributions involving only irrelevant composite operators. In the following we focus our attention solely on the relevant operators $\mathcal{O}_i (i = 1, \ldots, N)$.

As we have already pointed out, the rhs of (25) is covariant, and the lhs should be also. By inserting the expansions (27) and (30) into (25), we find

$$\left[\beta^l \frac{\partial}{\partial g^l}\Gamma^{\,k}_{i\,j} - \Gamma^{\,l}_{i\,j}\frac{\partial\beta^k}{\partial g^l} + \frac{\partial\beta^l}{\partial g^j}\Gamma^{\,k}_{l\,i} + \frac{\partial\beta^l}{\partial g^i}\Gamma^{\,k}_{l\,j} + \frac{\partial^2\beta^k}{\partial g^i\partial g^j}\right]\mathcal{O}_k = d^k_{ij}\mathcal{O}_k, \tag{31}$$

where we have kept only the terms involving relevant operators in the expansions (27) and (30). The lhs of (31) can be rewritten in a geometric fashion and, by selecting the term proportional to $\mathcal{O}_k$, we can write

$$\frac{1}{2}(\nabla_i\nabla_j + \nabla_j\nabla_i)\beta^k - \frac{1}{2}(R_{il}{}^k{}_j + R_{jl}{}^k{}_i)\beta^l = d^k_{ij}, \tag{32}$$

where the covariant derivatives are defined as usual as

$$\nabla_i\beta^j \equiv \partial_i\beta^j + \Gamma^{\,j}_{i\,k}\beta^k, \tag{33a}$$

$$\nabla_i\nabla_j\beta^k \equiv \partial_i(\nabla_j\beta^k) - \Gamma^{\,l}_{i\,j}\nabla_l\beta^k + \Gamma^{\,k}_{i\,l}\nabla_j\beta^l, \tag{33b}$$

and the curvature is defined by

$$R_{il}{}^k{}_j \equiv \partial_i\Gamma^{\,k}_{l\,j} - \partial_l\Gamma^{\,k}_{i\,j} + \Gamma^{\,k}_{i\,m}\Gamma^{\,m}_{l\,j} - \Gamma^{\,k}_{l\,m}\Gamma^{\,m}_{i\,j}. \tag{34}$$

Equation (32) is one of the main results of this paper. It shows that the flow equation for $\mathcal{P}_{ij}$ can be written in an inspiring covariant form thanks to the connection defined by Eq. (27). We also wish to point out that a relation very similar to our Eq. (32) was derived in a non-ERG context in [4]. (See also [12].) More details on the derivation of Eq. (32) are given in Appendix B.

Let us observe that we have constructed the connection $\Gamma^{\,k}_{i\,j}$ using the generating functional W. However, it can be checked that the same steps can be repeated both for the Wilson action [1,13] and for the effective average action (EAA) [8,14,15].

Before concluding this section, we wish to show explicitly that the curvature defined in (34) is generally nontrivial. To see this, let us first consider

$$\begin{aligned}\frac{\partial}{\partial g^k}\mathcal{P}_{ij} &= \partial_k\left(\sum_{l=1}^{N}\Gamma^{\,l}_{i\,j}\mathcal{O}_l + \sum_{a=N+1}^{\infty}\Gamma^{\,a}_{i\,j}\mathcal{O}_a\right)\\ &= \sum_{l=1}^{N}\left(\partial_k\Gamma^{\,l}_{i\,j}\mathcal{O}_l + \sum_{m=1}^{N}\Gamma^{\,l}_{i\,j}\Gamma^{\,m}_{k\,l}\mathcal{O}_m + \sum_{a=N+1}^{\infty}\Gamma^{\,l}_{i\,j}\Gamma^{\,a}_{k\,l}\mathcal{O}_a\right)\\ &\quad + \left(\sum_{a=N+1}^{\infty}\partial_k\Gamma^{\,a}_{i\,j}\mathcal{O}_a + \sum_{a=N+1}^{\infty}\Gamma^{\,a}_{i\,j}\partial_k\mathcal{O}_a\right).\end{aligned} \tag{35}$$

Moreover, it is convenient to consider the following expansion:

$$\partial_k\mathcal{O}_{a>N} = \sum_{j=1}^{N}\Gamma^{\,j}_{i\,a}\mathcal{O}_j + \sum_{b=N+1}^{\infty}\Gamma^{\,b}_{i\,a}\mathcal{O}_b. \tag{36}$$

