Chromopolarizabilities of a heavy quark at weak coupling

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We obtain the renormalization group improved expressions of the Wilson coefficients of the heavy quark effective theory Lagrangian with leading logarithmic approximation to $\mathcal{O}(1/m^3)$ for the spin-independent sector, which includes the heavy quark chromopolarizabilities. Our analysis includes the effects induced by spectator quarks. We observe that the numerical impact of these logarithms is very large in most cases.

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I. INTRODUCTION

The expansion in inverse powers of the heavy quark mass is a useful tool for the study of hadrons containing one (or more) heavy quarks. This expansion is formulated more systematically in terms of an effective theory and of its associated effective Lagrangian. For the one-heavy quark sector, this effective theory is the heavy quark effective theory (HQET) [1]. Once obtained, its Lagrangian can be applied to physical observables associated to the B or D mesons, such as their spectrum or decays. The HQET Lagrangian is also instrumental in the description of systems with more than one heavy quark, in particular if we fix our attention to the heavy quark-antiquark sector (i.e. heavy quarkonium), as the HQET Lagrangian corresponds to one of the building blocks of the nonrelativistic QCD (NRQCD) Lagrangian [2,3]. The Wilson coefficients of the HQET Lagrangian operators also enter into the Wilson coefficients of the operators (i.e. the potentials) of the potential NRQCD (pNRQCD) Lagrangian [4,5], an effective field theory optimized for the description of heavy quark-antiquark systems near threshold (for reviews, see Refs. [6,7]). Important for us is that the Wilson coefficients we will compute in this paper are necessary ingredients to obtain the pNRQCD Lagrangian with next-to-next-to-next-to-leading-log (NNNLL) accuracy, which in turn is the necessary precision to obtain the complete heavy quarkonium spectrum with NNNLL accuracy, and also necessary for the computation of the production and annihilation of heavy quarkonium with next-to-next-to-leading-log (NNLL) precision. Actually, this

is one of the main motivations we undertake this work. These results are also instrumental in the determination of higher order logarithms for Nonrelativistic QED (NRQED) bound states, like in hydrogen and muonic hydrogen-like atoms.

At present, the operator structure of the HQET Lagrangian, and the tree-level values of their Wilson coefficients, is known to $\mathcal{O}(1/m^3)$ in the case with no massless quarks [8]. The inclusion of light quarks has been considered in Ref. [9]. The Wilson coefficients with leading-log (LL) accuracy were computed in Refs. [10–12] to $\mathcal{O}(1/m^2)$ and in next-to-leading order in Ref. [8] to $\mathcal{O}(1/m^2)$ (without dimension-6 heavy-light operators). The LL running to $\mathcal{O}(1/m^3)$ has been considered in Refs. [9,13,14]. However, we find several discrepancies. We perform a detailed comparison in Sec. VA.

In this paper, we obtain the renormalization group (RG) improved expressions of the Wilson coefficients of the HQET Lagrangian with LL approximation to $\mathcal{O}(1/m^3)$ for the spin-independent sector, which includes the heavy quark chromopolarizabilities. Our analysis includes the effects induced by spectator quarks.

II. HQET LAGRANGIAN

A. HQET Lagrangian without light fermions

The HQET Lagrangian is defined uniquely up to field redefinitions. In this paper, we use the following HQET Lagrangian density for a quark of mass $m \gg \Lambda_{\text{QCD}}$ [8]:

$$\mathcal{L}_{\text{HQET}} = \mathcal{L}_g + \mathcal{L}_\psi, \quad (1)$$

$$\mathcal{L}_g = -\frac{1}{4} G^{\mu\nu a} G_{\mu\nu}^a + \frac{1}{4} \frac{c_1^g}{m^2} g f_{abc} G_{\mu\nu}^a G^{\mu b}{}_\alpha G^{\nu ac} + \mathcal{O}\left(\frac{1}{m^4}\right), \quad (2)$$

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$$\begin{aligned}
\mathcal{L}_\psi = \psi^\dagger \Big\{ & iD_0 + \frac{c_k}{2m} \mathbf{D}^2 + \frac{c_F}{2m} \boldsymbol{\sigma} \cdot g\mathbf{B} + \frac{c_D}{8m^2} (\mathbf{D} \cdot g\mathbf{E} - g\mathbf{E} \cdot \mathbf{D}) + i \frac{c_S}{8m^2} \boldsymbol{\sigma} \cdot (\mathbf{D} \times g\mathbf{E} - g\mathbf{E} \times \mathbf{D}) + \frac{c_4}{8m^3} \mathbf{D}^4 \\
& + ic_M g \frac{\mathbf{D} \cdot [\mathbf{D} \times \mathbf{B}] + [\mathbf{D} \times \mathbf{B}] \cdot \mathbf{D}}{8m^3} + c_{A_1} g^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8m^3} - c_{A_2} \frac{g^2 \mathbf{E}^2}{16m^3} + c_{W_1} g \frac{\{\mathbf{D}^2, \boldsymbol{\sigma} \cdot \mathbf{B}\}}{8m^3} - c_{W_2} g \frac{\mathbf{D}^i \boldsymbol{\sigma} \cdot \mathbf{B} \mathbf{D}^i}{4m^3} \\
& + c_{P'P} g \frac{\boldsymbol{\sigma} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \mathbf{D}}{8m^3} + c_{A_3} g^2 \frac{1}{N_c} \text{Tr} \left(\frac{\mathbf{B}^2 - \mathbf{E}^2}{8m^3} \right) - c_{A_4} g^2 \frac{1}{N_c} \text{Tr} \left(\frac{\mathbf{E}^2}{16m^3} \right) \\
& + ic_{B_1} g^2 \frac{\boldsymbol{\sigma} \cdot (\mathbf{B} \times \mathbf{B} - \mathbf{E} \times \mathbf{E})}{8m^3} - ic_{B_2} g^2 \frac{\boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{E})}{8m^3} \Big\} \psi + \mathcal{O} \left(\frac{1}{m^4} \right). \tag{3}
\end{aligned}$$

Here, ψ is the NR fermion field represented by a Pauli spinor. The components of the vector $\boldsymbol{\sigma}$ are the Pauli matrices. We define $iD^0 = i\partial^0 - gA^0$, $i\mathbf{D} = i\boldsymbol{\nabla} + g\mathbf{A}$, $\mathbf{E}^i = G^{i0}$, and $\mathbf{B}^i = -\epsilon_{ijk} G^{jk}/2$, where ϵ_{ijk} is the three-dimensional totally antisymmetric tensor¹ with $\epsilon_{123} = 1$ and $(\mathbf{a} \times \mathbf{b})^i \equiv \epsilon_{ijk} \mathbf{a}^j \mathbf{b}^k$. Note also that we have rescaled by a factor $1/N_c$ the coefficients $c_{A_{3,4}}$, compared with the definitions in Ref. [8]. In general, we will refer to the c_{A_i} as the chromopolarizabilities.

B. HQET Lagrangian with massless fermions

We now include n_f massless fermions to the HQET Lagrangian. The Lagrangian now has the following structure:

$$\mathcal{L}_{\text{HQET}} = \mathcal{L}_g + \mathcal{L}_\psi + \mathcal{L}_l, \tag{4}$$

$$\mathcal{L}_l = \sum_{i=1}^{n_f} \bar{q}_i \not{D} q_i + \frac{\delta\mathcal{L}_l^{(2)}}{m^2} + \frac{\delta\mathcal{L}_l^{(2)}}{m^2} + \frac{\delta\mathcal{L}_l^{(3)}}{m^3} + \frac{\delta\mathcal{L}_l^{(3)}}{m^3} + \mathcal{O} \left(\frac{1}{m^4} \right). \tag{5}$$

The complete set of operators at $\mathcal{O}(1/m^2)$ can be found in Ref. [12]. They read

$$\begin{aligned}
\delta\mathcal{L}_{\psi l}^{(2)} = & \frac{c_1^{hl}}{8} g^2 \sum_{i=1}^{n_f} \psi^\dagger T^a \psi \bar{q}_i \gamma_0 T^a q_i \\
& + \frac{c_2^{hl}}{8} g^2 \sum_{i=1}^{n_f} \psi^\dagger \gamma^\mu \gamma_5 T^a \psi_1 \bar{q}_i \gamma_\mu \gamma_5 T^a q_i \\
& + \frac{c_3^{hl}}{8} g^2 \sum_{i=1}^{n_f} \psi^\dagger \psi \bar{q}_i \gamma_0 q_i \\
& + \frac{c_4^{hl}}{8} g^2 \sum_{i=1}^{n_f} \psi^\dagger \gamma^\mu \gamma_5 \psi \bar{q}_i \gamma_\mu \gamma_5 q_i, \tag{6}
\end{aligned}$$

¹In dimensional regularization, several prescriptions are possible for the ϵ_{ijk} tensors and $\boldsymbol{\sigma}$, and the same prescription as for the calculation of the Wilson coefficients must be used.

$$\begin{aligned}
\delta\mathcal{L}_l^{(2)} = & \frac{c_D^l}{4} \bar{q}_i \gamma_\nu D_\mu G^{\mu\nu} q_i + \frac{c_1^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i T^a \gamma^\mu q_i \bar{q}_j T^a \gamma_\mu q_j \\
& + \frac{c_2^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i T^a \gamma^\mu \gamma_5 q_i \bar{q}_j T^a \gamma_\mu \gamma_5 q_j \\
& + \frac{c_3^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i \gamma^\mu q_i \bar{q}_j \gamma_\mu q_j \\
& + \frac{c_4^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i \gamma^\mu \gamma_5 q_i \bar{q}_j \gamma_\mu \gamma_5 q_j. \tag{7}
\end{aligned}$$

As we will discuss later, c_D^l and the light-light operators would contribute at next-to-next-to-leading log (NLL). Therefore, we will not consider them any further. We only have to discuss the $1/m^3$ heavy-light operators (i.e. those of dimension 7). These have been studied in Ref. [9]. We do not try to make an exhaustive analysis of all of them. We will only consider in detail those that have LL running and affect the running of the chromopolarizabilities.

