Backreaction on background fields: A coherent state approach

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There are many situations in which a strong electromagnetic field may be approximated as a fixed background. Going beyond this approximation, i.e. accounting for the backreaction of quantum processes on the field, is however challenging. Here we develop an approach to this problem which is a straightforward extension of background field methods. The approach follows from the observation that scattering in an on-shell background is equivalent to scattering between coherent states; we show that by deforming these states one can model backreaction. Focussing on intense laser-matter interactions, we provide examples which model beam depletion and, furthermore, introduce an extremization principle with which to determine the level of depletion in a given scattering process.

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I. INTRODUCTION

There are many physical phenomena which are naturally described within the framework of “quantum field theory in background fields.” For example, the physics of QCD in strong background magnetic fields [1], pair production in the fields of heavy-ion collisions [2], extreme magnetic fields in astrophysics [3] and QED in strong laser fields [4–6] have all received a great deal of attention in the last decade.

While the mathematical structure of background field theories can be very rich [7–9], there are inevitably physical regimes where the assumption of a background field breaks down. For example the backreaction of produced particles on pair-creating fields can be significant [10–13], as can beam depletion in laser-particle collisions at high laser field strengths and for dense particle bunches [14]. The breakdown of the background field approximation may be signaled by the breakdown of the perturbative expansion of the theory; see [15] for a review in the context of laser-matter interactions. For perturbations beyond background fields in heavy-ion collisions see e.g. [16].

Going beyond the background field approximation is in general a challenging problem. One can use classical-statistical approximations and lattice simulations [17], as has been applied to the screening of pair-producing fields

by the created particles [18]. Again in the context of beam depletion and laser-matter interactions, another (first quantized) approach is based on approximating the gauge field to a single (laser) mode in the Dirac equation, but treating this quantum mechanically [19–22]. We will return to this below.

Here we suggest an approach to backreaction in laser-matter interactions based on the observation that scattering in a background is equivalent to scattering between the same initial and final coherent states, with these states describing a prescribed field [23,24]. Beginning from this result, we take a step toward modeling backreaction by making a minimal adjustment; we consider scattering between different initial and final coherent states, and explore the consequences. We will see that this adjustment can indeed be used to model, for example, depletion of a laser field.

One motivation for our approach is that at high intensities particles emit large numbers of photons which may be better described as coherent states rather than discrete number states [25], hence the difference between initial and final scattering states would be at least partially coherent. A second motivation is that the coherent state origin of a background field is seldom exploited in calculations, even though it is essential for the physics—the quantum state describing a laser cannot be a number of state of photons, for example, as such states give zero expectation value for the electromagnetic fields. Finally, and as we will see, our approach has the advantage of being technically simple; it requires only a minor extension of the usual background field formalism.

This paper is organized as follows. In Sec. II A we review literature results on scattering amplitudes in background fields and how these are related to coherent states. In Sec. II B we give the physical motivation for our

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approach, based on using different initial and final coherent states, and derive reduction formulae for S -matrix elements in this setup. Several first examples of our formalism are outlined in Sec. III; we show that our methods can be used to model strong depletion of a weak field, weak depletion of a strong field, and strong depletion of a strong field. In Sec. IV we introduce an extremization principle in order to identify the “most likely” level of beam depletion and give a first investigation for the process of pair creation. We work with scalar field theories throughout in order to simplify the presentation. The extension to QED is given in Sec. V along with conclusions.

II. A COHERENT STATE MODEL OF DEPLETION

We begin by briefly reviewing the background field formalism and the connection to coherent states.

A. Review of QED in background fields

The action of QED with an additional background field described by the classical potential A^{ext} is

$$S = \int d^4x \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}(\not{A} + \not{A}^{\text{ext}})\psi + \text{gauge fixing} + \text{counterterms}, \quad (1)$$

in which the interaction terms are shifted by A_μ^{ext} . When the coupling $\sim eA^{\text{ext}}/m > 1$, interactions with the background cannot be treated in perturbation theory. Instead S -matrix elements should be calculated using the “Furry picture” expansion which treats the strong coupling to the background exactly. In this expansion the interaction with the quantized gauge field is treated in perturbation theory as normal, while the coupling to A_μ^{ext} is included in the “free” part of the action, such that the fermion propagator becomes the background-field dressed propagator G_ψ ,

$$G_\psi = (i\not{\partial} - e\not{A}^{\text{ext}} - m)^{-1}, \quad (2)$$

and the Feynman diagrams are as shown in Fig. 1, where the double line represents G_ψ . This propagator, and the corresponding asymptotic wave functions following from Lehmann-Symanzik-Zimmermann (LSZ) reduction, are what is used customarily in strong-field QED calculations; see [4–6,26–30] for reviews, and [31–49] for a selection of recent papers. (We will focus here on scalar theories; for investigations comparing scalar and spinor QED results see e.g. [50–53].)

Now let us make the connection to coherent states. Let the S -matrix be, highlighting only the dependency on the gauge potential, $\mathcal{S}[A + A^{\text{ext}}]$. We are then interested in the S -matrix elements

$$\langle \text{out} | \mathcal{S}[A + A^{\text{ext}}] | \text{in} \rangle, \quad (3)$$

FIG. 1. Feynman diagrams for the Furry picture expansion of correlation functions, in which the interaction between matter and the background field is, in principle, treated exactly. The double line is the background-dressed fermion propagator (2).

in which “in” and “out” represent injected, scattered or produced electrons, positrons and photons. We assume here that the background field is a solution of Maxwell’s equations in vacuum, $\partial_\mu F_{\text{ext}}^{\mu\nu} = 0$, which is a standard assumption for laser-matter interactions. Because the background is on-shell, it has the same Fourier modes as the quantum field, and hence the sum $A + A^{\text{ext}}$ is, in Fourier space, just a sum of translated mode operators $\sim a(\mathbf{k}) + z(\mathbf{k})$ in which $z(\mathbf{k})$ is the (appropriately normalized, see below) positive energy Fourier mode of A_μ^{ext} . We can therefore extract the background from the S -matrix using the standard displacement, or translation, operator

$$D(z) = \exp \left[\int d^3\mathbf{k} (a^\dagger(\mathbf{k})z(\mathbf{k}) - a(\mathbf{k})\bar{z}(\mathbf{k})) \right],$$

(suppressing polarization labels for the moment) which obeys the commutation relations

$$\begin{aligned} D^\dagger(z)a(\mathbf{k})D(z) &= a(\mathbf{k}) + z(\mathbf{k}), \\ D^\dagger(z)a^\dagger(\mathbf{k})D(z) &= a^\dagger(\mathbf{k}) + \bar{z}(\mathbf{k}). \end{aligned} \quad (4)$$

Then we have the long-established result [23,24,54,55]

$$\begin{aligned} \langle \text{out} | \mathcal{S}[A + A_{\text{ext}}] | \text{in} \rangle &= \langle \text{out} | D^\dagger(z)\mathcal{S}[A]D(z) | \text{in} \rangle \\ &= \langle \text{out}; z | \mathcal{S}[A] | \text{in}; z \rangle, \end{aligned} \quad (5)$$

where, in the last step, we used the fact that $D(z)$ acting on the vacuum¹ creates a normalized coherent state $|z\rangle$. Hence scattering in an on-shell background field is equivalent to scattering between asymptotic coherent states. Note that this derivation does not require any assumptions on the form of the S -matrix. Furthermore, the background field can now, using (5), be reinterpreted as the expectation value of the potential in the coherent state [54]:

$$A_\mu^{\text{ext}}(x) = \langle \text{out}; z | A_\mu(x) | \text{out}; z \rangle = \langle \text{in}; z | A_\mu(x) | \text{in}; z \rangle. \quad (6)$$

This makes it explicit that we have no backreaction, since the expectation value, and in particular the mean number of photons in each mode, $|z(\mathbf{k})|^2 = \langle z | a^\dagger(\mathbf{k})a(\mathbf{k}) | z \rangle$, are the same in both the initial and final states.