From the definition of $\mathcal{P}_{ij}$ we deduce

$$\partial_i\mathcal{P}_{kj} = \partial_k\mathcal{P}_{ij}. \tag{37}$$

Inserting (35) into (37) and extracting the coefficients of the relevant operator $\mathcal{O}_l$, we find

$$\left(\partial_i\Gamma^{\,l}_{k\,j} + \sum_{m=1}^{N}\Gamma^{\,m}_{k\,j}\Gamma^{\,l}_{i\,m}\right) - \left(\partial_k\Gamma^{\,l}_{i\,j} + \sum_{m=1}^{N}\Gamma^{\,m}_{i\,j}\Gamma^{\,l}_{k\,m}\right) = \sum_{a=N+1}^{\infty}(\Gamma^{\,a}_{i\,j}\Gamma^{\,l}_{k\,a} - \Gamma^{\,a}_{k\,j}\Gamma^{\,l}_{i\,a}), \tag{38}$$

which implies

$$R_{ik}{}^l{}_j = \sum_{a=N+1}^{\infty}(\Gamma^{\,a}_{i\,j}\Gamma^{\,l}_{k\,a} - \Gamma^{\,a}_{k\,j}\Gamma^{\,l}_{i\,a}). \tag{39}$$

Equation (39) implies that the curvature is generally nonzero because there is no reason that the rhs of (39) should vanish.

IV. A DIFFERENT APPROACH: RIEMANN NORMAL COORDINATE EXPANSION OF THE BETA FUNCTIONS

In this section we develop an approach different from the one considered in Sec. III, where the introduction of the connection is deeply related to the flow equation and its solution. Here, we wish to consider solely the theory space manifold and explore it in a covariant way. As we have argued in Sec. II, this is important in order to define physical, i.e., scheme-independent, quantities. We have already considered the example of the critical exponents. The critical exponents are calculated by considering linear perturbations around the fixed point. Nevertheless, information is contained also in the higher orders of the perturbation, although obtaining scheme invariant results is hindered by the use of a noncovariant expansion. Therefore, the purpose of this section is to introduce a covariant expansion around a fixed point.

Before discussing the nature of the covariant expansion around the fixed point, we remark that in order to define such an expansion we need a connection to start with. In Sec. III we have introduced a connection on the theory space, but this choice is by no means unique. How can we construct another connection? There is no canonically defined tensor like the metric and we have only the vector field defined by the beta function β^i. Given such a vector, it is straightforward to check that

$$\Gamma_i{}^k{}_j \equiv \frac{\partial g^k}{\partial\beta^l}\frac{\partial\beta^l}{\partial g^i\partial g^j} \tag{40}$$

transforms as a connection. [The connection (40) has been also recently proposed in [5].]

Let us comment on some features regarding this connection. First of all, the connection (40) is well defined only when $\frac{\partial g^k}{\partial \beta^l}$ actually is. For the connection (40) to be defined then, we need $\frac{\partial g^k}{\partial \beta^l}$ to be defined. In turn this implies that the inverse of the matrix $\partial_i \beta^j$ must exist. This inversion can be made locally provided that $\det \partial_i \beta^j \neq 0$. In our case of interest, i.e., in the vicinity of a fixed point, requiring $\det \partial_i \beta^j \neq 0$ is tantamount to having no exactly marginal direction. If an exactly marginal direction is present, another connection should be considered. Furthermore, the connection (40) is flat as its curvature vanishes identically. This is a striking difference from the connection introduced in Sec. III. We will come back to flat connections in Sec. V.

Let us now assume that we have some connection $\Gamma_i{}^k{}_j$ and discuss how to define a covariant expansion for the RG flow by employing this connection. The RG flow, as described by the beta function vector field, is a covariant quantity. In order to keep covariance in an expansion, however, special care must be taken.

Quite generally, we are given a vector, which we will later specify to be β^i, and we wish to express this vector at some point of the manifold via a covariant expansion around a different point, which we will eventually identify with the fixed point. This reminds us of the Riemann normal coordinate expansions: given a tensor at some point P (coordinatized by g^i), we can express this latter tensor via a covariant series expansion defined via tensorial quantities evaluated at the point Q (coordinatized by g^i_*, which eventually will be identified with the fixed point). More precisely, such an expansion is found by introducing the Riemann normal coordinates, which we denote ξ^i. The coordinates ξ^i cover a double role: they are a system of coordinates equivalent to g^i and represent a vector at the point Q coordinatized by g^i_*. In the ξ-coordinate system, the point Q is represented by $\xi^i = 0$. We refer the reader to [16] for more details.

Applying the Riemann normal coordinate expansion to the vector β^i, we obtain

$$\beta^i(g) = \beta^i(g_*) + \xi^j \nabla_j \beta^i(g_*) + \frac{1}{2}\xi^j \xi^k \nabla_j \nabla_k \beta^i(g_*) + \frac{1}{6} R_{jk}{}^i{}_l \beta^j(g_*) \xi^k \xi^l + \cdots . \tag{41}$$

Note that in order to write down the expansion (41) we need to have a connection that defines the covariant derivative and the curvature. The same expression holds for any connection.