The operators relevant for this calculation can be found in Eq. (10) of Ref. [9]. After disregarding some of them because of its spin dependence or because they are proportional to the energy of the heavy quark (so they become subleading after using the equations of motion), we find that, in QCD, the only relevant operators are

$$\mathcal{M}_{4\pm}^{(3h)s/o} = \pm g^2 [\bar{q} \gamma^\mu C_{s/o}^a q] [\bar{h}_v C_{s/o}^a iD_\mu^\pm h_v], \tag{8}$$

$$\mathcal{M}_{6\pm}^{(3h)s/o} = \pm g^2 [\bar{q} i\sigma^{\mu\lambda} v_\lambda C_{s/o}^a q] [\bar{h}_v C_{s/o}^a iD_\mu^\pm h_v], \tag{9}$$

$$\mathcal{M}_{2\pm}^{(3l)s/o} = \pm g^2 [\bar{q} C_{s/o}^a (ivD^\pm) q] [\bar{h}_v C_{s/o}^a h_v], \tag{10}$$

$$\mathcal{M}_{3\pm}^{(3l)s/o} = \pm g^2 [\bar{q} \not{v} C_{s/o}^a (ivD^\pm) q] [\bar{h}_v C_{s/o}^a h_v], \tag{11}$$

$$\mathcal{M}_{4\pm}^{(3l)s/o} = \pm g^2 [\bar{q} i\sigma^{\mu\nu} v_\lambda C_{s/o}^a iD_\nu^\pm q] [\bar{h}_v C_{s/o}^a h_v], \tag{12}$$

where $iD_\mu^+ = i\vec{\partial}_\mu - gA_\mu^a T^a$ and $iD_\mu^- = i\vec{\partial}_\mu + gA_\mu^a T^a$. The arrows over the derivatives indicate that the covariant derivatives act over fields in the left/right hand depending

on the direction of the arrow (they only act over heavy quark fields or over light quark fields), $C_s^a = 1$ and $C_o^a = T^a$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. In our case, we work in the rest frame, so that $v^\mu = (1, \mathbf{0})$ and $h_v \equiv \psi$. It is also understood that in the octet case the covariant derivative stands left/right of the color matrix when acting to the left/right. Moreover, we are in the heavy quark sector, and not in the antiquark one, so we can project to this sector. In practice, this is equivalent to taking the γ^0 which appears from $\bar{h}_v \equiv \bar{\psi}$ equal to 1, $\gamma^0 \rightarrow 1$. After all these simplifications, the previous operators can be written as

$$\mathcal{M}_{4\pm}^{(3h)s/o} = \pm g^2 [\bar{q} \gamma^\mu C_{s/o}^a q] [\psi^\dagger C_{s/o}^a i D_\mu^\pm \psi], \quad (13)$$

$$\mathcal{M}_{6\pm}^{(3h)s/o} = \pm i g^2 [\bar{q} \sigma^{i0} C_{s/o}^a q] [\psi^\dagger C_{s/o}^a i D_i^\pm \psi], \quad (14)$$

$$\mathcal{M}_{2\pm}^{(3l)s/o} = \pm g^2 [\bar{q} C_{s/o}^a (i D_0^\pm) q] [\psi^\dagger C_{s/o}^a \psi], \quad (15)$$

$$\mathcal{M}_{3\pm}^{(3l)s/o} = \pm g^2 [\bar{q} \gamma^0 C_{s/o}^a (i D_0^\pm) q] [\psi^\dagger C_{s/o}^a \psi], \quad (16)$$

$$\mathcal{M}_{4\pm}^{(3l)s/o} = \pm i g^2 [\bar{q} \sigma^{0i} C_{s/o}^a i D_i^\pm q] [\psi^\dagger C_{s/o}^a \psi]. \quad (17)$$

We then have

$$\delta \mathcal{L}_\psi^{(3)} = \sum_i d_i^{hl} O_i, \quad (18)$$

where the O_i are linear combinations of all possible $\mathcal{M}$ to be defined later. We are not interested in all of them, but only those that may get LL running and contribute to the chromopolarizabilities. We discuss this issue in Sec. IV B.

III. COMPTON SCATTERING

In order to prepare the computation of the anomalous dimensions of the heavy quark chromopolarizabilities, it is useful to consider the Compton scattering of a heavy quark with a gluon. In this section, we study the spin-independent part of the Compton effect in QCD, which is the scattering of a gluon with a heavy quark, $Qg \rightarrow Qg$. We will compute it at tree level up to $\mathcal{O}(1/m^3)$ in the mass expansion and in the Coulomb gauge (though obviously the Compton scattering is a gauge-independent object). We will consider the incoming and outgoing quarks with four-momentum $p = (E_1, \mathbf{p})$ and $p' = (E'_1, \mathbf{p}')$. Gluon four-momenta will be considered as being outgoing and labeled as k_1, i, a and k_2, j, b with respect to color and vector indices. This also implies the on-shell condition $k_1^0 = -|\mathbf{k}_1|$ and $k_2^0 = |\mathbf{k}_2|$. We work in the incoming quark rest frame, i.e. $E_1 = 0$ and $\mathbf{p} = 0$, so $\mathbf{p}' = -(\mathbf{k}_1 + \mathbf{k}_2)$ and $E'_1 = -(k_1^0 + k_2^0)$. In addition, let us define the unit vectors $\mathbf{n}_1 = \mathbf{k}_1/|\mathbf{k}_1|$ and $\mathbf{n}_2 = \mathbf{k}_2/|\mathbf{k}_2|$. The relation

$$\begin{aligned} |\mathbf{k}_2| &= \frac{|\mathbf{k}_1|}{1 + \frac{|\mathbf{k}_1|}{m}(1 + \mathbf{n}_1 \cdot \mathbf{n}_2)} \\ &= |\mathbf{k}_1| \left(1 - \frac{|\mathbf{k}_1|}{m}(1 + \mathbf{n}_1 \cdot \mathbf{n}_2) \right) \end{aligned} \quad (19)$$

also holds from four-momenta conservation.

With appropriate Wilson coefficients [to $\mathcal{O}(1/m^3)$], the topologies of the diagrams we have to consider for such computation can be found in Fig. 1. Overall, we obtain

$$\begin{aligned} \mathcal{A}^{ijab} &= -ig^2 \delta^{ij} \frac{1}{|\mathbf{k}_1|} \frac{1}{1 + \mathbf{n}_1 \cdot \mathbf{n}_2} [T^a, T^b]_{\alpha\beta} - c_k \frac{ig^2}{2m} \delta^{ij} \{T^a, T^b\}_{\alpha\beta} - \frac{ig^2}{2m} \delta^{ij} [T^a, T^b]_{\alpha\beta} \\ &+ \frac{ig^2}{4m^2} |\mathbf{k}_1| (\delta^{ij} (2 - c_F^2 \mathbf{n}_1 \cdot \mathbf{n}_2) + c_F^2 \mathbf{n}_2^i \mathbf{n}_1^j) [T^a, T^b]_{\alpha\beta} + c_1^g \frac{3ig^2}{2m^2} [T^a, T^b]_{\alpha\beta} |\mathbf{k}_1| \left(\delta^{ij} - \frac{\mathbf{n}_2^i \mathbf{n}_1^j}{1 + \mathbf{n}_1 \cdot \mathbf{n}_2} \right) \\ &+ \frac{ig^2}{16m^3} |\mathbf{k}_1|^2 (\delta^{ij} ((4c_4 + 4c_M - 2c_{A_1} - c_{A_2} + 2c_S c_F) + (4c_4 - 2c_{A_1}) \mathbf{n}_1 \cdot \mathbf{n}_2) + 2c_{A_1} \mathbf{n}_2^i \mathbf{n}_1^j) \{T^a, T^b\}_{\alpha\beta} \\ &- c_F^2 \frac{ig^2}{8m^3} |\mathbf{k}_1|^2 (\delta^{ij} \mathbf{n}_1 \cdot \mathbf{n}_2 - \mathbf{n}_2^i \mathbf{n}_1^j) (\mathbf{n}_1 \cdot \mathbf{n}_2) \{T^a, T^b\}_{\alpha\beta} - \frac{ig^2}{8m^3} |\mathbf{k}_1|^2 (\delta^{ij} (2 - c_F^2 \mathbf{n}_1 \cdot \mathbf{n}_2) \\ &+ c_F^2 \mathbf{n}_2^i \mathbf{n}_1^j) (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) [T^a, T^b]_{\alpha\beta} - c_1^g \frac{3ig^2}{4m^3} [T^a, T^b]_{\alpha\beta} |\mathbf{k}_1|^2 (\delta^{ij} (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) - \mathbf{n}_2^i \mathbf{n}_1^j). \end{aligned} \quad (20)$$

We first observe that c_D does not appear explicitly in the computation. It only appears implicitly through c_M (as c_M is related with c_D by reparametrization invariance). From the above result, we also observe that c_{A_2} and c_M always appear in the same combination, $\bar{c}_{A_2} \equiv c_{A_2} - 4c_M$, in observables. Since c_M is gauge dependent, only $\bar{c}_{A_2}$ can

be considered physical. Actually, for the observables we have studied, we see that c_{A_2} always appear in the combination $\bar{c}_{A_2} \equiv 2c_{A_1} + \bar{c}_{A_2}$. This happens for the Compton scattering but also for the heavy quarkonium. A similar analysis for the elastic scattering of a light and heavy quark shows that $\bar{c}_1^{hl} \equiv c_D + c_1^{hl}$ is physical. Note,

FIG. 1. Topologies of the tree-level diagrams contributing to the Compton scattering to $\mathcal{O}(1/m^3)$. Diagrams are generated from these topologies by considering all possible vertex and kinetic insertions contributing to $\mathcal{O}(1/m^3)$.

however, that c_D and c_1^{hl} individually are not; indeed, they are gauge dependent (for instance, see Ref. [15] for a discussion on this issue).

For QED, we obtain

$$\begin{aligned} \mathcal{A}^{ijab} = & -c_k \frac{ig^2}{m} \delta^{ij} + \frac{ig^2}{8m^3} |\mathbf{k}_1|^2 \delta^{ij} ((2 + 4c_4 - 2c_{A_1} + 2c_{SC_F} + 4c_M - c_{A_2} - 2c_F^2) \\ & + (2 + 4c_4^{(1)} - 2c_{A_1} + 2c_F^2 c_k - 2c_F^2) \mathbf{n}_1 \cdot \mathbf{n}_2) + \frac{ig^2}{4m^3} |\mathbf{k}_1|^2 \delta^{ij} (c_F^2 (1 - \mathbf{n}_1 \cdot \mathbf{n}_2) - 1) (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) \\ & + \frac{ig^2}{4m^3} |\mathbf{k}_1|^2 ((c_F^2 + c_{A_1} - c_F^2 c_k) + c_F^2 \mathbf{n}_1 \cdot \mathbf{n}_2) \mathbf{n}_2^i \mathbf{n}_1^j. \end{aligned} \quad (21)$$

Note that there is no $\mathcal{O}(1/m^0, 1/m^2)$ contribution. Setting the Wilson coefficients to their tree-level values, we obtain

$$\begin{aligned} \mathcal{A}^{ijab} = & -\frac{ig^2}{m} \delta^{ij} + \frac{ig^2}{2m^3} |\mathbf{k}_1|^2 \delta^{ij} (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) \\ & - \frac{ig^2}{4m^3} |\mathbf{k}_1|^2 (\delta^{ij} \mathbf{n}_1 \cdot \mathbf{n}_2 - \mathbf{n}_2^i \mathbf{n}_1^j) (1 + \mathbf{n}_1 \cdot \mathbf{n}_2). \end{aligned} \quad (22)$$

This expression agrees with Eq. (19) in Ref. [16].