¹This requires a mild assumption on the momentum support of the “in” and “out” states which we explain below.

B. Beam depletion

We saw above that we can associate a background with asymptotic coherent states which are unaffected by scattering processes, i.e. both the in and out states have the same coherent piece. This gives us a natural manner in which to include backreaction on the background field; we allow the coherent state to change under the scattering process. The physical situation is illustrated in Fig. 2. Our initial state contains incoming electrons and positrons (not shown), incoming photons (in a number state), and a coherent state of photons representing a strong field. In a real scattering situation, the initial coherent and number states would occupy well separated volumes of momentum space; e.g. when colliding high energy photons with an optical laser pulse the photons and laser would typically have different injection angles, in order to collide, and would by definition be of different energies. (This is certainly the case for e.g. planned birefringence experiments based on laser-laser scattering [6].) Hence if $z_i(\mathbf{k})$ is the profile function of the coherent state and p the momentum of a photon in the number-state part, it is safe to assume that $z_i(\mathbf{p}) = 0$. This assumption also means that there is no ambiguity in the definition of the initial state, for then $[D(z_i), a^\dagger(\mathbf{p})] = 0$ and

$$D(z)a^\dagger(\mathbf{p}_1)\dots|0\rangle = a^\dagger(\mathbf{p}_1)\dots D(z)|0\rangle \equiv a^\dagger(\mathbf{p}_1)\dots|z\rangle, \quad (7)$$

which is normalized provided the number-state part is normalized.²

Consider now the final state of the scattering process. During the interaction photons are emitted from particles, or absorbed from the initial coherent state. We describe scattered photons as a number state, as usual, and encode absorption from the initial state as a change in the coherent state profile, including a coherent piece $z_f(\mathbf{k}) \neq z_i(\mathbf{k})$ in the final state. We again assume that the momentum support of the final coherent state profile $z_f(\mathbf{k})$ is disjoint from that of the *final* number state. We do not make any assumption about the relative support of the initial coherent state and the final number state, or vice versa. It is natural to imagine that $|z_f|^2 < |z_i|^2$ for some or all modes $\mathbf{k}$, representing losses from the initial field (although we will not need to make this assumption explicitly). In addition, the phase of the coherent state profile can be affected by the scattering process; the phase $\varphi_{\mathbf{k}} := \arg z(\mathbf{k})$ resembles the spectral phase of the laser pulse, a change in which implies dispersion of the laser and could strongly affect the pulse shape if certain frequencies were significantly “delayed.” This suggests an analogy with electrodynamics in continuous media, in which the number states act as the “medium” causing absorption and phase-shifts of the incident light.

²States in which the support of the number and coherent profiles overlap are known in the literature as “photon-added coherent states” or “Agarwal states” [56]. These states could be employed if the separation condition were violated.

FIG. 2. The physical situation of interest; an initial state containing matter (not shown), photons (in a number state) and a coherent state of photons (representing a strong field) scatters into a different state also comprising matter, emitted photons, and a coherent state of typically reduced amplitude representing depletion of the initial strong field.

We are now ready to calculate scattering amplitudes with the above initial and final states. For clarity we will work mostly with scalar fields; this allows us to present a derivation uncluttered by spin, polarization or gauge degrees of freedom. The extension of our final result to QED will though be straightforward. Hence let ϕ be a complex scalar representing either the scalar in sQED or the spinor in QED proper, coupled to a real scalar A representing the photon. The action is

$$S = \int d^4x \partial\phi^\dagger \cdot \partial\phi - m^2\phi^\dagger\phi + \frac{1}{2}\partial A \cdot \partial A - \frac{1}{2}\mu^2 A^2 + S_{\text{int}}[A, \phi, \phi^\dagger], \quad (8)$$

in which we include, for generality, a “photon” mass μ . Although the interaction terms are in principle arbitrary, for concreteness we will typically use the scalar Yukawa interaction

$$S_{\text{int}}[A, \phi, \phi^\dagger] = -e \int d^4x A \phi^\dagger \phi, \quad (9)$$

in order to mock up the three-point vertex of QED, see also [57]. (Note that e then has mass dimension one in $3+1$ dimensions, and would be dimensionless in $5+1$ dimensions.) The shortcomings of the scalar Yukawa theory will not concern us, as our initial interest is in establishing the formalism rather than extracting phenomenological results.

We require reduction formulae for scattering of particles between different coherent states of photons. The mode expansion of the photon field reads

$$A(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} (a^\dagger(\mathbf{k})e^{ik \cdot x} + a(\mathbf{k})e^{-ik \cdot x}),$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{k})] = \delta^3(\mathbf{p} - \mathbf{k}). \quad (10)$$

For clarity we write the S -matrix as $S[A] \equiv S[a, a^\dagger]$, making the photon creation and annihilation operators

explicit but suppressing momentum labels. Then our goal is to calculate transitions of the form

$$S_{fi} := \langle \text{out}; z_f | \mathcal{S}[a, a^\dagger] | \text{in}; z_i \rangle. \quad (11)$$

First we extract the coherent pieces from the initial and final states. We write out the coherent part of the initial state explicitly, and commute the exponential operator to the left using the translation property (4):

$$\begin{aligned} S_{fi} &= \langle \text{out}; z_f | \mathcal{S}[a, a^\dagger] e^{a^\dagger z_i} | \text{in} \rangle e^{-|z_i|^2/2} \\ &= \langle \text{out}; z_f | e^{a^\dagger z_i} \mathcal{S}[a + z_i, a^\dagger] | \text{in} \rangle e^{-|z_i|^2/2}. \end{aligned} \quad (12)$$

(For clarity we write out only a single mode; the extension to all modes is trivial.) Now we do the same for the final coherent state, passing it to the right of the S -matrix using the translation property and the standard results for the commutator of two displacement operators,

$$\begin{aligned} S_{fi} &= e^{-|z_f|^2/2} \langle \text{out} | e^{\bar{z}_f a} e^{a^\dagger z_i} \mathcal{S}[a + z_i, a^\dagger] | \text{in} \rangle e^{-|z_i|^2/2} \\ &= e^{-\frac{1}{2}(|z_f|^2 + |z_i|^2 - 2\bar{z}_f z_i)} \langle \text{out} | e^{a^\dagger z_i} e^{\bar{z}_f a} \mathcal{S}[a + z_i, a^\dagger] | \text{in} \rangle \\ &= e^{-\frac{1}{2}(|z_f|^2 + |z_i|^2 - 2\bar{z}_f z_i)} \langle \text{out} | e^{a^\dagger z_i} \mathcal{S}[a + z_i, a^\dagger + \bar{z}_f] e^{\bar{z}_f a} | \text{in} \rangle. \end{aligned} \quad (13)$$

The effect of the exponential operators on the initial and final states is to shift $a^\dagger(\mathbf{p}) \rightarrow a^\dagger(\mathbf{p}) + \bar{z}_f(\mathbf{p})$ for each $a^\dagger$ in the in-state, and $a(\mathbf{p}) \rightarrow a(\mathbf{p}) + z_i(\mathbf{p})$ for each a in the out-state. We write these states schematically as “ $|\text{in} + \bar{z}_f\rangle$ ” and “ $|\text{out} + z_i\rangle$.” Finally, because $\mathcal{S}[a, a^\dagger] \sim \mathcal{S}[a^\dagger e^{ik.x} + a e^{-ik.x}]$, we see that the photon field in the S -matrix is once again shifted, but this time by a complex-valued background field $A_D \sim \bar{z}_f e^{ik.x} + z_i e^{-ik.x}$ [see (16) below] for which $\bar{z}_f$ and z_i are the negative and positive frequency modes respectively. Our key result is then

$$S_{fi} = e^{-\frac{1}{2}(|z_f|^2 + |z_i|^2 - 2\bar{z}_f z_i)} \langle \text{out} + z_i | \mathcal{S}[A + A_D] | \text{in} + \bar{z}_f \rangle, \quad (14)$$

which shows that scattering between different initial and final coherent states is equivalent to scattering in a complex-valued background, up to shifts in the number part of the initial and final states, and the leading factor in (14) which can be recognized as the overlap of the asymptotic coherent states. In the remainder of this paper we will investigate the consequences of (14), which follows from basic properties of quantum mechanics.