Coming back to physical quantities, it is interesting to consider what information is contained in the second order expansion of the beta functions. Let the couplings $\{\check{g}^i\}$ be conjugate to scaling operators in coordinate space with scaling dimensions $\Delta_i = D - y_i$, and denote the OPE coefficients $c_{jk}{}^i$. Cardy has shown that the beta functions around the fixed point can be written as [17]

$$\check{\beta}^i = y_i \check{g}^i - \sum_{j,k} c_{jk}{}^i \check{g}^j \check{g}^k + O(\check{g}^3), \tag{42}$$

where the couplings have been rescaled by an angular integral factor. One then deduces that

$$\frac{1}{2}\frac{\partial}{\partial \check{g}^j}\frac{\partial}{\partial \check{g}^k}\check{\beta}^i\bigg|_{\check{g}=0} = -c_{jk}{}^i. \tag{43}$$

It is natural to ask whether one can use a relation like (43) in the ERG context. In this section we make the first steps in this direction. [In Appendix C we also consider the connection of the ERG with the results of Wegner for the higher order terms in the expansion of the functional $W(g)$.]

As it has also been noted in [6], it is crucial to discuss the dependence of the OPE coefficients on the RG scheme employed to compute the running of the couplings. In order to arrive at a formula involving the scaling fields conjugate to $\{\check{g}^i\}$, we consider the eigendirections of the linearized RG flow and identify the relation between the couplings $\{\check{g}^i\}$ and $\{g^i\}$ via the matrix A^{-1} introduced in Eq. (3).

However, if we wish to compute the OPE coefficients via Eq. (43) in terms of g^i-dependent quantities, we see that we have to consider the second derivative $\partial_{g^j}\partial_{g^k}\beta^i$. More precisely, one has to consider the following expression: $c_{jk}{}^i \sim A^{(-1)}{}^i{}_l \partial_{g^m}\partial_{g^n}\beta^l A^m_j A^n_k$. From the transformation properties of A and β it is straightforward to check that the so defined $c_{jk}{}^i$ is invariant under coordinate transformations up to an additive term due to the fact that $\partial_{g^m}\partial_{g^n}\beta^l$ does not transform as a tensor (see also [6]).

To obviate this fact one may consider the covariant version of $\partial_{g^j}\partial_{g^k}\beta^i$, where the partial derivatives have been promoted to covariant derivatives: $\nabla_{g^m}\nabla_{g^n}\beta^l$. It is clear then that the expression $A^{(-1)}{}^i{}_l \nabla_{g^m}\nabla_{g^n}\beta^l A^m_j A^n_k$ is invariant under a change of scheme and thus it is a physical candidate to be considered. The purpose of the geometric expansion (41) is exactly to probe the vicinity of the fixed point in a covariant fashion, and it provides a natural introduction for the covariant expression $\nabla_{g^m}\nabla_{g^n}\beta^l$. Critical exponents are found by looking at the linear perturbation around the fixed point, which corresponds to the first term in (41), where ξ corresponds to the perturbation. The second term in (41) now contains the information regarding the second order perturbation around the fixed point in a covariant manner.

We conclude this section by stressing that the covariant expansion (41) can be used in the ERG context to define further physical quantities besides the critical exponents, such as the Wilson operator product coefficients. Nevertheless, employing different connections selects different quantities, and it is not straightforward to deduce their meaning.

However, the discussion of the previous section and its connection with the previous works in the literature, e.g. [12], suggest that OPE coefficients are found by employing the connection of Sec. III.

V. THE INFINITE-DIMENSIONAL THEORY SPACE

So far we have taken the theory space to be N dimensional, with N being the number of relevant directions. This is possible solely for renormalizable trajectories, that is, theories whose continuum limit is well defined. However, the ERG framework can be employed to test the theory space with its fullest content, i.e., taking into account also the infinitely many irrelevant directions. The aim of this section is to discuss how the machinery developed until now is modified when considering this more general theory space.

In actual applications of the ERG, the need for an ansatz or some truncation scheme generally requires us to consider a finite-dimensional approximation of the theory space, which is then parametrized by n couplings with N relevant and $n-N$ irrelevant directions. For the purposes of this section, let us consider n fixed and eventually take the formal limit $n\to\infty$.

The definition of the connection (40) can be straightforwardly extended by truncating the theory space to include the $n-N$ irrelevant directions. In a typical ERG computation, where an ansatz $S_\Lambda=\sum_{i=1}^n g^i\mathcal{O}_i$ is considered, we have n coordinates and beta functions, and a connection may be considered.