For completeness, we also define polarizabilities. The concept of polarizability is potentially ambiguous, as it is defined after subtracting what are called Born terms (which indeed can be defined in several ways) from the Compton scattering computation. Indeed, this discussion already appears in the context of QED and the elastic scattering of photons with protons (see, for instance, Refs. [17–19]). One possible definition is the one used in Ref. [20], which adapted to the notation of our paper reads

$$\frac{4m^3}{\alpha} \alpha_{E1} \equiv -c_{A_1} - \frac{\bar{c}_{A_2}}{2} + c_F^2 - c_F + 1, \quad (23)$$

$$\frac{4m^3}{\alpha} \beta_{M1} \equiv -1 + c_{A_1}. \quad (24)$$

Note that, even in QED, the polarizabilities are not low energy constants, as they depend on the renormalization scale.

The above analysis gives us the set of Wilson coefficients and its combinations that appear in physical observables. Those are the ones for which we will compute the anomalous dimensions: $\{c_{A_1}, \bar{c}_{A_2}, c_{A_3}, c_{A_4}, c_4\}$.

IV. COMPUTATION OF THE $1/m^3$ ANOMALOUS DIMENSION

We want to determine the anomalous dimension of the $1/m^3$ Wilson coefficient operators with $\mathcal{O}(\alpha)$ accuracy.

In principle, one would like to only compute irreducible diagrams. Nevertheless, in such a situation, we would need to consider a more extensive basis of operators, including those that vanish on shell. Therefore, we will then instead also consider reducible diagrams in a computation that resembles the one of an S-matrix element. In our case, we will compute the divergent part of the elastic scattering of the heavy quark with a transverse gluon at one loop. These divergences cancel with divergences of the Wilson coefficients determining the anomalous dimension. The computation is organized according to the powers of $1/m$ up to $\mathcal{O}(1/m^3)$ by considering all possible insertions of the HQET Lagrangian operators. This statement requires some qualifications once $1/m^3$ operators with light fermions fields are involved. For those, we do not seek the anomalous dimension of all their Wilson coefficients. To start with, we do not consider spin-dependent ones, but even for the spin-independent operators, we are not exhaustive in the search of a complete basis of operators. The reason is that most of them start to contribute at NLL, playing a subleading role in heavy quarkonium physics. We will only consider those that contribute at LL to the polarizabilities.

In order to check some parts of the computation, we will also compute the elastic scattering of the heavy quark with a longitudinal gluon. This allows us to check the combinations $2c_{A_1} + c_{A_2}$ and $2c_{A_3} + c_{A_4}$.

Please note that the matching coefficients of the kinetic term are protected by reparametrization invariance ($c_k = c_4 = 1$ to any order in perturbation theory) [21]. Nevertheless, we will often keep them explicit for tracking purposes. We will also compute the running of c_4 explicitly at one loop as a check.

In principle, c_M is also fixed by reparametrization invariance. It was originally determined in Ref. [8]. Recently, a new result, $2c_M = c_D - c_F$, was obtained [20], which differs by a sign of the old one. As we have already discussed, the computation of a physical observable, like the Compton scattering, does not allow us to determine the RG equation of c_M , as it always appears in the combination $\bar{c}_{A_2} = c_{A_2} - 4c_M$. Working off shell, our computation in QED in the next section indeed confirms (at one loop and in the Coulomb gauge) $2c_M = c_D - c_F$. Nevertheless, we cannot perform a similar check in QCD. The constraints imposed by reparametrization invariance once operators with light fermion fields are included have also been studied in Ref. [20] for the QED case where the relation $d_1^{hl} = c_1^{hl}/16$ [see Eq. (29)] was deduced. Nevertheless, such a Wilson coefficient does not enter into the running of the chromopolarizabilities.

We will perform the computation in the Coulomb gauge. On the one hand, this will significantly reduce the number of diagrams to compute. On the other hand, the complexity of each of them increases. It also makes it difficult to use standard routines for computations of diagrams designed for Feynman gauges and more relativisticlike setups. In the Coulomb gauge, the normalization of the quarks and gluon fields and g read (in this paper, we define $D = 4 + 2\epsilon$)

$$\begin{aligned} Z_{A^0}^{-1/2} &= Z_g = 1 + \frac{11}{6} C_A \frac{\alpha}{4\pi\epsilon} - \frac{2}{3} T_F n_f \frac{\alpha}{4\pi\epsilon}, \\ Z_A^{\frac{1}{2}} &= 1 - \frac{C_A}{2} \frac{\alpha}{4\pi\epsilon} - \frac{2}{3} T_F n_f \frac{\alpha}{4\pi\epsilon}, \\ Z_g^2 Z_A &= 1 + \frac{8}{3} C_A \frac{\alpha}{4\pi\epsilon}, \quad Z_l = 1 + C_F \frac{\alpha}{4\pi\epsilon}, \\ Z_h &= 1 + \frac{\mathbf{p}^2}{m^2} \frac{4}{3} C_F \frac{\alpha}{4\pi\epsilon}, \end{aligned} \quad (25)$$

where

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \quad C_A = N_c = 3. \quad (26)$$

A. QED case

As a warm-up, we first consider the pure QED case ($C_F = 1$, $C_A = 0$, and $n_f = 0$). The diagrams that contribute can be found in Fig. 2.

For the reducible diagrams, we have to keep in mind that the subirreducible part of the diagram can be Taylor expanded in powers of the energy. When computing, as

expected, we find that the nonlocal terms are finite and all divergences can be absorbed by local counterterms that correspond to operators of the Lagrangian.² The result is the following,

$$\begin{aligned} c_{M,B} &= c_M + c_k^3 \frac{2\alpha}{3\pi\epsilon}, \quad c_{A_1,B} = c_{A_1} - c_k^3 \frac{8\alpha}{3\pi\epsilon}, \\ c_{A_2,B} &= c_{A_2} + c_k^3 \frac{40\alpha}{3\pi\epsilon}, \end{aligned} \quad (27)$$

where the subscript B stands for the bare Wilson coefficient, whereas the renormalized Wilson coefficients do not have an associated subscript.

c_4 does not renormalize. This we explicitly check. Overall, we obtain the following RG equations:

$$\begin{aligned} \nu \frac{d}{d\nu} c_M &= -c_k^3 \frac{4\alpha}{3\pi}, \quad \nu \frac{d}{d\nu} c_{A_1} = +c_k^3 \frac{16\alpha}{3\pi}, \\ \nu \frac{d}{d\nu} c_{A_2} &= -c_k^3 \frac{80\alpha}{3\pi}. \end{aligned} \quad (28)$$

Note that in order to determine the running of c_M and c_{A_2} separately we had to consider the gluons to be off shell; otherwise, we could not have distinguished the Feynman rule of c_M and c_{A_2} but only of the physical quantity $\bar{c}_{A_2}$. In QED, this seems to be fine. For instance, the running of c_D in QED happens to be equal in the Coulomb and in the Feynman gauge (see the discussion in Ref. [15]). For QCD, working off shell produces divergences that violate reparametrization invariance for c_M . Therefore, in the following, we will restrict ourselves to the analysis of physical quantities, i.e. $\bar{c}_{A_2}$.

B. $1/m^3$ light fermions: LL running of d_i^{hl}

Before going to the complete QCD case, we need to study with some detail the inclusion of light fermions to the computation. The coefficients c^{ll} and c_D^l are NLL, and we will neglect them in the following. The c_i^{hl} were computed with LL accuracy in Ref. [12]. The d_i^{hl} are zero at leading-order at the hard scale (as there were the c_i^{hl}). There are no tree-level contributions to these operators. The only way they can get nonzero LL running is through mixing with nonvanishing operators. The way we determine such mixing is by computing the divergent contributions to the elastic scattering of a heavy with a light quark. The divergences get then absorbed in the d_i^{hl} . As we already anticipated in previous sections, not all Wilson coefficients get nonzero anomalous dimensions, but only a combination of some of them. We see in the calculation of the LL running of these operators that only three different

²If we only compute irreducible diagrams, we would need a larger number of operators, in particular those that vanish on shell.

FIG. 2. The first diagram is the tree-level diagram multiplied by the prefactor $Z_g^2 Z_A Z_h^{1/2}(\mathbf{p}) Z_h^{1/2}(\mathbf{p}')$. The rest are the one loop diagrams contributing to the anomalous dimensions of the $1/m^3$ Wilson coefficients in QED. Circles with crosses represent c_F vertices, and the other vertices are proportional to c_k . Crosses in the propagators represent kinetic insertions. Double-line propagators represent the heavy fermion. The wavy line stands for transverse photons only in this figure. Note that in some cases one has to Taylor expand in the energy.

structures appear. The other possible structures are irrelevant for our calculation because its associated Wilson coefficient does not run and the matching condition i.e. the Wilson coefficient evaluated at the hard scale is zero, at least they are $\mathcal{O}(\alpha)$. These two properties together make the contribution of these operators subleading. The three operators needed are

$$\mathcal{O}_1 = \mathcal{M}_{4+}^{(3h)o} + \mathcal{M}_{4-}^{(3h)o}, \quad (29)$$

$$\mathcal{O}_2 = \mathcal{M}_{3+}^{(3l)o} + \mathcal{M}_{3-}^{(3l)o}, \quad (30)$$

$$\mathcal{O}_3 = \mathcal{M}_{3+}^{(3l)s} + \mathcal{M}_{3-}^{(3l)s}. \quad (31)$$

The running of these operators is determined from the diagrams (topologies) drawn in Fig. 3. They produce around 57 diagrams to be computed (without counting crossed ones).