Let us briefly comment on unitarity, which can be a subtle issue in relation to background field theories [58]. That asymptotic coherent states lead to complex backgrounds has also been observed in [25]. Complex potentials are commonly used as phenomenological models describing decay or absorption: complex optical potentials are an established way to describe inelastic scattering, as well as

the energies and widths of resonances, see for example [59–61], while imaginary potentials are used as a computational tool in numerical time-dependent Schrödinger equation (TDSE) simulations in order to absorb outgoing wave packets at the boundaries of the simulation box [62]. However, because we began with (11), which is a transition amplitude of the ordinary, unitary, S -matrix between normalized states, unitarity is preserved despite the appearance of a complex background. As a check, we set $z_f = z_i$; we then recover the background field relation (5). For this we need to assume no scattering into the beam, but this is consistent with the assumptions behind (14); if $z_i(p_{\text{in}}) = z_f(p_{\text{out}}) = 0$ then because $z_i = z_f$ there can be no overlap between any of the number states and coherent states.

C. Feynman rules

The Feynman rules corresponding to the result (14) are as follows.

- (1) Correlation functions are derived from the shifted action

$$\begin{aligned} S &= \int d^4x \partial \phi^\dagger \cdot \partial \phi - m^2 \phi^\dagger \phi + \frac{1}{2} \partial A \cdot \partial A - \frac{1}{2} \mu^2 A^2 \\ &\quad + S_{\text{int}}[A + A_D, \phi, \phi^\dagger], \end{aligned} \quad (15)$$

in which the photon field in the interaction term is shifted by the complex valued field A_D which encodes the chosen incoming (z_i) and outgoing (z_f) coherent state profiles via

$$A_D = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} (\bar{z}_f(\mathbf{k}) e^{ik.x} + z_i(\mathbf{k}) e^{-ik.x}). \quad (16)$$

- (2) Define the usual free asymptotic photon states (external legs) by

$$\varepsilon_p(x) := \frac{e^{-ip.x}}{\sqrt{(2\pi)^3 2\omega_p}}, \quad \omega_p := \sqrt{\mathbf{p}^2 + \mu^2}. \quad (17)$$

To convert correlation functions into S -matrix elements insert the following into the path integral: for an incoming photon of momentum p ,

$$\bar{z}_f(\mathbf{p}) + i \int d^4x \varepsilon_p(x) (\partial_x^2 + \mu^2) A(x), \quad (18)$$

and for an outgoing photon of momentum q ,

$$z_i(\mathbf{q}) + i \int d^4y \bar{\varepsilon}_q(y) (\partial_y^2 + \mu^2) A(y). \quad (19)$$

The second terms in these expressions describe the standard replacement of external propagators with the asymptotic wave functions (17) (“amputation”). The first terms in (18) and (19) correspond to the insertions in (14) in which e.g. a final state photon is sourced from the initial coherent state. Practically this means that for each insertion one has to calculate one additional Feynman diagram with one fewer photon leg, and multiply it by the appropriate coherent state profile function. We will show in Sec. IV that these terms can sometimes play a crucial role. Similar terms occur in the infrared safe LSZ reduction of charged particles and their radiation fields in [25].

- (3) Multiply all *mod-squared* S -matrix elements by the real exponential

$$\exp\left[-\int d^3\mathbf{k} |z_f(\mathbf{k}) - z_i(\mathbf{k})|^2\right], \quad (20)$$

as follows from (14), and which accounts for the overlap and normalization of the coherent states. [The imaginary part of the prefactor in (14) drops out at the level of the probability.] When z_f and z_i are very different from each other it seems that this exponential will lead to a suppression of any probability or cross section; however, it should be remembered that the rest of the amplitude depends in a highly nontrivial way on z_f and z_i , so it is not possible to draw such general conclusions. We will look at examples below.

This establishes our formalism and offers a potential framework in which to investigate backreaction and depletion effects. Aside from the additional factors in LSZ reduction, the only essential difference between calculations in the above formalism and ordinary background field calculations is that the background has become complex valued.

III. BASIC EXAMPLES

In this section we consider some simple illustrative examples of the above prescription. The emphasis is on establishing the applicability of the formalism, rather than on providing phenomenological results.

A. Strong depletion of a weak field

Consider pair production, momenta p and p' , by a photon of momentum k' . This process is forbidden by energy-momentum conservation in vacuum, but not if there is a background field present; see e.g. [34,63,64]. We will compare the background field (no depletion) amplitude, $z_f = z_i$, with the case that the field is completely depleted during the scattering (the initial coherent state is completely absorbed) so $z_f = 0$. We assume first that the field is weak,

so that an entirely perturbative treatment is reasonable. Hence, rather than adopt the Furry expansion, we work for the moment in ordinary perturbation theory, expanding in powers of e such that the coupling of the matter field to the complex field A_D is treated on the same footing as the coupling to the quantized photon field.

Consider then the background field calculation with $z_f = z_i$. The background field is

$$A_{\text{ext}}(x) = \int d^3\mathbf{k} (\bar{e}_k(x) \bar{z}_i(\mathbf{k}) + \varepsilon_k(x) z_i(\mathbf{k})) \in \mathbb{R}, \quad (21)$$

with $\varepsilon_k(x)$ as in (17). The lowest order contribution to the pair production amplitude $\mathcal{A}$ in this background is of order e^2 , and is given by a superposition of standard “two photon to pair” or “one photon to pair plus photon” tree level Feynman diagrams $\mathcal{A}_0$ in which the one photon leg with momentum k is attached to the background field A_{ext} :

$$\begin{aligned} \mathcal{A}(k' \rightarrow p, p') &= \int d^3\mathbf{k} z_i(\mathbf{k}) \mathcal{A}_0(k, k' \rightarrow p, p') \\ &\quad + \int d^3\mathbf{k} \bar{z}_i(\mathbf{k}) \mathcal{A}_0(k' \rightarrow p, p', k) \\ &= \int d^3\mathbf{k} z_i(\mathbf{k}) \begin{array}{c} \text{Diagram 1: } k' \text{ (wavy)} \rightarrow \text{vertex} \rightarrow \text{vertex} \rightarrow p, p' \end{array} \\ &\quad + \int d^3\mathbf{k} \bar{z}_i(\mathbf{k}) \begin{array}{c} \text{Diagram 2: } k' \text{ (wavy)} \rightarrow \text{vertex} \leftarrow \text{vertex} \leftarrow k \text{ (wavy)} \rightarrow p, p' \end{array}. \end{aligned} \quad (22)$$

The second diagram above vanishes by momentum conservation. Hence to this order we may freely replace the background field (21) by, cf. (16),

$$A_{\text{ext}}(x) \rightarrow A_D(x) = \int d^3\mathbf{k} \varepsilon_k(x) z_i(\mathbf{k}), \quad (23)$$

which is equal to the A_D we would use in the case of complete depletion, $z_f = 0$. It follows that, to lowest order in perturbation theory, the only difference between the amplitudes for no depletion and complete depletion is the inclusion of the prefactor (20). Then we have the simple result

$$\mathbb{P}_{\text{complete depl}}(k' \rightarrow p, p') \simeq \exp\left[-\int d^3\mathbf{k} |z_i(\mathbf{k})|^2\right] \mathbb{P}_{\text{background}}(k' \rightarrow p, p'). \quad (24)$$

This result says that as the photon density $\sim |z|^2$ in the initial state increases, the probability of completely depleting the field during a perturbative process falls. This makes intuitive sense; we would not expect a field containing many photons to be entirely depleted by a single pair production event. That we obtain a physically sensible first result from our approach is encouraging.