Let us go back to the framework developed in Sec. III and adapt it to the present n-dimensional space. The expansion (27) of $\mathcal{P}_{ij}$ is no longer split in relevant and irrelevant parts, but we include all the operators in a single sum (possibly truncated, retaining only n operators). Extending the range of indices of the connection is not as innocuous as it may seem. Indeed, by repeating the reasoning at the end of Sec. III stemming from the relation $\partial_k\mathcal{P}_{ij}=\partial_i\mathcal{P}_{kj}$ we see that now the curvature identically vanishes. This is due to the inclusion of the rhs of (38) in the definition of the curvature.

Is there any obvious reason for this fact? Let us consider that we can view the theory space as a space of functionals, i.e., the Wilsonian actions S_Λ, and that there is *a priori* no need for this space to be flat. However, if we *assume* that such functionals can be expanded in couplings as $S_\Lambda=\sum_i g^i\mathcal{O}_i$, where the $\mathcal{O}_i$ are independent of g^i, we can check that this space enjoys the properties of a vector space, e.g., distributivity $\sum_i g^i\mathcal{O}_i+\sum_i\tilde{g}^i\mathcal{O}_i=\sum_i(g^i+\tilde{g}^i)\mathcal{O}_i$. Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$, which is a flat space. Thus, in this sense, it is appealing to consider the theory space as a flat manifold.

This is a striking difference from the "continuum theories subspace" considered in Sec. III. However, this is not a contradiction. Actually, even if the full theory space were flat, it would be generally possible to have a curved subspace expressed in the intrinsic coordinates provided by the relevant couplings g^i with $i=1,\ldots,N$.

In the "continuum theories subspace" one could possibly consider nontrivial topological invariants. For instance, for a subspace of dimension $N=2p$ one could consider the Euler invariant

$$E_{2p}=\frac{(-1)^p}{2^{2p}\pi^p p!}\int\epsilon_{i_1\cdots i_{2p}}R^{i_1 i_2}\wedge\cdots\wedge R^{i_{2p-1}i_{2p}}, \qquad (44)$$

which is defined via the exterior product of p curvature two-forms R defined in (34). It is not clear, though, if the above E_{2p} could be of any practical interest.

VI. CONCLUSIONS

In this work we have put forward a geometric viewpoint on the theory space inspired by the ERG flow equation. While viewing the theory space as a manifold, we have introduced further geometric structures. In particular we have shown it possible to define connections over the theory space. The theory space has been, for most of this work, restricted to the space where the continuum limit of the field theory is well defined.

Remarkably, we have been able to define explicitly two connections. One stems from the expansion of $\mathcal{P}_{ij}$ in composite operators $\mathcal{O}_k$; see Eqs. (27) and (28). The other exploits the transformation properties of the beta functions; see Eq. (40). In Sec. III we have also shown that the ERG equation associated with the expansion (27) can be written in a manifestly covariant way.

In Sec. IV we have discussed a different geometric view on the RG flow. Namely, we have looked at the RG flow around the fixed point via a covariant expansion by employing the Riemann normal coordinates. Furthermore, we have emphasized that our geometric framework allows us to possibly define further physical quantities directly from the RG flow. In this case, physical quantities are identified as scheme-independent quantities, such as the critical exponents.

In Sec. V we have considered the full (infinite-dimensional) theory space. We have noted that the full theory space is actually flat and that one may view the "renormalizable theories subspace" as a curved submanifold embedded in the full (flat) theory space.

Concluding this paper, we would like to remark that the geometric understanding of the theory space, introduced here, could be helpful in defining in a suitable manner further physical quantities, such as the operator product expansion coefficients, on top of the critical exponents. In the future, we hope to be able to come back to the formalism developed in this work and compute explicitly some of the quantities that we have introduced, like the connection $\Gamma_i{}^k{}_j$ and the associated curvature, in some approximation scheme (e.g., epsilon or $1/N$ expansion).

APPENDIX A: INCLUSION OF THE ANOMALOUS DIMENSION

In Sec. III we derived the geometric relation (32) while neglecting the coupling dependence of the anomalous dimension. Here we generalize Eq. (32) by including such dependence.