We will see later that d_1^{hl} does not contribute to the running of the chromopolarizabilities. Therefore, we will not consider it. For the other two Wilson coefficients, we obtain

$$\begin{aligned} d_{2,B}^{hl} = & d_2^{hl} - \frac{1}{\epsilon} \frac{\alpha}{\pi} \left(\frac{2}{3} C_F - \frac{17}{24} C_A + \frac{\beta_0}{4} \right) d_2^{hl} + 0 d_{1,3}^{hl} + \frac{1}{\epsilon} \frac{\alpha}{\pi} (8C_F - 3C_A) \left[\left[0c_{A_1} - \frac{1}{192} \bar{c}_{A_2} + 0c_{A_{3,4}} + \frac{1}{32} c_4 + 0c_M \right] \right. \\ & \left. + \left[\frac{1}{96} c_S c_F + 0c_D c_k \right] + \left[-\frac{5}{96} c_k^3 - \frac{5}{24} c_k c_F^2 + \frac{1}{32} \bar{c}_1^{hl} c_k \right] \right] + \frac{1}{\epsilon} \frac{\alpha}{\pi} \left[+\frac{1}{8} c_3^{hl} c_k + C_A \frac{1}{32} c_2^{hl} c_F + 0c_4^{hl} c_F \right], \end{aligned} \quad (32)$$

FIG. 3. In the first diagram, the counterterm of the external propagators and g^2 are included. The other diagrams are one loop topologies contributing to the anomalous dimensions of the $1/m^3$ Wilson coefficients in QCD of the heavy-light operators. All possible vertices and insertions with the right counting in $1/m$ should be considered to generate the diagrams. In general, the gluon propagator may be longitudinal or transverse.

$$d_{3,B}^{hl} = d_3^{hl} - \frac{1}{\epsilon} \frac{\alpha}{\pi} \left(\frac{2}{3} C_F + \frac{\beta_0}{4} \right) d_3^{hl} + 0 d_{1,2}^{hl} + \frac{1}{\epsilon} \frac{\alpha}{\pi} C_F (C_A - 2C_F) \left[+ \left[0c_{A_1} - \frac{1}{96} \bar{c}_{A_2} + 0c_{A_3} - \frac{1}{96} c_{A_4} + \frac{1}{16} c_4 + 0c_M \right] \right. \\ \left. + \left[\frac{1}{48} c_S c_F + 0c_D c_k \right] + \left[-\frac{5}{48} c_k^3 - \frac{5}{12} c_k c_F^2 \right] + \frac{1}{16} \bar{c}_1^{hl} c_k + 0c_3^{hl} c_k + 0c_{2,4}^{hl} c_F \right], \quad (33)$$

which produce the following RG equations:

$$\nu \frac{d}{d\nu} d_2^{hl} = -2 \frac{\alpha}{\pi} \left[-\left(\frac{2}{3} C_F - \frac{17}{24} C_A + \frac{\beta_0}{4} \right) d_2^{hl} + (8C_F - 3C_A) \left[-\frac{1}{192} \bar{c}_{A_2} + \frac{1}{32} c_4 + \frac{1}{96} c_S c_F - \frac{5}{96} c_k^3 - \frac{5}{24} c_k c_F^2 + \frac{1}{32} \bar{c}_1^{hl} c_k \right] \right. \\ \left. + \frac{1}{8} c_3^{hl} c_k + C_A \frac{1}{32} c_2^{hl} c_F \right], \quad (34)$$

$$\nu \frac{d}{d\nu} d_3^{hl} = -2 \frac{\alpha}{\pi} \left[-\left(\frac{2}{3} C_F + \frac{\beta_0}{4} \right) d_3^{hl} + C_F (C_A - 2C_F) \left[-\frac{1}{96} \bar{c}_{A_2} - \frac{1}{96} c_{A_4} + \frac{1}{16} c_4 + \frac{1}{48} c_S c_F - \frac{5}{48} c_k^3 - \frac{5}{12} c_k c_F^2 + \frac{1}{16} \bar{c}_1^{hl} c_k \right] \right]. \quad (35)$$

Quite remarkably, the RG equations depend only on gauge-independent combinations of Wilson coefficients: $\bar{c}_{A_2}$ and $\bar{c}_1^{hl}$. This is quite a strong check, as at intermediate steps we get contributions from c_D , c_M , c_{A_2} , c_1^{hl} , and d_1^{hl} , which only at the end of the computation combine in gauge-independent sets.

The other d coefficients have the structure ($i, j > 3$)

$$\nu \frac{d}{d\nu} d_i^{hl} = \frac{\alpha}{\pi} A_{i,j} d_j^{hl}. \quad (36)$$

Therefore, they are NLL.

C. QCD case

For the pure gluonic sector, we have that c_1^g is NLL. Therefore, we will neglect it in the following.

The running of $\{c_{A_i}, c_4\}$ is determined from the topologies drawn in Fig. 4. Out of these topologies, we generate all possible diagrams of order $1/m^3$ by considering all possible vertices to the appropriate order in $1/m$ and/or kinetic insertions. This generates around 200 diagrams (without taking into account permutations and crossing).

These topologies refer to the elastic scattering of the heavy quark with a transverse gluon. The topologies of the elastic scattering of the heavy quark with a longitudinal gluon (which we also compute) are the same (though, obviously, the external wave function counterterms change) and also generate around 200 diagrams (without taking into account permutations and crossing). Note that each of the internal gluon propagators may refer to a transverse or longitudinal gluon.

FIG. 4. The tree-level diagrams drawn should be understood to be multiplied by counterterms: either from low order in $1/m$, from external wave function counterterms, or from g . The remaining diagrams are the one loop topologies contributing to the anomalous dimensions of the $1/m^3$ Wilson coefficients in QCD of the heavy gluon operators. Final diagrams are generated from those topologies by considering all possible vertices and kinetic insertions to $\mathcal{O}(1/m^3)$.

FIG. 5. Running of the $1/m^3$ spin-independent Wilson coefficients. The continuous line is the LL result with $n_f = 4$, the dotted line is the LL result with $n_f = 0$, and the dashed line is the single leading-log result (it does not depend on n_f).

Since we work in the massless limit, the expression for d_i^{hl} does not depend on the specific light fermion, so the result of each diagram has to be multiplied by the number of light fermions.

For diagrams proportional to $1/m^3$ operators, only irreducible diagrams need to be considered. When one considers diagrams proportional to iterations of $1/m^2$ and/or $1/m$ operators, one also has to consider reducible diagrams, following the same line of argumentation as for the discussion of the QED case.

Overall, we obtain

$$c_{4,B} = c_4 + 0c_{A_1} + 0c_4 + 0c_M + 0c_D c_k + 0c_S c_F + 0c_k^3 + 0c_F^3 + 0c_k c_F^2 + 0c_{1,3}^{hl} c_k + 0c_{2,4}^{hl} c_F + 0d_{1,2,3}^{hl}, \quad (37)$$

$$\begin{aligned} c_{A_1,B} = & c_{A_1} - \frac{2}{3} c_{A_1} C_A \frac{\alpha}{\pi \epsilon} + \frac{1}{2} c_4 C_A \frac{\alpha}{\pi \epsilon} + \left[\frac{5}{12} c_{A_1} + \frac{11}{96} \bar{c}_{A_2} \right] C_A \frac{\alpha}{\pi \epsilon} + 0c_{A_{3,4}} + 0c_M + 0c_D c_k + \frac{1}{48} c_S c_F C_A \frac{\alpha}{\pi \epsilon} \\ & - \frac{1}{6} c_k^3 (16C_F + 11C_A) \frac{\alpha}{\pi \epsilon} - \frac{3}{4} c_F^3 C_A \frac{\alpha}{\pi \epsilon} - \frac{73}{48} c_F^2 c_k C_A \frac{\alpha}{\pi \epsilon} + 0c_{1,3}^{hl} c_k + 0c_{2,4}^{hl} c_F + 0d_1^{hl} + \frac{8}{3} T_F n_f \frac{\alpha}{\pi \epsilon} d_2^{hl} + 0d_3^{hl}, \end{aligned} \quad (38)$$

$$\begin{aligned} \bar{c}_{A_2,B} \equiv c_{A_2,B} - 4c_{M,B} = & \bar{c}_{A_2} - \frac{11}{24} \bar{c}_{A_2} C_A \frac{\alpha}{\pi \epsilon} - 4c_4 C_A \frac{\alpha}{\pi \epsilon} + \frac{1}{2} c_{A_1} C_A \frac{\alpha}{\pi \epsilon} + 0c_{A_{3,4}} + 0c_M + 0c_D c_k C_A \frac{\alpha}{\pi \epsilon} - \frac{1}{12} c_S c_F C_A \frac{\alpha}{\pi \epsilon} \\ & + \frac{4}{3} c_k^3 (8C_F + 7C_A) \frac{\alpha}{\pi \epsilon} + \frac{3}{2} c_F^3 C_A \frac{\alpha}{\pi \epsilon} + \frac{43}{12} c_F^2 c_k C_A \frac{\alpha}{\pi \epsilon} + 0c_{1,3}^{hl} c_k + 0c_{2,4}^{hl} c_F + 0d_1^{hl} - \frac{32}{3} T_F n_f \frac{\alpha}{\pi \epsilon} d_2^{hl} + 0d_3^{hl}, \end{aligned} \quad (39)$$

$$\begin{aligned} c_{A_3,B} = & c_{A_3} - \frac{2}{3} c_{A_3} C_A \frac{\alpha}{\pi \epsilon} + \left[\frac{1}{4} c_{A_1} + \frac{11}{96} \bar{c}_{A_2} + \frac{2}{3} c_{A_3} + \frac{11}{48} c_{A_4} \right] C_A \frac{\alpha}{\pi \epsilon} + 0c_M + 0c_D c_k + \frac{1}{48} c_S c_F C_A \frac{\alpha}{\pi \epsilon} \\ & + 0c_k^3 + \frac{3}{4} c_F^3 C_A \frac{\alpha}{\pi \epsilon} - \frac{73}{48} c_F^2 c_k C_A \frac{\alpha}{\pi \epsilon} + 0c_{1,3}^{hl} c_k + 0c_{2,4}^{hl} c_F + 0d_1^{hl} - \frac{8}{3} T_F n_f \frac{\alpha}{\pi \epsilon} d_2^{hl} + \frac{16}{3} T_F n_f \frac{\alpha}{\pi \epsilon} C_A d_3^{hl}, \end{aligned} \quad (40)$$

$$\begin{aligned} c_{A_4,B} = & c_{A_4} - \frac{2}{3} c_{A_4} C_A \frac{\alpha}{\pi \epsilon} - \frac{4}{3} c_4 C_A \frac{\alpha}{\pi \epsilon} - \left[\frac{1}{2} c_{A_1} + \frac{11}{24} \bar{c}_{A_2} + 0c_{A_3} + \frac{1}{4} c_{A_4} \right] C_A \frac{\alpha}{\pi \epsilon} + 0c_M + 0c_D c_k C_A \frac{\alpha}{\pi \epsilon} + \frac{17}{12} c_S c_F C_A \frac{\alpha}{\pi \epsilon} \\ & + \frac{4}{3} c_k^3 C_A \frac{\alpha}{\pi \epsilon} - \frac{3}{2} c_F^3 C_A \frac{\alpha}{\pi \epsilon} + \frac{31}{12} c_F^2 c_k C_A \frac{\alpha}{\pi \epsilon} + 0c_{1,3}^{hl} c_k + 0c_{2,4}^{hl} c_F + 0d_1^{hl} + \frac{32}{3} T_F n_f \frac{\alpha}{\pi \epsilon} d_2^{hl} - \frac{64}{3} T_F n_f \frac{\alpha}{\pi \epsilon} C_A d_3^{hl}. \end{aligned} \quad (41)$$