B. Diagrammatic expansion

In later examples we will go beyond perturbation theory and treat at least part of the field A_D exactly. In preparation for this we now look at the diagrammatic representation of depletion effects. Consider again a field which is fully depleted, so that A_D is again given by (23). Let $(A, \phi^\dagger \phi)$ represent the integral in the scalar Yukawa interaction (9). Then the path integral describing some chosen scattering amplitude depends on A_D through the term

$$e^{-ie(A_D, \phi^\dagger \phi)} = \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} (A_D, \phi^\dagger \phi)^n, \quad (25)$$

which, given that A_D in (23) has only positive frequency modes, describes a sum over all numbers of *incoming* photons (connecting to $\phi^\dagger \phi$ at vertices in the scattering amplitude). This is natural, as we have no photons in the outgoing coherent state. Now, we can rewrite A_D as the undepleted field (21) together with a correction,

$$A_D = A_{\text{ext}} - \delta A, \quad (26)$$

where δA has the form, comparing (21) and (23),

$$\delta A = \int d^3 \mathbf{k} \bar{\epsilon}_k(x) \bar{z}_i(\mathbf{k}), \quad (27)$$

and which describes only outgoing photons. With this (25) may be written as

$$\begin{aligned} e^{-ie(A_D, \phi^\dagger \phi)} &= e^{-ie(A_{\text{ext}}, \phi^\dagger \phi)} e^{ie(\delta A, \phi^\dagger \phi)} \\ &= e^{-ie(A_{\text{ext}}, \phi^\dagger \phi)} \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} (-\delta A, \phi^\dagger \phi)^n. \end{aligned} \quad (28)$$

This expression shows us how depletion effects look in the Furry expansion of a scattering process in which the

coupling to A_{ext} is strong; the presence of A_{ext} in the exponent here indicates that it is still treated exactly, as described in Sec. II A. Because this expansion is equal to (25), the series in δA describes the depletion of A_{ext} : these terms *subtract* from the background field process diagrams in which (all numbers of) photons are emitted into the final state [since δA has only negative frequency modes, see (27)]. These diagrams cancel contributions which are included by A_{ext} , which contains both positive and negative frequency modes. It is for this reason that we have included a negative sign in the definition of δA ; it emphasizes that something is being removed. This is illustrated in Fig. 3.

The series expansion (28) will look very similar if we consider the more general case of $z_f \neq 0$, with the only difference being that in this case we need to make the replacement, in (27) and (28), $\bar{z}_i \rightarrow \bar{z}_i - \bar{z}_f$ in δA . We will elaborate on this in the following section.

C. Weak depletion of a strong field

Having understood the diagrammatic appearance of depletion effects, we consider the example of a strong field which is only weakly depleted during some given process (the opposite of the situation in Sec. III A). We take as an ansatz for the depleted field

$$z_f(\mathbf{k}) = (1 - \delta(\mathbf{k})) z_i(\mathbf{k}), \quad (29)$$

in which $\delta(\mathbf{k}) \gtrsim 0$ is to describe a small reduction in amplitude of the final coherent state relative to the initial. We work to first order in δ . In the path integral for the scattering amplitude we then expand the action as in (28),

$$\begin{aligned} \exp(iS) &\rightarrow \exp(iS_{\text{ext}}) \left(1 + ie \int d^4 x \phi^\dagger \phi \int d^3 \mathbf{k} \bar{\epsilon}_k(x) \delta(\mathbf{k}) \bar{z}_i(\mathbf{k}) \right). \end{aligned} \quad (30)$$

Here S_{ext} is the background field action in which the matter fields are, as well as being coupled to the dynamical photon field, coupled to a strong background (21). This is treated in the Furry picture, as in Fig. 1, since we assume the coupling $\sim eA/m > 1$. The effect of the $\mathcal{O}(\delta)$ term in (30) on the path integral is to insert into any process an additional (consistently normalized) outgoing photon line

FIG. 3. Illustrative relation between two descriptions of the same process, in which an initial coherent state z_i is completely depleted. On the left, the coherent state is expanded as a sum of incoming photons, none of which survive to the final state. On the right, the same expression is rewritten in terms of an infinite series of corrections to the background field amplitude, in which the corrections remove background field diagrams with photons in the final state.

of momentum $\mathbf{k}$, and then to integrate this against $\delta(\mathbf{k})\bar{z}_i(\mathbf{k})$ which hence appears as a wave packet.

Using the rules in Sec. II C we construct the probability of some process $i \rightarrow f$ including the scattering of photons with momenta l_j to momenta l'_j . The insertions of $z_i(l')$ and $\bar{z}_f(l)$ in (18) and (19) may be dropped because e.g. $z_f(l) \propto z_i(l) = 0$ for all l_μ by assumption. The expansion (30) then gives us the scattering probability to lowest order in δ in terms of the Furry picture S -matrix element for the process in the *background* field (21), call this S_i^f , as

$$\mathbb{P} = e^{-\delta^2 \int |z_i|^2} \int d p_f \times \left| S_i^f - \int d^3 \mathbf{k} \delta(\mathbf{k}) \bar{z}_i(\mathbf{k}) S_i^{f,k} + \text{c.c.} + \mathcal{O}(\delta^2) \right|^2, \quad (31)$$

where $S_i^{f,k}$ is the S -matrix element for the same process as in S_i^f but with the emission of an additional photon into the final state. As illustrated in Fig. 3, this diagram describes depletion of the background field A_{ext} . To lowest order in δ , the exponential prefactor does not contribute, and the probability becomes

$$\mathbb{P} = \int d p_f |S_i^f|^2 - \int d p_f \int d^3 \mathbf{k} (\delta(\mathbf{k}) \bar{z}_i(\mathbf{k}) S_i^{f,k} \overline{S_i^f} + \text{c.c.}) + \mathcal{O}(\delta^2). \quad (32)$$

We thus see that a reduction in amplitude of the coherent state corresponds, at the level of the probability or cross section, to cross terms of different photon-number processes; one is the background field amplitude, the other is the amplitude for the same process but with an additional emission, integrated over the profile of the background field. This is in agreement with the discussion in Sec. III B and with Fig. (3). To be explicit, consider again pair production by a photon of momentum k' . Writing a double line for the matter propagator in the background (21) as in Fig. 1, and adopting the scalar-Yukawa interaction (9) we have

The diagram for S_i^f shows an incoming wavy line with momentum k' meeting a vertex. From this vertex, two double lines (representing matter propagators) emerge with momenta p and p' . The diagram for $S_i^{f,k}$ shows the same vertex and outgoing double lines, but with an additional wavy line (representing a photon) emerging with momentum k . The two diagrams are summed.

$$S_i^f = \text{diagram with } k' \text{ and } p, p' \quad (33)$$

$$S_i^{f,k} = \text{diagram with } k', k, p, p' \quad (33)$$

We comment that the inclusion of additional diagrams giving higher-order photon emission is reminiscent of the IR, though the interpretation here is different. In our case

the additional diagrams subtract photon emission contributions from the background field amplitude in order to describe a depletion of that field.