The anomalous dimension $\eta = \eta(g)$ is a scalar under coordinate transformations. It follows that a derivative $\partial_i \eta = \nabla_i \eta$ is a covariant quantity, whereas a second derivative is not. By taking a derivative with respect to g^j of (16) we obtain

$$\frac{\partial \beta^i}{\partial g^j}\mathcal{O}_i + \left(\beta \cdot \frac{\partial}{\partial g}\right)\mathcal{O}_j = \mathcal{D}\mathcal{O}_j + \int_p \frac{1}{2}\frac{\partial \eta}{\partial g^j} J(p)\frac{\delta W}{J(p)} + \frac{1}{2}\int_p \left(-\frac{\partial \eta}{\partial g^j} R(p^2)\right)\left[\frac{\delta W}{\delta J(p)}\frac{\delta W}{\delta J(-p)} + \frac{\delta^2 W}{\delta J(p)\delta J(-p)}\right], \tag{A1}$$

which is equivalent to Eq. (23) when η is a constant. Equation (A1) can be written in a more geometric fashion as follows:

$$\nabla_j \beta^i \mathcal{O}_i + \beta^i \nabla_i \mathcal{O}_j = \mathcal{D}\mathcal{O}_j + \nabla_j \eta \int_p \frac{1}{2} J(p)\frac{\delta W}{J(p)} - \frac{1}{2}\nabla_j \eta \int_p R(p^2)\left[\frac{\delta W}{\delta J(p)}\frac{\delta W}{\delta J(-p)} + \frac{\delta^2 W}{\delta J(p)\delta J(-p)}\right],$$

where we used the fact that the connection is symmetric.

By differentiating once again with respect to g^i we obtain

$$\beta \cdot \frac{\partial}{\partial g}\mathcal{P}_{ij} - \frac{\partial \beta^k}{\partial g^j}\mathcal{P}_{ki} + \frac{\partial \beta^k}{\partial g^i}\mathcal{P}_{kj} + \frac{\partial \beta^k}{\partial g^i \partial g^j} O_k = \mathrm{rhs} \tag{A2}$$

where

$$\begin{aligned}
\mathrm{rhs} = {} & \mathcal{D}P_{ij} + \int_p ((2-\eta)R(p^2) - p\cdot\partial_p R(p^2))\frac{\delta \mathcal{O}_i}{\delta J(p)}\frac{\delta \mathcal{O}_j}{\delta J(-p)} \\
& + \frac{1}{2}\frac{\partial \eta}{\partial g^i}\int_p J(p)\frac{\delta}{\delta J(p)}\frac{\partial W}{\partial g^j} + \frac{1}{2}\frac{\partial \eta}{\partial g^j}\int_p J(p)\frac{\delta}{\delta J(p)}\frac{\partial W}{\partial g^i} + \frac{1}{2}\frac{\partial^2 \eta}{\partial g^i \partial g^j}\int_p J(p)\frac{\delta W}{\delta J(p)} \\
& - \frac{\partial^2 \eta}{\partial g^i \partial g^j}\int_p R(p^2)\left[\frac{1}{2}\frac{\delta W}{\delta J(p)}\frac{\delta W}{\delta J(-p)} + \frac{1}{2}\frac{\delta^2 W}{\delta J(p)\delta J(-p)}\right] \\
& - \frac{\partial \eta}{\partial g^j}\int_p R(p^2)\left[\frac{\delta W}{\delta J(-p)}\frac{\delta \mathcal{O}_i}{\delta J(p)} + \frac{1}{2}\frac{\delta^2 \mathcal{O}_i}{\delta J(p)\delta J(-p)}\right] - \frac{\partial \eta}{\partial g^i}\int_p R(p^2)\left[\frac{\delta W}{\delta J(-p)}\frac{\delta \mathcal{O}_j}{\delta J(p)} + \frac{1}{2}\frac{\delta^2 \mathcal{O}_j}{\delta J(p)\delta J(-p)}\right].
\end{aligned}$$

Following the same steps as in Sec. III, using Eq. (A2), and dropping terms coming from irrelevant operators, we can rewrite (A2) as follows:

$$\begin{aligned}
& \left[\frac{1}{2}(\nabla_i \nabla_j + \nabla_j \nabla_i)\beta^k - \frac{1}{2}(R_{il}{}^k{}_j + R_{jl}{}^k{}_i)\beta^l\right]\mathcal{O}_k \\
& = d^k_{ij}\mathcal{O}_k + \frac{1}{2}\nabla_i \eta \int_p J(p)\frac{\delta}{\delta J(p)}\frac{\partial W}{\partial g^j} + \frac{1}{2}\nabla_j \eta \int_p J(p)\frac{\delta}{\delta J(p)}\frac{\partial W}{\partial g^i} + \frac{1}{2}\nabla_i \nabla_j \eta \int_p J(p)\frac{\delta W}{\delta J(p)} \\
& \quad - \nabla_i \nabla_j \eta \int_p R(p^2)\left[\frac{1}{2}\frac{\delta W}{\delta J(p)}\frac{\delta W}{\delta J(-p)} + \frac{1}{2}\frac{\delta^2 W}{\delta J(p)\delta J(-p)}\right] - \nabla_j \eta \int_p R(p^2)\left[\frac{\delta W}{\delta J(-p)}\frac{\delta \mathcal{O}_i}{\delta J(p)} + \frac{1}{2}\frac{\delta^2 \mathcal{O}_i}{\delta J(p)\delta J(-p)}\right] \\
& \quad - \nabla_i \eta \int_p R(p^2)\left[\frac{\delta W}{\delta J(-p)}\frac{\delta \mathcal{O}_j}{\delta J(p)} + \frac{1}{2}\frac{\delta^2 \mathcal{O}_j}{\delta J(p)\delta J(-p)}\right].
\end{aligned} \tag{A3}$$