This produces the following RG equations:

$$\nu \frac{d}{d\nu} c_4 = 0, \quad (42)$$

$$\begin{aligned} \nu \frac{d}{d\nu} c_{A_1} = & -2 \frac{\alpha}{\pi} \left[+ \frac{1}{2} c_4 C_A + \left[-\frac{1}{4} c_{A_1} + \frac{11}{96} \bar{c}_{A_2} \right] C_A - \frac{1}{6} c_k^3 (16C_F + 11C_A) - \frac{3}{4} c_F^3 C_A - \frac{73}{48} c_F^2 c_k C_A \right. \\ & \left. + \frac{1}{48} c_S c_F C_A + \frac{8}{3} T_F n_f d_2^{hl} \right], \end{aligned} \quad (43)$$

$$\nu \frac{d}{d\nu} \bar{c}_{A_2} = -2 \frac{\alpha}{\pi} \left[-\frac{11}{24} \bar{c}_{A_2} C_A - 4c_4 C_A + \frac{1}{2} c_{A_1} C_A - \frac{1}{12} c_S c_F C_A + \frac{4}{3} c_k^3 (8C_F + 7C_A) + \frac{3}{2} c_F^3 C_A + \frac{43}{12} c_F^2 c_k C_A - \frac{32}{3} T_F n_f d_2^{hl} \right], \quad (44)$$

$$\nu \frac{d}{d\nu} c_{A_3} = -2 \frac{\alpha}{\pi} \left[+ \left[\frac{1}{4} c_{A_1} + \frac{11}{96} \bar{c}_{A_2} + \frac{11}{48} c_{A_4} \right] C_A + \frac{1}{48} c_S c_F C_A + \frac{3}{4} c_F^3 C_A - \frac{73}{48} c_F^2 c_k C_A - \frac{8}{3} T_F n_f d_2^{hl} + \frac{16}{3} T_F n_f C_A d_3^{hl} \right], \quad (45)$$

$$\nu \frac{d}{d\nu} c_{A_4} = -2 \frac{\alpha}{\pi} \left[-\frac{4}{3} c_4 C_A - \left[\frac{1}{2} c_{A_1} + \frac{11}{24} \bar{c}_{A_2} + \frac{11}{12} c_{A_4} \right] C_A + \frac{17}{12} c_S c_F C_A + \frac{4}{3} c_k^3 C_A - \frac{3}{2} c_F^3 C_A + \frac{31}{12} c_F^2 c_k C_A \right. \\ \left. + \frac{32}{3} T_F n_f d_2^{hl} - \frac{64}{3} T_F n_f C_A d_3^{hl} \right]. \quad (46)$$

It is worth mentioning that the running of c_{A_1} and $\bar{c}_{A_2}$ can be obtained without determining the running of c_{A_3} and c_{A_4} .

Again, it is quite remarkable that the RG equations depend only on gauge-independent combinations of Wilson coefficients: $\bar{c}_{A_2}$ and $\bar{c}_1^{hl}$. This is quite a strong check, as at intermediate steps we get contributions from c_D , c_M , c_{A_2} , c_1^{hl} , and d_1^{hl} , which only at the end of the computation combine in gauge-independent sets.

V. SOLUTION AND NUMERICAL ANALYSIS

The RG equations obtained in the previous section depend on a list of seven Wilson coefficients $\mathbf{A} = \{c_{A_1}, \bar{c}_{A_2}, c_{A_3}, c_{A_4}, c_4, d_2^{hl}, d_3^{hl}\}$. The last two are only computed since they are nonzero at LL and contribute to the different c_{A_i} . The running of c_4 is zero, as predicted by reparametrization invariance. The remaining equations can be written, more compactly, in a matricial form:

$$\nu \frac{d}{d\nu} \mathbf{A} = \frac{\alpha}{\pi} \mathbf{M} \mathbf{A} + \mathbf{f}(\alpha). \quad (47)$$

The matrix $\mathbf{M}$ follows from the results of the previous section. We only need the LL running of α ,

$$\nu \frac{d}{d\nu} \alpha \equiv \beta(\alpha_s) = -2\alpha \left\{ \beta_0 \frac{\alpha}{4\pi} + \dots \right\}, \quad (48)$$

where

$$c_{A_1} = +\frac{75}{17} z^{-3C_A} - \frac{29}{3} z^{-2C_A} - \frac{z^{-C_A}}{11} + \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{-\frac{1}{12}(17+\sqrt{157})C_A} \\ + \left(-\frac{64C_F}{\sqrt{157}C_A} - \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{\frac{1}{12}(\sqrt{157}-17)C_A}, \quad (51)$$

$$\bar{c}_{A_2} = -\frac{216}{17} z^{-3C_A} + 34 z^{-2C_A} + \frac{2z^{-C_A}}{11} + \frac{128}{11} + \frac{256C_F}{11C_A} - \left(\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} + \frac{3096}{187} \right) z^{-\frac{1}{12}(17+\sqrt{157})C_A} \\ + \left(\frac{640C_F}{11\sqrt{157}C_A} - \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} - \frac{3096}{187} \right) z^{\frac{1}{12}(\sqrt{157}-17)C_A}, \quad (52)$$

$$c_{A_3} = \left(\frac{32C_F}{11C_A} + \frac{344}{55} \right) z^{-\frac{11C_A}{3}} - \frac{75}{17} z^{-3C_A} + \frac{88}{15} z^{-2C_A} + \frac{23z^{-C_A}}{11} - \frac{38}{11} - \frac{32C_F}{11C_A} \\ - \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{-\frac{1}{12}(17+\sqrt{157})C_A} + \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} - \frac{1780}{561} \right) z^{\frac{1}{12}(\sqrt{157}-17)C_A}, \quad (53)$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \quad (49)$$

and n_f is the number of dynamical (active) quarks.

Then, the above equation can be simplified to

$$\frac{d}{d\alpha} \mathbf{A} = -\frac{2}{\beta_0 \alpha} \mathbf{M} \mathbf{A} - \frac{2\pi}{\beta_0 \alpha^2} \mathbf{f}(\alpha). \quad (50)$$

We also need the initial matching conditions at the hard scale. The tree-level Wilson coefficients at the hard scale have been determined in Ref. [8]. They read $c_k = c_4 = c_F = c_D = c_S = c_{A_1} = 1$ and $c_M = c_{A_2} = c_{A_3} = c_{A_4} = 0$.

Please note that the matching coefficients of the kinetic term are protected by reparametrization invariance ($c_k = c_4 = 1$ to any order in perturbation theory) [21]. Nevertheless, even if we set them to 1 when solving the RG equations, we have kept them explicit in the RG equations for tracking purposes.

The initial matching conditions of the d_i^{hl} and c_i^{hl} are $\mathcal{O}(\alpha)$ unless the operators can be generated at tree level. This is not the case with the basis we consider, but it is if one eliminates the Darwin operator c_D (i.e. for $\bar{c}_1^{hl}$).

After solving the RG equations, we obtain the running of the different Wilson coefficients. For the case without light fermions, we can obtain analytic results. They read

$$(z = \frac{[\alpha(\nu)]^{\frac{1}{\beta_0}}}{[\alpha(m)]^{\frac{1}{\beta_0}}}) \simeq 1 - \frac{1}{2\pi} \alpha(\nu) \ln(\frac{\nu}{m})$$

$$\begin{aligned}
c_{A_4} = & -\left(\frac{128C_F}{11C_A} + \frac{1376}{55}\right)z^{-\frac{11C_A}{3}} + \frac{216}{17}z^{-3C_A} - \frac{64}{5}z^{-2C_A} - \frac{24z^{-C_A}}{11} - \frac{64}{11} - \frac{128C_F}{11C_A} \\
& + \left(\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} + \frac{3096}{187}\right)z^{-\frac{1}{12}(17+\sqrt{157})C_A} \\
& + \left(-\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} - \frac{29720}{187\sqrt{157}} + \frac{3096}{187}\right)z^{\frac{1}{12}(\sqrt{157}-17)C_A}.
\end{aligned} \tag{54}$$

After the inclusion of light fermions, the solution of the RG equations is numerical. We show the result for $n_f = 4$ light fermions where $\alpha(m)$ has n_f active light flavors:

$$\begin{aligned}
c_{A_1} = & 1.08838 \times 10^{-9} - \frac{4.34453}{z^{1.02367}} - \frac{1.14702 \times 10^1}{z^{2.87467}} + \frac{8.05227}{z^3} + \frac{5.54957 \times 10^{-1}}{z^6} + \frac{2.56988}{z^{6.5}} \\
& - \frac{2.07923}{z^{6.83333}} + \frac{1.77472}{z^{7.99055}} - \frac{4.61377 \times 10^{-19}}{z^{8.33333}} + \frac{5.9421}{z^9},
\end{aligned} \tag{55}$$

$$\begin{aligned}
\bar{c}_{A_2} = & 2.39049 \times 10^1 - \frac{8.08634}{z^{1.02367}} + \frac{2.19402 \times 10^1}{z^{2.87467}} - \frac{1.61045 \times 10^1}{z^3} + \frac{3.33513}{z^6} - \frac{7.90732}{z^{6.5}} \\
& + \frac{6.49126}{z^{6.83333}} - \frac{5.76625}{z^{7.99055}} + \frac{5.03321 \times 10^{-19}}{z^{8.33333}} - \frac{1.7807 \times 10^1}{z^9} - \frac{7.8439 \times 10^{-21}}{z^{1.45556 \times 10^1}},
\end{aligned} \tag{56}$$

$$\begin{aligned}
c_{A_3} = & \frac{4.34453}{z^{1.02367}} + \frac{1.14702 \times 10^1}{z^{2.87467}} - \frac{7.55227}{z^3} + \frac{1.50000}{z^3} - \frac{1.19344 \times 10^1}{z^6} + \frac{1.50000}{z^6} \\
& + \frac{5.52225 \times 10^{-21}}{z^{13/2}} - \frac{5.10498}{z^{6.5}} + \frac{6.21007 \times 10^{-1}}{z^{6.83333}} + \frac{1.53256 \times 10^{-21}}{z^{41/6}} + \frac{1.45822}{z^{6.83333}} \\
& - \frac{1.77472}{z^{7.99055}} + \frac{1.59173 \times 10^1}{z^{8.33333}} - \frac{5.9421}{z^9} + \frac{2.28973 \times 10^{-1}}{z^{1.45556 \times 10^1}} - 4.73174,
\end{aligned} \tag{57}$$

$$\begin{aligned}
c_{A_4} = & -1.29779 \times 10^1 + \frac{8.08634}{z^{1.02367}} - \frac{2.19402 \times 10^1}{z^{2.87467}} + \frac{1.41045 \times 10^1}{z^3} + \frac{4.21825 \times 10^1}{z^6} \\
& + \frac{1.80477 \times 10^1}{z^{6.5}} - \frac{6.49126}{z^{6.83333}} + \frac{5.76625}{z^{7.99055}} - \frac{6.36691 \times 10^1}{z^{8.33333}} + \frac{1.7807 \times 10^1}{z^9} - \frac{9.15892 \times 10^{-1}}{z^{1.45556 \times 10^1}},
\end{aligned} \tag{58}$$

$$\begin{aligned}
d_2^{hl} = & -1.24078 \times 10^{-1} + \frac{1.1871 \times 10^{-1}}{z^{1.02367}} - \frac{1.4815}{z^{2.87467}} + \frac{1.04971}{z^3} + \frac{5.39031 \times 10^{-1}}{z^6} \\
& + \frac{8.80307 \times 10^{-2}}{z^{6.5}} - \frac{8.02099 \times 10^{-2}}{z^{6.83333}} + \frac{3.54398 \times 10^{-2}}{z^{6.83333}} - \frac{4.35098 \times 10^{-2}}{z^{7.99055}} - \frac{1.01624 \times 10^{-1}}{z^9},
\end{aligned} \tag{59}$$

$$\begin{aligned}
d_3^{hl} = & 1.35186 \times 10^{-3} - \frac{1.52127 \times 10^{-20}}{z^{1.02367}} + \frac{1.92993 \times 10^{-19}}{z^{2.87467}} - \frac{2.26963 \times 10^{-12}}{z^3} - \frac{2.37477 \times 10^{-1}}{z^6} - \frac{8.91247 \times 10^{-2}}{z^{6.5}} \\
& + \frac{1.56003 \times 10^{-20}}{z^{6.83333}} + \frac{5.48043 \times 10^{-20}}{z^{7.99055}} + \frac{3.3161 \times 10^{-1}}{z^{8.33333}} + \frac{1.63677 \times 10^{-19}}{z^9} - \frac{6.36036 \times 10^{-3}}{z^{1.45556 \times 10^1}}.
\end{aligned} \tag{60}$$