D. Strong depletion of a strong field

In this section we investigate another consequence of the result (14), which is the possibility of treating the complex field A_D exactly, in a modified Furry expansion. Consider the action (15) together with the Scalar Yukawa interaction (9). Following the background field calculations in the literature, see Sec. II A, we ask whether it is possible to use a Furry picture expansion of a general correlation function in which the matter field propagator G_ϕ obeys

$$G_\phi^{-1} = \epsilon + i(\partial^2 + m^2 + eA_D). \quad (34)$$

If G_ϕ can be found explicitly then the coupling to both the initial and final coherent states can be treated without approximation. In this approach particles propagate in the complex background field A_D while the interaction (the three-point vertex) between the matter field and the quantized photon field is treated in perturbation theory as usual. The Feynman rules are illustrated in Fig. 4 and are precisely as for the usual Furry picture in Fig. 1 except that the background field in the matter propagator is complex. Applying LSZ amputation [analogous to (18) and (19) but without the additional terms] to external G_ϕ lines gives the asymptotic particle wave functions.

We stress that this is only an initial investigation, as there are some conceptual issues to be addressed. These are best illustrated by example. Assume that both the initial and final coherent states are plane waves depending on $n.x$ for $n^2 = 0$. With this we can write down G_ϕ in (34) exactly

$$G_\phi(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{\epsilon - i(p^2 - m^2)} \times \exp \left[-ip \cdot (x - y) - \frac{i}{2n \cdot p} \int_{n \cdot y}^{n \cdot x} eA_D \right], \quad (35)$$

which we recognize as being very similar to that in (scalar) QED; see e.g. [5,30,65] and references therein. Applying the standard analogues of (18) and (19) to an external leg of a correlation function transforms the propagator into the following “asymptotic wave functions” analogous to the

FIG. 4. Position-space Feynman rules in the Furry picture; the interaction between the matter fields and the coherent states (described by A_D) is accounted for exactly by using the propagator G_ϕ in (34).

Volkov solutions in QED, albeit with a complex classical field:

$$\begin{aligned}\sqrt{(2\pi)^3 n \cdot p} e_{\text{out}}^-(x) &= \int d^4 y e^{ip \cdot y} (\epsilon + i(\partial_y^2 + m^2)) G_\phi(y, x) \\ &= \exp \left[ip \cdot x - \frac{i}{2n \cdot p} \int_{n \cdot x}^\infty e A_D \right], \\ \sqrt{(2\pi)^3 n \cdot p} e_{\text{in}}^-(x) &= \int d^4 y e^{-ip \cdot y} (\epsilon + i(\partial_y^2 + m^2)) G_\phi(x, y) \\ &= \exp \left[-ip \cdot x - \frac{i}{2n \cdot p} \int_{-\infty}^{n \cdot x} e A_D \right],\end{aligned}\quad (36)$$

and similarly for e^+ . These functions obey the Klein-Gordon equation in the background A_D . Their interpretation as one-particle wave functions is though unclear. The physical meaning of a particle propagating in a complex potential is not obvious, and the wave functions are not normalizable due to the imaginary part of A_D . Note also $\bar{A}_D$ does not appear, because the S -matrix depends only on A_D . At the level of the probability, of course, $\bar{A}_D$ reappears through the conjugate S -matrix element.

It seems clear though that a perturbative solution of (34), resummed to all orders, would yield precisely (35) and from there (36), just as it does for real plane waves. Thus we could attempt to perform scattering calculations in this Furry-like picture. Due to the assumption of plane wave coherent profiles there would be some divergent factors to assign a meaning to in e.g. (20) corresponding to longitudinal and transverse volumes and numerical integration might be made harder by the presence of real exponentials rather than phases. For these reasons we defer a full calculation in QED proper to future research, though we will briefly look at this plane wave model again, below.

IV. EXTREMIZATION

In Sec. II B we introduced the final state profile z_f as a new degree of freedom describing the depletion of the laser. In Sec. III we fixed z_f in order to model a chosen level of depletion, and explored how scattering probabilities depended on this. Now, in background field calculations there are two couplings, one between the quantized fields and one to the background field $\sim e A_{\text{ext}}/m$. The inclusion of depletion effects introduces (at least) a third parameter, the amount of depletion, which may be reinterpreted as the coupling to $A_D - A_{\text{ext}}$. In fixing z_f as we did above, we made both explicit and implicit assumptions on the relative strengths of the three couplings.

Here we take a different view, and instead try to determine the (most likely) level of depletion in a given process using an extremization principle, much like the Hamiltonian action principle in classical mechanics (where the classical path is obtained by extremizing the action) or the principle of extremization of entropy. We do not imply

that there is a strict analogy, noting only that extremization principles are useful in many areas of physics [66].

We will fix the initial coherent z_i and the scattered particle content, as above. We will then maximize the probability of a process with respect to the hitherto unknown z_f . The determined value of z_f then represents the physically relevant “most likely” level of depletion. (In doing this we eliminate the dependency on a “third coupling.”) The eventual goal is to apply this to QED processes and learn how depletion affects observables such as final state particle spectra. Our aim here is more modest; we only wish to investigate the feasibility of an extremization approach. We begin with a simple, but exactly solvable, model.

A. Exactly soluble toy model

Consider a real scalar field $A(x)$ (the “photon” from the scalar Yukawa model) coupled to a given (fixed) external source $J(x)$; this is a well-known exactly solvable theory; see e.g. [67] or [[68], Sec. 3]. The action is, including the complex field A_D describing the asymptotic coherent states of the A -field, cf. (15),

$$S = \frac{1}{2} \int d^4 x \partial A \cdot \partial A + \int d^4 x J(A + A_D), \quad (37)$$

with A_D as in (21). Since the scalar field couples linearly to the source the presence of A_D in the action leads only to a further exponential prefactor in front of S -matrix elements. There are however the insertions of (18) and (19) to account for. Consider then the simple scattering process shown in the left-hand diagram of Fig. 5, in which an incoming photon of momentum k'_μ is absorbed by the source J . Let the initial state contain (as well as the coherent state z_i accounted for in A_D) a single photon in a wave packet ψ ,

$$\begin{aligned}|\psi\rangle &= \int d^3 \mathbf{k} \psi(\mathbf{k}) a^\dagger(\mathbf{k}) |0\rangle, \\ \langle \psi | \psi \rangle &= \int d^3 \mathbf{k} |\psi(\mathbf{k})|^2 = 1.\end{aligned}\quad (38)$$

We assume as before that the wave packet ψ is disjoint from z_i in momentum space. The probability that the photon is absorbed by the source, together with a transition

FIG. 5. On the left, a photon is absorbed by the external source $J(x)$. This source mocks up the matter current on the right, in which a (scalar) photon produces a (scalar) pair in the Yukawa theory. The triple line indicates that the coherent states can in principle be accounted for exactly in the modified Furry picture discussed in Sec. III D. (The blob represents loop corrections.)

from the coherent state z_i to the coherent state z_f is now easily calculated [67,68]. The total photon absorption probability is, without approximation,

$$\mathbb{P}(z_f) = e^{-\int d^3\mathbf{k} |z_f - z_i - iJ|^2} \left| \int d^3\mathbf{k} \psi(\mathbf{k}) [\bar{z}_f(\mathbf{k}) + i\bar{J}(\mathbf{k})] \right|^2, \quad (39)$$

in which the insertion of $\bar{z}_f$ arises from the additional LSZ term in (18), and where we have defined the on-shell Fourier transform of J ,

$$J(\mathbf{k}) := \int d^4x \bar{\epsilon}_k(x) J(x) \Leftrightarrow \bar{J}(\mathbf{k}) := \int d^4x \epsilon_k(x) J(x). \quad (40)$$