The first line in (A3) corresponds to (32) for the case of constant η. As in the case of Eq. (30), the η-dependent lines in (A3) can be expanded in the $\mathcal{O}_k$ basis, retaining only the relevant operators.

APPENDIX B: THE ROLE OF IRRELEVANT OPERATORS IN (32)

In deriving Eq. (32) we truncated the expansion (27) for $\mathcal{P}_{ij}$ by retaining only the relevant operators. One may wonder if any effect is to be expected from the irrelevant operators, since the RG flow of irrelevant operators mixes in general with relevant ones. In this appendix we discuss this point in detail.

Let us first introduce irrelevant composite operators. From the transformation property (29) we deduce that an irrelevant operator is a scalar quantity labeled by an index $a \in [N+1, \infty)$. Such index then cannot be traced back to a coordinate index; rather it can be thought of as an "internal index." For this reason, in this section we shall denote the composite operators via greek indices $\mu = a \in [N+1, \infty)$. Adopting this notation we can write the coordinate transformation property (29) as

$$\Gamma'^{\,\mu}_{i\ j}\mathcal{O}'_\mu = \frac{\partial g^k}{\partial g'^i}\frac{\partial g^l}{\partial g'^j}\Gamma^{\ \mu}_{k\ l}\mathcal{O}_\mu, \tag{B1}$$

where the sum over μ is intended. An operator $\mathcal{O}_\mu$ transforms as a scalar, and $\Gamma^{\ \mu}_{i\ j}$ transforms as a tensor in the two lower indices. Furthermore, an operator $\mathcal{O}_\mu$ satisfies the following ERG equation:

$$\left(\beta\cdot\frac{\partial}{\partial g} - \mathcal{D}\right)\mathcal{O}_\mu + y_\mu\mathcal{O}_\mu = M_\mu{}^i\mathcal{O}_i + M_\mu{}^\nu\mathcal{O}_\nu, \tag{B2}$$

where we split the mixing into relevant and irrelevant operators in the rhs. From the transformation properties of $\mathcal{O}_i$ and $\mathcal{O}_\mu$, we deduce that the matrix $M_\mu{}^i$ transforms as a vector. Moreover, at the fixed point, the ERG Eq. (B2) reduces to

$$(y_\mu - \mathcal{D})\mathcal{O}_\mu = 0, \tag{B3}$$

where $-y_\mu \geq 0$ is the scaling dimension of $\mathcal{O}_\mu$ in momentum space.

Employing the notation introduced so far, we can rewrite the expansion (27) as follows:

$$\mathcal{P}_{ij} = \Gamma^{\ k}_{i\ j}\mathcal{O}_k + \Gamma^{\ \mu}_{i\ j}\mathcal{O}_\mu. \tag{B4}$$

Then, plugging the expansion (B4) into (25), it is straightforward to check that a new term appears in (32). Such a term arises due to the following contribution:

$$\begin{aligned}\left(\beta^k\frac{\partial}{\partial g^k} - \mathcal{D}\right)\mathcal{P}_{ij} &\supset \Gamma^{\ \mu}_{i\ j}\left(\beta^k\frac{\partial}{\partial g^k} - \mathcal{D}\right)\mathcal{O}_\mu \\ &= \Gamma^{\ \mu}_{i\ j}(-y_\mu\mathcal{O}_\mu + M_\mu{}^k\mathcal{O}_k + M_\mu{}^\nu\mathcal{O}_\nu).\end{aligned}$$

Thus we see that also a term proportional to the relevant operator $\mathcal{O}_k$ is generated and that Eq. (32) is generalized to

$$\frac{1}{2}(\nabla_i\nabla_j + \nabla_j\nabla_i)\beta^k - \frac{1}{2}(R_{il}{}^k{}_j + R_{jl}{}^k{}_i)\beta^l + \Gamma^{\ \mu}_{i\ j}M_\mu{}^k = d^k_{ij}, \tag{B5}$$

where the last term on the lhs transforms also as a tensor. Note that at a fixed point Eq. (B5) reads

$$\frac{1}{2}(\nabla_i\nabla_j + \nabla_j\nabla_i)\beta^k|_{\mathrm{FP}} = d^k_{ij}|_{\mathrm{FP}},$$

since the last term in (B5) does not contribute to the fixed point formula.