Let us finally notice that the running of c_{A_2} can be deduced from the above results using the expression of c_M determined from reparametrization invariance, which in turn depends on c_D . The result will depend on the gauge, though.

If we expand the above solutions in powers of α , we can explicitly write the single log (it can also be obtained by trivial inspection of the RG equations in Sec. IV). We obtain

$$c_{A_1} = 1 + \left(\frac{16}{3}C_F + \frac{23}{3}C_A\right)\frac{\alpha}{\pi}\ln\left(\frac{\nu}{m}\right) + \mathcal{O}(\alpha^2), \tag{61}$$

$$\bar{c}_{A_2} = 0 - \left(\frac{64}{3} C_F + \frac{65}{3} C_A \right) \frac{\alpha}{\pi} \ln \left(\frac{\nu}{m} \right) + \mathcal{O}(\alpha^2), \quad (62)$$

$$c_{A_3} = 0 + C_A \frac{\alpha}{\pi} \ln \left(\frac{\nu}{m} \right) + \mathcal{O}(\alpha^2), \quad (63)$$

$$c_{A_4} = 0 - 4C_A \frac{\alpha}{\pi} \ln \left(\frac{\nu}{m} \right) + \mathcal{O}(\alpha^2), \quad (64)$$

$$d_2^{hl} = 0 + \left(3C_F - \frac{9}{8} C_A \right) \frac{\alpha}{\pi} \ln \left(\frac{\nu}{m} \right) + \mathcal{O}(\alpha^2), \quad (65)$$

$$d_3^{hl} = 0 + \frac{3}{4} C_F (C_A - 2C_F) \frac{\alpha}{\pi} \ln \left(\frac{\nu}{m} \right) + \mathcal{O}(\alpha^2). \quad (66)$$

We draw in Fig. 5 the above results when applied to the bottom heavy quark case to illustrate the importance of the incorporation of large logarithms in heavy quark physics (for physical processes where these Wilson coefficients appear). We run the Wilson coefficients from the heavy quark mass to 1 GeV for zero and four massless fermions. For illustrative purposes, we take $m_b = 4.73$ GeV and $\alpha(m_b) = 0.215943$. Besides the running of the Wilson coefficients, we also consider some specific combinations that appear in physical observables. In heavy quarkonium physics applications (like the NNNLL running of the spectrum and the NNLL running of Wilson coefficient of the electromagnetic current), we observe that only the combinations $\tilde{c}_{A_2} \equiv 2c_{A_1} + \bar{c}_{A_2}$, $\tilde{c}_{A_4} \equiv 2c_{A_3} + c_{A_4}$ appear [22]. For the Compton scattering discussed in Sec. III, we observe that $c_{A_{1,3}}$ and, again, $\tilde{c}_{A_{2,4}}$ appear. We remind the reader that those can be understood as linear combinations of the (chromo)polarizabilities of the heavy quark.

For the numerical analysis, we observe the following. For most cases, the incorporation of light fermions plays a minor role in the result. The effect due to the logarithms is very large in most cases. This is due to very large coefficients multiplying the logs (even in the Abelian limit, the coefficient is quite large). We also observe that the LL resummation is basically saturated by the single log in all cases except for c_{A_4} and $\tilde{c}_{A_4}$. Let us now discuss in more detail each individual Wilson coefficient. We observe the following: c_{A_1} changes from 1 to -2 after running. The case of $\bar{c}_{A_2}$ is even more dramatic. It goes from 0 to 10 after running. The change after running of c_{A_3} is more moderate though certainly sizable and so is for c_{A_4} . In this last case, the resummation of logarithms happens to be important. Comparatively, the running of d_2^{hl} and d_3^{hl} is much smaller, confirming that light fermions associated corrections are subleading. It is interesting to note that the running of $\tilde{c}_{A_2}$ is smaller than the running of $\bar{c}_{A_2}$ but still rather large; it

changes by nearly a factor of 3. Finally, the qualitative behavior of $\tilde{c}_{A_4}$ and c_{A_4} is similar.

A. Comparison with earlier work

The LL running of the Wilson coefficients of the $1/m^3$ operators of the HQET Lagrangian was first addressed in Refs. [9,13,14]. For the case with no light fermions, expressions for the anomalous dimension matrix and explicit expressions for the Wilson coefficients with single log accuracy are given. We find that these results are mutually inconsistent, as his anomalous dimension matrix produces a different expression for the explicit single log expression written in these references for the Wilson coefficients (except for $c_{12}^{(3)}$).

The basis of operators these results have been obtained is different from the basis we use in our paper. This makes the comparison difficult. In order to compare our results we have to move from one basis to the other. This is possible by the use of field redefinitions (at the order we are working it is equivalent to the use of the (full) equations of motion to order $1/m$). We obtain the following relation between the Wilson coefficients (spin-independent) in the two basis:

$$c_1^{(3)} = -2c_M + \frac{1}{2}c_{A_2} + c_k c_F^2 + c_S c_F, \quad (67)$$

$$c_2^{(3)} = c_4 + 2c_M - c_k^3 - c_D c_k, \quad (68)$$

$$c_3^{(3)} = 2c_M + c_{A_1} - c_k c_F^2 - c_S c_F, \quad (69)$$

$$c_4^{(3)} = -4c_M - c_{A_1} + c_k c_F^2 + c_S c_F, \quad (70)$$

$$c_{12}^{(3)} = \frac{1}{12} c_{A_3}, \quad (71)$$

$$c_{13}^{(3)} = \frac{1}{12} c_{A_4}, \quad (72)$$

and these relations between the Wilson coefficients at $\mathcal{O}(1/m)$ and $\mathcal{O}(1/m^2)$:

$$c_1^{(1)} = c_k, \quad c_2^{(1)} = c_F, \quad c_1^{(2)} = -c_D, \quad c_2^{(2)} = c_S. \quad (73)$$

In order to use these expressions we need to give values for c_D and c_M . We use the LL resummed expressions of c_D obtained in the Feynman gauge (for c_M we use reparametrization invariance), as the computation in those references was done in the Feynman gauge. Note that this makes several of these coefficients gauge dependent. We can now produce expressions for the anomalous dimension matrix in Balzereit basis from our result. We obtain (we use the same ordering and notation as in Ref. [13])

$$\hat{\gamma}_l^{(3)A} = \begin{pmatrix} -11/12 & 0 & 11/24 & -11/24 & 11/288 & -11/72 \\ -4 & 0 & 1 & -1 & 0 & -2/9 \\ -3 & 5/6 & 5/6 & -5/3 & 1/12 & -7/18 \\ -7/2 & 5/6 & 4/3 & -13/6 & 1/24 & -11/36 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11/24 & -11/6 \end{pmatrix} \quad (74)$$

$$\hat{\gamma}_l^{(111)A} = \begin{pmatrix} 16/3 & 0 & -5/2 & 7/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2/3 & -11/6 & 5/3 & -1/4 & 1/2 \\ 3/2 & 0 & -3/2 & 3/2 & 1/8 & -1/4 \end{pmatrix} \quad (75)$$

$$\hat{\gamma}_l^{(111)F} = \begin{pmatrix} 32/3 & 0 & -8/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (76)$$

$$\hat{\gamma}_l^{(12)A} = \begin{pmatrix} 4 & -5/6 & -3/2 & 2 & 0 & 2/9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5/6 & 1/3 & -1/12 & -1/4 & 1/144 & 11/36 \end{pmatrix} \quad (77)$$

$$\hat{\gamma}_l^{(12)F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (78)$$

where we only include the contribution of spin-independent operators: $\mathcal{O}_{1-4,12,13}$. In all matrices (except the last one), we find discrepancies with the entries in the Appendix of Ref. [13]. The differences do not follow a clear pattern. On the other hand, remarkably enough, our anomalous dimension matrix produces the same single logs as those in Table II of Ref. [14] (note that the expression for $c_1^{(3)p}$ is different from the one one can find in Table I of Ref. [13]).

It is also interesting to make the comparison backward and try to produce results for our Wilson coefficients from the results obtained in Refs. [13,14]. The inverse relations read

$$c_M = -\frac{1}{2}(c_3^{(3)} + c_4^{(3)}), \quad (79)$$

$$c_4 = c_2^{(3)} + c_3^{(3)} + c_4^{(3)} + c_1^{(1)3} - c_1^{(1)}c_1^{(2)}, \quad (80)$$

$$c_{A_1} = 2c_3^{(3)} + c_4^{(3)} + c_1^{(1)}c_2^{(1)2} + c_2^{(1)}c_2^{(2)}, \quad (81)$$

$$\bar{c}_{A_2} = 2c_1^{(3)} - 2c_1^{(1)}c_2^{(1)2} - 2c_2^{(2)}c_2^{(1)}, \quad (82)$$

$$c_{A_3} = 12c_{12}^{(3)}, \quad (83)$$

$$c_{A_4} = 12c_{13}^{(3)}. \quad (84)$$

The running of c_M agrees with the result predicted by reparametrization invariance (in Feynman gauge). For the gauge invariant combinations of Wilson coefficients we have computed in our paper, the anomalous dimension matrix given in Ref. [13] yields different RG equations than those we found in Sec. IV C (nor even the running of c_4 is zero), and also different logs as those we found in Eqs. (61)–(64) [except for (63)]. On the other hand, the single logs given in Table II of Ref. [14] yield results in agreement with our single log results in Eqs. (61)–(64).