The coherent-state normalizations and the contribution of A_D from the action (and from the conjugate S -matrix element) combine into a single exponential. From this we can identify that final z_f which maximizes the probability by extremizing (39) with respect to $\bar{z}_f(\mathbf{p})$. This yields

$$\begin{aligned} \frac{\delta \mathbb{P}}{\delta \bar{z}_f(\mathbf{k})} = 0 &\Rightarrow z_f(\mathbf{k}) = iJ(\mathbf{k}) \text{ or} \\ (z_f(\mathbf{p}) - z_i(\mathbf{p}) - iJ(\mathbf{p})) \int d^3\mathbf{k} \psi(\mathbf{k}) [\bar{z}_f(\mathbf{k}) + i\bar{J}(\mathbf{k})] &= \psi(\mathbf{p}). \end{aligned} \quad (41)$$

The first condition minimizes the probability, setting it to zero. The solution to the second condition is (remembering that there is no overlap between z_i and ψ),

$$z_f(\mathbf{k}) = z_i(\mathbf{k}) + iJ(\mathbf{k}) + e^{i\theta} \psi(\mathbf{k}), \quad (42)$$

in which θ is an arbitrary, but constant, phase, implying a possible spectral phase shift in the final coherent state profile. (Recall the discussion above.) The iJ -term simply describes (as we will make explicit below) the standard instability of the empty vacuum state in the presence of the external source J [67]. The ψ term comes from the LSZ insertion in (18) and means, interestingly, that the support of the final coherent state profile overlaps with that of the *initial* number state. If we associate J with some coupling, then (42) shows us that the final coherent state is corrected not only at first order in the coupling, due to the interaction, but also at zeroth order through the initial state ψ . Inserting the solution (42) into (39) gives a maximal probability of

$$\mathbb{P} \rightarrow \mathbb{P}_\theta = e^{-\int d^3\mathbf{k} |\psi|^2} \left| \int d^3\mathbf{k} |\psi|^2 \right|^2 = e^{-1}, \quad (43)$$

independent of θ . Despite the simplicity of the theory, this example illustrates that it is possible to maximize the probability with respect to the final coherent state profile.

The implied choice of final state depends both on the interaction and on the properties of the initial state.

Note that we have not insisted that the final coherent state amplitude be reduced with respect to the initial; the final state which maximizes the probability could contain, as well as depletion in the form of absorption from the initial coherent state, coherent emission into the final state. Indeed, the assumption that the initial number and coherent state parts are disjoint in momentum space allows us to rewrite (42) as

$$z_f(\mathbf{k}) = \begin{cases} z_i(\mathbf{k}) + iJ(\mathbf{k}) & \text{when } z_i(\mathbf{k}) \neq 0, \\ e^{i\theta} \psi(\mathbf{k}) + iJ(\mathbf{k}) & \text{when } z_i(\mathbf{k}) = 0, \end{cases} \quad (44)$$

which means that populated initial number state modes [$\mathbf{k}$ for $\psi(\mathbf{k}) \neq 0$] become populated final coherent state modes [$z_f(\mathbf{k}) \neq 0$ for such $\mathbf{k}$]. (There is also a population of final coherent state modes due to J , which is particular to this model.) It seems clear that there can be physical situations in which emission is more significant than absorption, and vice versa. The expectation value of the number of photons in the final state, N_f , is given by

$$\begin{aligned} N_f &= \int d^3\mathbf{k} |z_f(\mathbf{k})|^2 \\ &= 1 + N_i + \int d^3\mathbf{k} |J(\mathbf{k})|^2 + \int d^3\mathbf{k} iJ(\mathbf{k}) (\bar{z}_i(\mathbf{k}) \\ &\quad + e^{-i\theta} \bar{\psi}(\mathbf{k})) - i\bar{J}(\mathbf{k}) (z_i(\mathbf{k}) + e^{i\theta} \psi(\mathbf{k})), \end{aligned} \quad (45)$$

where, in contrast to the probability above, the phase survives. If we average over the phase then we can write

$$\int_0^{2\pi} \frac{d\theta}{2\pi} N_f = 1 + \int d^3\mathbf{k} |z_i(\mathbf{k}) + iJ(\mathbf{k})|^2. \quad (46)$$

Whether the final state corresponds to an overall reduction or increase in the average number of photons therefore depends on the distance (in the complex plane) between the initial state coherent z and the source $J(x)$.

We comment that the expectation value of z_f in the final state is

$$\langle z_f(\mathbf{k}) \rangle = \int \mathcal{D}z_f z_f(\mathbf{k}) \mathbb{P}(z_f) = z_i(\mathbf{k}) + iJ(\mathbf{k}), \quad (47)$$

from which it follows that the expectation value of $A(x)$ in the asymptotic future is (remembering to convert back from the interaction picture)

$$\begin{aligned} \langle A(x) \rangle &= A_i(x) + i \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} (e^{-ik \cdot (x-y)} - e^{ik \cdot (x-y)}) J(y) \\ &= A_i(x) + \int d^4y G_{\text{rad}}(x-y) J(y), \end{aligned} \quad (48)$$

in which $G_{\text{rad}} = G_{\text{ret}} - G_{\text{adv}}$ gives the radiation field [25]. This comes entirely from the iJ term in (42), i.e. is independent of ψ , and is the same as the result obtained if starting from vacuum. This is independent of the initial number state ψ only due to the simplicity of the model, as coherent states evolve directly to coherent states when the only interaction is with an external source [67,68]. In general we would expect that if, for example, large numbers of particles were present then interactions, and hence absorption, would be more likely. Such a dependence seems to be reflected in (42).

B. Pair production in scalar Yukawa theory

We return to scalar Yukawa and consider again stimulated pair production by an initial photon. This is the natural generalization of the absorption process in the toy model above; see Fig. 5. Let the initial state be a single photon with wave packet $\psi(\mathbf{k}')$, while the final state contains a pair of matter particles with momenta p and p' . Write $\text{Amp}_{\mathbf{k}'}[A]$ for the amputation (18), and define $\text{Amp}_{\mathbf{p}}[\phi]$ similarly for the matter fields but without additional insertions corresponding to z_i or z_f as the matter states contain no coherent pieces. Then the probability of stimulated pair production is

$$\begin{aligned} \mathbb{P} &= \int dp_f e^{-\int dk |z_f - z_i|^2} |\mathcal{M}|^2, \\ \mathcal{M} &= \int d^3\mathbf{k}' \psi(\mathbf{k}') \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \mathcal{D}A \text{Amp}_{\mathbf{k}'}[A] \text{Amp}_{\mathbf{p}}[\phi^\dagger] \\ &\quad \times \text{Amp}_{\mathbf{p}'}[\phi] e^{iS[\phi, \phi^\dagger, A, A_D]}, \end{aligned} \quad (49)$$

in which $dp_f = d^3\mathbf{p}' d^3\mathbf{p}$ is the final state integral over outgoing pair momenta.

Note that the exponential factor containing the difference of asymptotic coherent state profiles stands under the phase space integral. The reason is the following. Scattering to a final state with disjoint number and coherent pieces has been shown in [24] to be equivalent, at the level of the probability, to an *inclusive* process in which one sums over all numbers of photons emitted into a “forward” direction given by the support of the coherent state. With this in mind, it makes sense for the level of depletion in each mode of $z(\mathbf{k})$ to be allowed to depend on the momenta of the final state particles, and for this reason the exponential stands under the final state integrals.