Now let us discuss in more detail the presence of the term $\Gamma^{\ \mu}_{i\ j}M_\mu{}^k$ in (B5). In particular, we wish to make two observations which reveal that $\Gamma^{\ \mu}_{i\ j}M_\mu{}^k$ constitutes a subleading contribution to (B5).

The first observation is based on an explicit estimate of the cutoff dependence in the dimensionful convention. A careful analysis, based on the choice of coordinates found in [18,19], shows that the contribution due to the irrelevant operators in (B4) is subleading in the large Λ limit. More precisely, denoting $y_O \equiv d - \Delta_O$, where Δ_O is the scaling dimension of an operator $O(x)$, the leading contributions scale like $\Lambda^{y_k - y_i - y_j}$. For $y_k > y_i + y_j$, this leads to a singular behavior that can be put in correspondence with the nonintegrable short distance singularities in the OPE via dimensional analysis arguments. The term $\Gamma^{\ \mu}_{i\ j}M_\mu{}^k$ does not contribute to the singular behavior and can be dropped in (B5) when considering nonintegrable short distance singularities as it scales like $\Lambda^{(y_\mu - y_i - y_j)<0}$. This observation makes evident a link with some previous works in the literature (see in particular [3,20–22]), where the nonintegrable short distance singularities are considered, and a geometric formula fully analogous to (32) is derived.

As a second observation, we note that in order to write down (B5) a certain basis of irrelevant operators has been selected. If we limit ourselves to consider nonintegrable short distance singularities, i.e., scaling dimensions such that $y_k > y_i + y_j$, then the term $\Gamma^{\ \mu}_{i\ j}M_\mu{}^k$ is dismissed. Hence this truncation has the nice feature of being independent of the convention chosen for the irrelevant operators.

APPENDIX C: CARDY'S FORMULA

Let us consider a generic fixed point with N relevant directions. Following [23] we construct the Wilson action perturbatively around the fixed point. Let us denote the relevant parameters with scale dimension $y_i > 0$ by $g^i (i = 1, \ldots, N)$. The generating functional $W(g)$ with an IR cutoff is determined by

$$\sum_{i=1}^{N}\beta^i(g)\frac{\partial}{\partial g^i}e^{W(g)[J]} = \int_p \left[\left(p\cdot\partial_p + \frac{D-2}{2}+\gamma\right)J(p)\cdot\frac{\delta}{\delta J(p)} + (-p\cdot\partial_p + 2-2\gamma)R(p)\cdot\frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}\right]e^{W(g)[J]}. \tag{C1}$$

Denoting the fixed point functional $W^* = W(g=0)$, we rewrite this in a form more convenient for perturbative calculations:

$$\sum_{i=1}^{N}\beta^i(g)\frac{\partial}{\partial g^i}e^{W(g)-W^*} = \int_p \left[\left(p\cdot\partial_p + \frac{D-2}{2}+\gamma\right)J(p)\frac{\delta}{\delta J(p)} + (-p\cdot\partial_p + 2-2\gamma)R(p)\cdot\left(\frac{\delta W^*[J]}{\delta J(-p)}\frac{\delta}{\delta J(p)} + \frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}\right)\right]e^{W(g)-W^*}. \tag{C2}$$

We assume a constant anomalous dimension γ for simplicity. We wish to solve this perturbatively by expanding the functional as

$$W(g) = W^* + \sum_{i=1}^{N} g_i W_i + \sum_{i,j=1}^{N}\frac{1}{2}g_i g_j W_{ij} + \sum_{i,j,k=1}^{N}\frac{1}{3!}g_i g_j g_k W_{ijk} + \cdots \tag{C3a}$$

and the beta functions as

$$\beta^i(g) = y_i g^i + \frac{1}{2}\sum_{j,k=1}^{N}\beta^i_{jk}g^j g^k + \frac{1}{3!}\sum_{j,k,l=1}^{N}\beta^i_{jkl}g^j g^k g^l + \cdots. \tag{C3b}$$