The case including light fermions was analyzed in Ref. [9]. In this case, no anomalous dimension matrix was given, but only the result for the single logs. For the operators we consider in our paper, such results disagree with Eqs. (65) and (66).

VI. CONCLUSIONS

We have computed the LL running of the Wilson coefficients of the spin-independent $1/m^3$ operators of the HQET Lagrangian in the case without light fermions. These include the heavy quark chromopolarizabilities induced by the strong interactions. We have also computed the LL running of the Wilson coefficients of the spin-independent $1/m^3$ operators of the HQET Lagrangian in the case with light fermions except for the heavy-light operators that do not contribute to the chromopolarizabilities.

We have performed a numerical analysis of these results. We observe that the running produces a very large effect. For combinations that appear in physical cases such as in heavy quarkonium dynamics or in Compton scattering, the running is more moderate but still quite large.

These results are necessary building blocks for the complete determination of the production (and annihilation) of the heavy quarkonium with NNLL accuracy near threshold and of the heavy quarkonium mass with NNNLL precision.

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APPENDIX: HQET FEYNMAN RULES

Here, we collect some Feynman rules (in Coulomb gauge) that we need for our computation and complement those that can be found in Ref. [7]. We also profit to correct a misprint of Eq. (205) of that reference in Eq. (A8).

The heavy quark four-momentum $p = (E_1, \mathbf{p})$ is incoming with associated color index β , whereas the heavy quark four-momentum $p' = (E'_1, \mathbf{p}')$ is outgoing with associated color index α .

Incoming light quarks have four-momentum $-k_1$, color index γ , and gamma matrix index A , whereas outgoing light quarks have four-momentum k_2 , color index δ , and gamma matrix index B . Gluons appearing in Feynman rules with light quarks have outgoing four-momentum k_3 with color index b and spatial vector index i in case the gluon is transverse.

For gluons in Feynman rules with no light quarks, the notation is the following. All four-momentums of gluons, k_i , are outgoing. If more than one gluon appears, let us say n , then they are labeled with four-momentum k_i ($i = 1, \dots, n$) and by four-momentum conservation $k = \sum_{i=1}^n k_i = p - p'$. We start labeling transverse gluons first and longitudinal gluons after with the labels $a, b, c, \dots$ referring to color indices in the adjoint representation, $i, j, k, \dots$ referring to space vector indices, and $k_1, k_2, k_3, \dots$ referring to four-momentum.

1. Proportional to c_i^{hl}

$$\mathcal{V} = c_1^{hl} \frac{ig^2}{8m^2} (T^a)_{\alpha\beta} (T^a)_{\delta\gamma} (\gamma^0)_{BA} \quad (\text{A1})$$

$$\mathcal{V} = -c_2^{hl} \frac{ig^2}{8m^2} \boldsymbol{\sigma}^i (T^a)_{\alpha\beta} (T^a)_{\delta\gamma} (\gamma^i \gamma_5)_{BA} \quad (\text{A2})$$

$$\mathcal{V} = c_3^{hl} \frac{ig^2}{8m^2} (I_{N_c})_{\alpha\beta} (I_{N_c})_{\delta\gamma} (\gamma^0)_{BA} \quad (\text{A3})$$

$$\mathcal{V} = -c_4^{hl} \frac{ig^2}{8m^2} \boldsymbol{\sigma}^i (I_{N_c})_{\alpha\beta} (I_{N_c})_{\delta\gamma} (\gamma^i \gamma_5)_{BA} \quad (\text{A4})$$

2. Proportional to c_D

$$\mathcal{V}_{c_D}^{iab} = c_D \frac{ig^2}{4m^2} \mathbf{k}_2^i [T^a, T^b]_{\alpha\beta} \quad (\text{A5})$$

$$\mathcal{V}_{c_D}^{ijab} = c_D \frac{ig^2}{8m^2} \delta^{ij} (k_1^0 - k_2^0) [T^a, T^b]_{\alpha\beta} \quad (\text{A6})$$

$$\mathcal{V}_{c_D}^{ijabc} = c_D \frac{ig^3}{8m^2} \delta^{ij} ([T^a, [T^b, T^c]]_{\alpha\beta} + [T^b, [T^a, T^c]]_{\alpha\beta}) \quad (\text{A7})$$

3. Proportional to c_S

$$\mathcal{V}_{c_S}^a = c_S \frac{g}{4m^2} \boldsymbol{\sigma} \cdot (\mathbf{p}' \times \mathbf{p}) (T^a)_{\alpha\beta} \quad (\text{A8})$$

$$\mathcal{V}_{c_S}^{ia} = -c_S \frac{g}{8m^2} k^0 (\boldsymbol{\sigma} \times (\mathbf{p} + \mathbf{p}'))^i (T^a)_{\alpha\beta} \quad (\text{A9})$$

$$\begin{aligned} \mathcal{V}_{c_S}^{iab} &= c_S \frac{g^2}{8m^2} [(\boldsymbol{\sigma} \times (\mathbf{p} + \mathbf{p}'))^i [T^a, T^b]_{\alpha\beta} \\ &\quad + (\boldsymbol{\sigma}_1 \times \mathbf{k}_2)^i \{T^a, T^b\}_{\alpha\beta}] \end{aligned} \quad (\text{A10})$$

$$\mathcal{V}_{c_S}^{ijab} = c_S \frac{g^2}{8m^2} \boldsymbol{\sigma}^k \epsilon^{kij} (k_2^0 - k_1^0) \{T^a, T^b\}_{\alpha\beta} \quad (\text{A11})$$

$$\begin{aligned} \mathcal{V}_{c_S}^{ijabc} &= -c_S \frac{g^3}{8m^2} \boldsymbol{\sigma}^k \epsilon^{kij} (\{T^a, [T^b, T^c]\}_{\alpha\beta} \\ &\quad - \{T^b, [T^a, T^c]\}_{\alpha\beta}) \end{aligned} \quad (\text{A12})$$

4. Proportional to c_4

$$\mathcal{V}_{c_4}^{ia} = -c_4 \frac{ig}{8m^3} (\mathbf{p}^2 + \mathbf{p}'^2) (\mathbf{p} + \mathbf{p}')^i (T^a)_{\alpha\beta} \quad (\text{A13})$$

$$\begin{aligned} \mathcal{V}_{c_4}^{ijab} &= c_4 \frac{ig^2}{8m^3} (\delta^{ij} (\mathbf{p}^2 + \mathbf{p}'^2) \{T^a, T^b\}_{\alpha\beta} \\ &\quad + 4\mathbf{p}^i \mathbf{p}^j (T^a T^b)_{\alpha\beta} + 4\mathbf{p}^i \mathbf{p}'^j (T^b T^a)_{\alpha\beta}) \end{aligned} \quad (\text{A14})$$

$$\begin{aligned} \mathcal{V}_{c_4}^{ijkabc} &= -c_4 \frac{ig^3}{4m^3} [\delta^{ij} ((T^c \{T^a, T^b\})_{\alpha\beta} \mathbf{p}'^k \\ &\quad + (\{T^a, T^b\} T^c)_{\alpha\beta} \mathbf{p}^k) + \delta^{jk} ((T^a \{T^b, T^c\})_{\alpha\beta} \mathbf{p}'^i \\ &\quad + (\{T^b, T^c\} T^a)_{\alpha\beta} \mathbf{p}^i) \end{aligned} \quad (\text{A15})$$

$$+ \delta^{ik} ((T^b \{T^a, T^c\})_{\alpha\beta} \mathbf{p}'^j + (\{T^a, T^c\} T^b)_{\alpha\beta} \mathbf{p}^j)] \quad (\text{A16})$$

5. Proportional to c_M

$$\mathcal{V}_{c_M}^{ia} = -c_M \frac{ig}{8m^3} \mathbf{k}^2 (\mathbf{p} + \mathbf{p}')^i (T^a)_{\alpha\beta} \quad (\text{A17})$$

$$\mathcal{V}_{c_M CG}^{ijab} = c_M \frac{ig^2}{8m^3} (\delta^{ij} \{T^a, T^b\}_{\alpha\beta} (\mathbf{k}_1^2 + \mathbf{k}_2^2) - \delta^{ij} [T^a, T^b]_{\alpha\beta} ((\mathbf{p} + \mathbf{p}') \cdot (\mathbf{k}_1 - \mathbf{k}_2))) + 4[T^a, T^b]_{\alpha\beta} (\mathbf{p}^i \mathbf{k}_1^j - \mathbf{k}_2^i \mathbf{p}^j) \quad (\text{A18})$$

$$\begin{aligned} \mathcal{V}_{c_M CG}^{ijkabc} = & c_M \frac{ig^3}{8m^3} ([T^a, [T^b, T^c]]_{\alpha\beta} (\delta^{ik} (\mathbf{p}' + \mathbf{p})^j - \delta^{ij} (\mathbf{p}' + \mathbf{p})^k) + [T^b, [T^a, T^c]]_{\alpha\beta} (\delta^{jk} (\mathbf{p}' + \mathbf{p})^i - \delta^{ij} (\mathbf{p}' + \mathbf{p})^k) \\ & + [T^c, [T^a, T^b]]_{\alpha\beta} (\delta^{jk} (\mathbf{p}' + \mathbf{p})^i - \delta^{ik} (\mathbf{p}' + \mathbf{p})^j) + \{T^a, [T^b, T^c]\}_{\alpha\beta} (2\delta^{ik} \mathbf{k}_3^j - 2\delta^{ij} \mathbf{k}_2^k + \delta^{jk} (\mathbf{k}_2 - \mathbf{k}_3)^i) \\ & + \{T^b, [T^a, T^c]\}_{\alpha\beta} (2\delta^{jk} \mathbf{k}_3^i - 2\delta^{ij} \mathbf{k}_1^k + \delta^{ik} (\mathbf{k}_1 - \mathbf{k}_3)^j) + \{T^c, [T^a, T^b]\}_{\alpha\beta} (2\delta^{jk} \mathbf{k}_2^i - 2\delta^{ik} \mathbf{k}_1^j + \delta^{ij} (\mathbf{k}_1 - \mathbf{k}_2)^k)) \end{aligned} \quad (\text{A19})$$