Differentiating with respect to $\bar{z}_f(\mathbf{k})$ we obtain

$$\begin{aligned} \frac{\delta \mathbb{P}}{\delta \bar{z}_f(\mathbf{k})} &= \int dp_f e^{-\int dk |z_f - z_i|^2} \left(\bar{\mathcal{M}} \frac{\delta \mathcal{M}}{\delta \bar{z}_f(\mathbf{k})} - (z_f(\mathbf{k}) - z_i(\mathbf{k})) |\mathcal{M}|^2 \right). \end{aligned} \quad (50)$$

Demanding that this expression be extremized for each momentum mode requires us to solve the deceptively simple equation

$$z_f(\mathbf{k}) = z_i(\mathbf{k}) + \frac{1}{\mathcal{M}} \frac{\delta \mathcal{M}}{\delta \bar{z}_f(\mathbf{k})}, \quad (51)$$

which generalizes (41). Here, however, (51) is a complicated, highly nonlinear equation for the final coherent state profile z_f , since the matrix element $\mathcal{M}$ depends on z_f through the complex background A_D . Clearly we cannot solve this new equation exactly, so the simplicity of the extremal results (42) and (43) will not extend to this system. We will though still be able to write down a series solution to (51). The extension to QED will also be apparent.

We write $A_D = A_{\text{ext}} - \delta A$ as in (26), and consider the situation where the coupling to the background field A_{ext} is strong, but where the coupling to the quantized field A is treated in perturbation theory, along with the fluctuation δA , see (28). In other words we work in the ordinary Furry picture and assume weak depletion. Before proceeding to solve (51) in this expansion, let us establish some notation. Define the two background field amplitudes

$$\begin{aligned} \mathcal{M}_{\psi}^{pp'} &= \int d\mathbf{k}' \psi(\mathbf{k}') \quad \text{diagram: } k' \text{ wavy line splitting into } p \text{ and } p' \text{ solid lines} \\ \mathcal{M}_{\psi}^{pp'k} &= \int d\mathbf{k}' \psi(\mathbf{k}') \quad \text{diagram: } k' \text{ wavy line splitting into } p \text{ and } p' \text{ solid lines, with } k \text{ wavy line attached to the } p \text{ line} \end{aligned} \quad (52)$$

which differ from the diagrams considered earlier only in the inclusion of the initial wave packet. It is clear why the first is relevant; the second arises here because the effect of the z_f -derivative in (51) is to insert into the path integral defining $\mathcal{M}$ a term

$$-ie \int d^4x \bar{\phi} \phi \bar{e}_k(x), \quad (53)$$

which just attaches a new, correctly normalized, outgoing photon line to our process; we saw such terms previously in (30). Since depletion is here being caused by an $\mathcal{O}(e)$ process we make the ansatz $z_f = z_i + \delta z_0 + e\delta z + \dots$ and neglect higher powers in e . The zeroth order correction here is suggested by the exactly solvable model above. Things simplify somewhat if we assume that the background field A_{ext} is incapable of spontaneous (i.e. non-perturbative, or Schwinger) pair production. In that case the LSZ insertion $\bar{z}_f$ in $\text{Amp}[A]$ does not contribute, and further one finds that $\delta z_0 = 0$. Thus we have $z_f = z_i + e\delta z$, and one finds

$$e\delta z(\mathbf{k}) = \frac{\mathcal{M}_{\psi}^{pp'k}}{\mathcal{M}_{\psi}^{pp'}}, \quad (54)$$

where both the background field amplitudes from (52) appear. This demonstrates that the extremization principle can be successfully applied to scalar Yukawa theory. One can proceed to higher orders.

We remark briefly that one could also look for an “overall” level of depletion which depends only on integrated emission rates, not on the differential rates. Here one returns to (49) but places the exponential outside of the final-state integrations. Then extremising with respect to $\bar{z}_f(\mathbf{k})$ yields the equation

$$(z_f(\mathbf{k}) - z_i(\mathbf{k})) \int dp_f |\mathcal{M}|^2 = \int dp_f \frac{\delta \mathcal{M}}{\delta z_f(\mathbf{k})} \bar{\mathcal{M}}. \quad (55)$$

Proceeding as above, the zeroth order correction vanishes while the first nontrivial correction, $e\delta z_1$ is

$$e\delta z_1(\mathbf{k}) = \frac{\int dp_f \overline{\mathcal{M}_{\psi}^{pp'}} \mathcal{M}_{\psi}^{pp'k}}{\int dp_f |\mathcal{M}_{\psi}^{pp'}|^2}. \quad (56)$$

Here we see the same cross terms arise as in earlier sections.

Thus we have shown that there are (at least) two extremization approaches which can be used to identify “most likely” final coherent states, depending on whether one focuses on unintegrated or integrated observables. We observe that the corresponding results and (54) and (56) are related as follows. Let $\mathbb{P}_0$ be the external field ($z_f \rightarrow z_i$) limit of the probability (49), so that

$$\mathbb{P}_0 = \int dp_f |\mathcal{M}_{\psi}^{pp'}|^2. \quad (57)$$

Then starting from (56) we can write

$$\begin{aligned} e\delta z_1(\mathbf{k}) &= \frac{1}{\mathbb{P}_0} \int dp_f \overline{\mathcal{M}_{\psi}^{pp'}} \mathcal{M}_{\psi}^{pp'k} \\ &= \frac{1}{\mathbb{P}_0} \int dp_f \overline{\mathcal{M}_{\psi}^{pp'}} \mathcal{M}_{\psi}^{pp'} \frac{\mathcal{M}_{\psi}^{pp'k}}{\mathcal{M}_{\psi}^{pp'}} \\ &= \int dp_f \frac{1}{\mathbb{P}_0} \frac{d\mathbb{P}_0}{dp_f} e\delta z(\mathbf{k}). \end{aligned} \quad (58)$$

This gives an interpretation of the integrated result, $e\delta z_1$, as describing “inclusive depletion” of the initial coherent mode $z(\mathbf{k})$, in which we integrate $e\delta z$ over the final state phase space, i.e. we integrate out the scattered particles. The weight for this integral is the normalized, background field, differential scattering probability ($1/\mathbb{P}_0$)($d\mathbb{P}_0/dp_f$). That this properly normalized distribution appears naturally in the integrated depletion result (56) suggests that we are correctly capturing the infrared physics.

C. Depletion of a plane wave

Finally, let us take another step toward including depletion effects in the strong field QED calculations of the literature, by considering the case that the initial coherent state is a plane wave, with support only on momentum modes in a single (lightlike) direction n_μ . We have seen that the first correction to the initial state profile involves the amplitude $\mathcal{M}_{\psi}^{pp'k}$, in which a photon is emitted along with the created pair. Overall momentum conservation for this amplitude takes the form (with the same momentum assignments as immediately above)

$$k'_\mu + \omega n_\mu = p_\mu + p'_\mu + k_\mu, \quad (59)$$

where ω is some frequency scale determined by the mass-shell condition. The point to make is that there are solutions in which the emitted photons can have momenta $k_\mu \not\propto n_\mu$ and hence the final coherent state profile $z_f = z_i + e\delta z$ does not necessarily depend only on $n.x$, i.e. is not necessarily a plane wave. This shows that including higher order depletion effects will ultimately also require going beyond plane waves models of the background, which is another challenging problem [69,70].