We can regard g^i as the coefficient of an external source with zero momentum. Hence,

$$\mathcal{O}_{i_1,\ldots,i_n} = e^{-W^*}\frac{\partial^n}{\partial g^{i_1}\partial g^{i_2}\cdots\partial g^{i_n}}e^{W(g)}\Big|_{g=0} \tag{C4}$$

is the nth order product of composite operators $W_i = \frac{\partial}{\partial g^i}W(g)|_{g=0}$ with zero momentum. We obtain, up to third order,

$$\mathcal{O}_i = W_i, \tag{C5a}$$

$$\mathcal{O}_{ij} = [\mathcal{O}_i\mathcal{O}_j] = \mathcal{O}_i\mathcal{O}_j + W_{ij}, \tag{C5b}$$

$$\mathcal{O}_{ijk} = [\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k] = \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k + W_{ij}\mathcal{O}_k + W_{ik}\mathcal{O}_j + W_{jk}\mathcal{O}_i + W_{ijk}. \tag{C5c}$$

$\mathcal{O}_{i_1,\ldots,i_n}$ satisfies the ERG equation

$$\left(-\sum_{j=1}^{n}y_{i_j} + \mathcal{D}\right)\mathcal{O}_{i_1\cdots i_n} = \sum_{j=1}^{N}\left[\sum_{1\le\alpha<\beta\le n}\beta^j_{i_\alpha i_\beta}\mathcal{O}_{j i_1\cdots\widehat{i_\alpha}\cdots\widehat{i_\beta}\cdots i_n} + \sum_{1\le\alpha_1<\alpha_2<\alpha_3\le n}\beta^j_{i_{\alpha_1}i_{\alpha_2}i_{\alpha_3}}\mathcal{O}_{j i_1\cdots\widehat{i_{\alpha_1}}\cdots\widehat{i_{\alpha_2}}\cdots\widehat{i_{\alpha_3}}\cdots i_n} + \cdots + \beta_{j,i_1\cdots i_n}\mathcal{O}_j\right], \tag{C6}$$

where $\mathcal{D}$ is the functional differential operator defined by the right-hand side of (C2). We have thus shown that the higher order derivatives of the beta functions give mixing of the operator products.

We only consider the first two cases: $n = 1, 2$. Taking $n = 1$ in (C6), we obtain

$$(y_i - \mathcal{D})W_i = 0, \qquad (i = 1,\ldots,N), \tag{C7}$$

implying that W_i is a composite operator of scale dimension $-y_i$. (This was actually taken for granted.) Taking $n = 2$ in (C6), we obtain

$$(y_j + y_k - \mathcal{D})W_{jk} = -\sum_{i=1}^{N}W_i\beta^i_{jk} + \int_p (-p\cdot\partial_p + 2-2\gamma)R(p)\cdot\frac{\delta W_j}{\delta J(p)}\frac{\delta W_k}{\delta J(-p)}. \tag{C8}$$

The integral is local, and we can expand

$$\int_p (-p\cdot\partial_p + 2-2\gamma)R(p)\cdot\frac{\delta W_j}{\delta J(p)}\frac{\delta W_k}{\delta J(-p)} = \sum_{i=1}^{\infty}d^i_{jk}\mathcal{O}_i, \tag{C9}$$

where $\mathcal{O}_i = W_i (i = 1,\ldots,N)$, and $\mathcal{O}_{i>N}$ are irrelevant operators of scale dimension $-y_i \ge 0$. Hence, we obtain

$$(y_j + y_k - \mathcal{D})W_{jk} = \sum_{i=1}^{N}W_i(d^i_{jk} - \beta^i_{jk}) + \sum_{i>N}d^i_{jk}\mathcal{O}_i. \tag{C10}$$

In the absence of degeneracy, i.e.,

$$y_j + y_k \neq y_i \tag{C11}$$

for any $i, j, k \leq N$, we can choose

$$\beta^i_{jk} = 0 \tag{C12}$$

so that

$$W_{jk} = \sum_{i=1}^{\infty} \frac{d^i_{jk}}{y_j + y_k - y_i} \mathcal{O}_i. \tag{C13}$$

Hence, the beta functions are linear up to second order. This is expected from the old result of Wegner [23]. (In the absence of degeneracy, the parameters can be chosen to satisfy linear RG equations.)

Alternatively, we can demand W_{jk} be free of $W_i (i = 1, \ldots, N)$. We must then choose

$$\beta^i_{jk} = d^i_{jk}. \tag{C14}$$

We obtain

$$W_{jk} = \sum_{i>N} \frac{d^i_{jk}}{y_j + y_k - y_i} \mathcal{O}_i \quad (j, k = 1, \ldots, N). \tag{C15}$$

Let $g'^i (i = 1, \ldots, N)$ be the choice of parameters for this alternative convention. These are related to g's satisfying (C12) as

$$g'^i = g^i + \frac{1}{2} \sum_{j,k=1}^{N} \frac{d^i_{jk}}{y_j + y_k - y_i} g^j g^k$$

to order g^2. (C14) is a relation very much like what Cardy has obtained using UV regularization in coordinate space [17].

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