6. Proportional to c_{A_1}

$$\mathcal{V}_{c_{A_2}}^{ab} = c_{A_2} \frac{ig^2}{16m^3} \mathbf{k}_1 \cdot \mathbf{k}_2 \{T^a, T^b\}_{\alpha\beta} \quad (\text{A29})$$

$$\mathcal{V}_{c_{A_1}}^{ijab} = c_{A_1} \frac{ig^2}{8m^3} (\delta^{ij} (k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2) + \mathbf{k}_2^i \mathbf{k}_1^j) \{T^a, T^b\}_{\alpha\beta} \quad (\text{A20})$$

$$\begin{aligned} \mathcal{V}_{c_{A_2}}^{iabc} = & c_{A_2} \frac{ig^3}{16m^3} (\mathbf{k}_2^i \{T^b, [T^a, T^c]\}_{\alpha\beta} \\ & + \mathbf{k}_3^i \{T^c, [T^a, T^b]\}_{\alpha\beta}) \end{aligned} \quad (\text{A30})$$

$$\begin{aligned} \mathcal{V}_{c_{A_1}}^{ijkabc} = & c_{A_1} \frac{ig^3}{8m^3} [(\delta^{ij} \mathbf{k}_1^k - \delta^{ik} \mathbf{k}_1^j) \{[T^b, T^c], T^a\}_{\alpha\beta} \\ & + (\delta^{ij} \mathbf{k}_2^k - \delta^{jk} \mathbf{k}_2^i) \{[T^a, T^c], T^b\}_{\alpha\beta} \\ & + (\delta^{ik} \mathbf{k}_3^j - \delta^{jk} \mathbf{k}_3^i) \{[T^a, T^b], T^c\}_{\alpha\beta}] \end{aligned} \quad (\text{A21})$$

8. Proportional to c_{A_3}

$$\mathcal{V}_{c_{A_1}}^{iab} = -c_{A_1} \frac{ig^2}{8m^3} k_1^0 \mathbf{k}_2^i \{T^a, T^b\}_{\alpha\beta} \quad (\text{A22})$$

$$\mathcal{V}_{c_{A_3}}^{ijab} = c_{A_3} \frac{ig^2}{4m^3} \frac{T_F}{N_c} (\delta^{ij} (k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2) + \mathbf{k}_2^i \mathbf{k}_1^j) \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A31})$$

$$\begin{aligned} \mathcal{V}_{c_{A_1}}^{iabc} = & -c_{A_1} \frac{ig^3}{8m^3} \delta^{ij} (k_1^0 \{T^a, [T^b, T^c]\}_{\alpha\beta} \\ & + k_2^0 \{T^b, [T^a, T^c]\}_{\alpha\beta}) \end{aligned} \quad (\text{A23})$$

$$\begin{aligned} \mathcal{V}_{c_{A_3}}^{ijkabc} = & c_{A_3} \frac{g^3}{4m^3} \frac{T_F}{N_c} f^{abc} [\delta^{ik} (\mathbf{k}_1 - \mathbf{k}_3)^j + \delta^{jk} (\mathbf{k}_3 - \mathbf{k}_2)^i \\ & + \delta^{ij} (\mathbf{k}_2 - \mathbf{k}_1)^k] (I_{N_c})_{\alpha\beta} \end{aligned} \quad (\text{A32})$$

$$\mathcal{V}_{c_{A_1}}^{ab} = c_{A_1} \frac{ig^2}{8m^3} \mathbf{k}_1 \cdot \mathbf{k}_2 \{T^a, T^b\}_{\alpha\beta} \quad (\text{A24})$$

$$\mathcal{V}_{c_{A_3}}^{iab} = -c_{A_3} \frac{ig^2}{4m^3} \frac{T_F}{N_c} k_1^0 \mathbf{k}_2^i \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A33})$$

$$\mathcal{V}_{c_{A_1}}^{iabc} = c_{A_1} \frac{ig^3}{8m^3} (\mathbf{k}_2^i \{T^b, [T^a, T^c]\}_{\alpha\beta} + \mathbf{k}_3^i \{T^c, [T^a, T^b]\}_{\alpha\beta}) \quad (\text{A25})$$

$$\mathcal{V}_{c_{A_3}}^{ijabc} = c_{A_3} \frac{g^3}{4m^3} \frac{T_F}{N_c} \delta^{ij} f^{abc} (k_1^0 - k_2^0) (I_{N_c})_{\alpha\beta} \quad (\text{A34})$$

$$\mathcal{V}_{c_{A_3}}^{ab} = c_{A_3} \frac{ig^2}{4m^3} \frac{T_F}{N_c} \mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A35})$$

$$\mathcal{V}_{c_{A_3}}^{iabc} = -c_{A_3} \frac{g^3}{4m^3} \frac{T_F}{N_c} f^{abc} (\mathbf{k}_3 - \mathbf{k}_2)^i (I_{N_c})_{\alpha\beta} \quad (\text{A36})$$

7. Proportional to c_{A_2}

$$\mathcal{V}_{c_{A_2}}^{ijab} = c_{A_2} \frac{ig^2}{16m^3} \delta^{ij} k_1^0 k_2^0 \{T^a, T^b\}_{\alpha\beta} \quad (\text{A26})$$

$$\mathcal{V}_{c_{A_2}}^{iab} = -c_{A_2} \frac{ig^2}{16m^3} k_1^0 \mathbf{k}_2^i \{T^a, T^b\}_{\alpha\beta} \quad (\text{A27})$$

$$\begin{aligned} \mathcal{V}_{c_{A_2}}^{ijkabc} = & -c_{A_2} \frac{ig^3}{16m^3} \delta^{ij} (k_1^0 \{T^a, [T^b, T^c]\}_{\alpha\beta} \\ & + k_2^0 \{T^b, [T^a, T^c]\}_{\alpha\beta}) \end{aligned} \quad (\text{A28})$$

9. Proportional to c_{A_4}

$$\mathcal{V}_{c_{A_4}}^{ijab} = c_{A_4} \frac{ig^2}{8m^3} \frac{T_F}{N_c} \delta^{ij} k_1^0 k_2^0 \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A37})$$

$$\mathcal{V}_{c_{A_4}}^{iab} = -c_{A_4} \frac{ig^2}{8m^3} \frac{T_F}{N_c} k_1^0 \mathbf{k}_2^i \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A38})$$

$$\mathcal{V}_{c_{A_4}}^{ijabc} = c_{A_4} \frac{g^3}{8m^3} \frac{T_F}{N_c} \delta^{ij} f^{abc} (k_1^0 - k_2^0) (I_{N_c})_{\alpha\beta} \quad (\text{A39})$$

$$\mathcal{V}_{c_{A_4}}^{ab} = c_{A_4}^{(1)} \frac{ig^2}{8m^3} \frac{T_F}{N_c} \mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{ab} (I_{N_c})_{\alpha\beta} \quad (\text{A40})$$

$$\mathcal{V}_{c_{A_4}}^{iabc} = -c_{A_4}^{(1)} \frac{g^3}{8m^3} \frac{T_F}{N_c} f^{abc} (\mathbf{k}_3 - \mathbf{k}_2)^i (I_{N_c})_{\alpha\beta} \quad (\text{A41})$$

10. Proportional to d_1^{hl}

$$\mathcal{V} = -d_1^{hl} \frac{ig^2}{m^3} (T^a)_{\alpha\beta} (T^a)_{\delta\gamma} (\gamma^i)_{BA} (\mathbf{p} + \mathbf{p}')^i \quad (\text{A42})$$

$$\mathcal{V}^{ib} = d_1^{hl} \frac{ig^3}{m^3} \{T^a, T^b\}_{\alpha\beta} (T^a)_{\delta\gamma} (\gamma^i)_{BA} \quad (\text{A43})$$

11. Proportional to d_2^{hl}

$$\mathcal{V} = -d_2^{hl} \frac{ig^2}{m^3} (T^a)_{\alpha\beta} (T^a)_{\delta\gamma} (\gamma^0)_{BA} (k_1^0 - k_2^0) \quad (\text{A44})$$

$$\mathcal{V}^b = -d_2^{hl} \frac{ig^3}{m^3} (T^a)_{\alpha\beta} \{T^a, T^b\}_{\delta\gamma} (\gamma^0)_{BA} \quad (\text{A45})$$

12. Proportional to d_3^{hl}

$$\mathcal{V} = -d_3^{hl} \frac{ig^2}{m^3} (I_{N_c})_{\alpha\beta} (I_{N_c})_{\delta\gamma} (\gamma^0)_{BA} (k_1^0 - k_2^0) \quad (\text{A46})$$

$$\mathcal{V}^b = -d_3^{hl} \frac{2ig^3}{m^3} (I_{N_c})_{\alpha\beta} (T^b)_{\delta\gamma} (\gamma^0)_{BA} \quad (\text{A47})$$

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- [1] M. B. Voloshin and M. A. Shifman, *Sov. J. Nucl. Phys.* **45**, 292 (1987); H. D. Politzer and M. B. Wise, *Phys. Lett. B* **206**, 681 (1988); N. Isgur and M. B. Wise, *Phys. Lett. B* **232**, 113 (1989); E. Eichten and B. Hill, *Phys. Lett. B* **234**, 511 (1990); H. Georgi, *Phys. Lett. B* **240**, 447 (1990); B. Grinstein, *Nucl. Phys.* **B339**, 253 (1990).
- [2] W. E. Caswell and G. P. Lepage, *Phys. Lett.* **167B**, 437 (1986).
- [3] G. T. Bodwin, E. Braaten, and G. P. Lepage, *Phys. Rev. D* **51**, 1125 (1995); **55**, 5853(E) (1997).
- [4] A. Pineda and J. Soto, *Nucl. Phys. B, Proc. Suppl.* **64**, 428 (1998).
- [5] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, *Nucl. Phys.* **B566**, 275 (2000).
- [6] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, *Rev. Mod. Phys.* **77**, 1423 (2005).
- [7] A. Pineda, *Prog. Part. Nucl. Phys.* **67**, 735 (2012).
- [8] A. V. Manohar, *Phys. Rev. D* **56**, 230 (1997).
- [9] C. Balzereit, *Phys. Rev. D* **59**, 094015 (1999).
- [10] M. Finkemeier and M. McIrvine, *Phys. Rev. D* **55**, 377 (1997).
- [11] B. Blok, J. G. Korner, D. Pirjol, and J. C. Rojas, *Nucl. Phys.* **B496**, 358 (1997).
- [12] C. W. Bauer and A. V. Manohar, *Phys. Rev. D* **57**, 337 (1998).
- [13] C. Balzereit, *Phys. Rev. D* **59**, 034006 (1999).
- [14] C. Balzereit, *arXiv:hep-ph/9809226*.
- [15] A. Pineda, *Phys. Rev. D* **65**, 074007 (2002).
- [16] S. Balk, J. G. Korner, and D. Pirjol, *Nucl. Phys.* **B428**, 499 (1994).
- [17] S. Scherer, A. Y. Korchin, and J. H. Koch, *Phys. Rev. C* **54**, 904 (1996).
- [18] H. W. Fearing and S. Scherer, *Few Body Syst.* **23**, 111 (1998).
- [19] D. Nevado and A. Pineda, *Phys. Rev. C* **77**, 035202 (2008).
- [20] R. J. Hill, G. Lee, G. Paz, and M. P. Solon, *Phys. Rev. D* **87**, 053017 (2013).
- [21] M. E. Luke and A. V. Manohar, *Phys. Lett. B* **286**, 348 (1992).
- [22] C. Anzai, D. Moreno, A. Penin, A. Pineda, and M. Steinhauser (to be published).