Out of interest, though, to see what kind of structures may arise in a more complete investigation, we impose the assumption of Sec. III D that both the initial and final coherent states are plane waves, characterized by the lightlike direction n_μ . This allows us to make a little more analytic progress. Because both the initial and final state profiles depend on the same lightlike direction only, the variation of $\mathcal{M}$ with regard to $\bar{z}_f$ simplifies, and reduces to the variation of $\mathcal{M}_{\psi}^{pp'}$ itself. If we write $z(\mathbf{k}) = \delta^2(\mathbf{k}_\perp) \theta(k_3) \zeta(k_3)$ for each coherent state profile, then the equation to solve for the first correction to ζ becomes

$$e\delta \zeta(k) V_\perp = \frac{1}{\mathcal{M}_{\psi}^{pp'}} \frac{\delta \mathcal{M}_{\psi}^{pp'}}{\delta \bar{\zeta}(k)} \Big|_{\zeta=\zeta_i}, \quad (60)$$

where the volume factor arises from differentiating the exponential in (49). So, we first calculate the matrix element using the Volkov-like states (36) with a background field A_{ext} . As is usual in plane wave calculations, momentum is conserved in three directions which, in this case, allows us to perform the three integrals over the wave packet momenta k' . Let us collect the resulting wave packet factor together with all normalization factors (e.g. factors of $\sqrt{2\omega}$ and so on), into a single factor $\mathcal{C}$. The nontrivial part of $\mathcal{M}_{\psi}^{pp'}$ is an integral over the lightlike direction $n.x \equiv x^+$ on which the external field depends,

$$\mathcal{M}_{\psi}^{pp'} = \mathcal{C} \int dx^+ \exp \left[iK_+ x^+ - i \left(\frac{e}{2n.p} + \frac{e}{2n.p'} \right) \int_{x^+}^{\infty} ds A_{\text{ext}}(s) \right] \quad (61)$$

$$\equiv \mathcal{C} \mathcal{J}(K_+), \quad (62)$$

in which we have defined $K_+ = p_+ + p'_+ - k'_+$ (all evaluated on-shell), and $\mathcal{J}$ is a “transition current” which may be thought of as a generalization of the classical source (J) above. Taking the derivative in (60), the final result becomes (observing that for our chosen plane wave $k_+ = k_3$ and so $k_\mu = k_+ n_\mu$),

$$e\delta\zeta(k_+) = \frac{1}{V_\perp} \left(\frac{e}{2k.p} + \frac{e}{2k.p'} \right) \frac{1}{\sqrt{(2\pi)^3 2k_+}} \frac{\mathcal{J}(K_+ + k_+)}{\mathcal{J}(K_+)}. \quad (63)$$

We see explicitly from (63) that the change in laser mode k_+ depends also on the momenta of the final number state particles, through K_+ . The terms in round brackets follow from taking the functional derivative and correspond, recall (52), to the emission of the additional photon from either the final electron or positron line. Interestingly, these terms are strongly reminiscent of infrared factors in QED. Although the interpretation may be different, it seems that the appearance of depletion effects in QED may have a lot in common with infrared structures and higher-order processes. While the infrared is often assumed to be well understood, there is still a

great deal of interesting physics to explore there; see e.g. [71–74] and references therein. For example, the asymptotic behavior of particles in plane waves and their associated IR divergences [65,75] has a reinterpretation as an electromagnetic analogue of gravitational memory effects [76,77].

V. DISCUSSION AND CONCLUSIONS

A. Extension to QED

The extension of our basic formalism to QED is straightforward. One needs to keep track of signs coming from the metric and vector structure of the photon field; as a result the prefactor multiplying mod-squared S -matrix elements becomes, in say Lorenz gauge $\partial_\mu A^\mu = 0$,

$$\exp \left[+ \int d^3\mathbf{k} (z_f - z_i)_\mu (\bar{z}_f - \bar{z}_i)^\mu \right]. \quad (64)$$

[This prefactor can be written in terms of the difference of two potentials, $A_{\text{out}}^\mu(x) - A_{\text{in}}^\mu(x)$, and is therefore gauge invariant.] The QED action itself becomes

$$S = \int d^4x \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e\bar{\psi}A\psi + \text{gauge fixing} + \text{counterterms},$$

$$\mathcal{D}^\mu := \partial_\mu + ieA_D^\mu(x), \quad (65)$$

in which A_D^μ is now the complex valued field

$$A_D^\mu = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} (\bar{z}_f^\mu(\mathbf{k}) e^{ik.x} + z_i^\mu(\mathbf{k}) e^{-ik.x}). \quad (66)$$

As a first investigation in QED it would seem natural to follow the ideas of Secs. III C and IV C in which we perturb around external field calculations assuming the plane wave model, as that is a well-established formalism. This will be investigated elsewhere.

B. Conclusions

Within the framework of “quantum field theory in background fields” one can identify (at least) two primary theoretical challenges. The first is to accurately account for realistic, typically complicated, background field profiles in the strong field regime. The second is to go beyond the

background field model itself, i.e. beyond the assumption that the background is unaffected by backreaction in the form of, for example, depletion.

In this paper we have put forward a framework in which backreaction can be accounted for using a minimal extension of the background field formalism; essentially the only change is that the background field is allowed to become complex, which corresponds to considering scattering between different initial and final coherent states. In contrast to phenomenological models, though, unitarity is not violated. Further, our approach is exact in the sense that no approximation of the basic theory is needed, which may be an advantage regarding renormalization [78].

Rules such as (18) and (19) mean that additional (but lower order) diagrams must be calculated in order to completely describe depletion effects. However this is rather a (usually neglected) feature of strong field effects

than a drawback of the method. Indeed diagrams which are usually dropped in ordinary perturbation theory can contain rich physics in background fields; the *disconnected* product of two tadpole diagrams, for example, contributes non-trivially to the two-loop two-point function in a background, as it can describe absorption by, and spontaneous emission from, that background. Such diagrams have been the focus of some attention recently; see [79] and [80,81] for related results.

We have given several elementary applications of our method which account for both weak and strong back-reaction effects. We have seen that depletion is described, at the level of the probability, by cross terms between scattering amplitudes with different numbers of final state particles. We have also introduced an extremization principle in order to determine the “most likely” final coherent state profile given an initial state.

Another approach to depletion is considered in [19–22]. One begins with the Dirac equation in a background, monochromatic (single mode) plane wave. Replacing that field with a single quantized mode the Dirac equation can still be solved exactly. This yields “single-electron wave functions” describing the interaction of an electron with a single photonic mode. One can in principle use these states to model depletion of a monochromatic (laser) field. The photon coherence which appears naturally in that approach certainly suggests a connection with our methods, but there are also differences. We saw in Sec. IV C, for example, that a depleted plane wave is typically no longer plane. Thus a detailed comparison of our results and the first-quantized literature approach will be addressed elsewhere [82].

An interesting challenge for future research is to extend our approach to background fields which are sourced; the immediate problem to overcome there is that the field will then have off-shell Fourier modes, so that it cannot be so

simply extracted from the S -matrix using the displacement operators $D(z)$. It would also be interesting to investigate the classical limit of our methods. How does the appearance of a complex potential relate to the dynamical change in a field interacting with a classical particle, as could be solved for using the classical Maxwell and Lorentz equations?

We have seen several features reminiscent of infrared structures. As described above, scattering to a coherent state can be regarded as an inclusive process [24]. It would be interesting to investigate the infrared structure of our coherent state prescription in QED proper, with massless photons, especially in light of e.g. [25] where complex backgrounds arise to account for the asymptotic dressing [71] of particles due to infrared effects.

While modern terrestrial experiments may not yet typically probe physical regimes in which beam depletion is significant, this situation will change [13,83]. The next step is therefore to apply the approach described above to phenomenological processes of interest. Consider for example multiple scattering events, such as photon emission or pair production; including depletion due to the first event will impact the second scattering event as the “background” field structure will have changed, and so on. Such questions will be pursued elsewhere.

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Correction: The license statement “Funded by SCOAP³” was missing on the first page of the article and has been inserted.